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using addition and transfer invariance

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October 2013

**Working paper No. 2013–08**

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# A Decomposition of the Space of TU-games Using Addition and Transfer Invariance

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## Abstract

In Béal et al. (2013) two new axioms of invariance, called Addition invariance and Transfer invariance respectively, are introduced to design allocation rules for TU-games. Here, we derive direct sum decompositions of the linear space of TU-games by using the TU-games selected to construct the operations of Addition and Transfer. These decompositions allow us to recover previous characterization results obtained by Béal et al. (2013), to provide new characterizations of well-known (class of) of allocation rules and also to design new allocation rules.

*Keywords:* Addition invariance, Transfer invariance, Direct-sum decomposition, Allocation rules  
*2010 MSC:* 91A12

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## 1. Introduction

A situation in which a finite set of players can obtain certain payoffs by cooperation can be described by a cooperative game with transferable utility, shortly a TU-game. A (point-valued) solution, also called an allocation rule, on a class of TU-games assigns a payoff vector to every game in the class. The axioms employed to design allocation rules in TU-games can be divided up into punctual and relational axioms. A punctual axiom applies to each game separately and a relational axiom relates payoff vectors of games that are related in a certain way. The class of relational axioms includes as a subclass the axioms of invariance. Such axioms specify either the same payoff vector or the same payoff for some specific players across TU-games that are somehow linked. For instance, the well-known axiom of Marginality (Young, 1985) requires to attribute the same payoff to a player in two games where his contributions to coalitions are identical. In Béal et al. (2013) two new axioms of invariance are introduced. The first axiom of invariance, called Addition invariance, requires that if the worths of all coalitions of a given size, except the grand coalition, fluctuate by the same amount, then the allocation rule should prescribe the same payoff vector. The second axiom of invariance, called Transfer invariance, requires that if the worth of two coalitions of the same size is affected by opposite amounts, then the payoff of a player belonging to both coalitions should not change. It is shown that the class of allocation rules satisfying the combination of Addition invariance and Transfer invariance strictly contains the class of Efficient,

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Linear and Anonymous allocation rules. This class includes the most popular allocation rules, among which the Shapley value (1953), the Equal Division rule, the Equal Surplus Division rule, the Consensus value (Ju, Borm, Ruys, 2007), the Solidarity value (Nowak, Radzik, 1994). Béal et al. (2013) establish that the five above-mentioned allocation rules can be characterized by Addition invariance, Transfer invariance plus one punctual axiom and possibly one another relation axiom. Each punctual axiom used in these characterizations specifies the payoff obtained by some specific players in certain TU-games. These specific players are the well-known dummy players, nullifying players, quasi-dummy players, and average dummy players respectively.

The set  $V_N$  of TU-games on a fixed player set  $N$  forms a linear space. In this paper, we exploit this structure to derive direct sum decompositions of  $V_N$  through the TU-games used to construct the operations of Addition and Transfer respectively. Each decomposition is indexed by a player selected arbitrarily in  $N$ . The reason is that the operation of Transfer between two coalitions of the same size concerns only the players lying in both coalitions. So, for each player, we consider all pairs of coalitions of the same size to which it belongs.

In a first step, we show that, for each player, the subspaces of TU-games used for the operation of Addition and the operation of Transfer respectively are in direct sum. For each player  $i \in N$ , let us denote by  $W_i$  the resulting direct sum. This result is important because it provides a method to characterize allocation rules using Addition invariance, Transfer invariance and one punctual axiom. Indeed, assume that, for each player  $i \in N$ ,  $V_N$  is the direct sum of  $W_i$  and some subspace  $P_i$ , where  $P_i$  is the subspace of TU-games for which the punctual axiom is applied on player  $i$ . Assume further that the punctual axiom determines player  $i$ 's payoff in  $P_i$ . Then, it is straightforward to verify that there exists exactly one allocation rule satisfying these two axioms of invariance and the punctual axiom.

In a second step, we apply this method to recover previous results obtained by Béal et al. (2013), to provide new characterizations of well-known (class of) rules and also to design new allocation rules. We proceed as follows. First, we show that, for each  $i \in N$ ,  $V_N$  is the direct sum of  $W_i$  and the subspace of the TU-games for which player  $i$  is a dummy player. Because the Dummy player axiom is a punctual axiom and the Shapley value satisfies Addition invariance, Transfer invariance and the Dummy player axiom, we directly get the characterization of the Shapley value obtained by Béal et al. (2013). Second, we show that the subspace of TU-games for which player  $i$  is dummy is the direct sum of the subspace on which player  $i$  is a null player and the subspace of TU-games generated by the unanimity TU-game on player  $i$ . This new decomposition allows us to provide a new characterization of the Shapley value by using Addition invariance, Transfer invariance, the null game axiom, another axiom of invariance and a new axiom of covariance. The null game axiom requires that players get zero in the null game. The added axiom of invariance, called null player invariance, requires that adding a TU-game in which player  $i$  is null to any TU-game does not change his payoff. It turns out that null player invariance is equivalent to the Marginality axiom introduced by Young (1985). Our axiom of covariance requires that if one adds a multiple of the unanimity TU-game on player  $i$  to any TU-game, then player  $i$ 's payoff is equal to his original payoff plus this multiple. Third, this decomposition of  $V_N$  not only serves to provide a new characterization of the Shapley value but also to deal with the inverse problem of determining the set of TU-games for which the Shapley value of player  $i$  is equal to a fixed scalar. We show that the kernel of the Shapley value for  $i$  is equal to the direct sum of  $W_i$  and the subspace for which player  $i$  is null. Due to the above decomposition, we are able to solve this inverse problem.

We then apply this method to the Equal Division rule, to a subclass of Efficient, Linear and

Anonymous allocation rules including the most popular allocation rules used in the literature, and finally to allocation rules which do not satisfy any of these three axioms. In particular, we show that a large subclass of Efficient, Linear and Anonymous allocation rules as well as a large class of allocation rules which do not verify Efficiency, Linearity and Anonymity can be characterized by Addition invariance, Transfer invariance and a punctual axiom adapted from the Dummy player axiom. This means that the manner in which one modifies the Dummy player axiom allows to delineate a large subclass of Efficient, Linear and Anonymous allocation rules from a class of allocation rules which do not verify any of these three axioms. An instance of such allocation rules is the exponential Shapley value which distributes to each player the exponential of the Shapley value divided by a scalar.

In a third step, we revisit the approach by Yokote (2013) who introduces a new axiom of invariance, called Strong addition Invariance. This axiom is related to the axiom of Addition invariance. It requires that the payoffs distributed to the players in a TU-game do not change if some specific TU-games are added. The set of TU-games that can be added to a TU-game forms a basis of the kernel of the Shapley value. Yokote (2013) shows that the combination of Strong addition invariance and the Dummy player axiom characterizes the Shapley value. We show that the Dummy player axiom is stronger than necessary in the sense that the kernel of the Shapley value and the subspace of TU-games for which a player is dummy are not in direct sum. We then substitute the Dummy player axiom by the weaker axiom of Inessential game requiring that the players get their stand-alone worth in each inessential (or additive) TU-games. It turns out that the linear space of TU-games  $V_N$  is the direct sum of the space of inessential TU-games and the kernel of the Shapley value. Thus, we obtain a stronger characterization of the Shapley value by Strong invariance and the Inessential game axiom. In fact, our characterization is optimal in presence of Strong addition invariance in the sense that the Inessential game axiom can not be weakened. We also offer an alternative proof of the fact that the union of the set of TU-games generating the subspace of inessential TU-games and the set of TU-games used to construct the axiom of Strong addition invariance form a basis of  $V_N$ . This new proof reveals some interesting properties of this basis.

In a last step, we show that the decomposition method using the axioms of invariance and covariance proves very useful to show logical independence of the axioms used in the various characterizations.

The rest of the paper is organized as follows. Section 2 provides preliminaries. Section 3 introduces the axioms of (Strong) Addition invariance and Transfer invariance and summarizes the main results obtained in Béal et (2013) and Yokote (2013). Section 4 is devoted to the decomposition method. Section 5 gives the main characterizations. Section 6 revisits the approach developed by Yokote (2013). Section 7 deals with the logical independence of the axioms used in the characterizations.

## 2. Preliminaries

Throughout this article, the cardinality of a finite set  $S$  will be denoted by  $|S|$ , the collection of all subsets of  $S$  will be denoted by  $2^S$ , and weak set inclusion will be denoted by  $\subseteq$ . Also for notational convenience, we will write singleton  $\{i\}$  as  $i$ . Let  $V$  be a real linear space equipped with an inner product “.”. Its additive identity element is denoted by  $\mathbf{0}$  and its dimension by  $\dim(V)$ . Given a linear subspace  $U$  of  $V$ , we denote by  $U^\perp$  its orthogonal complement. If  $V$  is the direct

sum of the subspaces  $V^1$  and  $V^2$ , i.e.  $V = V^1 + V^2$  and  $V^1 \cap V^2 = \{\mathbf{0}\}$ , we write  $V = V^1 \oplus V^2$ . If  $X$  is a non-empty subset of  $V$ , then  $\text{Sp}(X)$  denotes the smallest subspace containing  $X$ .

Let  $N = \{1, 2, \dots, n\}$  be a fixed and finite set of  $n$  players. Subsets of  $N$  are called *coalitions*, while  $N$  is called the *grand coalition*. A cooperative game with transferable utility or simply a *TU-game* on  $N$  is a function  $v : 2^N \rightarrow \mathbb{R}$  such that  $v(\emptyset) = 0$ . For each coalition  $S \subseteq N$ ,  $v(S)$  describes the *worth* of the coalition  $S$  when its members cooperate. The set  $H(v)$ , called the *support* of  $v$ , is the set of coalitions  $S \subseteq N$  such that  $v(S) \neq 0$ . For any two TU-games  $v$  and  $w$  in  $V_N$  and any  $\alpha \in \mathbb{R}$ , the TU-game  $\alpha v + w \in V_N$  is defined as follows: for each  $S \subseteq N$ ,  $(\alpha v + w)(S) = \alpha v(S) + w(S)$ . The inner product  $v \cdot w$  is defined as  $\sum_{S \subseteq N} v(S)w(S)$ . Let  $V_N$  be the linear space of all TU-games on  $N$ , where  $\dim(V_N) = 2^n - 1$ .

For any non-empty coalition  $T \subseteq N$ , the *Dirac TU-game*  $\delta_T \in V_N$  is defined as:  $\delta_T(T) = 1$ , and  $\delta_T(S) = 0$  for each other  $S$ . Clearly, the collection of all Dirac TU-games is a basis for  $V_N$ . For  $k \in \{1, \dots, n\}$  let the TU-game  $1_k$  defined as:  $1_k(S) = 1$  if  $|S| = k$  and  $1_k(S) = 0$  if  $|S| \neq k$ . We have  $1_n = \delta_N$ . For any non-empty coalition  $T \subseteq N$ , the *unanimity TU-game* on  $u_T$  on  $T$  is defined as:  $u_T(S) = 1$  if  $S \supseteq T$  and  $u_T(S) = 0$  otherwise. For any non-empty coalition  $T \subseteq N$ , the *intersection TU-game*  $\chi_T$  is defined as:  $\chi_T(S) = 1$  if  $|T \cap S| = 1$  and  $\chi_T(S) = 0$  if  $|T \cap S| \neq 1$ .

A TU-game  $v \in V_N$  is *zero-normalized* if, for each  $i \in N$ ,  $v(i) = 0$ . A TU-game  $v \in V_N$  is *inessential* if, for each  $S \subseteq N$  such that  $S \neq \emptyset$ ,  $w(S) = \sum_{i \in S} w(i)$ . An inessential TU-game  $w$  is *constant* if  $w(1) = w(2) = \dots = w(n)$ . A permutation  $\sigma$  on  $N$  assigns a position  $\sigma(i)$  to each player  $i \in N$ . Let  $\Sigma_N$  be the set of  $n!$  permutations on  $N$ . Given  $v \in V_N$  and  $\sigma \in \Sigma_N$ ,  $\sigma v \in V_N$  is defined as: for each  $S \subseteq N$  such that  $S \neq \emptyset$ ,  $\sigma v(\cup_{i \in S} \sigma(i)) = v(S)$ .

Two distinct players  $i \in N$  and  $j \in N$  are *symmetric* in  $v \in V_N$  if for each  $S \subseteq N \setminus \{i, j\}$ , it holds that  $v(S \cup i) = v(S \cup j)$ . Player  $i \in N$  *nullifies* coalition  $S \ni i$  in  $v$  if  $v(S) = 0$ . Player  $i \in N$  is a *nullifying player* in  $v$  if he nullifies each coalition  $S \ni i$  in  $v$ . Player  $i$  is a *dummy player* in  $v$  if, for each coalition  $S \ni i$ ,  $v(S) = v(S \setminus i) + v(i)$ . Player is *null* in  $v$  if, for each coalition  $S \ni i$ ,  $v(S) = v(S \setminus i)$ .

An *allocation rule*  $\Phi$  on  $V_N$  is a mapping  $\Phi : V_N \rightarrow \mathbb{R}^n$  which uniquely determines, for each  $v \in V_N$  and each  $i \in N$ , a payoff  $\Phi_i(v) \in \mathbb{R}$  for participating to  $v \in V_N$ . Below, we list a set classical of axioms for an allocation rule that are referred to later on.

**Efficiency.** An allocation rule  $\Phi$  is efficient on  $V_N$  if for each  $v \in V_N$ , it holds that:  $\sum_{i \in N} \Phi_i(v) = v(N)$ .

**Anonymity.** An allocation rule  $\Phi$  is anonymous on  $V_N$  if for each  $v \in V_N$  and each  $\sigma \in \Sigma_N$ , it holds that:  $\Phi_i(v) = \Phi_{\sigma(i)}(\sigma v)$ .

**Symmetry.** An allocation rule  $\Phi$  is symmetric on  $V_N$  if for each  $v \in G$  and each pair  $\{i, j\} \subseteq N$  of symmetric players in  $v$ , it holds that:  $\Phi_i(v) = \Phi_j(v)$ .

**Null game axiom.** An allocation rule  $\Phi$  satisfies the Null game axiom if  $\Phi(\mathbf{0}) = (0, \dots, 0)$ .

**Dummy player axiom.** An allocation rule  $\Phi$  satisfies the Dummy player axiom if for each  $v \in G$  and each dummy player  $i \in N$  in  $v$ , it holds that:  $\Phi_i(v) = v(i)$ .

**Null player axiom.** An allocation rule  $\Phi$  satisfies the Null player axiom if for each  $v \in G$  and each null player  $i \in N$  in  $v$ , it holds that:  $\Phi_i(v) = 0$ .

**Nullifying player axiom.** An allocation rule  $\Phi$  satisfies the Nullifying player axiom if for each  $v \in G$  and each nullifying player  $i \in N$  in  $v$ , it holds that:  $\Phi_i(v) = v(i)$ .

**Linearity.** An allocation rule  $\Phi$  is linear if for each  $v$  and  $w$  in  $V_N$  and each  $\alpha \in \mathbb{R}$ , it holds that:  $\Phi(\alpha v + w) = \alpha \Phi(v) + \Phi(w)$ .

**Additivity.** An allocation rule  $\Phi$  is additive if for each  $v$  and  $w$  in  $V_N$ , it holds that:  $\Phi(v + w) = \Phi(v) + \Phi(w)$ .

**Marginality.** An allocation rule  $\Phi$  satisfies the Marginality axiom if for each  $v$  and  $w$  in  $V_N$  and each  $i \in N$  such that, for each  $S \ni i$ ,  $v(S) - v(S \setminus i) = w(S) - w(S \setminus i)$ , it holds that:  $\Phi_i(v) = \Phi_i(w)$ .

**Covariance.** An allocation rule  $\Phi$  is covariant if for each  $v$ , each inessential  $w$  in  $V_N$  and each  $\alpha \in \mathbb{R}$ , it holds that:  $\Phi(\alpha v + w) = \alpha \Phi(v) + (w(1), \dots, w(n))$ .

**Weak covariance.** An allocation rule  $\Phi$  is weak covariant if for each  $v$ , each inessential and constant  $w$  in  $V_N$ , each  $\alpha \in \mathbb{R}$ , it holds that:  $\Phi(\alpha v + w) = \alpha \Phi(v) + (w(1), \dots, w(1))$ .

Note that Anonymity implies Symmetry, Linearity implies Additivity, and Dummy player axiom implies Null player axiom. Below is a set of well known allocation rules that we will discuss in this article. They all satisfy Efficiency, Linearity and Anonymity.

The *Equal Division rule*, ED, is defined on  $V_N$  as:

$$\forall i \in N, \quad \text{ED}_i(v) = n^{-1}v(N).$$

A characterization due to van den Brink (2007) establishes that ED is the unique allocation rule which satisfies Additivity, Efficiency, Symmetry and the Nullifying player axiom.

The *Equal Surplus Division rule*, ESD, is defined on  $V_N$  as:

$$\forall i \in N, \quad \text{ESD}_i(v) = v(i) + n^{-1}(v(N) - \sum_{j \in N} v(j)).$$

ESD is the unique rule on  $V_N$  which satisfies Efficiency, Symmetry, Additivity, the Nullifying player axiom for zero-normalized TU-games and covariance (van den Brink, 2007).

The *Shapley value* (Shapley, 1953), Sh, is defined on  $V_N$  as:

$$\begin{aligned} \forall i \in N, \quad \text{Sh}_i(v) &= n^{-1} \sum_{k=1}^n \hat{M}c_i^k(v), \quad \text{where for each } k \in \{1, \dots, n\}, \\ \hat{M}c_i^k(v) &= \binom{n-1}{k-1}^{-1} \sum_{\substack{S \subseteq N: \\ |S|=k, S \ni i}} (v(S) - v(S \setminus i)). \end{aligned} \tag{1}$$

Since Shapley (1953), it is known that Sh is the unique allocation rule on  $V_N$  which satisfies Efficiency, Symmetry, Additivity and the Dummy player axiom.

### 3. Addition invariance and transfer invariance

Béal et al. (2013) introduce two new axioms of invariance under the assumptions that the worth of the grand coalition is left unchanged.

Firstly, suppose that the same variation of worth is applied to all coalitions of a given size. For any such coalition, the axiom of Addition invariance requires that each player should be affected similarly. The rationale behind this axiom is that any two players belong to the same number of coalitions of a given size, so that they should bear the same impact from a uniform variation of worth in all such coalitions. Since, in addition, the worth of the grand coalition is unchanged, it makes sense to require that the overall effect is null for every player regardless of the fact that the allocation rule under consideration satisfies efficiency or not.

Consider any TU-game  $v \in V_N$ , any coalition size  $k \in \{1, \dots, n-1\}$  and any real number  $\alpha \in \mathbb{R}$ . We say that TU-game  $v + \alpha 1_k \in V_N$  is obtained from  $v \in V_N$  through the  $(k, \alpha)$ -addition.

**Addition invariance.** An allocation rule  $\Phi$  is invariant under additions if for each  $v \in V_N$  and each  $(k, \alpha)$ -addition, it holds that:  $\Phi(v) = \Phi(v + \alpha 1_k)$ .

Secondly, suppose that two coalitions with the same number of players are affected by opposite changes in worth. Which payoff variation can a player belonging to both coalitions expect? The axiom of Transfer invariance requires a null overall effect for any such player. The intuition is that the per-capita impact within the two involved coalitions should be of opposite magnitudes. Thus, it is reasonable to think that an exact compensation will occur for each player belonging to both coalitions.

Consider any TU-game  $v \in V_N$ , any pair of distinct coalitions  $S^+ \subseteq N$  and  $S^- \subseteq N$  such that  $|S^+| = |S^-| = k$ ,  $k \in \{2, \dots, n-1\}$ , and any real number  $\alpha \in \mathbb{R}$ . We say that the TU-game  $v + \alpha(\delta_{S^+} - \delta_{S^-}) \in V_N$  is obtained from  $v \in V_N$  through the  $(S^+, S^-, \alpha)$ -transfer.

**Transfer invariance.** An allocation rule  $\Phi$  is invariant under transfers if for each  $v \in V_N$  and each  $(S^+, S^-, \alpha)$ -transfer, it holds that: for each  $i \in S^+ \cap S^-$ ,  $\Phi_i(v) = \Phi_i(v + \alpha(\delta_{S^+} - \delta_{S^-}))$ .

Note that Addition invariance does not imply nor is implied by Transfer invariance.

Béal et al. (2013) establish that the class of allocation rules satisfying Addition invariance and Transfer invariance properly contains the class of allocation rules obeying to Efficiency, Additivity, and Anonymity. To be precise, we have:

**Proposition 1** (Béal et al. 2013, Proposition 1)

Consider any allocation rule  $\Phi$  on  $V_N$ .

1. If  $\Phi$  satisfies Efficiency, Additivity and Symmetry, then it satisfies Addition invariance.
2. If  $\Phi$  satisfies Additivity and Anonymity, then it satisfies Transfer invariance.
3. If  $\Phi$  satisfies Addition invariance and Transfer invariance, then it does not necessarily satisfy any of the following axioms: Efficiency, Additivity, Anonymity and Symmetry.

Because the allocation rules ED, ESD and Sh satisfy Efficiency, Additivity and Anonymity, Proposition 1 leads to the following corollary.

**Corollary 1** The allocation rules ED, ESD and Sh satisfy the combination of Addition invariance and Transfer invariance on  $V_N$ .

Béal et al. (2013) show that the combination of these two axioms together with a small set of other axioms allow to axiomatically characterize ED, ESD and Sh. Furthermore, these characterizations are obtained without the classical axioms of Efficiency, Linearity/Additivity, Anonymity/Symmetry.

**Proposition 2** (Béal et al., 2013, Theorem 1)

Let  $\Phi$  be an allocation rule on  $V_N$  that satisfies Addition invariance and Transfer invariance. Then, it holds that:

1.  $\Phi$  satisfies Weak covariance and the Nullifying player axiom if and only if  $\Phi = ED$ .

2.  $\Phi$  satisfies Covariance and the Nullifying player axiom for zero-normalized TU-games if and only if  $\Phi = \text{ESD}$ .
3.  $\Phi$  satisfies the Dummy player axiom if and only if  $\Phi = \text{Sh}$ .

Yokote (2013) introduces a new axiom of addition invariance. This axiom, called Strong addition invariance, states that if the worths of all coalitions whose intersection with a fixed coalition is a singleton, change by the same amount, then the payoff vector should not change. Yokote shows that Transfer invariance and Addition invariance can be replaced by this new version of Addition invariance in point 3. of Proposition 2.

Consider any TU-game  $v \in V_N$ , any coalition  $S \subseteq N$  such that  $|S| \geq 2$  and any real number  $\alpha \in \mathbb{R}$ . We say that the TU-game  $v + \alpha\chi_S \in V_N$  is obtained from  $v \in V_N$  through the  $(S, \alpha)$ -addition. As for the previous operations of transfer and addition, this new operation of addition from a TU-game  $v$  keeps the worth  $v(N)$  unchanged.

**Strong addition invariance.** An allocation rule  $\Phi$  is strongly invariant under additions if for each  $v \in V_N$  and each  $(S, \alpha)$ -addition, it holds that:  $\Phi(v) = \Phi(v + \alpha\chi_S)$ .

**Proposition 3** (Yokote, 2013, Theorem 1)

A solution  $\Phi$  on  $V_N$  satisfies Strong addition invariance and the Dummy player axiom if and only if  $\Phi = \text{Sh}$ .

#### 4. Decomposition of $V_N$

Set  $V^1 = \text{Sp}(X)$  the subspace of  $V_N$  spanned by the set of TU-games  $X = \{1_k : k \in \{1, \dots, n-1\}\}$ . For any  $i \in N$ , set  $V_i^2 = \text{Sp}(Y_i)$  the subspace of  $V_N$  spanned by the set of TU-games  $Y_i = \{\delta_{S^+} - \delta_{S^-} : S^+ \neq S^-, |S^+| = |S^-|, i \in S^+ \cap S^-\}$ . Define by  $W_i$  the sum  $V^1 + V_i^2$ .

**Proposition 4** For each  $i \in N$ , it holds that  $W_i = V^1 \oplus V_i^2$ , where  $\dim(V^1) = n-1$  and  $\dim(V_i^2) = 2^{n-1} - n$ .

*Proof.* First note that the elements of  $X$  are linearly independent, from which we conclude that  $\dim(V^1) = n-1$ . Next, define the subspace  $U_i$  of  $V_N$  as:

$$U_i = \text{Sp}\left(\left\{\delta_S : S \not\ni i\right\} \cup \left\{\sum_{\{S \subseteq N : |S|=k, S \ni i\}} \delta_S : k \in \{1, 2, \dots, n\}\right\}\right).$$

We first prove that  $V_i^2 = U_i^\perp$ , where

$$U_i^\perp = \left\{v \in V_N : \forall S \not\ni i, \delta_S \cdot v = \mathbf{0} \wedge \forall k \in \{1, \dots, n\}, \left(\sum_{\{S \subseteq N : |S|=k, S \ni i\}} \delta_S\right) \cdot v = \mathbf{0}\right\}.$$

Pick any  $v \in V_i^2$ . Then there exist real numbers, not necessarily unique,  $\alpha_{(S^+, S^-)}$  for  $S^+ \neq S^-, |S^+| = |S^-|$  and  $i \in S^+ \cap S^-$ , such that:

$$v = \sum_{(S^+, S^-)} \alpha_{(S^+, S^-)} (\delta_{S^+} - \delta_{S^-}).$$



Therefore, for each  $S \not\ni i$ ,  $v(S) = 0$ , and so  $\delta_S \cdot v = \mathbf{0}$ . For each pair  $(S^+, S^-)$  such that  $S^+ \neq S^-$ ,  $|S^+| = |S^-|$  and  $i \in S^+ \cap S^-$ ,  $v(S^+) - v(S^-) = 0$  which implies

$$\left( \sum_{\{S \subseteq N: |S|=k, S \ni i\}} \delta_S \right) \cdot v = \mathbf{0} \text{ for each } k \in \{1, \dots, n\}.$$

Thus,  $V_i^2 \subseteq U_i^\perp$ . To prove the converse inclusion, we proceed by induction on the size of the support  $H(v)$  of  $v \in U_i^\perp$ . If  $|H(v)| = 0$ , then  $v = \mathbf{0}$  and so  $v \in V_i^2$ . Assume  $v \in V_i^2$  for all  $v \in U_i^\perp$  such that  $|H(v)| \leq h$ , and pick any  $v \in U_i^\perp$  such that  $|H(v)| = h + 1$ . Consider any  $S^+ \in H(v)$ . By definition of  $U_i^\perp$ ,  $i \in S^+$  and there exists a coalition  $S^- \in H(v)$ , distinct from  $S^+$  such that  $|S^+| = |S^-|$ . Consider the TU-game  $w = v - v(S^+)(\delta_{S^+} - \delta_{S^-})$ . We have  $w \in U_i^\perp$  and  $H(w) < H(v)$ . By the induction hypothesis  $w \in V_i^2$ . It is also clear that  $-v(S^+)(\delta_{S^+} - \delta_{S^-}) \in V_i^2$  so that  $v = w + v(S^+)(\delta_{S^+} - \delta_{S^-}) \in V_i^2$ . Thus,  $U_i^\perp \subseteq V_i^2$ . Conclude that  $U_i^\perp = V_i^2$ .

To determine the dimension of  $V_i^2$ , we use the fact that  $U_i \oplus U_i^\perp = 2^n - 1$  and that the elements of

$$\left\{ \delta_S : S \not\ni i \right\} \cup \left\{ \sum_{\{S \subseteq N: |S|=k, S \ni i\}} \delta_S : k \in \{1, 2, \dots, n\} \right\}$$

are linearly independent. Indeed, take a linear combination

$$\sum_{S: S \not\ni i} \alpha_S \delta_S + \sum_{k=1}^n \alpha_k \sum_{\{S \subseteq N: |S|=k, S \ni i\}} \delta_S = \mathbf{0}. \quad (2)$$

Take any non-empty coalition  $T$  of size  $k \in \{1, \dots, n\}$ . If  $i \notin T$ , then the left-hand side of (2) writes:

$$\sum_{S: S \not\ni i} \alpha_S \delta_S(T) + \sum_{k=1}^n \alpha_k \sum_{\{S \subseteq N: |S|=k, S \ni i\}} \delta_S(T) = \alpha_T \delta_T(T) = \alpha_T.$$

So,  $\alpha_T = 0$ . If  $i \in T$ , then the left-hand side of (2) writes:

$$\sum_{S: S \not\ni i} \alpha_S \delta_S(T) + \sum_{k=1}^n \alpha_k \sum_{\{S \subseteq N: |S|=k, S \ni i\}} \delta_S(T) = \sum_{k=1}^n \alpha_k \sum_{\{S \subseteq N: |S|=k, S \ni i\}} \delta_S(T) = \alpha_k \delta_T(T) = \alpha_k.$$

So,  $\alpha_k = 0$ . Thus the vectors in (2) are linearly independent. It follows that  $\dim(U_i) = 2^{n-1} + n - 1$  and so

$$\dim(U_i^\perp) = 2^n - 1 - (2^{n-1} + n - 1) = 2^{n-1} - n = \dim(V_i^2).$$

It remains to verify that  $V^1 \cap V_i^2 = \{\mathbf{0}\}$ . Pick  $v \in V^1 \cap V_i^2$ . On the one hand,  $v \in V^1$  implies  $v(N) = 0$  and, for each  $S \subseteq N$  such that  $|S| = k$ ,  $k \in \{1, \dots, n-1\}$ ,  $v(S) = \alpha_k$  for some real number  $\alpha_k$ . On the other hand  $v \in V_i^2$  implies  $v(S) = 0$  for each  $S \not\ni i$  and, for each  $k \in \{1, \dots, n\}$ ,

$$\sum_{\{S \subseteq N: |S|=k, S \ni i\}} v(S) = \mathbf{0} = \sum_{\{S \subseteq N: |S|=k, S \ni i\}} \alpha_k,$$

which forces  $\alpha_k = 0$ . Thus,  $v = \mathbf{0}$ , which completes the proof of Proposition 4. ■

Consider the following class of axioms. Let  $i \in N$  and let  $P_i$  be a subspace of  $V_N$ .

**$P_i$ -axiom.** For each  $v \in P_i$ ,  $\Phi_i(v)$  is uniquely determined.

We have the material to state the main result of this section.

**Proposition 5** *Assume that  $V_N = W_i \oplus P_i$  and that the allocation rule  $\Phi$  satisfies Addition invariance, Transfer invariance and the  $P_i$ -axiom. Then, for each  $v \in V_N$ , there exist exactly one  $v^1 \in W_i$  and exactly one  $v^2 \in P_i$  such that  $v = v^1 + v^2$ , and  $\Phi_i(v)$  is uniquely determined by  $\Phi_i(v^2)$ .*

*Proof.* Assume that  $V_N$  is the direct sum of  $W_i$  and  $P_i$ . As a consequence, for each  $v \in V_N$ , there exist exactly one  $v^1 \in W_i$  and exactly one  $v^2 \in P_i$  such that  $v = v^1 + v^2$ . Because  $v^1 \in W_i$ , we conclude that  $v$  is obtained from  $v^2$  by additions and transfers, i.e. there exist unique real numbers  $\alpha_k$ ,  $k \in \{1, \dots, n-1\}$ , and real numbers, not necessarily unique,  $\alpha_{(S^+, S^-)}$ ,  $S^+ \neq S^-$ ,  $|S^+| = |S^-|$ ,  $i \in S^+ \cap S^-$ , such that

$$v = \sum_{k=1}^{n-1} \alpha_k 1_k + \sum_{(S^+, S^-)} \alpha_{(S^+, S^-)} (\delta_{S^+} - \delta_{S^-}) + v^2.$$

By Addition invariance and Transfer invariance, we have  $\Phi_i(v) = \Phi_i(v^2)$ . By the  $P_i$ -axiom, we conclude that  $\Phi_i(v)$  is uniquely determined.  $\blacksquare$

Proposition 5 means that from any TU-game in  $V_N$  there exists exactly one TU-game in  $W_i$  leading to a TU-game in  $P_i$  in which the payoff of player  $i$  is uniquely determined. The TU-game in  $W_i$  is obtained through a sequence of additions and transfers from the original TU-game in  $V_N$ . Applying this decomposition of  $V_N$  to any  $i$ , one concludes that there is exactly one  $\Phi$  which satisfies Addition invariance, Transfer invariance, and the  $P_i$ -axiom for each  $i \in N$ .

## 5. Characterizations

A leading instance of the  $P_i$ -axiom is the Dummy player axiom. Fix player  $i$ , and denote by  $D_i$  the set of TU-games for which  $i$  is dummy.

**Proposition 6** *For each  $i \in N$ , it holds that  $W_i \oplus D_i$ .*

*Proof.* Define by  $L_i$  the linear subspace

$$\text{Sp} \left( \left\{ \delta_S - \delta_{S \setminus i} - \delta_i : S \ni i \right\} \right).$$

Clearly,  $D_i = L_i^\perp$ . The subspace  $L_i$  is generated by  $2^{n-1} - 1$  linearly independent vectors, one for each  $S$  such that  $i \in S$  and  $S \neq i$ . It follows that:

$$\dim D_i = \dim L_i^\perp = 2^n - 1 - (2^{n-1} - 1) = 2^{n-1},$$

as imposed by the direct sum  $W_i \oplus D_i$  and  $\dim W_i = 2^{n-1} - 1$ . It remains to prove that  $W_i \cap D_i = \{0\}$ . Pick any  $v \in W_i \cap D_i$ . By Proposition 4, there exist exactly one  $w \in V_i^2$  and unique real numbers  $\alpha_k$ ,  $k \in \{1, \dots, n-1\}$  such that:

$$v = w + \sum_{k=1}^{n-1} \alpha_k 1_k.$$

Because  $v$  also belongs to  $D_i$ , it holds that  $v(N) - v(N \setminus i) - v(i) = 0$ , i.e.

$$w(N) + \sum_{k=1}^{n-1} \alpha_k 1_k(N) - w(N \setminus i) - \sum_{k=1}^{n-1} \alpha_k 1_k(N \setminus i) - w(i) - \sum_{k=1}^{n-1} \alpha_k 1_k(i) = 0.$$

Because  $w \in V_i^2$ , it holds that  $w(N) = w(N \setminus i) = w(i) = 0$ , which forces  $\alpha_{n-1} = -\alpha_1$ . Next, for any other  $S \ni i$  such that  $|S| = k$ , the equation  $v(S) - v(S \setminus i) - v(i) = 0$  writes

$$w(S) + \alpha_k - \alpha_{k-1} - \alpha_1 = 0. \quad (3)$$

By definition of  $V_i^2$ , we know that the sum of worths  $w(S)$  over all  $S \ni i$  such that  $|S| = k$  yields 0. Thus, summing all equations of type (3), yields  $\alpha_k - \alpha_{k-1} = \alpha_1$ . It follows that  $\alpha_k = k\alpha_1$  for all  $k \in \{2, \dots, n-1\}$ . Together with  $\alpha_{n-1} = -\alpha_1$ , we conclude that  $\alpha_1 = 0$ , which forces  $\alpha_k = 0$  for all other  $k$ . So,  $v = w$ . On the one hand,  $w \in V_i^2$  leads to  $w(S) = 0$  for each  $S \not\ni i$  and  $w(N) = w(i) = 0$ . On the other hand,  $v \in D_i$ , forces  $w = 0$ , as desired.  $\blacksquare$

The combination of Proposition 5, Proposition 6 and Corollary 1 leads immediately to the result 3. of Proposition 2 stating that Sh is the unique allocation rule on  $V_N$  obeying to Addition invariance, Transfer invariance and the Dummy player axiom. Another consequence of Proposition 5 and Proposition 6 is that if  $v \in V_N$  and  $w \in V_N$  decompose as  $v = v^1 + v^2$  and  $w = w^1 + v^2$ , where  $v^1, w^1 \in W_i$  and  $v^2 \in D_i$ , then  $\text{Sh}_i(v) = \text{Sh}_i(w)$ .

Sometimes, Sh is characterized by using the Null player axiom. We can follow the same approach as in Proposition 6 in order to obtain a new characterization of Sh by using Addition invariance, Transfer invariance, the Null game axiom, a modified version of the Null player axiom and Individual covariance defined below.

**Null player invariance.** An allocation rule  $\Phi$  satisfies Null player invariance on  $V_N$  if, for each  $v \in V_N$ , each  $w \in V_N$  such that  $i \in N$  is null it holds that:  $\Phi_i(v + w) = \Phi_i(v)$ .

Null player invariance is equivalent to Marginality.<sup>1</sup> Assume that the allocation rule  $\Phi$  obeys to Marginality. Pick any  $v \in V_N$  and any  $w \in V_N$  such player  $i \in N$  is null in  $w$ . By definition of the null player, we get for each  $S \ni i$ ,  $v(S) - v(S \setminus i) = (v + w)(S) - (v + w)(S \setminus i)$ . By Marginality,  $\Phi_i(v) = \Phi_i(v + w)$ , which proves that  $\Phi$  satisfies Null player invariance. Reciprocally, assume that  $\Phi$  obeys to Null player invariance. Consider  $v, w \in V_N$  and  $i \in N$  such that, for each  $S \ni i$ ,

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<sup>1</sup>Null player invariance is called coalitional strategic equivalence in van den Brink (2007). It is shown that Null player invariance is equivalent to the axiom introduced by Chun (1989) stating that for each  $v \in V_N$ , each non-empty  $S \in 2^N$  and each  $\alpha \in \mathbb{R}$ ,  $f_i(v + \alpha 1_S) = f_i(v)$  for each  $i \in N \setminus S$ . Consequently the axiom introduced by Chun is equivalent to Marginality.

$v(S) - v(S \setminus i) = w(S) - w(S \setminus i)$ . This is equivalent to say that  $i$  is a null player in  $v - w$ . Therefore,  $v = w + (v - w)$  and by Null player invariance,  $\Phi_i(v) = \Phi_i(w + (v - w)) = \Phi_i(w)$ , which proves that  $\Phi$  satisfies Marginality. Marginality is used by Young (1985) to characterize Sh. Young (1985) shows that Sh is the unique allocation rule on  $V_N$  satisfying Efficiency, Symmetry and Marginality. Chun (1989) obtains an equivalent result using the formulation inserted in footnote 1. The interest of our formulation of Marginality appears in the proof of Proposition 7 below and also in section 7 for proving the logical independence of the chosen axioms.

**Individual covariance.** An allocation rule  $\Phi$  satisfies individual covariance on  $V_N$  if, for each  $v \in V_N$ , each  $\alpha \in \mathbb{R}$ , and each  $i \in N$ , it holds that:  $\Phi_i(v + \alpha u_i) = \Phi_i(v) + \alpha$ .

**Proposition 7** *Let  $\Phi$  be an allocation rule on  $V_N$  that satisfies Addition invariance and Transfer invariance. Then  $\Phi = \text{Sh}$  if and only if it satisfies Null player invariance, Individual covariance, and the Null game axiom.*

*Proof.* By Corollary 1, Sh satisfies Addition invariance and Transfer invariance. Because Sh is a linear map, it satisfies the Null game axiom. Because the Dummy player axiom implies the Null player and that Sh is a linear map, Sh satisfies Null player invariance. By linearity of Sh,  $\Phi_i(v + \alpha u_i) = \Phi_i(v) + \alpha \Phi_i(u_i)$ . Since Sh distributes 1 to player  $i$  in  $u_i$ , Sh satisfies Individual covariance. To prove the uniqueness part, pick any  $i \in N$  and let  $D_i^0$  be the subspace of TU-games where player  $i$  is null. To show:  $D_i = D_i^0 \oplus \mathbb{R}u_i$ . We have  $\dim \mathbb{R}u_i = 1$ , and proceeding as in Proposition 6, we get  $\dim(D_i^0) = 2^{n-1} - 1$ . It remains to verify that  $D_i^0 \cap \mathbb{R}u_i = \{\mathbf{0}\}$ . Pick any  $\alpha u_i \in D_i^0 \cap \mathbb{R}u_i$ . For each  $S \ni i$ ,  $\alpha u_i(S) - \alpha u_i(S \setminus i) = \alpha u_i(S) = \alpha$ . Since  $\alpha u_i \in D_i^0$ , this forces  $\alpha = 0$  and so  $\alpha u_i = \mathbf{0}$ . It follows that  $V_N = W_i \oplus D_i^0 \oplus \mathbb{R}u_i$ . Pick any  $v \in V$ , there exist exactly one  $v^1 \in W_i$ , exactly one  $v^2 \in D_i^0$  and exactly one  $\alpha \in \mathbb{R}$  such that  $v = v^1 + v^2 + \alpha u_i$ . Applying successively Addition and Transfer invariance, Null player invariance, Individual covariance and the Null game axiom, we get:

$$\Phi_i(v) = \Phi_i(v^1 + v^2 + \alpha u_i) = \Phi_i(v^2 + \alpha u_i) = \Phi_i(\alpha u_i) = \Phi_i(\mathbf{0} + \alpha u_i) = \Phi_i(\mathbf{0}) + \alpha = \alpha.$$

This proves that  $\Phi_i$  is uniquely determined on  $V_N$ . Because player  $i \in N$  has been chosen arbitrarily, the proof is complete. ■

From (the proof of) Proposition 7, one determines the kernel of  $\text{Sh}_i$  and so the set of TU-games for which player  $i$  obtains the same Shapley payoffs on  $V_N$ .

**Proposition 8** *Fix any player  $i \in N$ . A direct-sum decomposition of the kernel of the linear form  $\text{Sh}_i : V_N \rightarrow \mathbb{R}$  is  $W_i \oplus D_i^0$ . Therefore, for  $\alpha \in \mathbb{R}$ ,  $\text{Sh}_i(v) = \alpha$  if and only if  $v = v^1 + \alpha u_i$  for some  $v^1 \in W_i \oplus D_i^0$ .*

*Proof.* Because  $\text{Sh}_i$  is different from the null solution, the dimension of its kernel is  $\dim(V_N) - \dim(\mathbb{R}) = 2^n - 2$ . By the null player axiom, the kernel of  $\text{Sh}_i$  contains  $D_i^0$ . It also contains  $W_i$  since for  $v \in W_i$ ,  $\text{Sh}_i(\mathbf{0} + v) = \text{Sh}_i(\mathbf{0}) = 0$  by addition and transfer invariance, and linearity of  $\text{Sh}_i$ . On the other hand,  $\dim(W_i \oplus D_i^0) = \dim(W_i) + \dim(D_i^0) = 2^n - 2$ , which proves that  $W_i \oplus D_i^0$  spans the kernel of  $\text{Sh}_i$ . By the proof of Proposition 7, we know that  $V_N = W_i \oplus D_i^0 \oplus \mathbb{R}u_i$ . It follows that the set of TU-games for which  $i$  obtains the payoff  $\alpha \in \mathbb{R}$  coincides with the set of TU-games  $v$  whose decomposition are of the form  $v = v^1 + \alpha u_i$  for some  $v^1 \in W_i \oplus D_i^0$ . Indeed, in  $\alpha u_i$  each player  $j \neq i$  is null. By the above arguments, the null player axiom and efficiency,  $\text{Sh}_i(v^1 + \alpha u_i) = \text{Sh}_i(\alpha u_i) = \alpha$ . ■

Point 1. of Proposition 2 indicates that ED is characterized by Addition invariance, Transfer invariance, Weak covariance and the Nullifying player axiom. Based on a decomposition of the subspace in which a player nullifies all coalitions he belongs to, except the grand coalition, we are able to show that ED can also be characterized by deleting Weak covariance and adding a new axiom, called Covariance with respect to the grand coalition.

**Covariance with respect to the grand coalition.** An allocation rule  $\Phi$  satisfies covariance with respect to the grand coalition on  $V_N$  if, for each  $v \in V_N$ , each  $\alpha \in \mathbb{R}$ , it holds that  $\Phi(v + \alpha\delta_N) = \Phi(v) + \alpha n^{-1}$ .

**Proposition 9** *Let  $\Phi$  be an allocation rule on  $V_N$  that satisfies Addition invariance and Transfer invariance. Then  $\Phi = \text{ED}$  if and only if it satisfies the Nullifying player axiom and Covariance with respect to the grand coalition.*

*Proof.* By Corollary 1, ED satisfies Addition invariance and Transfer invariance. Clearly, ED satisfies Covariance with respect to the grand coalition. Fix player  $i$ , and denote by  $N_i$  the subspace of TU-games for which  $i$  nullifies all coalitions  $S \ni i$  except the grand coalition  $N$ . It corresponds to the orthogonal complement of

$$\text{Sp}\left(\left\{\delta_S : S \ni i, S \neq N\right\}\right),$$

and so its dimension is  $\dim(N_i) = 2^n - 1 - (2^{n-1} - 1) = 2^{n-1}$ . Next pick any  $v \in W_i \cap N_i$ . On the one hand,  $v(S) = 0$  for all  $S \ni i, S \neq N$  since  $v \in N_i$ . On the other hand,  $v \in W_i = V^1 \oplus V_i^2$  implies that there exist exactly one  $v^1 \in V^1$  and exactly one  $v^2 \in V_i^2$  such that  $v = v^1 + v^2$ . Consider any  $S \ni i$ . If  $S = N$ , by definition of  $V^1$  and  $V_i^2$ , we have  $v^2(N) = v^1(N) = 0$  and so  $v(N) = 0$ . If  $S \neq N$ , we have  $v(S) = v^1(S) = \alpha_k 1_k(S) = \alpha_k$  for a unique real number  $\alpha_k$ . Since the value of  $1_k(S)$  does not depend on the presence of  $i$  in  $S$  but only of its size,  $v(S) = 0$  for  $S \ni i$  forces  $\alpha_k = 0$  and so  $v = \mathbf{0}$ . Therefore, we have  $V_N = W_i \oplus N_i$ . Consider the subspace of TU-games  $N_i^*$  such that  $i$  nullifies all coalitions  $S \ni i$ . Clearly, we have  $N_i = N_i^* \oplus \mathbb{R}\delta_N$  and so  $V_N = W_i \oplus N_i^* \oplus \mathbb{R}\delta_N$ . Pick any  $v \in V_N$ . There exist exactly one  $w \in W_i$ , exactly one  $v^* \in N_i^*$  and exactly one real number  $\alpha$  such that  $v = w + v^* + \alpha 1_N$ . Note that  $w(N) = v^*(N) = 0$  so that  $v(N) = \alpha$ . By Addition and Transfer invariance,  $\Phi_i(v) = \Phi_i(v^* + v(N)\delta_N)$ . By Covariance with respect to the grand coalition,  $\Phi_i(v) = \Phi_i(v^*) + n^{-1}v(N)$ . And by the Nullifying player property,  $\Phi_i(v) = n^{-1}v(N)$ . As  $i$  has been chosen arbitrarily, the proof is complete.  $\blacksquare$

Let us compare the characterization of ED contained in point 1. of Proposition 2 with the characterization contained in Proposition 9. First, note that, in both characterizations, Addition invariance, Transfer invariance and the Nullifying player axiom distributes the null payoff on  $W_i \oplus N_i^*$  to each player  $i \in N$ . Then, it is necessary and sufficient to add an axiom constructed on a line not lying in  $W_i \oplus N_i^*$ . In Proposition 9, we consider the line generated by  $\delta_N$ . By slightly modifying the axiom of Weak covariance, we can instead consider the line generated by the inessential TU-game  $\mathbf{1} = (1, \dots, 1)$ . It suffices to impose  $\alpha = 1$  in the axiom of Weak covariance. This way, for each  $i \in N$ , we get the direct-sum decomposition  $V_N = W_i \oplus N_i^* \oplus \mathbb{R}\mathbf{1}$  and obtain a characterization of ED very similar the characterization contained in point 1. of Proposition 2. From this point of view, the characterization of ED contained in point 1. of Proposition 2 and the characterization contained in Proposition 9 are close in spirit. Other similar characterizations of ED can be derived

from Addition invariance, Transfer invariance and the Nullifying player axiom by adding a fourth axiom constructed from a line not lying in  $W_i \oplus N_i^*$ . From this discussion and proceeding as in Proposition 8, we derive the following result.

**Proposition 10** *Fix any player  $i \in N$ . A direct-sum decomposition of the kernel of the linear form  $ED_i : V_N \rightarrow \mathbb{R}$  is  $W_i \oplus N_i^*$ . Therefore, for  $\alpha \in \mathbb{R}$ ,  $ED_i(v) = \alpha$  if and only if  $v = v^1 + \alpha \delta_N$  for some  $v^1 \in W_i \oplus N_i^*$ .*

Proposition 1 indicates that the set of allocation rules obeying to Addition invariance and Transfer invariance strictly contains the set of allocation rules obeying to Efficiency, Linearity, and Anonymity. This set of allocation rules has been characterized in the following way by Radzik and Driessen (2003). See also Radzik and Driessen (2013).

**Proposition 11** *An allocation rule  $\Phi$  on  $V_N$  satisfies Efficiency, Linearity and Anonymity if and only if there exists a unique collection of constants  $B = \{b_s : s \in \{0, 1, \dots, n\}\}$  such that  $b_0 = 0$ ,  $b_n = 1$  and  $\Phi(v) = Sh(Bv)$ , where  $(Bv)(S) = b_s v(S)$  for each coalition  $S$  of size  $s$ ,  $s \in \{0, 1, \dots, n\}$ .*

Proposition 11 generalizes Sh in the sense that each marginal contribution  $v(S \cup i) - v(S)$  has been replaced by  $b_{s+1}v(S \cup i) - b_s v(S)$ . Assume that one restricts our attention to the subset of Efficient, Linear and Anonymous allocation rules such that  $b_s \neq 0$  for each  $s = \{1, \dots, n-1\}$ . Among this subset of Efficient, Linear and Anonymous allocation rules, one finds Sh, the Solidarity value (Novak, Radzik, 1994), the Least square prenucleolus (Ruiz et al., 1998), the Consensus value (Ju et al., 2007). On the contrary, ED and ESD do not belong to this subset of allocation rules. By considering a richer subspace than  $D_i$  and proceeding as in Proposition 6, one obtains a characterization of the above subset of allocation rules. Given a collection  $B = \{b_s : s \in \{0, 1, \dots, n\}\}$  such that  $b_0 = 0$ ,  $b_n = 1$  and  $b_s \neq 0$  for each other  $s$ , define for  $i \in N$ ,  $BD_i$  the subspace of TU-games for which  $i$  is  $B$ -dummy, i.e. the subset of TU-games  $v$  such that:

$$\forall S \ni i, \quad b_{s+1}v(S \cup i) - b_s v(S) = b_1 v(i).$$

Because  $b_s \neq 0$  for each  $s = \{1, \dots, n\}$ ,  $BD_i$  has the same dimension as  $D_i$  and, proceeding as in Proposition 6, we can prove that  $W_i \oplus BD_i$ . Therefore, using the following variant of the Dummy player axiom, we obtain a characterization of the subset of Efficient, Linear and Anonymous allocation rules such that  $b_s \neq 0$  for each  $s \in \{1, \dots, n-1\}$ .

**$B$ -dummy player axiom.** An allocation rule  $\Phi$  on  $V_N$  satisfies the  $B$ -dummy player axiom if for all  $v \in V_N$  and all  $B$ -dummy players  $i \in N$  in  $v$ , it holds that:  $\Phi_i(v) = b_1 v(i)$ .

**Proposition 12** *Let a collection of real numbers  $B = \{b_s : s \in \{0, 1, \dots, n\}\}$  such that  $b_0 = 0$ ,  $b_n = 1$  and  $b_s \neq 0$  for each other  $s$ . There exists a unique allocation rule  $\Phi$  satisfying Addition invariance, Transfer invariance and the  $B$ -dummy axiom. This allocation rule is given by  $\Phi(v) = Sh(Bv)$  for all  $v \in V_N$ .*

Point 3. of Proposition 1 indicates that the set of allocation rules obeying to Addition invariance and Transfer invariance contains allocation rules that are not necessarily Linear, Efficient or Anonymous. An instance of such rules is the  $\alpha$ -exponential Shapley value defined as follows. Given

a collection of positive real numbers  $\alpha_i > 0$ ,  $i \in \{1, \dots, n\}$ , the  $\alpha$ -exponential Shapley value  $\text{Sh}^\alpha$  is given by:

$$\forall v \in V_N, \forall i \in N, \quad \text{Sh}_i^\alpha(v) = \exp\left(\frac{\text{Sh}_i(v)}{\alpha_i}\right).$$

It is straightforward to verify that  $\text{Sh}^\alpha$  satisfies Addition invariance and Transfer invariance and the following variant of the Dummy player axiom.

**Exponential-dummy player axiom.** An allocation rule  $\Phi$  on  $V_N$  satisfies the Exponential-dummy player axiom if for all  $v \in V_N$  and all dummy players  $i \in N$  in  $v$ , it holds that:

$$\Phi_i(v) = \exp\left(\frac{v(i)}{\alpha_i}\right).$$

A direct consequence of Proposition 2 and Proposition 6 is the following result.

**Proposition 13** *Let  $\Phi$  be an allocation rule on  $V_N$  that satisfies Addition invariance and Transfer invariance. Then,  $\Phi = \text{Sh}^\alpha$  if and only if it satisfies the Exponential-dummy player axiom.*

## 6. Yokote's approach

Proposition 3 in Yokote (2013) is based of the following result.

**Proposition 14** (Yokote, 2013, Lemma 3)

1. The collection of TU-games  $B = \{\chi_S : S \subseteq N, S \neq \emptyset\}$  is a basis for  $V_N$ .
2. The subspace

$$Z = \text{Sp}\left(\left\{\chi_S : S \subseteq N, |S| \geq 2\right\}\right)$$

is the kernel of  $\text{Sh}$ .

Point 2. of Proposition 14, also contained in Yokote et al. (Lemma 5, 2013), follows from Efficiency, Symmetry and the Null player axiom, from the fact that  $\text{Sh}$  is onto and from point 1 of Proposition 14. Then, Proposition 3 in Yokote (2013) is proved as follows. By point 2. of Proposition 14 and the linearity of  $\text{Sh}$ , the latter satisfies Strong addition invariance. Next, consider any  $v \in V_N$ . By point 1 of Proposition 14, there exist unique real numbers  $\alpha_S$ ,  $S \subseteq N$ ,  $S \neq \emptyset$ , such that:

$$\Phi(v) = \Phi\left(\sum_{S \subseteq N, S \neq \emptyset} \alpha_S \chi_S\right).$$

By Strong addition invariance,

$$\Phi(v) = \Phi\left(\sum_{j=1}^n \alpha_j \chi_j\right).$$

Since the TU-game  $\sum_{j=1}^n \alpha_j \chi_j$  is the inessential TU-game given by  $(\alpha_1, \dots, \alpha_n)$ , the result follows by applying the Dummy player property. Two remarks are in order. Firstly, the Dummy player axiom can be weakened by using the Inessential game axiom.

**Inessential game axiom.** An allocation rule  $\Phi$  satisfies the inessential game axiom if for all inessential TU-game  $(v(1), \dots, v(n))$  of  $V_N$ , it holds that:  $\Phi(v(1), \dots, v(n)) = (v(1), \dots, v(n))$ .

Obviously, Sh satisfies the Inessential game axiom. Note that  $V_N$  is not the direct sum of  $D_i$  and  $Z$  since  $\dim(D_i) + \dim(Z) > 2^n - 1$ . The Dummy player axiom is thus stronger than necessary. Secondly, Yokote offers a combinatorial proof of part 1 of Proposition 14 by linking the intersection TU-games  $\chi_S$  with the unanimity TU-games  $u_S$ ,  $S \subseteq N$ ,  $S \neq \emptyset$ . Here, we offer a direct proof by using a simple induction argument, which highlights another property of this basis.

*Proof.* (Part 1. of Proposition 14). Because  $B$  contains  $2^n - 1$  elements, it suffices to prove that they are linearly independent. Thus, take any linear combination of the elements of  $B$ ,

$$\sum_{S \subseteq N: S \neq \emptyset} \alpha_S \chi_S.$$

Consider any ordered pair  $(S^1, S^2)$  of coalitions such that  $S^1 \neq \emptyset$  et  $S^1 \cap S^2 = \emptyset$ . Define by  $b_{S^1, S^2}$  the following quantity:

$$b_{S^1, S^2} = \sum_{S^1 \subseteq S \subseteq N \setminus S^2} \alpha_S.$$

**Remark 1:**

1. For each  $S^1 \neq \emptyset$ ,  $b_{S^1, N \setminus S^1} = \alpha_{S^1}$ .
2. If  $i \notin S^1 \cup S^2$ , then  $b_{S^1 \cup i, N \setminus S^2} + b_{S^1, N \setminus (S^2 \cup i)} = b_{S^1, N \setminus S^2}$ .

Assume that:

$$\sum_{S \subseteq N: S \neq \emptyset} \alpha_S \chi_S = \mathbf{0}.$$

To show:  $\alpha_S = 0$  for all  $S \subseteq N, S \neq \emptyset$ . We prove a stronger result:  $b_{S^1, N \setminus S^1} = 0$  for all feasible pairs  $(S^1, S^2)$ . We proceed by induction on the number of elements in  $S^1 \cup S^2$ .

**Initial step.** If  $S^1 \cup S^2$  contains only one element, say  $i$ , then  $S^1 = \{i\}$  and  $S^2 = \emptyset$ . On the one hand,

$$b_{S^1, S^2} = \sum_{S \subseteq N: S \ni i} \alpha_S. \quad (4)$$

On the other hand, by assumption,

$$\sum_{S \subseteq N: S \neq \emptyset} \alpha_S \chi_S(i) = 0. \quad (5)$$

Because  $\chi_S(i) = 1$  if and only if  $i \in S$ , (5) becomes:

$$\sum_{S \subseteq N: S \ni i} \alpha_S = 0 \quad \text{and (4) is such that } b_{S^1, S^2} = 0,$$

as desired.



**Induction hypothesis.** Assume that  $b_{S^1, S^2} = 0$  for all feasible pair  $(S^1, S^2)$  such that  $S^1 \cup S^2$  contains  $k$  elements.

**Induction step.** Pick any feasible pair  $(S^1, S^2)$  such that  $S^1 \cup S^2$  contains  $k + 1$  elements. Let  $i \in S^2$ . By point 2. of Remark 1, we have:

$$b_{S^1 \cup i, S^2 \setminus i} + b_{S^1, S^2} = b_{S^1, S^2 \setminus i}.$$

By the induction hypothesis,  $b_{S^1, S^2 \setminus i} = 0$ . Therefore,  $b_{S^1, S^2} = -b_{S^1 \cup i, S^2 \setminus i}$ . Continuing in this fashion by picking all elements of  $S^2$ , we get:

$$b_{S^1, S^2} = (-1)^{|S^2|} b_{S^1 \cup S^2, \emptyset}. \quad (6)$$

Next, consider two distinct elements, say  $i$  and  $j$ , of  $S^1 \cup S^2$ . We have:

$$b_{S^1 \cup S^2, \emptyset} + b_{(S^1 \cup S^2) \setminus j, j} = b_{((S^1 \cup S^2) \setminus j) \cup j, \emptyset} + b_{(S^1 \cup S^2) \setminus j, j},$$

and by By point 2. of Remark 1,

$$b_{((S^1 \cup S^2) \setminus j) \cup j, \emptyset} + b_{(S^1 \cup S^2) \setminus j, j} = b_{(S^1 \cup S^2) \setminus j, \emptyset}.$$

Hence,

$$b_{S^1 \cup S^2, \emptyset} + b_{(S^1 \cup S^2) \setminus j, j} = b_{(S^1 \cup S^2) \setminus j, \emptyset}.$$

By the induction hypothesis,  $b_{(S^1 \cup S^2) \setminus j, \emptyset} = 0$ , and so  $b_{S^1 \cup S^2, \emptyset} = -b_{(S^1 \cup S^2) \setminus j, j}$ . Proceeding as above by picking all elements of  $S^1 \cup S^2$  except  $i$ , we get:

$$b_{S^1 \cup S^2, \emptyset} = (-1)^{|S^1 \cup S^2| - 1} b_{i, (S^1 \cup S^2) \setminus i}. \quad (7)$$

The above equality does not depend on the choice of player  $i$ . By assumption:

$$\sum_{S \subseteq N: S \neq \emptyset} \alpha_S \chi_S(S^1 \cup S^2) = 0.$$

By definition of  $\chi_S$ ,  $\chi_S(S^1 \cup S^2) = 1$  if and only if it exists  $i \in S^1 \cup S^2$  such that  $(S^1 \cup S^2) \cap S = i$ , which implies that  $S \subseteq N \setminus ((S^1 \cup S^2) \setminus i)$ . Therefore,

$$0 = \sum_{S \subseteq N: S \neq \emptyset} \alpha_S \chi_S(S^1 \cup S^2) = \sum_{i \in S^1 \cup S^2} \sum_{S \subseteq N \setminus ((S^1 \cup S^2) \setminus i)} \alpha_S = \sum_{i \in S^1 \cup S^2} b_{i, (S^1 \cup S^2) \setminus i}.$$

As  $b_{i, (S^1 \cup S^2) \setminus i}$  does not depend on the choice of  $i$ , conclude that  $b_{i, (S^1 \cup S^2) \setminus i} = 0$ . By (7), conclude that  $b_{S^1 \cup S^2, \emptyset} = 0$ , and by (6) we obtain  $b_{S^1, S^2} = 0$  as desired. All feasible  $b_{S^1, S^2}$  are thus equal to zero, which implies that  $\alpha_S = 0$  for all  $S \subseteq N$ ,  $S \neq \emptyset$ . Therefore, the elements of  $B$  are linearly independent, and so  $B$  is basis for  $V_N$ .  $\blacksquare$

As a corollary of Proposition 14, we obtain  $V_N = Z \oplus I$ , where  $I = \text{Sp}(\{\chi_i : i \in N\})$ . The collection  $\{\chi_i : i \in N\}$  is a basis for the subspace of inessential TU-games. From  $V_N = Z \oplus I$ , Proposition 14 and the fact that Sh satisfies the Inessential game axiom, we obtain a stronger characterization of Sh in presence of Strong addition invariance.

**Proposition 15** *A solution  $\Phi$  on  $V_N$  satisfies Strong addition invariance and the Inessential game axiom if and only if  $\Phi = \text{Sh}$ . Moreover, for  $\alpha = (\alpha_1, \dots, \alpha_n) \in I$ ,  $\text{Sh}(v) = (\alpha_1, \dots, \alpha_n)$  if and only if  $v = v^1 + (\alpha_1, \dots, \alpha_n)$  for some  $v^1 \in Z$ .*

## 7. Logical independence of the axioms

The decomposition of the space of  $V_N$  is also very useful to prove the logical independence of the axioms used in the different characterizations of the allocation rules. A common method can be applied to prove that an axiom is independent of the other axioms used in the characterization.

Note that most of the axioms used are of the following types. We have considered a linear subspace  $U$ , possibly induced by some player  $i$ , of  $V_N$ , a linear mapping  $L$  from  $U$  to  $\mathbb{R}^n$  and a relation between TU-games and payoffs of the following types:

1.  $\forall v \in V_N, \forall w \in U, \Phi(v + w) = \Phi(v) + L(w)$ .
2.  $\forall v \in V_N, \forall w \in U, \Phi_i(v + w) = \Phi_i(v) + L_i(w)$  for some players  $i$  in  $w$ .
3.  $\forall v \in U, \Phi_i(v) = L_i(v)$  for some players  $i$  in  $w$ .

Addition invariance and Covariance with respect to the grand coalition are of the first type. Transfer invariance, Individual covariance, Null player invariance are of the second type. The Null player axiom, the Nullifying player axiom, the Null game axiom, the  $B$ -dummy axiom, the Inessential game axiom are of the third type. We have also invoked the extra axiom of exponential-dummy player.

Consider first Proposition 7. The Shapley value  $\text{Sh}$  is characterized by Addition invariance and Transfer invariance, Null player invariance, Individual covariance, and the Null game axiom. To show that Addition invariance is logically independent of Transfer invariance, Null player invariance, Individual covariance, and the Null game axiom, we proceed as follows. Consider any player  $i \in N$ . By Proposition 4, Proposition 6 and the proof of Proposition 7, we have:

$$V_N = V^1 \oplus V_i^2 \oplus D_i^0 \oplus \mathbb{R}u_i.$$

For any  $v$ , let  $v^1 + v^2 + v^3 + \alpha u_i$  be its unique decomposition along  $V^1$ ,  $V_i^2$ ,  $D_i^0$  and  $\mathbb{R}u_i$ . Define the allocation rule  $\Phi$  as follows. For any  $i \in N$ , consider any mapping  $\Psi_i$  from  $V^1$  to  $\mathbb{R}$  such that  $\Psi_i(\mathbf{0}) = 0$ , and

$$\forall i \in N, \quad \Phi_i(v) = \Psi_i(v^1) + \text{Sh}_i(v^2 + v^3 + \alpha u_i).$$

Let us verify that  $\Phi$  satisfies Transfer invariance. Consider any  $v \in V_N$  and any  $w \in V_i^2$ . Using the decomposition of  $v$  above and the definition of  $\Phi$ , we get:

$$\begin{aligned} \Phi_i(v + w) &= \Psi_i(v^1) + \text{Sh}_i((v^2 + w) + v^3 + \alpha u_i) \\ &= \Psi_i(v^1) + \text{Sh}_i(v^2 + v^3 + \alpha u_i) + \text{Sh}_i(w) \\ &= \Phi_i(v) + \text{Sh}_i(w) \\ &= \Phi_i(v) + \text{Sh}_i(\mathbf{0} + w) \\ &= \Phi_i(v), \end{aligned}$$

where the last equality follows from the linearity of  $\text{Sh}$  and the fact that  $\text{Sh}$  satisfies Transfer invariance. We conclude that  $\Phi$  satisfies Transfer invariance.

Proceeding in the same way for the other axioms, we conclude that  $\Phi$  also satisfies Null player invariance, Individual covariance and the Null game axiom.

In return,  $\Phi$  does not satisfy Addition invariance as soon as there exists  $w$  in  $V^1$  such that  $\Psi_i(w) \neq 0$ . Indeed, we have:

$$\Phi_i(\mathbf{0} + w) = \Psi_i(w) + \text{Sh}_i(\mathbf{0}) = \Psi_i(w) \neq 0 = \Phi_i(\mathbf{0}),$$

which contradicts Addition invariance.

The same method can be used to show that each axiom of invariance is logically independent of the three other axioms. It suffices to invert the roles of the subspaces of the decomposition in the definition of  $\Phi$ . Regarding the null game axiom, one easily sees that, for each non-null vector  $\alpha = (\alpha_1, \dots, \alpha_n)$  of  $\mathbb{R}^n$ , the allocation  $\text{Sh} + \alpha$  satisfies all invariance axioms but not the Null game axiom. This proves the logical independence of the axioms contained in the statement of Proposition 7.

Proposition 9 provides a characterization of ED using three axioms of invariance and the Nullifying player axiom. Proceeding as above by using, for each  $i \in N$ , the decomposition (see Proposition 4, Proposition 6 and the proof of Proposition 9):

$$V_N = V^1 \oplus V_i^2 \oplus N_i^* \oplus \mathbb{R}\delta_N,$$

we conclude that the four axioms are logically independent.

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