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CRESE
30, avenue de l'Observatoire
25009 Besançon
France
http://crese.univ-fcomte.fr/

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The proportional Shapley value and an application[☆]

Sylvain Béal[a,*], Sylvain Ferrières[a,b], Eric Rémila[c], Philippe Solal[c]

[a] *CRESE EA3190, Univ. Bourgogne Franche-Comté, F-25000 Besançon, France*
[b] *Chair of Economics and Information Systems, HHL Leipzig Graduate School of Management, Jahnallee 59, 04109 Leipzig, Germany*
[c] *Université de Saint-Etienne, CNRS UMR 5824 GATE Lyon Saint-Etienne, France*

Abstract

We introduce a non linear weighted Shapley value for cooperative games with transferable utility, in which the weights are endogenously given by the players' stand-alone worths. We call it the proportional Shapley value since it distributes the Harsanyi dividend (Harsanyi, 1959) of all coalitions in proportion to the stand-alone worths of its members. We show that this value recommends an appealing payoff distribution in a land production economy introduced in Shapley and Shubik (1967). Although the proportional Shapley value does not satisfy the classical axioms of linearity and consistency (Hart and Mas-Colell, 1989), the main results provide comparable axiomatic characterizations of our value and the Shapley value by means of weak versions of these two axioms. Moreover, our value inherits several well-known properties of the weighted Shapley values.

Keywords: (Weighted) Shapley value, proportionality, Harsanyi dividends, potential, land production economy.

1. Introduction

The Shapley value (Shapley, 1953b) is a central tool in game theory, and has received considerable attention in numerous fields and applications. Moretti and Patrone (2008) and other articles in the same issue survey several examples. Many axiomatic characterizations have helped to understand the mechanisms underlying the Shapley value, and compare it to other types of values. Shapley's original characterization (Shapley, 1953b) and the one in Shubik (1962) rely on the axiom of additivity/linearity. In Myerson (1980), the axiom of balanced contribution requires that if a player leaves a game, then the payoff variation for another player is identical to his/her own payoff variation if this other player leaves the game. Young (1985) invokes an invariance principle: a player should obtain the same payoff in two games in which all his/her contributions to

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[*]Corresponding author

Email addresses: sylvain.beal@univ-fcomte.fr (Sylvain Béal), sylvain.ferrieres@univ-fcomte.fr (Sylvain Ferrières), eric.remila@univ-st-etienne.fr (Eric Rémila), philippe.solal@univ-st-etienne.fr (Philippe Solal)

coalitions are identical. Harsanyi (1959) proposes an interpretation of the Shapley value in terms of the coalitions' dividends. Roughly speaking, the Harsanyi dividend of a coalition measures the coalition's contribution to the worth of the grand coalition. The Shapley value splits equally the dividend of each coalition among its members. This interpretation has given rise to other solution concepts related to the Shapley value such as the selectope (Hammer et al., 1977) and the weighted Shapley values, originally introduced in Shapley (1953a) but popularized later by Kalai and Samet (1987). The selectope is the convex hull of the payoff vectors obtained by assigning the Harsanyi dividends to the associated coalitions' members. A weighted Shapley value splits the Harsanyi dividends in proportional to the exogenously given weights of its members. Both solution concepts are linear. The Harsanyi dividends are also often employed to compare different values (see section 5 in Herings et al., 2008; van den Brink et al., 2011, for instance).

In this article, we introduce a value based on another distribution of the Harsanyi dividends. It is similar in spirit to the weighted Shapley values, except that the weights are endogenous: they are given by the stand-alone worths of the players. Thus it coincides with the Shapley value whenever all such worths are equal. We call our value the proportional (weighted) Shapley value. The proportional principle incorporated to this value is often considered as intuitive in various classes of sharing problems (see Moulin, 1987, for instance).[1] Although the proportional Shapley value is non linear, it admits a close form and operational expression. It also satisfies many classical axioms such as efficiency and the dummy player property, and preserves the equal treatment property contrary to the asymmetric weighted Shapley values. The proportional Shapley value is well-defined for games in which the worths of all singleton coalitions have the same sign. This (not so) restrictive class of games includes several applications, such as airport games (Littlechild and Owen, 1973), auction games (Graham et al., 1990), carpool problems (Naor, 2005) and data sharing games (Dehez and Tellone, 2013). In airport games, a player is characterized by a positive real number (his/her "cost"), and the worth of the associated singleton coalition is equal to this number. So, it makes sense to use these numbers to define weights. In this article, we focus on the land production economies introduced by Shapley and Shubik (1967) in order to underline that the proportional Shapley value prescribes particularly relevant payoff distributions, especially compared to the (weighted) Shapley value(s). An expression of the Shapley value for land production economies is also given.

The rest of our contributions can be described as follows.

Firstly, the proportional Shapley value inherits some of the results concerning the weighted Shapley values. In particular, we can easily adapt the characterization in Myerson (1980) by using an axiom of proportional balanced contributions, and the characterization in Hart and Mas-Colell (1989) by constructing a proportional potential function. This part also includes a recursive formula inspired by the recursive formula of the Shapley value in Maschler and Owen (1989) and underlines that the proportional Shapley value of any convex game is in the core as a corollary of a result in Monderer et al. (1992). The proofs of these benchmark results are straightforward and omitted. In fact, any result stated for the weighted Shapley values on a class of games built from a fixed

[1]Other values incorporating some degree of proportionality are the proportional value (Ortmann, 2000) and the proper Shapley values (van den Brink et al., 2015).

characteristic function and its subgames also holds for the proportional Shapley value. A similar result is pointed out by Neyman (1989), who shows that Shubik (1962)'s axiomatic characterization of the Shapley value still holds if the axioms are applied to the additive group generated by the considered game and the games obtained from it after the nullification of any coalition (called subgames by Neyman).

Secondly, as soon as we consider a class of games with varying characteristic functions, the immediate transposition of existing results is no longer possible. For instance, the proportional Shapley value does not satisfy the classical axioms of linearity and consistency (Hart and Mas-Colell, 1989). Nevertheless, weak versions of these two axioms can be invoked (and even combined) to provide comparable axiomatic characterizations of the proportional Shapley value and the Shapley value, in the sense that these results only differ with respect to one axiom. Both characterizations have in common the well-known axioms of efficiency and dummy player out (Derks and Haller, 1999), which states that the payoff of a player is not affected if a dummy player leaves the game, and our weak version of linearity. More specifically, Proposition 5 shows that if two values satisfy efficiency, dummy player out and weak linearity, and if they coincide on games that are additive except, possibly, for the grand coalition, then they must be equal. In other words, there exists a unique extension of a value defined on these almost additive games to the set of all games in the much larger class we consider. The proof of this result emphasizes that tools from linear algebra can still be used on a class of games that is not a vector space.

Thirdly, Proposition 7 invokes the weak version of the axiom of consistency in addition to the three axioms appearing in the previous result. It turns out that a value satisfying these four axioms is the Shapley value if and only if it also satisfies the classical axiom of standardness, and is the proportional Shapley value if and only if it also satisfies a natural proportional version of standardness. The later axioms requires, in two-player games, that each player obtains first his/her stand alone worth plus a share of what remains of the worth of the grand coalition that is proportional to his/her stand-alone worth. It is worth noting that the two values are distinguished by axioms on two-player games only. Similarly, in addition to the three axioms appearing in Proposition 5, Proposition 8 invokes two new axioms inspired by the axiom of aggregate monotonicity in Megiddo (1974). More specifically, these axioms examine the consequences of a change in the worth of the grand coalition, *ceteris paribus*. Equal aggregate monotonicity requires equal payoff variations for all players, while proportional aggregate monotonicity requires payoff variations in proportion to the players' stand-alone worths. Among the values satisfying efficiency, dummy player out and weak linearity, the Shapley value is the only one that also satisfies equal aggregate monotonicity, and the proportional Shapley value is the only one that also satisfies proportional aggregate monotonicity.

Fourthly, the results presented so far all involve variable player sets, since they invoke axioms such as dummy player out and consistency. It is however possible to characterize the proportional Shapley value on a class of games with a fixed player set. In order to do so, we introduce another variant of the axiom of balanced contributions in which the removal of a player is replaced by his/her dummification. A player's dummification refers to his/her complete loss of synergy, in the sense that the worth of any coalition containing this player is now identical to that of the same coalition without this player plus his/her stand alone worth. In other words, the player becomes dummy, while the worth of any coalition not containing him/her remains unchanged. The dummification is in essence similar to the nullification of a player studied by Gómez-Rúa and Vidal-Puga (2010), Béal et al. (2014) and Béal et al. (2016). The new axiom of balanced contributions under dummification requires, for any two players, equal allocation variation after the dummification of the other player. Combined with efficiency and the classical axiom of inessential game property (each player obtains his/her stand-alone worth in case the game is additive), this axiom characterizes the proportional Shapley value value.

The rest of the article is organized as follows. Section 2 provides definitions and introduces the proportional Shapley value. It also states a first result for the case of land production economies. Section 3 briefly states properties of the proportional Shapley value that are inherited from the literature on the weighted Shapley values. Section 4 contains the main axiomatic characterizations, relying on the weak version of linearity, consistency, and on balanced contributions under dummification. Section 5 concludes. The appendix contains the results on the land production economies that are not stated in section 2, some technical proofs and the proofs of logical independence of the axioms used in some results.

2. Definitions, notation and motivation

2.1. Notation

We denote by $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$ and $\mathbb{R}^*$ the sets of all real numbers, non-negative real numbers, positive real numbers and non-null real numbers respectively. For a real number $b \in \mathbb{R}$ we shall also use notation $|b|$ for the absolute value of b. In order to denote the cardinality of any finite set S, the same notation $|S|$ will sometimes be used without any risk of confusion, but we shall often write s for simplicity.

2.2. Cooperative games with transferable utility

Let $\mathcal{U} \subseteq \mathbb{N}$ be a fixed and infinite universe of players. Denote by U the set of all finite subsets of $\mathcal{U}$. A **cooperative game with transferable utility**, or simply a **game**, is a pair (N, v) where $N \in U$ and $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\varnothing) = 0$. For a game (N, v), we write (S, v) for the **subgame** of (N, v) induced by $S \subseteq N$ by restricting v to 2^S. For $N \in U$ and $a \in \mathbb{R}^N$, denote by (N, v_a) the **additive game on** N **induced by** a, i.e. $v_a(S) = \sum_{j \in S} a_j$ for all $S \in 2^N$.

Define $\mathcal{C}$ as the class of all games with a finite player set in U and $\mathcal{C}_N$ as the subclass of $\mathcal{C}$ containing the games with player set N. A game (N, v) is **individually positive** if $v(\{i\}) > 0$ for all $i \in N$ and **individually negative** if $v(\{i\}) < 0$ for all $i \in N$. Let $\mathcal{C}^0$ denote the class containing all individually positive and individually negative games, and $\mathcal{C}^0_N$ the intersection of $\mathcal{C}^0$ and $\mathcal{C}_N$. For $N \in U$ and $a \in \mathbb{R}^N_{++}$, define the subclass of $\mathcal{C}^a_N$ containing all games such that the singleton worths are obtained by multipliying vector a by some non-null real number, that is:

$$\mathcal{C}^a_N = \{(N, v) \in \mathcal{C}_N | \exists c \in \mathbb{R}^* : \forall i \in N, v(\{i\}) = c a_i\}.$$

Thus, if $a' \in \mathbb{R}^N_{++}$ is multiple of $a \in \mathbb{R}^N_{++}$, then $\mathcal{C}^a_N = \mathcal{C}^{a'}_N$, and $\mathcal{C}^0_N = \bigcup_{a \in \mathbb{R}^N_{++}} \mathcal{C}^a_N$. Finally, let $\mathcal{A}^0$ and $\mathcal{A}^0_N$ denote the subclasses of additive games in $\mathcal{C}^0$ and $\mathcal{C}^0_N$ respectively.

For all $b \in \mathbb{R}$, all $(N,v),(N,w) \in \mathcal{C}$, the game $(N, bv + w) \in \mathcal{C}$ is defined as $(bv + w)(S) = bv(S) + w(S)$ for all $S \in 2^N$. The **unanimity game** on N induced by a nonempty coalition S, denoted by (N, u_S), is defined as $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise. Since Shapley (1953b), it is well-known that each function v admits a unique decomposition into unanimity games:

$$v = \sum_{S \in 2^N \setminus \{\varnothing\}} \Delta_v(S) u_S$$

where $\Delta_v(S)$ is the **Harsanyi dividend** (Harsanyi, 1959) of S, defined as $\Delta_v(S) = v(S) - \sum_{T \in 2^S \setminus \{\varnothing\}} \Delta_v(T)$. The Harsanyi dividend of S represents what remains of $v(S)$ once the dividends of all nonempty subcoalitions of S have been distributed. A player $i \in N$ is **dummy** in (N,v) if $v(S) - v(S \setminus \{i\}) = v(\{i\})$ for all $S \in 2^N$ such that $S \ni i$. Let $D(N,v)$ be the set of dummy players in (N,v). Two distinct players $i,j \in N$ are **equal** in (N,v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \in 2^{N \setminus \{i,j\}}$.

2.3. Values

A **value** on $\mathcal{C}$ (respectively on $\mathcal{C}^0$) is a function f that assigns a payoff vector $f(N,v) \in \mathbb{R}^N$ to any $(N,v) \in \mathcal{C}$ (respectively any $(N,v) \in \mathcal{C}^0$). In this article, we call upon values that admit intuitive formulations in terms of the distribution of the Harsanyi dividends.

The **Shapley value** (Shapley, 1953b) is the value Sh on $\mathcal{C}$ defined as:

$$Sh_i(N,v) = \sum_{S \in 2^N : S \ni i} \frac{1}{s} \Delta_v(S), \quad \forall (N,v) \in \mathcal{C}, \forall i \in N.$$

For each $i \in \mathcal{U}$ let $w_i \in \mathbb{R}_{++}$, and $w = (w_i)_{i \in \mathcal{U}}$. The **(positively) weighted Shapley value** (Shapley, 1953b) with weights w is the value Sh^w on $\mathcal{C}$ defined as:[2]

$$Sh_i^w(N,v) = \sum_{S \in 2^N : S \ni i} \frac{w_i}{\sum_{j \in S} w_j} \Delta_v(S), \quad \forall (N,v) \in \mathcal{C}, \forall i \in N.$$

The **proportional Shapley value** is the value PSh on $\mathcal{C}^0$ defined as:

$$PSh_i(N,v) = \sum_{S \in 2^N : S \ni i} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S), \quad \forall (N,v) \in \mathcal{C}^0, \forall i \in N.$$

Thus, the Harsanyi dividend of a coalition S is shared equally among its members in the Shapley value, in proportion to exogenous weights in a positively weighted Shapley value, and in proportion to the stand-alone worths of its members in the PSh value. As a consequence, the Shapley and PSh values coincide whenever all singleton worths are equal.

[2]Weighted Shapley values with possibly null weights are defined in Shapley (1953a), and studied in Kalai and Samet (1987) and Monderer et al. (1992), among others.

2.4. A motivating example: Land production economies

Consider a set $N = \{1, \dots, n\}$ of peasants and an amount of land $L \in \mathbb{R}_{++}$. Shapley and Shubik (1967, section VI) model the production process of several laborers working together by a function $\phi : N \times \mathbb{R}_{++} \longrightarrow \mathbb{R}_{+}$ which specifies the output

$$\phi(s, l) = \frac{l}{L} z(s) \tag{1}$$

achieved by s identical laborers from an area of land $l \leq L$, where $z(s) := \phi(s, L)$, and $z(s) > 0$ whenever $s > 0$.[3] They assume that each farmer owns the same share of L, which leads to an associated game by setting $v(S) = \phi(s, sL/n) = sz(s)/n$ for all $S \in 2^N$. A consequence of the symmetry in this model is that both PSh and Sh yield an equal split of the total output.

As suggested by Shapley and Shubik (1967), it makes sense to introduce some heterogeneity by considering that each peasant owns an amount of land $a_i \in \mathbb{R}_{++}$, such that $\sum_{i \in N} a_i = L$. Let $a := (a_i)_{i \in N}$. Since the output only depends on function z, a **land production economy** can be described by a triple (N, a, z). For any land production economy (N, a, z), the associated game $(N, v_{a,z})$ assigns to each coalition S a worth

$$v_{a,z}(S) = \phi\left(s, \sum_{i \in S} a_i\right) = \frac{\sum_{i \in S} a_i}{L} z(s)$$

for any coalition of farmers S. Note that $(N, v_{a,z}) \in \mathcal{C}_N^a$. In this asymmetric version of the model, Shapley and Shubik (1967) do not provide a formulation of the Shapley value, which is not easy to compute. In the appendix, we provide a close form expression of the Shapley value, which is nonetheless much less interpretable and appealing than the expression of the proportional Shapley value below. Proposition 1 shows that PSh can be considered as a relevant alternative to the Shapley value in the asymmetric land production economy. The proof is also relegated to the appendix.

Proposition 1. *For any land production economy (N, a, z) and any $i \in N$, it holds that*

$$PSh_i(N, v_{a,z}) = \frac{a_i}{L} z(n).$$

The meaning of Proposition 1 is clear. The output $z(n)$ produced by the n farmers altogether is shared in proportion to the landholdings. Proposition 1 also emphasizes situations in which the proportional Shapley value can be more suitable than the weighted Shapley values. After all, it is true that $PSh(N, v_{a,z})$ in Proposition 1 can be obtained as a weighted Shapley value by choosing the landholdings $(a_i)_{i \in N}$ as weights. But now, suppose that a farmer buys a part, but not all, of the landholding of another farmer, *ceteris paribus*. The new production economy is characterized by the same player set. Because the weights in a weighted Shapley value do not change with the characteristic function, they must remain the same in the new land production economy. As a consequence, the original weighted Shapley value applied to this new problem is likely to be less suitable. To the contrary, if the proportional Shapley value is applied to both situations, the weights associated to the players adjust accordingly to account for the new landholdings.

[3] Rather than ϕ, Shapley and Shubik (1967) use function ϕ^* defined as $\phi^*(s, l) = \max_{t \in \{1, \dots, s\}} \phi(t, l)$ for all pairs $(s, l) \in N \times \mathbb{R}_{++}$. The result in this section and in the appendix holds if ϕ is replaced by ϕ^*.

3. Legacy results

All the results in this section are based on the following useful property of PSh.

Lemma 1. *For each game $(N,v) \in \mathcal{C}^0$, define the weights $w(v) = (w_i(v))_{i\in N}$ such that $w_i(v) = |v(\{i\})|$ for all $i \in N$. Then, it holds that $PSh(N,v) = Sh^{w(v)}(N,v)$.*

The proof is obvious and omitted. Lemma 1 does not mean that PSh is a weighted Shapley value since the weights $w(v)$ can be different for two games defined on the same player set. Nevertheless, Lemma 1 is sufficient to adapt well-known results in the literature that involve a game with a fixed characteristic function and its subgames. As a first example, we consider the characterizations obtained by Myerson (1980) and Hart and Mas-Colell (1989) by means of the next axioms.

Balanced contributions (BC). For all $(N,v) \in \mathcal{C}$, all $i,j \in N$,

$$f_i(N,v) - f_i(N\backslash\{j\},v) = f_j(N,v) - f_j(N\backslash\{i\},v).$$

w-balanced contributions (w-BC). For all $w = (w_i)_{i\in U}$ with $w_i \in \mathbb{R}_{++}$ for all $i \in \mathcal{U}$, all $(N,v) \in \mathcal{C}$, all $i,j \in N$,

$$\frac{f_i(N,v) - f_i(N\backslash\{j\},v)}{w_i} = \frac{f_j(N,v) - f_j(N\backslash\{i\},v)}{w_j}.$$

Myerson (1980) characterizes the Shapley value by **BC** and **E**.

Efficiency (E). $\sum_{i\in N} f_i(N,v) = v(N)$.

Hart and Mas-Colell (1989) demonstrate that the class of weighted Shapley values coincides with the values satisfying **E** and **w-BC** for all possible weights w. A natural variant of **w-BC** requires, for any two players, an allocation variation for each of them after the other player has left that is proportional to their stand-alone worth.

Proportional balanced contributions (PBC). For all $(N,v) \in \mathcal{C}^0$, all $i,j \in N$,

$$\frac{f_i(N,v) - f_i(N\backslash\{j\},v)}{v(\{i\})} = \frac{f_j(N,v) - f_j(N\backslash\{i\},v)}{v(\{j\})}.$$

The main notable difference between **PBC** and **w-BC** is that our weights are endogenous, i.e. they can vary across games. The consequence is that the system of equations generated by **PBC** together with **E** is not linear. Nevertheless, it gives rise to a unique non-linear value.

Proposition 2. *The proportional Shapley value is the unique value on $\mathcal{C}^0$ that satisfies **E** and **PBC**.*

Using **PBC** and the fact that PSh satisfies **E**, it is possible to obtain a recursive formula of PSh very close to the recursive formula of the Shapley value provided by Maschler and Owen (1989). For all $(N,v) \in \mathcal{C}^0$ and all $i \in N$,

$$PSh_i(N,v) = \sum_{j\in N\setminus\{i\}} \frac{v(\{j\})}{\sum_{k\in N} v(\{k\})} PSh_i(N\setminus\{j\},v) + \frac{v(\{i\})}{\sum_{k\in N} v(\{k\})}\big(v(N) - v(N\setminus\{i\})\big).$$

The latter expression is similar to following recursive formula for the Shapley value (Maschler and Owen, 1989):

$$Sh_i(N,v) = \sum_{j\in N\setminus\{i\}} \frac{1}{n} Sh_i(N\setminus\{j\},v) + \frac{1}{n}\big(v(N) - v(N\setminus\{i\})\big).$$

Connected to Myerson's approach is the fundamental notion of potential introduced by Hart and Mas-Colell (1989). For any system of weights w, the unique w-potential function P_w is defined as $P_w(\varnothing,v) = 0$ and as $\sum_{i\in N} w_i\big(P_w(N,v) - P_w(N\setminus\{i\},v\big) = v(N)$. They show that the weighted Shapley value with weights w in game (N,v) coincides with payoffs $w_i\big(P_w(N,v) - P_w(N\setminus\{i\},v\big)$, $i \in N$. Below is an adaptation for PSh. A **proportional potential function** is a function $Q : \mathcal{C}^0 \longrightarrow \mathbb{R}$ such that $Q(\varnothing,v) = 0$ and for all $(N,v) \in \mathcal{C}^0$,

$$\sum_{i\in N} |v(\{i\})|\big(Q(N,v) - Q(N\setminus\{i\},v)\big) = v(N). \tag{2}$$

The following proposition mimics Theorem 5.2 in Hart and Mas-Colell (1989).

Proposition 3. *There exists a unique proportional potential function Q on $\mathcal{C}^0$. Moreover, for each game $(N,v) \in \mathcal{C}^0$, it holds that $Q(N,v) = P_{w(v)}(N,v)$. Thus, for each game $(N,v) \in \mathcal{C}^0$ and each $i \in N$,*

$$PSh_i(N,v) = |v(\{i\})|\big(Q(N,v) - Q(N\setminus\{i\},v)\big).$$

Finally Q can be computed recursively by the following formula:

$$Q(N,v) = \frac{1}{\sum_{i\in N} |v(\{i\})|}\Bigg(v(N) + \sum_{j\in N} |v(\{j\})| Q(N\setminus\{j\},v)\Bigg).$$

It suffices to define Q on $\mathcal{C}^0$ as $Q(N,v) = P_{w(v)}(N,v)$ for all $(N,v) \in \mathcal{C}^0$. In a sense, Q is a normalized (or dimensionless) potential because for each $i \in N$, $Q(\{i\},v) = 1$ if $v(\{i\}) > 0$ and $Q(\{i\},v) = -1$ if $v(\{i\}) < 0$. Dimensionless numbers are often desirable, in particular in economics (elasticities). The proportional potential will play a key role in some of the main results in the next section. We conclude this section by stating a sufficient condition under which the PSh value lies in the core. The core of a game $(N,v) \in \mathcal{C}^0$ is the (possibly empty) set $C(N,v) = \big\{x \in \mathbb{R}^N : \sum_{i\in S} x_i \geq v(S) \text{ and } \sum_{i\in N} x_i = v(N)\big\}$. Shapley (1971) shows that the Shapley value belongs to the core of a convex game. Monderer et al. (1992) generalize this result and prove that the core of a convex game coincides with the set of weighted Shapley values (with possibly null weights). Building on this result, it is immediate to prove the PSh value lies in the core of a convex game.

Proposition 4. *If* $(N,v) \in \mathcal{C}^0$ *is convex, then* $PSh(N,v) \in C(N,v)$*.*

Not surprisingly, PSh is not necessarily a core imputation in a non-convex game with a nonempty core as shown in the following example.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	4	4	1	12	−5	15	22
$\Delta_v(S)$	4	4	1	4	−10	10	9

The core of (N,v) is not empty since it contains, for instance, the Shapley value $Sh(N,v) = (4,14,4)$. However, $PSh(N,v) = (2,18,2)$ is not in the core.

4. Main results

The results in this section rely on weak versions of the axioms of linearity and consistency as proposed by Hart and Mas-Colell (1989), and on another variant of **PBC**.

4.1. Potential, linearity and consistency

Contrary to the potential approach, the well-known axioms of linearity and consistency require less evident modifications in order to account for PSh, even if these axioms are satisfied by any weighted Shapley value. We examine each case separately before combining them in order to characterize PSh.

Linearity (L). For all $b \in \mathbb{R}$, all $(N,v),(N,w) \in \mathcal{C}$, $f(N,bv+w) = bf(N,v) + f(N,w)$.

The class $\mathcal{C}^0$ is not a vector space. Even if it is required that the game $(N,bv+w)$ constructed in the previous definition still belongs to $\mathcal{C}^0$, it is clear that PSh violates this adaptation of **L** on $\mathcal{C}^0$. Nonetheless, PSh satisfies the following weaker version of the axiom.

Weak linearity (WL). For all $a \in \mathbb{R}^N_{++}$, all $b \in \mathbb{R}$, all $(N,v),(N,w) \in \mathcal{C}^a_N$, if $(N,bv+w) \in \mathcal{C}^a_N$ then $f(N,bv+w) = bf(N,v) + f(N,w)$.

So **WL** only applies to games belonging to the same subclass $\mathcal{C}^a_N$ of $\mathcal{C}^0_N$, i.e. pair of games for which the ratio $w(\{i\})/v(\{i\})$ is the same for all players. The requirement that $(N,bv+w) \in \mathcal{C}^a_N$ is necessary: in case b is equal to the opposite of the above-mentioned ration $(N,bv+w)$ would not belong to $\mathcal{C}^a_N$ (and neither to $\mathcal{C}^0_N$). Before showing that PSh satisfies **WL**, we present a key result in which **WL** is combined with the classical axioms **E** and dummy player out.

Dummy player out (DPO). For all $(N,v) \in \mathcal{C}^0$, if $i \in N$ is a dummy player in (N,v), then for all $j \in N\backslash\{i\}$, $f_j(N,v) = f_j(N\backslash\{i\},v)$.

DPO was suggested first in Tijs and Driessen (1986, section V) and is closely related to the widely-used null player out axiom (Derks and Haller, 1999). Proposition 5 below requires the following definition. A game (N,v) is **quasi-additive** if $v(S) = \sum_{i\in S} v(\{i\})$ for all $S \in 2^N\backslash\{N\}$. Let $\mathcal{QA}^0$ denote the class of all quasi-additive games in $\mathcal{C}^0$. In a quasi-additive game, the worths of all coalitions are additive except, possibly, for the grand coalition for which there can be some surplus or loss compared to the sum of the stand-alone worths of its members. So $\mathcal{QA}^0$ includes the class $\mathcal{A}^0$. Proposition 5 essentially states that a value satisfies **E**, **DPO** and **WL** is completely determined by what it prescribes on quasi-additive games.

Proposition 5. *Consider two values f and g satisfying* **E**, **DPO** *and* **WL** *on $\mathcal{C}^0$ such that $f = g$ on $\mathcal{QA}^0$. Then $f = g$ on $\mathcal{C}^0$.*

The non-trivial and lengthy proof of Proposition 5 relies on two Lemmas. In order to lighten the exposition, this material is relegated to the appendix. A similar result can be stated on the larger class $\mathcal{C}$ by replacing **WL** by **L**. The only change would be to consider all quasi-additive games in $\mathcal{C}$ and not just those in $\mathcal{C}^0$. At this point, remark also that both the Shapley value and PSh satisfy the three axioms invoked in Proposition 5.[4] In order to compare and distinguish the two values, we present extra axioms below. The next axiom relies on the reduced game proposed by Hart and Mas-Colell (1989). Let f be a value on $\mathcal{C}$, $(N,v)\in\mathcal{C}$ and $S\in 2^N\backslash\{\emptyset\}$. The **reduced game** (S, v_S^f) induced by S and f is defined, for all $T\in 2^S$, by:

$$v_S^f(T) = v(T\cup(N\backslash S)) - \sum_{i\in N\backslash S} f_i(T\cup(N\backslash S), v), \tag{3}$$

and resumes to $v_S^f(T) = \sum_{i\in T} f_i(T\cup(N\backslash S), v)$ if f satisfies **E**.

Consistency (C). For all $(N,v)\in\mathcal{C}$, all $S\in 2^N$, and all $i\in S$, $f_i(N,v) = f_i(S, v_S^f)$.

The Shapley value satisfies **C** on $\mathcal{C}$. If **C** is enunciated on $\mathcal{C}^0$, the extra condition that the considered reduced game (S, v_S^f) remains in $\mathcal{C}^0$ must be added. Such a condition is, however, not sufficient for our objective: PSh fails to satisfy the axiom as illustrated by the following example.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	1	2	3	9	4	5	18
$\Delta_v(S)$	1	2	3	6	0	0	6

Consider player 1. It is easy to check that $PSh_1(\{1,2,3\},v) = 4$. Now consider the reduced game $\big(\{1,3\}, v_{\{1,3\}}^{PSh}\big)$, where $v_{\{1,3\}}^{PSh}(\{1\}) = PSh_1(\{1,2\},v) = 3$, $v_{\{1,3\}}^{PSh}(\{3\}) = PSh_3(\{2,3\},v) = 3$, and $v_{\{1,3\}}^{PSh}(\{1,3\}) = PSh_1(\{1,2,3\},v) + PSh_3(\{1,2,3\},v) = 4+6 = 10$. Note that $\big(\{1,3\}, v_{\{1,3\}}^{PSh}\big)\in\mathcal{C}^0$ and is symmetric, which implies that $PSh_1\big(\{1,3\}, v_{\{1,3\}}^{PSh}\big) = 5 \neq PSh_1(\{1,2,3\},v) = 4$, proving that

[4] Formal proofs are given later on.

PSh does not satisfy **C** in this context.

It is possible to weaken **C** by imposing consistency of the value on the particular subclass of quasi-additive games $\mathcal{QA}^0$. To this end, we begin by a Lemma stating that $\mathcal{QA}^0$ is almost close under the reduction operation for values satisfying the following mild condition.

Inessential game property (IGP). For all $(N, v) \in \mathcal{A}^0$, all $i \in N$, $f_i(N, v) = v(\{i\})$.[5]

Lemma 2. *Consider any value f that satisfies **IGP** on $\mathcal{QA}^0$. Then, for each $(N, v) \in \mathcal{QA}^0$ and each $S \in 2^N$ such that $s \geq 2$, it holds that $(S, v_S^f) \in \mathcal{QA}^0$. Furthermore, for each $T \in 2^S \backslash \{S\}$, $v_S^f(T) = \sum_{i \in T} v(\{i\})$.*

Proof. Consider any value f that satisfies **IGP** on $\mathcal{QA}^0$, any $(N, v) \in \mathcal{QA}^0$ and any $S \in 2^N$ such that $s \geq 2$. If $S = N$, the result is trivial. So let $S \neq N$, and consider any coalition $T \in 2^S \backslash \{S\}$. To show $v_S^f(T) = \sum_{i \in T} v_S^f(\{i\})$. By definition (3) of the reduced game (S, v_S^f), the worth $v_S^f(T)$ only relies on the subgame $(T \cup (N \backslash \{S\}), v)$ of (N, v). Since $T \neq S$, $(T \cup (N \backslash \{S\}), v)$ is a strict subgame of (N, v), and since $(N, v) \in \mathcal{QA}^0$, we get that $(T \cup (N \backslash \{S\}), v) \in \mathcal{A}^0$. By **IGP**, f satisfies **E** in $(T \cup (N \backslash \{S\}), v)$ and $f_i(T \cup (N \backslash \{S\}), v) = v(\{i\})$ for each $i \in T \cup (N \backslash \{S\})$. This implies that $v_S^f(T) = \sum_{i \in T} f_i(T \cup (N \backslash \{S\}), v) = \sum_{i \in T} v(\{i\})$. The proof is complete since T was an arbitrary coalition in $2^S \backslash \{S\}$. ∎

Remark 1. *Lemma 2 excludes coalitions of size 1, i.e. reduced games with a unique player. Such reduced games may not belong to $\mathcal{QA}^0$ as suggested by the following generic example. For any game $(N, v) \in \mathcal{QA}^0$ such that $v(N) = 0$, we get $PSh_i(N, v) = 0$ for all $i \in N$. Therefore, for each $i \in N$, the reduced game $(\{i\}, v_{\{i\}}^{PSh})$ is such that $v_{\{i\}}^{PSh}(\{i\}) = 0$ and does not belong to $\mathcal{QA}^0$.* □

Next, we introduce our weak version of **C**.

Weak consistency (WC). For all $(N, v) \in \mathcal{QA}^0$, all $S \in 2^N$ such that $(S, v_S^f) \in \mathcal{QA}^0$, and all $i \in S$, $f_i(N, v) = f_i(S, v_S^f)$.

Finally, we invoke the following axioms.

Proportional standardness (PS). For all $(\{i, j\}, v) \in \mathcal{C}^0$, $f_i(\{i, j\}, v) = \frac{v(\{i\})}{v(\{i\}) + v(\{j\})} v(\{i, j\})$.

This axiom is called proportionality for two person games in Ortmann (2000) and two-player games proportionality in Huettner (2015). It can be considered as the proportional counterpart of the classical axiom of standardness (Hart and Mas-Colell, 1989).

[5] This axiom is also called the projection axiom in Aumann and Shapley (1974).

Standardness (S). For all $(\{i,j\},v) \in \mathcal{C}^0$, $f_i(\{i,j\},v) = v(\{i\}) + \frac{1}{2}\big(v(\{i,j\}) - v(\{i\}) - v(\{j\})\big)$.

Both axioms first assign their stand-alone worths to the two players. Then, proportional standardness splits the remaining surplus in proportion to these stand-alone worths, while standardness shares the surplus equally, if any. To understand this interpretation, note that

$$\frac{v(\{i\})}{v(\{i\}) + v(\{j\})} v(\{i,j\}) = v(\{i\}) + \frac{v(\{i\})}{v(\{i\}) + v(\{j\})}\big(v(\{i,j\}) - v(\{i\}) - v(\{j\})\big).$$

Proportional aggregate monotonicity (PAM). For all $b \in \mathbb{R}$, all $(N,v) \in \mathcal{C}^0$ such that $n \geq 2$, and all $i,j \in N$,

$$\frac{f_i(N,v) - f_i(N, v + bu_N)}{v(\{i\})} = \frac{f_j(N,v) - f_j(N, v + bu_N)}{v(\{j\})}.$$

The axiom compares two games that only differ with respect to the worth of the grand coalition. It states that the players enjoy payoff variations that are proportional to their stand-alone worths. Note that **PAM** is well-defined since $(N,v) \in \mathcal{C}^0$ implies that $(N, v + bu_N) \in \mathcal{C}^0$ for all $b \in \mathbb{R}$. Without the further requirement of **E**, **PAM** is not related to Aggregate monotonicity (Megiddo, 1974), which requires that none of the players should be hurt if the worth of the grand coalition increases. In fact, the Shapley value satisfies Aggregate monotonicity but not **PAM**, while the value f on $\mathcal{C}^0$, which assigns to each game $(N,v) \in \mathcal{C}^0$ and to each $i \in N$, the payoff $f_i(N,v) = -v(\{i\})/\sum_{j\in N} v(\{j\}) \times v(N)$ satisfies **PAM** but not Aggregate monotonicity. However, if a value satisfies **PAM** and **E** on $\mathcal{C}^0$, then it also satisfies Aggregate monotonicity on $\mathcal{C}^0$.

The next result lists which of these axioms are satisfied by PSh.

Proposition 6. *PSh satisfies **E**, **DPO**, **WL**, **PAM**, **WC** and **PS** on $\mathcal{C}^0$.*

Proof. The proof follows from Proposition 2 for **E**.

Regarding **DPO**, observe that if a player $i \in N$ is dummy in (N,v), then $\Delta_v(S) = 0$ for all $S \in 2^N$ such that $S \ni i$ and $s \geq 2$. So, for any $j \in N\backslash\{i\}$, it holds that

$$PSh_j(N,v) = \sum_{S\in 2^N : S\ni j} \frac{v(\{j\})}{\sum_{k\in S} v(\{k\})}\Delta_v(S) = \sum_{S\in 2^{N\backslash\{i\}} : S\ni j} \frac{v(\{j\})}{\sum_{k\in S} v(\{k\})}\Delta_v(S) = PSh_j(N\backslash\{i\}, v)$$

as desired.

Regarding **WL**, consider any two games $(N,v),(N,w) \in \mathcal{C}^a_N$, which means that, for all $i \in N$, $w(\{i\}) = cv(\{i\})$ for some $c \in \mathbb{R}^*$. Note that for all nonempty $S \in 2^N$, this implies

$$\frac{v(\{i\})}{\sum_{j\in S} v(\{j\})} = \frac{w(\{i\})}{\sum_{j\in S} w(\{j\})}. \tag{4}$$

Choose any $b \in \mathbb{R}$ such that $(N, bv + w) \in \mathcal{C}^a_N$ in order to compute $PSh_i(N, bv + w)$. The claim is trivial for $b = 0$. So suppose $b \in \mathbb{R}^*$. By linearity of function Δ (third equality) and equation (4) (fourth equality), we have

$$\begin{aligned}
PSh_i(N, bv + w) &= \sum_{S \in 2^N : S \ni i} \frac{(bv + w)(\{i\})}{\sum_{j \in S}(bv + w)(\{j\})} \Delta_{bv+w}(S) \\
&= \sum_{S \in 2^N : S \ni i} \frac{(b + c)v(\{i\})}{\sum_{j \in S}(b + c)v(\{j\})} \Delta_{bv+w}(S) \\
&= \sum_{S \in 2^N : S \ni i} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Big(b\Delta_v(S) + \Delta_w(S) \Big) \\
&= b \sum_{S \in 2^N : S \ni i} \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \Delta_v(S) + \sum_{S \in 2^N : S \ni i} \frac{w(\{i\})}{\sum_{j \in S} w(\{j\})} \Delta_w(S) \\
&= bPSh_i(N, v) + PSh_i(N, w).
\end{aligned}$$

Regarding **PAM**, pick any game $(N, v) \in \mathcal{C}^0$ and any $b \in \mathbb{R}$. Note that $\Delta_{v+bu_N}(S) = \Delta_v(S)$ for all $S \in 2^N \backslash \{N\}$, and that $\Delta_{v+bu_N}(N) = \Delta_v(N) + b$. As a consequence,

$$\frac{PSh_i(N, v) - PSh_i(N, v + bu_N)}{v(\{i\})} = \frac{-b}{\sum_{j \in N} v(\{j\})}$$

does not depend on $i \in N$, which proves that PSh satisfies **PAM**.

Regarding **WC**, consider any game $(N, v) \in \mathcal{QA}^0$, any nonempty $S \in 2^N$ and any $i \in S$. The assertion that $PSh_i(N, v) = PSh_i(S, v_S^{PSh})$ is trivial if $s = n$. Since $(N, v) \in \mathcal{QA}^0$, note that $\Delta_v(T) = 0$ for all T such that $t \in \{2, \dots, n-1\}$. As a consequence, PSh admits the following simple formulation:

$$PSh_i(N, v) = \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N). \tag{5}$$

The assertion is thus also obvious for $s = 1$ in case $(S, v_S^{PSh}) \in \mathcal{QA}^0$ (see Lemma 2). Now, let us assume that $s \in \{2, \dots, n-1\}$. By Lemma 2, we know that $(S, v_S^{PSh}) \in \mathcal{QA}^0$. Furthermore, as noted in Lemma 2, $v_S^{PSh}(T) = \sum_{i \in T} v(\{i\})$ for each $T \in 2^S \backslash \{S\}$, and $v_S^{PSh}(S) = \sum_{i \in S} PSh_i(N, v)$. As a consequence, for each $i \in S$,

$$\begin{aligned}
PSh_i(S, v_S^{PSh}) &= \frac{v_S^{PSh}(\{i\})}{\sum_{j \in N} v_S^{PSh}(\{j\})} v_S^{PSh}(S) \\
&= \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \sum_{k \in S} PSh_k(N, v) \\
&= \frac{v(\{i\})}{\sum_{j \in S} v(\{j\})} \sum_{k \in S} \frac{v(\{k\})}{\sum_{j \in N} v(\{j\})} v(N) \\
&= \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} v(N) \\
&= PSh_i(N, v).
\end{aligned}$$

Finally, since any two-player game in $\mathcal{C}^0$ is quasi-additive, PSh satisfies **PS** by applying (5) to the two-player case. ■

Building on Propositions 5 and 6, we offer two characterizations of PSh that are comparable to two new characterizations of Sh in the sense that they only differ with respect to one axiom.

Both characterizations have in common the three axioms in Proposition 5: **E**, **DPO** and **WL**. The first one invokes **WC**.

Proposition 7. *A value f on $\mathcal{C}^0$ satisfies **E**, **DPO**, **WL**, **WC** and*

*(i) **PS** if and only if $f = PSh$;*

*(ii) **S** if and only if $f = Sh$.*

Proof. By Proposition 6 and the fact that Sh satisfies **E**, **DPO**, **WL**, **WC** and **S**, it suffices to show the uniqueness parts. Consider any value f on $\mathcal{C}^0$ that satisfies **E**, **DPO**, **WL** and **WC**. By Proposition 5, it only remains to show that f is uniquely determined on $\mathcal{QA}^0$. Pick any quasi-additive game $(N,v) \in \mathcal{QA}^0$, which means that $v = v_a + bu_N$ for some $a \in \mathbb{R}^N_{++}$ or $-a \in \mathbb{R}^N_{++}$ and $b \in \mathbb{R}$. In case $b = 0$, $v = v_a$ so that (N,v) is additive. All players are dummy, which implies that $f(N,v_a)$ is completely determined by combining **E** and **DPO**. In case $b \neq 0$, we distinguish two cases.

Firstly, assume as in point *(i)* that f satisfies **PS**. The proof borrows some steps of the proof of Theorem B in Hart and Mas-Colell (1989). We show that f admits a proportional potential on $\mathcal{QA}^0$. To do so, define the function R for games $(N,v) \in \mathcal{QA}^0$ with at most two players by setting $R(\varnothing,v) = 0$, $R(\{i\},v) = v(\{i\})/|v(\{i\})|$, and

$$R(\{i,j\},v) = \frac{v(\{i,j\} + v(\{i\}) + v(\{j\})}{|v(\{i\}) + v(\{j\})|}.$$

By **E** and **PS**, for all $(N,v) \in \mathcal{QA}^0$ such that $n \leq 2$, and all $i \in N$, it holds that

$$f_i(N,v) = |v(\{i\})|\big(R(N,v) - R(N\backslash\{i\},v)\big). \tag{6}$$

We now prove, by induction on the size of the player set, that R can be extended to all games in $\mathcal{QA}^0$, i.e. that R is the proportional potential Q on $\mathcal{QA}^0$, and in turn that $f = PSh$ on $\mathcal{QA}^0$.
Initialization. As noted before, the assertion holds for games $(N,v) \in \mathcal{QA}^0$ with $n \leq 2$.
Induction hypothesis. Assume that R has been defined and satisfies (6) for all games $(N,v) \in \mathcal{QA}^0$ such that $n \leq q$, $q \geq 2$.
Induction step. Consider any $(N,v) \in \mathcal{QA}^0$ with $n = q+1$. We have to show that $f_i(N,v)/|v(\{i\})| + R(N\backslash\{i\},v)$ is independent of $i \in N$. Pick a triple of distinct players, which is always possible since $n \geq 3$. By Lemma 2 (since **DPO** and **E** imply **IGP**), **WC** and (6), we can write:

$$\begin{aligned}
\frac{f_i(N,v)}{|v(\{i\})|} - \frac{f_j(N,v)}{|v(\{j\}|} &= \frac{f_i(N\backslash\{k\},v^f_{N\backslash\{k\}})}{|v(\{i\})|} - \frac{f_j(N\backslash\{k\},v^f_{N\backslash\{k\}})}{|v(\{j\})|} \\
&= \frac{f_i(N\backslash\{k\},v^f_{N\backslash\{k\}})}{|v^f_{N\backslash\{k\}}(\{i\})|} - \frac{f_j(N\backslash\{k\},v^f_{N\backslash\{k\}})}{|v^f_{N\backslash\{k\}}(\{j\})|} \\
&= \big(R(N\backslash\{k\},v^f_{N\backslash\{k\}}) - R(N\backslash\{i,k\},v^f_{N\backslash\{k\}})\big) \\
&\quad -\big(R(N\backslash\{k\},v^f_{N\backslash\{k\}}) - R(N\backslash\{j,k\},v^f_{N\backslash\{k\}})\big) \\
&= \big(R(N\backslash\{j,k\},v^f_{N\backslash\{k\}}) - R(N\backslash\{i,j,k\},v^f_{N\backslash\{k\}})\big) \\
&\quad -\big(R(N\backslash\{i,k\},v^f_{N\backslash\{k\}}) - R(N\backslash\{i,j,k\},v^f_{N\backslash\{k\}})\big).
\end{aligned}$$

Another application of **WC** and two applications of (6) yield that the preceding equality becomes:

$$\begin{aligned}
&= \frac{f_i(N\backslash\{j,k\}, v^f_{N\backslash\{k\}})}{|v^f_{N\backslash\{k\}}(\{i\})|} - \frac{f_j(N\backslash\{i,k\}, v^f_{N\backslash\{k\}})}{|v^f_{N\backslash\{k\}}(\{j\})|} \\
&= \frac{f_i(N\backslash\{j,k\}, v^f_{N\backslash\{k\}})}{|v(\{i\})|} - \frac{f_j(N\backslash\{i,k\}, v^f_{N\backslash\{k\}})}{|v(\{j\})|} \\
&= \frac{f_i(N\backslash\{j\}, v)}{|v(\{i\})|} - \frac{f_j(N\backslash\{i\}, v)}{|v(\{j\})|} \\
&= \big(R(N\backslash\{j\}, v) - R(N\backslash\{i,j\}, v)\big) \\
&\quad -\big(R(N\backslash\{i\}, v) - R(N\backslash\{i,j\}, v)\big) \\
&= R(N\backslash\{j\}, v) - R(N\backslash\{i\}, v)
\end{aligned}$$

as desired. Remark 1 points out that one-player reduced games of a quasi-additive game in $\mathcal{QA}^0$ may not belong to $\mathcal{QA}^0$. In case $n = 3$, $(N\backslash\{j,k\}, v^f_{N\backslash\{k\}})$ and $(N\backslash\{j,k\}, v^f_{N\backslash\{k\}})$ are one-player games, but they both belong to $\mathcal{QA}^0$. To see this, $(N,v) \in \mathcal{QA}^0$ and $n = 3$ imply that $(N\backslash\{k\}, v^f_{N\backslash\{k\}}) \in \mathcal{QA}^0$ by Lemma 2, and thus that for each nonempty $S \in 2^{N\backslash\{k\}}$, $(S, v^f_{N\backslash\{k\}}) \in \mathcal{QA}^0$ too.

Secondly, assume as in point *(ii)* that f satisfies **S**. The result follows from Hart and Mas-Colell (1989, Theorem B), where the preceding steps are developed on the basis of **S** and the classical potential function. ∎

The logical independence of the axioms in Proposition 7 is demonstrated in appendix. The second result in this section compares once again the Shapley value and PSh by keeping axioms **E**, **DPO** and **WL**, and by adding either **PAM** or the following axiom.

Equal aggregate monotonicity (EAM). For all $b \in \mathbb{R}$, all $(N,v) \in \mathcal{C}^0$, and all $i, j \in N$,

$$f_i(N,v) - f_i(N, v + bu_N) = f_j(N,v) - f_j(N, v + bu_N).$$

Replacing **WC** and **PS** (resp. **S**) by **PAM** (resp. **EAM**) in Proposition 7 yields a characterization of PSh (resp. Sh) on $\mathcal{C}^0$.

Proposition 8. *A value f on $\mathcal{C}^0$ satisfies **E**, **DPO**, **WL**, and*

*(i) **PAM** if and only if $f = PSh$;*

*(ii) **EAM** if and only if $f = Sh$.*

Proof. Regarding point (i), by Proposition 6, it suffices to show uniqueness. Consider any value f on $\mathcal{C}^0$ that satisfies **E**, **DPO**, **WL** and **PAM**. By Proposition 5, it only remains to show that f is uniquely determined on $\mathcal{QA}^0$. Pick any quasi-additive game $(N,v) \in \mathcal{QA}^0$, which means that $v = v_a + bu_N$ for some $a \in \mathbb{R}^N_{++}$ or some $-a \in \mathbb{R}^N_{++}$ and $b \in \mathbb{R}$. In case $b = 0$, $v = v_a$ is an additive function. All players are dummy, which implies $f(N, v_a)$ is completely determined by **E** and **DPO**.

If $b \neq 0$, remark that (N, v) with $v = v_a + bu_N$ and (N, v_a) only differ with respect to the worth of the grand coalition N. By **PAM**, we have, for all $i, j \in N$,

$$\frac{f_i(N, v) - f_i(N, v_a)}{v(\{i\})} = \frac{f_j(N, v) - f_j(N, v_a)}{v(\{j\})}.$$

Summing on all $j \in N$ and using **E** in both games, we get

$$f_i(N, v) = f_i(N, v_a) + \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})}\big(v(N) - v_a(N)\big) = f_i(N, v_a) + \frac{v(\{i\})}{\sum_{j \in N} v(\{j\})} b,$$

for all $i \in N$, and so $f_i(N, v)$ is uniquely determined, as desired.

Regarding point (ii), it is easy to check that Sh satisfies **EAM**. For the uniqueness part, mimics the proof of point (i) except in the very last part where the combination **EAM** and **E** implies that $f_i(N, v) = f_i(N, v_a) + b/n$ for all $i \in N$. ∎

The logical independence of the axioms in Proposition 8 is demonstrated in appendix. It should be noted that replacing either **PS** or **S** in Proposition 7 or either **PAM** and **EAM** in Proposition 8 by Aggregate monotonicity (Megiddo, 1974) does not yield the set of (positively) weighted Shapley values. The weighted Shapley values satisfy all axioms, but they are not the only one. In fact, consider a value f on $\mathcal{C}^0$ such that, for any $N \in U$, any $a \in \mathbb{R}^N_{++}$ and any $(N, v) \in \mathcal{C}^a_N$ it holds that $f(N, v) = Sh^w(N, v)$ for some weights w. Whenever two games belonging to disjoint sets $\mathcal{C}^a_N$ are associated to different weights, the value f satisfies all axioms but is not a (positively) weighted Shapley value.

As a final remark, it is worth noting that strengthening, for each $N \in U$, **WL** on $\mathcal{C}^0_N$ by **L** on $\mathcal{C}_N$ in Proposition 8 (ii) yields a characterization of the Shapley value on the full domain $\mathcal{C}$ by **E**, **DPO**, **WL**, and **EAM**. In case **WC** is further invoked on $\mathcal{QA}$ instead of $\mathcal{QA}^0$, then from Proposition 7 (ii), the Shapley value can be characterized on the full domain $\mathcal{C}$ by this axiom together with **E**, **DPO**, **WL**, and **S**.

4.2. Proportional Balanced contributions under dummification

Contrary to the results in section 4, we illustrate here that PSh can be characterized on a fixed player set by means of the following definition. For a game $(N, v) \in \mathcal{C}^0$ and a player $i \in N$, we denote by $(N, v^i) \in \mathcal{C}^0$ the game obtained from (N, v) if player i is **dummified**: $v^i(S) = v(\{i\}) + v(S \backslash \{i\})$ for all $S \in 2^N$ such that $S \ni i$ and $v^i(S) = v(S)$ for all $S \not\ni i$.[6] The dummification operation is similar to the nullification operation studied in Béal et al. (2016), among others. The dummification arises naturally in the so-called Myerson (graph) restricted game (Myerson, 1977), where, for a given graph on the player set, the worth of a coalition is the sum of the worths of its connected parts. If a player is deprived of his or her links then he or she becomes dummified in the resulting new Myerson restricted game. Below, we introduce a variant of **PBC** in which the subgame induced when a player leaves the game is replaced by the game in which this player is dummified.

[6] An equivalent, and perhaps shorter, definition is that $v^i(S) = v(S \backslash \{i\}) + v(S \cap \{i\})$ for all $S \in 2^N$.

Remark 2. *Note that* $(v^i)^i = v^i$ *and* $(v^i)^j = (v^j)^i$. *From the latter property, for each nonempty* $S \in 2^N$, *the function* v^S *in which the players in* S *are (successively) dummified is well-defined. For any* $(N,v) \in \mathcal{C}^0_N$ *and* $i \in N$, $v^{N\setminus\{i\}} = v^N$ *and* $(N, v^{N\setminus\{i\}}) \in \mathcal{A}^0_N$, *i.e. this game is additive. Regarding the set of dummy players* $D(N,v)$, *note also that* $D(N,v) = \{i \in N : v^i = v\}$, *and that for any* $(N,v) \in \mathcal{C}^0_N$ *and* $i \in N$, $D(N,v^i) \supseteq D(N,v) \cup \{i\}$, *where this inclusion may be strict.* □

Proportional balanced contributions under dummification, PBCD. For all $(N,v) \in \mathcal{C}^0_N$, all $i,j \in N$,

$$\frac{f_i(N,v) - f_i(N,v^j)}{v(\{i\})} = \frac{f_j(N,v) - f_j(N,v^i)}{v(\{j\})}.$$

For a fixed player set N, Proposition 9 below indicates that a value satisfying **E** and **PBCD** is completely determined by what it prescribes on additive games with player set N.

Proposition 9. *Consider two values* f *and* g *satisfying* ***E*** *and* ***PBCD*** *on* $\mathcal{C}^0_N$ *such that* $f = g$ *on* $\mathcal{A}^0_N$. *Then* $f = g$ *on* $\mathcal{C}^0_N$.

Proof. Consider two values f and g satisfying **E** and **PBCD** on $\mathcal{C}^0_N$ such that $f = g$ on $\mathcal{A}^0_N$. The proof that $f = g$ on $\mathcal{C}^0_n$ is done by (descending) induction on the number of dummy players.

INITIALIZATION. For a game $(N,v) \in \mathcal{C}^0_N$, if $|D(N,v)| = n$, *i.e.* all players are dummy. Then v is additive and $f = g$ by hypothesis. By Remark 2, there is no game in which $|D(N,v)| = n-1$.

INDUCTION HYPOTHESIS. Assume that $f(N,v) = g(N,v)$ for all games $(N,v) \in \mathcal{C}^0_N$ such that $|D(N,v)| \geq d$, $0 < d \leq n-1$.

INDUCTION STEP. Choose any game $(N,v) \in \mathcal{C}^0_N$ such that $|D(N,v)| = d-1$. Because $|D(N,v)| < n-1$, there exists $i \in N\setminus D(N,v)$, which implies $D(N,v^i) \supseteq D(N,v)\cup\{i\}$ and $|D(N,v^i)| \geq |D(N,v)| + 1 = d$. Now pick any $j \in D(N,v)$. It holds that $v = v^j$, so that **PBCD** and the induction hypothesis imply that

$$\begin{aligned} f_j(N,v) &= f_j(N,v^i) + \frac{v(\{j\})}{v(\{i\})}\big(f_i(N,v) - f_i(N,v^j)\big) \\ &= f_j(N,v^i) \\ &= g_j(N,v^i) \\ &= g_j(N,v^i) + \frac{v(\{j\})}{v(\{i\})}\big(g_i(N,v) - g_i(N,v^j)\big) \\ &= g_j(N,v). \end{aligned} \tag{7}$$

Conclude that the assertion is proved for dummy players in (N,v). Next, pick any $j \in N\setminus(D(N,v)\cup\{i\})$. Note that $N\setminus(D(N,v)\cup\{i\}) \neq \varnothing$ since $|D(N,v)| < n+1$. Applied to i and j, **PBCD** can be rewritten as follows:

$$f_j(N,v) = f_j(N,v^i) + \frac{v(\{j\})}{v(\{i\})} f_i(N,v) - \frac{v(\{j\})}{v(\{i\})} f_i(N,v^j). \tag{8}$$

Reformulation (8) can be done for g too. Note that $|D(N, v^j)| \geq d$. Using **E** for f and g gives:

$$v(N) = f_i(N, v) + \sum_{j \in D(N,v)} f_j(N, v) + \sum_{j \in N \setminus (D(N,v) \cup \{i\})} f_j(N, v)$$
$$v(N) = g_i(N, v) + \sum_{j \in D(N,v)} g_j(N, v) + \sum_{j \in N \setminus (D(N,v) \cup \{i\})} g_j(N, v)$$

Subtracting the lower equation to the upper, using (7), (8) and the induction hypothesis yield:

$$\begin{aligned} 0 &= f_i(N,v) - g_i(N,v) + \sum_{j \in N \setminus (D(Nv) \cup \{i\})} \Bigg[f_j(N, v^i) - g_j(N, v^i) + \frac{v(\{j\})}{v(\{i\})} \big(f_i(N,v) - g_i(N,v) \big) \\ &\quad - \frac{v(\{j\})}{v(\{i\})} \big(f_i(N, v^j) - g_i(N, v^j) \big) \Bigg] \\ &= (f_i(N,v) - g_i(N,v)) \times \left(1 + \sum_{j \in N \setminus (D(N,v) \cup \{i\})} \frac{v(\{j\})}{v(\{i\})} \right) \end{aligned} \tag{9}$$

Since $(N, v) \in \mathcal{C}_N^0$, the right term in (9) is positive, and so $f_i(N, v) = g_i(N, v)$ for any non-dummy player $i \in N \setminus D(N, v)$. This completes the proof. ∎

In order to characterize PSh, we invoke **IGP**.

Proposition 10. *The proportional Shapley value is the unique value on $\mathcal{C}_N^0$ that satisfies **E**, **PBCD** and **IGP**.*

Proof. Clearly, PSh satisfies the three axioms. So consider any value f on $\mathcal{C}_N^0$ satisfying the three axioms. By **IGP**, f is uniquely determined on $\mathcal{A}_N^0$. Since f also satisfies **E** and **PBCD**, Proposition 9 implies that f is also uniquely determined on $\mathcal{C}_N^0$. ∎

5. Conclusion

The promising results obtained for the land production economies reveal that the proportional Shapley value can outperform the (weighted) Shapley value(s) in specific cases. A challenging extension of our work would be to confirm or invalidate this assessment by study the proportional Shapley value in the other applications listed in the introduction of the article. Finally, let us conclude by mentioning a recurrent weakness related to weighted values. Haeringer (2006) argue that some information is contained in the Harsanyi dividends since it can be interpreted as the coalitions' contribution to the worth of the grand coalition. He advocates that the distribution of a Harsanyi dividends among its members should depend on its sign, i.e. whether the associated coalition contributes negatively or positively to the worth of the grand coalition. Coming back to the proportional Shapley value, in a game with negative stand-alone worths (which is not so common in applications), positive dividends are distributed in inverse proportion to these stand-alone worths: the players with the worse stand-alone worth get the best shares of the dividend. This difficulty may be overcome by considering the approach developed in Haeringer (2006), thus ensuring that the associated payoff to a player is always increasing with respect to his or her initial weight. This is left for future work.

Appendix

Land production economies

We start by stating a Lemma which is essential to prove Proposition 1.

Lemma 3. *For any land production economy (N, a, z), there exists a unique function $g : N \longrightarrow \mathbb{R}$ such that $\Delta_{v_{a,z}} = v_{a,g}$, and defined, for each $s = \{1, \ldots, n\}$, by:*

$$g(s) = \sum_{k=0}^{s-1} (-1)^{s-1-k} \binom{s-1}{k} z(k+1). \tag{10}$$

Moreover, for each $s = \{1, \ldots, n\}$, it holds that:

$$z(s) = \sum_{k=0}^{s-1} \binom{s-1}{k} g(k+1). \tag{11}$$

Proof. Consider any land production economy (N, a, z). For each $S \in 2^N$, we have:

$$\begin{aligned}
\Delta_{v_{a,z}}(S) &= \sum_{T \subseteq S} (-1)^{s-t} v_{a,z}(T) \\
&= \sum_{T \subseteq S} (-1)^{s-t} z(t) \frac{1}{L} \sum_{i \in T} a_i \\
&= \frac{1}{L} \sum_{i \in S} a_i \sum_{T \subseteq S, T \ni i} (-1)^{s-t} z(t) \\
&= \frac{1}{L} \sum_{i \in S} a_i \underbrace{\sum_{k=0}^{s-1} (-1)^{s-1-k} \binom{s-1}{k} z(k+1)}_{g(s)}.
\end{aligned}$$

The last equation defines g and we have $\Delta_{v_{a,z}} = v_{a,g}$. Conversely, we may recover z:

$$\begin{aligned}
v_{a,z}(S) &= \sum_{T \subseteq S} \Delta_{v_{a,z}}(S) \\
&= \sum_{T \subseteq S} g(t) \frac{1}{L} \sum_{i \in T} a_i \\
&= \frac{1}{L} \sum_{i \in S} a_i \sum_{T \subseteq S, T \ni i} g(t) \\
&= \frac{1}{L} \sum_{i \in S} a_i \underbrace{\sum_{k=0}^{s-1} \binom{s-1}{k} g(k+1)}_{=z(s)},
\end{aligned}$$

which completes the proof. ∎

Proof. **(Proposition 1)** Consider any land production economy (N, a, z). For each $i \in N$, by Lemma 3, we obtain:

$$\begin{aligned} PSh_i(N, v_{a,z}) &= \sum_{S \in 2^N : S \ni i} \frac{v_{a,z}(\{i\})}{\sum_{j \in S} v_{a,z}(\{j\})} \Delta_{v_{a,z}}(S) \\ &= \sum_{S \in 2^N : S \ni i} \frac{a_i}{\sum_{j \in S} a_j} \left(g(s) \frac{1}{L} \sum_{k \in S} a_k \right) \\ &= a_i \frac{1}{L} \sum_{S \in 2^N : S \ni i} g(s) \\ &= a_i \frac{1}{L} \sum_{k=0}^{n-1} \binom{n-1}{k} g(k+1) \\ &= \frac{a_i}{L} z(n), \end{aligned}$$

as desired. ■

Next, we provide a formulation of the Shapley value for land production economies. To this end, we rely on generating functions (see chapter 4 in Stanley, 1986, for an introduction), which have been widely used to compute power indices, for instance in Alonso-Meijide et al. (2014). For any function $z : N \longrightarrow \mathbb{R}$, let us define the (exponential) generating functions $Z(x) = \sum_{k \geq 0} z(k+1) x^k / k!$ and the corresponding $G(x) = \sum_{k \geq 0} g(k+1) x^k / k!$, where $g : N \longrightarrow \mathbb{R}$ is defined by formula (10). Here too, a Lemma is useful.

Lemma 4. *For any $z : N \longrightarrow \mathbb{R}$, one has $G(x) = \mathrm{e}^{-x} Z(x)$.*

Proof. Let us show that formula (11) is translated into $Z(x) = \mathrm{e}^x G(x)$ in the context of exponential generating functions:

$$\begin{aligned} \mathrm{e}^x G(x) &= \left(\sum_{l \geq 0} \frac{x^l}{l!} \right) \times \left(\sum_{k \geq 0} g(k+1) \frac{x^k}{k!} \right) \\ &= \sum_{k,l \geq 0} g(k+1) \frac{x^{l+k}}{l! k!} \\ &= \sum_{n \geq 0} \sum_{k=0}^{n} g(k+1) \frac{x^n}{(n-k)! k!} \\ &= \sum_{n \geq 0} \left(\sum_{k=0}^{n} \binom{n}{k} g(k+1) \right) \frac{x^n}{n!} \\ &= \sum_{n \geq 0} z(n+1) \frac{x^n}{n!} \\ &= Z(x), \end{aligned} \tag{12}$$

as desired. ■

For two exponential generating functions Z_1, Z_2, define the convolution operation $(Z_1 \star Z_2)(x) = \int_0^x Z_1(t) Z_2(x-t) \mathrm{d}t$. Recall that $(Z_1 \star Z_2)'(x) = (Z_1 \star Z_2')(x) + Z_1(x) Z_2(0)$ for instance.

Proposition 11. *For any land production economy (N, a, z), it holds that:*

$$Sh_i(N, v_{a,z}) = \frac{1}{L}\left(s(n)a_i + h(n) \sum_{j \in N \setminus i} a_j \right) \tag{13}$$

where function s comes from the exponential generating function $S(x) = \sum_{k \geq 0} s(k+1)x^k/k! = (Z \star \exp)(x)/x$ and $H(x) = \sum_{k \geq 0} h(k+2)x^k/k! = (xZ(x) - (Z \star \exp)(x))/x^2$.

Proof. Firstly, let us show that the Shapley value may be written as (13):

$$\begin{aligned}
Sh_i(N, v_{a,z}) &= \sum_{S \ni i} \frac{\Delta_{v_{a,z}}(S)}{s} \\
&= \sum_{S \ni i} \frac{v_{a,g}(S)}{s} \\
&= \sum_{S \ni i} \left(\frac{g(s)}{s} \frac{1}{L} \sum_{j \in S} a_j \right) \\
&= \frac{1}{L} \sum_{j=1}^{n} a_j \left(\sum_{S \ni i,j} \frac{g(s)}{s} \right) \\
&= \frac{1}{L}\left(\left(\sum_{S \ni i} \frac{g(s)}{s} \right) a_i + \sum_{j \in N \setminus i} a_j \left(\sum_{S \ni i,j} \frac{g(s)}{s} \right) \right) \\
&= \frac{1}{L}\left(\left(\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{g(k+1)}{k+1} \right) a_i + \sum_{j \in N \setminus i} a_j \left(\sum_{k=0}^{n-2} \binom{n-2}{k} \frac{g(k+2)}{k+2} \right) \right) \\
&= \frac{1}{L}\Bigg(\underbrace{\left(\sum_{k=0}^{n-1} \binom{n-1}{k} \frac{g(k+1)}{k+1} \right)}_{s(n)} a_i + \sum_{j \in N \setminus i} a_j \underbrace{\left(\sum_{k=0}^{n-2} \binom{n-2}{k} \frac{g(k+2)}{k+2} \right)}_{h(n)} \Bigg) \qquad (14)
\end{aligned}$$

Secondly, having defined $S(x) = \sum_{k\geq 0} s(k+1)x^k/k!$, let us connect $S(x)$ with $Z(x)$.

$$\begin{aligned}
S(x) &= \sum_{n\geq 0} s(n+1)\frac{x^n}{n!} \\
&= \sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}\frac{g(k+1)}{k+1}\frac{x^n}{n!} \\
&= \sum_{n\geq 0}\sum_{k=0}^{n}\left(\frac{g(k+1)}{k+1}\frac{x^k}{k!}\right)\left(\frac{x^{n-k}}{(n-k)!}\right) \\
&= \sum_{k,l\geq 0}\left(\frac{g(k+1)}{k+1}\frac{x^k}{k!}\right)\frac{x^l}{l!} \\
&= \left(\sum_{k\geq 0} g(k+1)\frac{x^k}{k+1}\frac{1}{k!}\right)\times\left(\sum_{l\geq 0}\frac{x^l}{l!}\right) && (15) \\
&= \left(\sum_{k\geq 0} g(k+1)\frac{1}{x}\int_0^x t^k \mathrm{d}t\frac{1}{k!}\right)\times \mathrm{e}^x \\
&= \frac{\mathrm{e}^x}{x}\int_0^x\left(\sum_{k\geq 0} g(k+1)\frac{t^k}{k!}\right)\mathrm{d}t \\
&= \frac{\mathrm{e}^x}{x}\int_0^x G(t)\mathrm{d}t \\
&= \frac{\mathrm{e}^x}{x}\int_0^x Z(t)\mathrm{e}^{-t}\mathrm{d}t \\
&= \frac{1}{x}\int_0^x Z(t)\mathrm{e}^{x-t}\mathrm{d}t \\
&= \frac{(Z\star\exp)(x)}{x} && (16)
\end{aligned}$$

Lastly, we find a closed expression for $H(x) = \sum_{k\geq 0} h(k+2)x^k/k!$. We will need to define $\hat{G}(x) = \sum_{k\geq 0} g(k+1)/(k+1)\times(x^k/k!)$ so that by (15), $\hat{G}(x) = S(x)\mathrm{e}^{-x}$. Moreover $\hat{G}'(x) = \sum_{k\geq 0} g(k+2)/(k+$

$2) \times (x^k/k!)$.

$$\begin{aligned} H(x) &= \sum_{n\geq 0} h(n+2)\frac{x^n}{n!} \\ &= \sum_{n\geq 0}\sum_{k=0}^{n}\binom{n}{k}\frac{g(k+2)}{k+2}\frac{x^n}{n!} \\ &= \left(\sum_{k\geq 0} g(k+2)\frac{x^k}{k+2}\frac{1}{k!}\right)\times\left(\sum_{l\geq 0}\frac{x^l}{l!}\right) \\ &= \mathrm{e}^x\left(\sum_{k\geq 0} g(k+2)\frac{x^k}{k+2}\frac{1}{k!}\right) \\ &= \mathrm{e}^x\hat{G}'(x) \\ &= \mathrm{e}^x(S'(x)\mathrm{e}^{-x} - S(x)\mathrm{e}^{-x}) \\ &= S'(x) - S(x) \\ &= \frac{x(F\star\exp)(x) + xZ(x)\exp(0) - (Z\star\exp)(x)}{x^2} - \frac{(Z\star\exp)(x)}{x} \\ &= \frac{xZ(x) - (Z\star\exp)(x)}{x^2} \end{aligned} \tag{17}$$

■

Below are some examples of specification of function z.

- if $z(s) = 1$, then $Z(x) = \mathrm{e}^x$ so that $G(x) = 1$. Moreover $S(x) = \mathrm{e}^x$ and $H(x) = 0$.
- if $z(s) = s$, then $Z(x) = (1+x)\mathrm{e}^x$ so that $G(x) = 1+x$. Moreover $S(x) = \mathrm{e}^x(1+x/2)$ and $H(x) = \mathrm{e}^x/2$.
- if $z(s) = 1/s$, then $Z(x) = (\mathrm{e}^x - 1)/x$ so that $G(x) = (1-\mathrm{e}^{-x})/x = F(-x)$.
- if $z(s) = 2^{s-1}$, then $Z(x) = \mathrm{e}^{2x}$ so that $G(x) = \mathrm{e}^x$. Moreover $S(x) = \mathrm{e}^x(\mathrm{e}^x - 1)/x$ and $H(x) = \mathrm{e}^{2x}(\mathrm{e}^{-x} + x - 1)/x^2$.

Proof of Proposition 5

Throughout this section, we consider fixed $N \in U$ and $a \in \mathbb{R}^N_{++}$. Several definitions will be useful. Firstly, define the class $\mathcal{C}^{a+}_N$ as

$$\mathcal{C}^{a+}_N = \{(N,v)\in\mathcal{C}_N | \exists c\in\mathbb{R} : \forall i\in N, v(\{i\}) = ca_i\},$$

so that $\mathcal{C}^a_N = \mathcal{C}^{a+}_N \cap \mathcal{C}^0_N$, i.e. $\mathcal{C}^a_N$ contains all games in $\mathcal{C}^{a+}_N$, except those with null stand-alone worths. Obviously, $\mathcal{C}^{a+}_N$ is a real vector space. Furthermore, the dimension of $\mathcal{C}^{a+}_N$ is $2^n - 1 - n + 1 = 2^n - n$. Note also that $\mathcal{C}^{a+}_N$ is the smallest vector space that contains $\mathcal{C}^a_N$. Secondly, for all $S \in 2^N$ such that $s \geq 2$, define the game (N, r_S) such that $r_S = u_S + v_a$. Lemma 5 essentially states that all games in $\mathcal{C}^a_N$ admit a unique decomposition via the collection $\{(N,v_a),(N,r_S)_{S\in 2^N : s\geq 2}\}$ and enunciates properties of the associated coefficients. More specifically, $\{(N,v_a),(N,r_S)_{S\in 2^N : s\geq 2}\}$ is a basis for the vector space $\mathcal{C}^{a+}_N$, and this basis is composed of games in $\mathcal{C}^a_N$ only.

Lemma 5. *Consider any* $(N, v) \in \mathcal{C}_N^{a+}$*, and let* $v(\{i\}) = ca_i$ *for all* $i \in N$*,* $c \in \mathbb{R}$*. Then,*

(i) there are unique coefficients $\gamma_v(S) \in \mathbb{R}$*,* $S \in 2^N$*,* $s \geq 2$*, and* $\gamma_v(0) \in \mathbb{R}$ *such that*

$$v = \sum_{S \in 2^N : s \geq 2} \gamma_v(r_S) r_S + \gamma_v(v_a) v_a;$$

(ii) for all $S \in 2^N$*,* $s \geq 2$*,* $\gamma_v(r_S) = \Delta_{v - cv_a}(S)$*, and* $\gamma_v(v_a) = c - \sum_{S \in 2^N : s \geq 2} \Delta_{v - cv_a}(S)$*;*

(iii) $\sum_{S \in 2^N : s \geq 2} \gamma_v(r_S) + \gamma_v(v_a) = c$*;*

(iv) $(N, v) \in \mathcal{C}_N^a$ *if and only if* $\sum_{S \in 2^N : s \geq 2} \gamma_v(r_S) + \gamma_v(v_a) \neq 0$.

Proof. Consider any $(N, v) \in \mathcal{C}_N^{a+}$, and let $v(\{i\}) = ca_i$ for all $i \in N$, $c \in \mathbb{R}$. We can write v as $(v - cv_a) + cv_a$, where $(v - cv_a)$ is a characteristic function on N that vanishes for singletons. Therefore, $v - cv_a$ can be written as

$$(v - cv_a) = \sum_{S \in 2^N : s \geq 2} \Delta_{v - cv_a}(S) u_S.$$

From this, we get

$$\begin{aligned}
v &= (v - cv_a) + cv_a \\
&= \sum_{S \in 2^N : s \geq 2} \Delta_{v - cv_a}(S) u_S + cv_a \\
&= \sum_{S \in 2^N : s \geq 2} \Delta_{v - cv_a}(S)(u_S + v_a) - \sum_{S \in 2^N : s \geq 2} \Delta_{v - cv_a}(S) v_a + cv_a \\
&= \sum_{S \in 2^N : s \geq 2} \Delta_{v - cv_a}(S)(u_S + v_a) + \left(c - \sum_{S \in 2^N : s \geq 2} \Delta_{v - cv_a}(S) \right) v_a \\
&= \sum_{S \in 2^N : s \geq 2} \Delta_{v - cv_a}(S) r_S + \left(c - \sum_{S \in 2^N : s \geq 2} \Delta_{v - cv_a}(S) \right) v_a \qquad (18)
\end{aligned}$$

Letting $\gamma_v(r_S) = \Delta_{v - cv_a}(S)$ and $\gamma_v(v_a) = c - \sum_{S \in 2^N : s \geq 2} \Delta_{v - cv_a}(S)$, we obtain

$$v = \sum_{S \in 2^N : s \geq 2} \gamma_v(r_S) r_S + \gamma_v(v_a) v_a.$$

So, the collection of games $\{(N, v_a), (N, r_S)_{S \in 2^N : s \geq 2}\}$ spans $\mathcal{C}_N^{a+}$, and this collection contains $2^n - n$ elements, *i.e.* as many elements as the dimension of $\mathcal{C}_N^{a+}$. Conclude that $\{(N, v_a), (N, r_S)_{S \in 2^N : s \geq 2}\}$ is a basis for the vector space $\mathcal{C}_N^{a+}$. Therefore, any game $(N, v) \in \mathcal{C}_N^{a+}$ is uniquely decomposed as in (18), proving claim (i). Claim (ii) follows from (18), claim (iii) is obvious via claim (ii), and claim (iv) is obvious from claim (iii) . ■

Lemma 6 is technical and will be used on the coefficients exhibited in Lemma 5 (i) so as to ensure the property highlighted in Lemma 5 (iv).

Lemma 6. *Let $(x_1, \dots, x_q)$ be a sequence of $q \geq 1$ real numbers such that $x_k \in \mathbb{R}^*$ for all $k \in \{1, \dots, q\}$ and $\sum_{k=1}^{q} x_k \in \mathbb{R}^*$. Then, there exists an ordering $(x_{(1)}, \dots, x_{(q)})$ of $(x_1, \dots, x_q)$ such that, for all $k \in \{1, \dots, q\}$, $\sum_{l=1}^{k} x_{(l)} \in \mathbb{R}^*$.*

Proof. The proof is by induction on q.
Initialization. The claim is trivial for the case $q = 1$, and any of the two possible orderings can be used to prove easily the case $q = 2$.
Induction hypothesis. Assume that there exists a desired ordering for all allowed sequences $(x_1, \dots, x_q)$ such that $q \leq \bar{q}$, $\bar{q} \geq 2$.
Induction step. Consider a sequence $(x_1, \dots, x_q)$, $q = \bar{q} + 1$, such that $x_k \in \mathbb{R}^*$ for all $k \in \{1, \dots, q\}$ and $\sum_{k=1}^{q} x_k \in \mathbb{R}^*$. We distinguish two cases. Firstly, suppose that $\sum_{k=1}^{q-1} x_k \in \mathbb{R}^*$, then the induction hypothesis can be applied to the sub-sequence $(x_1, \dots, x_{q-1})$. The desired ordering on $(x_1, \dots, x_q)$ is constructed by considering a desired ordering on the sub-sequence $(x_1, \dots, x_{q-1})$ and by adding number x_q in position q. Secondly, suppose that $\sum_{k=1}^{q-1} x_k = 0$. Since the numbers x_k, $k \in \{1, \dots, q-1\}$, are all non-null, there exists a number x_k, $k \in \{1, \dots, q-1\}$, such that $sign(x_k) = -sign(x_q)$. Thus $\sum_{l \in \{1, \dots, k-1, k+1, \dots, q\}} x_l \in \mathbb{R}^*$, which means that the induction hypothesis can be applied to the sub-sequence $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_q)$. Similarly as before, the desired ordering on $(x_1, \dots, x_q)$ is constructed by considering a desired ordering on the sub-sequence $(x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_q)$ and by adding number x_k in position q. ■

Proof. (Proposition 5) We shall show that if a value on $\mathcal{C}^0$ satisfies **E**, **DPO** and **WL**, and is uniquely determined on $\mathcal{QA}^0$, then f is uniquely determined on $\mathcal{C}^0$. So consider such a value f. Fix some $N \in \mathcal{U}$ and some $a \in \mathbb{R}^N_{++}$. Pick any $(N, v) \in \mathcal{C}^a_N$. By Lemma 5, we have that

$$v = \sum_{S \in 2^N : s \geq 2} \gamma_v(r_S) r_S + \gamma_v(v_a) v_a.$$

Let $q \in \{1, \dots, 2^n - n\}$ be the number of non-null coefficients in the above decomposition, where $q > 0$ by definition of $\mathcal{C}^a_N$ and Lemma 5 (iii). Denote by $(\gamma_v(v_1), \dots, \gamma_v(v_q))$ the associated sequence of coefficients. By Lemma 5 (iii), it holds that $\sum_{k=1}^{q} \gamma_v(v_k) = c \neq 0$. Thus, we can apply Lemma 6: there is an ordering $(\gamma_v(v_{(1)}), \dots, \gamma_v(v_{(q)}))$ of $(\gamma_v(v_1), \dots, \gamma_v(v_q))$ such that, for all $k \in \{1, \dots, q\}$,

$$\sum_{l=1}^{k} \gamma_v(v_{(l)}) \neq 0. \tag{19}$$

Now, denote by (N, v^k) the game such that

$$v^k = \sum_{l=1}^{k} \gamma_v(v_{(l)}) v_{(l)}.$$

By (19) and Lemma 5 (iv), $(N, v^k) \in \mathcal{C}^a_N$. Successive applications of **WL** to games (N, v^k) and $(N, \gamma_v(v_{k+1}) v_{(k+1)})$ for all $k \in \{1, \dots, q-1\}$ according to ordering $(\gamma_v(v_{(1)}), \dots, \gamma_v(v_{(q)}))$ imply that

$$f(N,v) = \sum_{S\in 2^N : s\geq 2} \gamma_v(r_S) f(N,r_S) + \gamma_v(v_a) f(N,v_a).$$

Note that $(N,v_a) \in \mathcal{QA}^0$. Moreover, consider any $S \in 2^N$ such that $s \geq 2$. Each player $i \in N\backslash S$ is dummy in (N,r_S), which means that $f_i(N,r_S)$ is uniquely determined by **E** and **DPO** for such players. Then $n-s$ successive applications of **DPO** yield that $f_i(N,r_S) = f_i(S,r_S)$ for all $i \in S$. Remark that $(S,r_S) \in \mathcal{QA}^0$. By assumption f is uniquely determined on $\mathcal{QA}^0$, so that f is uniquely determined in games (N,r_S), $S \in 2^N$, $s \geq 2$, and (N,v_a). This completes the proof. ∎

Logical independence of the axioms in Propositions 7 and 8

Proposition 7:

- The Shapley value Sh on $\mathcal{C}^0$ satisfies **E**, **DPO**, **WL**, **WC** but not **PS**.
- PSh on $\mathcal{C}^0$ satisfies **E**, **DPO**, **WL**, **WC** but not **S**.
- The value on $\mathcal{C}^0$ which assigns to each $(N,v) \in \mathcal{C}^0$ and each $i \in N$, the payoff

  $$\Delta_v(\{i\}) + \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}\Delta_v(N)$$

  satisfies **DPO**, **WL**, **WC**, **PS** but not **E**.
- The value on $\mathcal{C}^0$ which assigns to each $(N,v) \in \mathcal{C}^0$ and each $i \in N$, the payoff $\Delta_v(\{i\}) + \Delta_v(N)/n$ satisfies **DPO**, **WL**, **WC**, **S** but not **E**.
- The proportional value on $\mathcal{C}^0$, which assigns to each $(N,v) \in \mathcal{C}^0$ and each $i \in N$, the payoff

  $$\frac{v(\{i\})}{\sum_{j\in N} v(\{j\})} v(N)$$

  satisfies **E**, **WL**, **WC**, **PS** but not **DPO**.
- The equal surplus division ESD on $\mathcal{C}^0$, which assigns to each $(N,v) \in \mathcal{C}^0$ and each $i \in N$, the payoff

  $$ESD_i(N,v) = v(\{i\}) + \frac{1}{n}\left(v(N) - \sum_{j\in N} v(\{j\})\right)$$

  satisfies **E**, **WL**, **WC**, **PS**, but not **DPO**.
- The value f on $\mathcal{C}^0$ defined for each $(N,v) \in \mathcal{C}^0$ and each $i \in N$, by

  $$f_i(N,v) = v(\{i\}) + \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}\Delta_v(N) + \sum_{S\in 2^N : S\ni i, s\in\{2,\ldots,n-1\}} \frac{v(\{i\})^2}{\sum_{j\in S} v(\{j\})^2}\Delta_v(S).$$

  satisfies **E**, **DPO**, **WC**, **PS** but not **WL**.

- The value f on $\mathcal{C}^0$ defined for each $(N,v)\in\mathcal{C}^0$ and each $i\in N$, by

$$f_i(N,v) = v(\{i\}) + \frac{1}{n}\Delta_v(N) + \sum_{S\in 2^N: S\ni i, s\in\{2,\ldots,n-1\}} \frac{v(\{i\})^2}{\sum_{j\in S} v(\{j\})^2}\Delta_v(S).$$

  satisfies **E**, **DPO**, **WC**, **PS** but not **WL**.

- For a given integer $k\geq 2$, the value f on $\mathcal{C}^0$ defined for each $(N,v)\in\mathcal{C}^0$ and each $i\in N$, by

$$f_i(N,v) = \sum_{S\in 2^N: S\ni i, s\leq k} \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})}\Delta_v(S) + \sum_{S\in 2^N: S\ni i, s>k} \frac{\Delta_v(S)}{s}.$$

  satisfies **E**, **DPO**, **WL**, **PS** but not **WC**.

- For a given integer $k\geq 2$, the value f on $\mathcal{C}^0$ defined for each $(N,v)\in\mathcal{C}^0$ and each $i\in N$, by

$$f_i(N,v) = \sum_{S\in 2^N: S\ni i, s>k} \frac{v(\{i\})}{\sum_{j\in S} v(\{j\})}\Delta_v(S) + \sum_{S\in 2^N: S\ni i, s\leq k} \frac{\Delta_v(S)}{s}.$$

  satisfies **E**, **DPO**, **WL**, **S** but not **WC**.

Proposition 8:

- The null solution on $\mathcal{C}^0$ satisfies **DPO**, **WL**, **PAM**, **EAM**, but not **E**.
- The Shapley value Sh on $\mathcal{C}^0$ satisfies **E**, **DPO**, **WL**, **EAM** but not **PAM**.
- PSh on $\mathcal{C}^0$ satisfies **E**, **DPO**, **WL**, **PAM** but not **EAM**.
- The proportional value on $\mathcal{C}^0$, which assigns to each $(N,v)\in\mathcal{C}^0$ and each $i\in N$, the payoff

$$\frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}v(N)$$

  satisfies **E**, **WL**, **PAM**, but not **DPO**.

- The equal surplus division ESD on $\mathcal{C}^0$, which assigns to each $(N,v)\in\mathcal{C}^0$ and each $i\in N$, the payoff

$$ESD_i(N,v) = v(\{i\}) + \frac{1}{n}\left(v(N) - \sum_{j\in N} v(\{j\})\right)$$

  satisfies **E**, **WL**, **EAM**, but not **DPO**.

- The value f on $\mathcal{C}^0$ defined for each $(N,v)\in\mathcal{C}^0$ and each $i\in N$, by

$$f_i(N,v) = v(\{i\}) + \frac{v(\{i\})}{\sum_{j\in N} v(\{j\})}\Delta_v(N) + \sum_{S\in 2^N: S\ni i, s\in\{2,\ldots,n-1\}} \frac{v(\{i\})^2}{\sum_{j\in S} v(\{j\})^2}\Delta_v(S).$$

  satisfies **E**, **DPO**, **PAM**, but not **WL**.

- The value f on $\mathcal{C}^0$ defined for each $(N,v)\in\mathcal{C}^0$ and each $i\in N$, by

$$f_i(N,v) = v(\{i\}) + \frac{1}{n}\Delta_v(N) + \sum_{S\in 2^N: S\ni i, s\in\{2,\ldots,n-1\}} \frac{v(\{i\})^2}{\sum_{j\in S} v(\{j\})^2}\Delta_v(S).$$

  satisfies **E**, **DPO**, **EAM** but not **WL**.

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