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Characterization of the Average Tree solution and its kernel

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Abstract

In this article, we study cooperative games with limited cooperation possibilities, represented by a tree on the set of agents. Agents in the game can cooperate if they are connected in the tree. We first derive direct-sum decompositions of the space of TU-games on a fixed tree, and two new basis for these spaces of TU-games. We then focus our attention on the Average (rooted)-Tree solution (see Herings, P., van der Laan, G., Talman, D., 2008. The Average Tree Solution for Cycle-free Games. *Games and Economic Behavior* 62, 77-92). We provide a basis for its kernel and a new axiomatic characterization by using the classical axiom for inessential games, and two new axioms of invariance, namely Invariance with respect to irrelevant coalitions and Weighted addition invariance on bi-partitions.

Keywords: Average Tree solution, Direct-sum decomposition, Kernel, Weighted addition invariance on bi-partitions, Invariance to irrelevant coalitions
JEL: C71.

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1. Introduction

In a cooperative game, cooperation is not always possible for all coalitions of agents. The limited possibilities of cooperation can often be represented by an undirected graph in which cooperation is only possible if agents are connected to each other. The study of the so-called graph games was initiated by Myerson (1977) who introduced as solution concept the Myerson value, which equals the Shapley value of the induced graph-restricted game. For the subclass of cycle-free graph games, Herings et al. (2008) introduced the average tree solution (AT solution), being the average of the marginal contribution vectors corresponding to all rooted spanning trees of the graph. They characterize the AT solution by the classical axiom of component efficiency, and by component fairness, which requires that deleting a link between two agents yields for both resulting components the same average change in payoff, where the average is taken over the agents in the component.

Ever since, this solution has received considerable attention in the literature. Other characterizations on the class of cycle-free graph games have been provided by van den Brink (2009), Mishra and Talman (2010), Béal et al. (2010, 2012b) and Ju and Park (2012). Generalizations of the AT

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solution to the class of all graph games have been examined by Herings et al. (2010) and Baron et al. (2011). The average tree solution has also been implemented by van den Brink et al. (2013), and applied to and characterized in the richer frameworks of multichoice communication games by Béal et al. (2012a) and of games with a permission tree by van den Brink et al. (2014).

In this article, we provide a new characterization of the AT solution. Because we work with a fixed graph, there is no loss of generality to assume that this graph is connected. We invoke the classical axiom for inessential games as well as two other axioms of invariance. The axiom of Invariance to irrelevant coalitions is based on the set of cones of a tree, which contains the grand coalition, the empty coalition, and every pair of components obtained by removing a link from the original tree. Only coalitions in this set are involved in the computation of the AT solution, and, more generally, in the computations of marginalist tree solutions in the sense of Béal et al. (2010). The axiom of Invariance to irrelevant coalitions requires that the solution should prescribe the same payoff vector in two tree games where the worths of all cones are the same.[1] The axiom of Weighted addition invariance on bi-partitions requires that the solution is not affected if the worths of each of the two coalitions of a bi-partition of the agent set changes in proportion to their respective size. On the class of tree games, we show that the AT solution is characterized by Invariance to irrelevant coalitions, Weighted addition invariance on bi-partitions and the Inessential game axiom (Proposition 4).

Our characterization of the AT solution does not rely on the classical axioms of efficiency, contrary to all previous characterizations in the literature, and linearity, contrary to the characterizations in Mishra and Talman (2010) and Béal et al. (2010) where the graph is also fixed. It is based on two axioms of invariance plus a punctual axiom. The proof of this characterization is made easier by our second set of results. More specifically, we provide a basis for the kernel of the AT solution for a given tree (Proposition 1), and two direct-sum decompositions of the set of all cooperative games on a given agent set that rely on the tree under consideration (Proposition 2). Direct-sum decompositions are obtained through the games used to define Weighted addition invariance on bi-partitions and Invariance to irrelevant coalitions respectively. These results clearly underline new structural properties of the space of cooperative games on a fixed agent set and of the AT solution. First, thanks to Proposition 1 and 2, we are able to provide two new basis for the space of cooperative on a fixed agent set. These basis are indexed by the tree under consideration. Second, knowing a basis of the kernel of the AT solution, it becomes easiest to understand how our two axioms of invariance operate, and also to solve the following inverse problem. Given a payoff vector, find the set of all games such that the AT solution allocates this payoff vector. We solve this problem by showing that a game belongs to this solution set if and only if the total payoff allocated to the members of a cone is equal to the worth of this cone plus a share, proportional to the size of this cone, of the surplus created by the link incident to this cone and its complement (Proposition 3). At last, another advantage of our method is that it proves very useful to show logical independence of the axioms used in the characterizations. Although this approach has already been undertaken for cooperative games with no restriction on the cooperation possibilities – see, for instance, Kleinberg and Weiss (1985), and more recently, Béal et al., (2013) and Yokote (2014) –, to the best of our knowledge, this is the first time it is applied to graph games.

The rest of the article is organized as follows. Section 2 is devoted to the definitions and notations. Section 3 contains all the results. Precisely, section 3.1 provides a basis for the kernel

[1] A similar axiom, called cone equivalence, is used in Béal et al. (2010) to characterize the class of marginalist tree solutions.

of AT and two direct-sum decompositions of the set of TU-games with respect to a fixed tree. In section 3.2, we exploit these results to axiomatically characterize AT. Section 4 concludes.

2. Preliminaries

Throughout this article, the cardinality of a finite set S will be denoted by the lower case s, the collection of all subsets of S will be denoted by 2^S, and weak set inclusion will be denoted by $\subseteq$. The complement $S \setminus T$ of a subset T of S is denoted by T^c. Also for notational convenience, we will write singleton $\{i\}$ as i. Let V be a real linear space equipped with an inner product "$\cdot$". Its additive identity element is denoted by $\mathbf{0}_V$ and its dimension by $\dim(V)$. Given a linear subspace U of V, we denote by $U^{\perp}$ its orthogonal complement. If V is the direct sum of the subspaces V^1 and V^2, i.e. $V = V^1 + V^2$ and $V^1 \cap V^2 = \{\mathbf{0}_V\}$, we write $V = V^1 \oplus V^2$. If X is a non-empty subset of V, then $\mathrm{Sp}(X)$ denotes the smallest subspace containing X. If $f : V \longrightarrow U$ is a linear mapping, then denote by $\mathrm{Ker}(f)$ its kernel, i.e. the set of vectors $v \in V$ such that $f(v) = \mathbf{0}_U$.

Let $N = \{1, 2, \ldots, n\}$ be a fixed and finite set of n agents. Each subset S of N is called a *coalition* while N is called the *grand coalition*. A cooperative game with transferable utility or simply a *TU-game* on a fixed agent set N is a function $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. The set of TU-games v on N, denoted by V_N, forms a linear space where $\dim(V_N) = 2^n - 1$. For each coalition $S \subseteq N$, $v(S)$ describes the *worth* of the coalition S when its members cooperate. For any two TU-games v and w in V_N and any $\alpha \in \mathbb{R}$, the TU-game $\alpha v + w \in V_N$ is defined as follows: for each $S \subseteq N$, $(\alpha v + w)(S) = \alpha v(S) + w(S)$. The inner product $v \cdot w$ is defined as $\sum_{S \subseteq N} v(S)w(S)$. For any nonempty coalition $T \subseteq N$, the *Dirac TU-game* $\delta_T \in V_N$ is defined as: $\delta_T(T) = 1$, and $\delta_T(S) = 0$ for each other S. Clearly, the collection of all Dirac TU-games is a basis for V_N such that $v = \sum_{\{S \in 2^N : S \neq \emptyset\}} v(S)\delta_S$ for each $v \in V_N$. We will also consider the TU-games $\delta_T^t \in V_N$, $T \subseteq N$, $T \neq \emptyset$, defined as: $\delta_T^t(T) = t$, and $\delta_T^t(S) = 0$ for each other S. Define the *dictator* TU-game u_i, for $i \in N$, as $u_i(S) = 1$ if $S \ni i$, and $u_i(S) = 0$ otherwise. A TU-game $v \in V_N$ is *inessential* if, for each $S \subseteq N$ such that $S \neq \emptyset$, $v(S) = \sum_{i \in S} v(i)$. The subset of inessential TU-games, denoted by I, forms a subspace of V_N such that $\dim(I) = n$. A basis for I is the collection of dictator TU-games $\{u_i : i \in N\}$.

Assume that the set of agents N face restrictions on communication. The bilateral communication possibilities between the agents are represented by an *undirected graph* on N. An undirected graph on N is a pair (N, L), where the set of nodes coincides with the set of agents N, and the set of links L is a subset of the set L_N of all unordered pairs of elements of N. As N is assumed to be fixed in this article, we will denote without any risk of confusion (N, L) by L. A sequence of distinct agents $(i_1, i_2, \ldots, i_p)$, $p \geq 2$, is a *path* in L if $\{i_q, i_{q+1}\} \in L$ for $q = 1, \ldots, p-1$. Two agents i and j are *connected* in L if $i = j$ or there exists a path from i to j. A maximally set (with respect to set inclusion) of connected agents is called a *component* of the graph. A graph L is *connected* if N is the only component of the graph. A *tree* is a minimally connected graph L in the sense that if a link is removed from L, L ceases to be connected. Equivalently, a tree is a connected graph such that only one path connects any two agents. A *leaf* of L is an agent in N who is incident to only one link. Following Béal et al. (2010), the set of *cones* of L consists of N, $\emptyset$ and, for each $\{i, j\} \in L$, of the two connected components that are obtained after deletion of link $\{i, j\}$. Every cone except N is called a *proper cone*. The unique agent of a nonempty proper cone K who has a link with the complementary cone $K^c = N \setminus K$ is called the *head* of the cone and is denoted by $h(K)$. Thus, the tree L contains $2(n-1) + 2 = 2n$ cones. Denote by Δ_L the set of cones of L, and by Δ_L^0 the subset of nonempty proper cones of L.

The combination of a tree L and a TU-game $v \in V_N$ is called a *tree TU-game*, and it is denoted by (v, L). Following Béal et al. (2012b), we say that (v, L) is *cone-additive* if:

$$\forall \{h(K), h(K^c)\} \in L, \quad v(N) = v(K) + v(K^c), \text{ and, } \forall S \in 2^N \setminus \Delta_L, \quad v(S) = 0.$$

Denote by A_L the subspace of TU-games $v \in V_N$ which are cone-additive on L.

A *payoff vector* $x \in \mathbb{R}^n$ for a tree TU-game (v, L) is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to each agent $i \in N$. A *(single-valued) solution* on the set of tree TU-games on N is a function Φ that assigns to every tree TU-game (v, L) a payoff vector $\Phi(v, L) \in \mathbb{R}^n$. Such a solution Φ represents a method for measuring the value of playing a particular role in a tree TU-game. In this article, the set of links L is also assumed to be fixed, so that we consider solutions of the form $\Phi(., L) : V_N \longrightarrow \mathbb{R}^n$. Here is a list of classical axioms for a solution $\Phi(., L)$.

Efficiency A solution $\Phi(., L)$ is efficient if, for each $v \in V_N$, it holds that:

$$\sum_{i \in N} \Phi_i(v, L) = v(N).$$

Linearity A solution $\Phi(., L)$ is linear if, for each $v \in V_N$, each $w \in V_N$ and each $\alpha \in \mathbb{R}$, it holds that:

$$\Phi(\alpha v + w, L) = \alpha \Phi(v, L) + \Phi(w, L).$$

Inessential game axiom A solution $\Phi(., L)$ satisfies the inessential game axiom if, for each $v \in I$, it holds that:

$$\Phi(v, L) = (v(1), \ldots, v(n)).$$

In order to calibrate the importance of each agent in the different coalitions, we define specific contribution vectors as in Demange (2004) and Herings, van der Laan and Talman (2008). To describe these contribution vectors we first give some definitions concerning rooted trees. By a *rooted tree* t_i, we mean a directed tree that arises from the tree L by selecting agent $i \in N$, called the *root*, and directing all links away from the root. Each agent $i \in N$ is the root of exactly one rooted tree t_i on L. Note also that for any rooted tree t_i on L, any agent $k \in N \backslash \{i\}$, there is exactly one directed link (j, k); agent j is the unique *predecessor* of k and k is a *successor* of j in t_i. Denote by $s_i(j)$ the possibly empty set of successors of agent $j \in N$ in t_i. An agent k is a *subordinate* of j in t_i if there is a directed path from j to k, i.e. if there is a sequence of distinct agents $(i_1, i_2, \ldots, i_p)$ such that $i_1 = j$, $i_p = k$ and for each $q = 1, 2, \ldots, p-1$, $i_{q+1} \in s_i(i_q)$. The set $S_i[j]$ denotes the union of the set of all subordinates of j in t_i and coalition formed by j. Note that:

$$\Delta_L = \left\{ S_i[j] : \{i, j\} \in L_N \right\} \cup \left\{ \emptyset \right\}. \tag{1}$$

In particular, for each $i \in N$, $S_i[i] = N$.

Pick any $v \in V_N$, any rooted tree t_i of L, and consider the marginal contribution vector $m^i(v, L)$ on $\mathbb{R}^n$ defined as:

$$\forall j \in N, \quad m^i_j(v, L) = v(S_i[j]) - \sum_{k \in s_i(j)} v(S_i[k]). \tag{2}$$

The marginal contribution $m^i_j(v, L)$ of $j \in N$ in t_i is thus equal to the worth of the coalition consisting of agent j and all his subordinates in t_i minus the sum of the worths of the coalitions consisting of any successor of j and all subordinates of this successor in t_i. Note that for each tree TU-game (v, L), each rooted tree t_i of L, and each agent $j \in N$, we have:

$$\sum_{k \in S_i[j]} m^i_k(v) = v(S_i[j]). \tag{3}$$

Herings, van der Laan and Talman (2008) introduce the *Average Tree solution*, denoted by AT, which assigns to every tree TU-game (v, L) the average of all marginal contribution vectors $m^i(v, L)$, i.e.

$$\mathrm{AT}(v, L) = \frac{1}{n} \sum_{i \in N} m^i(v, L). \tag{4}$$

Remark From (2), (3) and (4), we easily conclude that $\mathrm{AT}(., L)$ is linear, efficient, and satisfies the Inessential game axiom. From the Inessential game axiom, we deduce that $\mathrm{AT}(., L)$ is onto.

In order to characterize $\mathrm{AT}(\cdot, L)$, we introduce two new axioms.

Weighted addition invariance on bi-partitions A solution $\Phi(., L)$ on V_N is invariant to addition on bi-partitions if, for each $v \in V_N$, each $\alpha \in \mathbb{R}$, and each nonempty coalition $S \subseteq N$, it holds that:

$$\Phi(v + \alpha(\delta^s_S + \delta^{s^c}_{S^c}), L) = \Phi(v, L).$$

The use of bi-partitions of N has been suggested by von Neumann and Morgenstern (1953) and exploited by Evans (1996) who considers a bargaining procedure between two representative agents of a randomly chosen bi-partition. We also exploit this idea here by assuming that in a game $v \in V_N$ the grand coalition N splits into two coalitions S and S^c that bargain on the surplus $v(N) - v(S) - v(S^c)$ they can create by cooperating. In a sense, the worths $v(S)$ and $v(S^c)$ are the bargaining powers of these two bargaining coalitions. The axiom of Weighted addition invariance on bi-partitions indicates that if the worths of S and its complement S^c vary by an amount proportional to their respective size, then this change should not affect the resulting payoff vector. Observe that the per-capita variation is the same within the two elements of the bi-partition, which is important since their respective size can be different.

The last axiom states that the payoffs only depend on the worths of cones.

Invariance to irrelevant coalitions A solution $\Phi(., L)$ is invariant to irrelevant coalitions if, for each $v \in V_N$, each $S \in 2^N \setminus \Delta_L$, and each $\alpha \in \mathbb{R}$, it holds that:

$$\Phi(v + \alpha\delta_S, L) = \Phi(v, L).$$

This type of axioms is used to characterize solutions in TU-games on combinatorial structures (see, for instance, van den Brink et al., 2011). From (1), (2), and (4), we easily see that $\mathrm{AT}(., L)$ satisfies Invariance to irrelevant coalitions.

3. Results

Section 3.1 provides a basis for the kernel of $\mathrm{AT}(\cdot, L)$ and two direct-sum decompositions of V_N with respect to L. We also obtain, as a by-product of these results, new basis for V_N with respect to L. In section 3.2, we exploit all these results to axiomatically characterize $\mathrm{AT}(\cdot, L)$.

3.1. A basis for the kernel of $AT(\cdot, L)$, and two decompositions of V_N

Denote by B_L the subset of TU-games defined as:

$$B_L = \left\{ \delta_K^k + \delta_{K^c}^{k^c} : K \in \Delta_L^0, K \ni 1 \right\} \bigcup \left\{ \delta_S : S \in 2^N \setminus \Delta_L \right\}. \tag{5}$$

Proposition 1 *For any tree L on N, it holds that B_L is a basis for Ker$(AT(.,L))$.*

Proof. Because AT is onto, we have:

$$\dim(\mathrm{Ker}(\mathrm{AT}(.,L))) = 2^n - 1 - n.$$

Consider the subset of TU-games B_L given by (5). Its cardinality is:

$$(n-1) + 2^n - 2n = 2^n - 1 - n.$$

It is straightforward to see that that the elements of this subset of TU-games are linearly independent. Indeed, take any linear combination equal to the identity element:

$$\sum_{K \in \Delta_L^0, K \ni 1} \alpha_{K,K^c}(\delta_K^k + \delta_{K^c}^{k^c}) + \sum_{S \in 2^N \setminus \Delta_L} \alpha_S \delta_S = \mathbf{0}_{V_N}.$$

Pick any nonempty coalition T. If $T \in \Delta_L^0$ and $T \ni 1$, we have:

$$\sum_{K \in \Delta_L^0, K \ni 1} \alpha_{K,K^c}(\delta_K^k + \delta_{K^c}^{k^c})(T) + \sum_{S \in 2^N \setminus \Delta_L} \alpha_S \delta_S(T) = \alpha_{T,T^c}(\delta_T^t + \delta_{T^c}^{t^c})(T) = t\alpha_{T,T^c} = 0,$$

Thus, $\alpha_{T,T^c} = 0$. If $T \in 2^N \setminus \Delta_L$, we have:

$$\sum_{K \in \Delta_L^0, K \ni 1} \alpha_{K,K^c}(\delta_K^k + \delta_{K^c}^{k^c})(T) + \sum_{S \in 2^N \setminus \Delta_L} \alpha_S \delta_S(T) = \alpha_T \delta_T(T) = \alpha_T = 0.$$

Therefore, the vectors of the set (5) are linearly independent. Next, from (2) and (4), we easily verify that $\mathrm{AT}(v,L)$ does not depend on the worths $v(S)$, $S \in 2^N \setminus \Delta_L$. Consider first any TU-game δ_S, $S \in 2^N \setminus \Delta_L$. By the above remark and the linearity of $\mathrm{AT}(v,L)$, we have:

$$\mathrm{AT}(\delta_S, L) = \mathrm{AT}(\mathbf{0}_{V_N} + \delta_S, L) = \mathrm{AT}(\mathbf{0}_{V_N}, L) = \mathbf{0}_{\mathbb{R}^n}.$$

Now, pick any TU-game $\delta_K^k + \delta_{K^c}^{k^c}$, for some nonempty proper cone $K \in \Delta_L^0$. Consider the head $h(K)$ of the proper cone K, and pick any root i of L. Two cases arise. If $i \in K$, by (2), we have:

$$m_{h(K)}^i(\delta_K^k + \delta_{K^c}^{k^c}) = -\delta_{K^c}^{k^c}(K^c) = -k^c$$

If $i \in K^c$, by (2) we have:

$$m_{h(K)}^i(\delta_K^k + \delta_{K^c}^{k^c}) = \delta_K^k(K) = k.$$

By (4), this gives:

$$\begin{aligned}
\mathrm{AT}_{h(K)}(\delta_K^k + \delta_{K^c}^{k^c}, L) &= \frac{1}{n}\sum_{i\in N} m_{h(K)}^i(\delta_K^k + \delta_{K^c}^{k^c}, L)\\
&= \frac{1}{n}\sum_{i\in K} m_{h(K)}^i(\delta_K^k + \delta_{K^c}^{k^c}, L) + \frac{1}{n}\sum_{i\in K^c} m_{h(K)}^i(\delta_K^k + \delta_{K^c}^{k^c}, L)\\
&= \frac{1}{n}(-kk^c + k^c k)\\
&= 0.
\end{aligned}$$

By a symmetric argument we obtain:

$$\mathrm{AT}_{h(K^c)}(\delta_K^k + \delta_{K^c}^{k^c}, L) = 0.$$

For each agent i distinct from $h(K)$ and $h(K^c)$, we easily get from (2) and (4):

$$\mathrm{AT}_i(\delta_K^k + \delta_{K^c}^{k^c}, L) = 0, \quad \text{and so} \quad \mathrm{AT}(\delta_K^k + \delta_{K^c}^{k^c}, L) = \mathbf{0}_{\mathbb{R}^n}.$$

Therefore, $B_L \subseteq \mathrm{Ker}(\mathrm{AT})$. Because the $2^n - 1 - n$ elements of B_L are linearly independent, they constitute a basis for $\mathrm{Ker}(\mathrm{AT}(.,L))$, and so $\mathrm{Ker}(\mathrm{AT}(.,L)) = \mathrm{Sp}(B_L)$, as desired. ■

For each $i \in N$, define the TU-game z^i as:

$$z^i = \sum_{\{K\in\Delta_L : K\ni i\}} \delta_K. \tag{6}$$

Define Z_L as:

$$Z_L = \Big\{ z^i \in V_N : i \in N \Big\}. \tag{7}$$

Proposition 2 *Consider any tree L on N, and the sets B_L and Z_L given by (5) and (7), respectively. The following statements hold:*

1. *The set Z_L forms a basis for the subspace of cone-additive tree-games A_L.*
2. *$V_N = Sp(B_L) \oplus A_L$, and $V_N = Sp(B_L) \oplus I$.*
3. *The sets $B_L \cup Z_L$ and $B_L \cup \{u_i : i \in N\}$ form two basis for V_N with respect to L.*

Proof. Statement 1. We proceed in three steps. First we show that the n elements of Z_L are linearly independent. In a second step, we show that $Z_L \subseteq A_L$. In a last step, we show that $\dim(A_L) = n$.

Step 1. Pick any linear combination of the elements of Z_L equal to the identity element:

$$\sum_{i\in N} \alpha_i z^i = \mathbf{0}_{V_N}. \tag{8}$$

Assume, without loss of generality, that the elements of N are labeled in the following way: each $i \in N$ is a leaf of the (sub)tree obtained by deleting the agents $1, 2, \ldots, i-1$ in L. We proceed by induction on N.

Initial step: Because the singleton formed by agent 1 is a cone, by (8), we obtain:

$$\sum_{i\in N}\alpha_i z^i(1)=\alpha_1=0.$$

Induction hypothesis: Assume that $\alpha_j=0$ for $j<i\leq n$.
Induction step: Consider agent i. By definition of L, there exists a unique cone, say K, such that $h(K)=i$ and $K\subseteq\{1,2,\ldots,i\}$. By (8), we have:

$$\sum_{i\in N}\alpha_i z^i(K)=\sum_{j\in K}\alpha_j=0.$$

Since $K\subseteq\{1,2,\ldots,i\}$, the induction hypothesis leads to: $\alpha_j=0$ for each $j\in K\setminus i$. This forces $\alpha_i=0$.

Conclude that the elements of Z_L are linearly independent. This completes the proof of Step 1.

Step 2. Pick any element $z^i\in Z_L$ and any link $\{h(K),h(K^c)\}\in L$. By definition of proper a cone, either $i\in K$ or $i\in K^c$. Assume, without loss of generality, that $i\in K$. By definition of z^i, we have:

$$z^i(K)=1,\ z^i(K^c)=0 \text{ and } z^i(N)=1,$$

which proves that $z^i\in A_L$, and so $Z_L\subseteq A_L$, as desired.

Step 3. Define the subspace W of V_N as $W=\mathrm{Sp}(C)$ where

$$C=\left(\left\{\delta_S:S\in 2^N\setminus\Delta_L\right\}\bigcup\left\{\ \delta_K+\delta_{K^c}-\delta_N:K\in\Delta_L^0,K\ni 1\right\}\right).$$

The generating set C of W contains $2^n-2n+(n-1)=2^n-n-1$ elements which are linearly independent. We also have : $A_L=W^\perp$, where

$$W^\perp=\left\{v\in V_N:\forall w\in C,\ w\cdot v=0\right\}.$$

Because $W\oplus W^\perp=V_N$, we obtain:

$$\dim(A^L)=\dim(W^\perp)=\dim(V_N)-\dim(W)=2^n-1-(2^n-n-1)=n.$$

Combining Steps 1-3, we obtain that Z_L is a basis for A_L, which completes the proof of Statement 1.

Statement 2. We first prove that $V_N=\mathrm{Sp}(B_L)\oplus A_L$, where $A_L=\mathrm{Sp}(Z_L)$ by Statement 1. By Proposition 1 and Statement 1, we already know that:

$$\dim(\mathrm{Sp}(B_L))+\dim(A_L)=2^n-1-n+n=2^n-1=\dim(V_N).$$

It remains to prove that $\mathrm{Sp}(B_L)\cap A_L=\{\mathbf{0}_{V_N}\}$. So, pick any $v\in\mathrm{Sp}(B_L)\cap A_L$. Because Z_L is a basis for A_L, if $v\in A_L$, then there exist unique real numbers α_i, $i\in N$, such that:

$$v=\sum_{i\in N}\alpha_i z^i.$$

Applying $\mathrm{AT}(.,L)$ to v, we get by linearity of $\mathrm{AT}(.,L)$:

$$\begin{aligned}\mathrm{AT}(v,L) &= \mathrm{AT}\Big(\sum_{i\in N}\alpha_i z^i, L\Big)\\ &= \sum_{i\in N}\alpha_i \mathrm{AT}(z^i,L).\end{aligned}$$

Consider the payoff vector $\mathrm{AT}(z^i,L)$ for some $i\in N$. Note that z^i and the dictator TU-game u_i (an inessential TU-game) are cone equivalent in the sense that $u_i(K)=z^i(K)$ for each $K\in\Delta_N$. Because $\mathrm{AT}(.,L)$ satisfies Invariance to irrelevant coalitions and the Inessential game property, we immediately get $\mathrm{AT}_i(z^i,L)=\mathrm{AT}_i(u_i,L)=1$ and $\mathrm{AT}_j(z^i,L)=\mathrm{AT}_j(u_i,L)=0$ for any $j\in N\setminus i$. Therefore, from (9), we obtain:

$$\mathrm{AT}(v,L)=\sum_{i\in N}\alpha_i\mathrm{AT}(z^i,L) \quad = \quad (\alpha_1,\ldots,\alpha_n).$$

On the other hand, $v\in \mathrm{Sp}(B_L)$. By Proposition 1, $\mathrm{Sp}(B_L)=\mathrm{Ker}(\mathrm{AT}(.,L))$, which implies $\mathrm{AT}(v,L)=\mathbf{0}_{\mathbb{R}^n}$, and so $0=\alpha_i$ for each $i\in N$. Thus, $v=\mathbf{0}_{V_N}$, as desired. Conclude that $V_N=\mathrm{Sp}(B_L)\oplus A_L$.
To show that $V_N=\mathrm{Sp}(B_L)\oplus I$, it suffices to proceed in a similar way and to apply the Inessential game axiom to $\mathrm{AT}(v,L)$, $v\in I$.

Statement 3 is a direct consequence of Statements 1-2, Proposition 1 and the fact the collection $\{u_i : i\in N\}$ is a basis for I.

■

Thanks to Proposition 1 and Proposition 2, we are able to solve the following inverse problem: given the payoff vector $(\alpha_1,\ldots,\alpha_n)\in\mathbb{R}^n$, find all $v\in V_N$ such that:

$$\mathrm{AT}(v,L)=(\alpha_1,\ldots,\alpha_n).$$

This problem is solved by using the decompositions of V_N contained in Proposition 2. Consider, without loss of generality, the decomposition $V_N=\mathrm{Sp}(B_L)\oplus I$, where $I=\mathrm{Sp}(\{u_i : i\in V_N\})$. Because $V_N=\mathrm{Sp}(B_L)\oplus I$, each $v\in V_N$ is the sum of exactly one TU-game in $\mathrm{Sp}(B_L)$ and exactly one inessential TU-game in I. Therefore, from the equality $\mathrm{Sp}(B_L)=\mathrm{Ker}(\mathrm{AT}(v,L))$ and the fact that $\mathrm{AT}(v,L)$ satisfies the Inessential game property:

$$\mathrm{AT}(v,L)=(\alpha_1,\ldots,\alpha_n) \text{ if and only if } v=v^1+\sum_{i\in N}\alpha_i u_i \text{ for some } v^1\in\mathrm{Sp}(B_L).$$

The TU-game v^1 admits a unique decomposition along the elements of B_L :

$$v^1=\sum_{\{K\in\Delta_L^0:K\ni 1\}}\alpha_{K,K^c}(\delta_K^k+\delta_{K^c}^{k^c})+\sum_{S\in 2^N\setminus\Delta_L}v^1(S)\delta_S.$$

Regarding the coordinates α_{K,K^c} note that:

$$\forall K\in\Delta_L^0, K\ni 1,\; v(K)=\alpha_{K,K^c}k+\sum_{i\in K}\alpha_i,\; v(K^c)=\alpha_{K,K^c}(n-k)+\sum_{i\in K^c}\alpha_i, \text{ and } v(N)=\sum_{i\in N}\alpha_i.$$

Combining these equalities, we obtain:

$$\forall K \in \Delta_L^0 : K \ni 1, \quad \alpha_{K,K^c} = -n^{-1}(v(N) - v(K) - v(K^c)).$$

In such a TU-game $v \in V_N$, it holds that:

$$\forall K \in \Delta_L, \quad v(K) = -n^{-1}k(v(N) - v(K) - v(K^c)) + \sum_{i \in K} \alpha_i.$$

From this, we obtain the following characterization.

Proposition 3 *Given the payoff vector $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$, we have:*

$$AT(v, L) = (\alpha_1, \ldots, \alpha_n) \;\; \text{if and only if, } \forall K \in \Delta_L, \; \sum_{i \in K} \alpha_i = v(K) + n^{-1}k(v(N) - v(K) - v(K^c)).$$

Proposition 3 indicates that $v \in V_N$ belongs to the solution set if and only if the total payoff distributed to the members of the cone N is equal to $v(N)$ by Efficiency of $\text{AT}(v, L)$, and the total payoff distributed to the members of each proper cone K is equal to its worth plus a share, proportional to the size k of K, of the surplus created by the link $\{h(K), h(K^c)\} \in L$.

3.2. Characterization

Using the results contained in section 3.1 we easily derive the following new characterization of $\text{AT}(., L)$.

Proposition 4 *Let L be any tree on N. The Average Tree solution $AT(., L)$ is the unique solution which satisfies Invariance to irrelevant coalitions, Addition invariance on bi-partitions, and the Inessential game axiom.*

Proof. We have already underlined that $\text{AT}(., L)$ satisfies Invariance to irrelevant coalitions and the Inessential game axiom. It remains to verify that it satisfies Weighted addition invariance on bi-partitions. Pick any nonempty $S \in 2^N$, $S \neq N$. If $S \notin \Delta_L$,

$$\text{AT}(v + \alpha(\delta_S^s + \delta_{S^c}^{s^c}), L) = \text{AT}(v, L),$$

by Invariance to irrelevant coalitions. If $S \in \Delta_L$,

$$\text{AT}(v + \alpha(\delta_S^s + \delta_{S^c}^{s^c}), L) = \text{AT}(v, L),$$

by linearity of $\text{AT}(., L)$ and the fact that $\delta_S^s + \delta_{S^c}^{s^c} \in \text{Ker}(\text{AT}(., L))$ by Proposition 1. So, $\text{AT}(., L)$ satisfies Addition invariance on bi-partitions.
Regarding the uniqueness part, assume that the solution $\Phi(., L)$ satisfies the four axioms in the statement of Proposition 4. By Proposition 2 we have:

$$V_N = \text{Sp}(B_L) \oplus I, \text{ and } \; B_L \cup \{u_i : i \in N\} \; \text{ is a basis for } \; V_N.$$

Thus any $v \in V_N$ admits a unique linear decomposition along the elements of $B_L \cup \{u_i : i \in N\}$:

$$v = \sum_{K \in \Delta_L^0, K \ni 1} \alpha_{K,K^c}(\delta_K^k + \delta_{K^c}^{k^c}) + \sum_{S \in 2^N \setminus \Delta_L} \alpha_S \delta_S + \sum_{i \in N} \beta_i u_i.$$

By successive applications of Invariance to irrelevant coalitions, we get:

$$\begin{aligned}\Phi(v,L) &= \Phi\Bigg(\sum_{K\in\Delta_L^0, K\ni 1}\alpha_{K,K^c}(\delta_K^k+\delta_{K^c}^{k^c}) + \sum_{S\in 2^N\setminus\Delta_L}\alpha_S\delta_S + \sum_{i\in N}\beta_i u_i\Bigg)\\ &= \Phi\Bigg(\sum_{K\in\Delta_L^0, K\ni 1}\alpha_{K,K^c}(\delta_K^k+\delta_{K^c}^{k^c}) + \sum_{i\in N}\beta_i u_i\Bigg).\end{aligned}$$

By successive applications of Weighted addition invariance on bi-partitions, we get:

$$\Phi(v,L) = \Phi\Bigg(\sum_{K\in\Delta_L^0, K\ni 1}\alpha_{K,K^c}(\delta_K^k+\delta_{K^c}^{k^c}) + \sum_{i\in N}\beta_i u_i\Bigg) = \Phi\Bigg(\sum_{i\in N}\beta_i u_i\Bigg).$$

By the Inessential game axiom, we obtain:

$$\Phi(v,L) = \Phi\Bigg(\sum_{i\in N}\beta_i u_i\Bigg) = (\beta_1,\ldots,\beta_n),$$

which proves that $\Phi(v)$ is uniquely determined. This completes the proof of Proposition 4. ■

To prove the logical independence of the axioms used in the two characterizations, we proceed as follows.

• To prove that Invariance to irrelevant coalitions is logically independent of the two other axioms, we proceed as follows. Define the subspace U of V_N as:

$$U = \mathrm{Sp}\Bigg(\Big\{\delta_S^s+\delta_{S^c}^{s^c} : S\in 2^N, S\ni 1, S\neq N\Big\}\Bigg).$$

We have: $\dim(U) = 2^{n-1}-1$. Note that $U\cap I$ contains the subspace of constant inessential TU-games I_C:

$$I_C = \mathrm{Sp}\Bigg(\Big\{\sum_{i\in N}u_i\Big\}\Bigg), \quad \text{where} \quad \dim(I_C)=1.$$

From this remark, we obtain:

$$\dim(U+I)\leq \dim(U)+\dim(I)-\dim(I_C) = 2^{n-1}-1+n-1 < 2^n-1 = \dim(V_N).$$

Combining this inequality with the fact that

$$U+I+\mathrm{Sp}\Bigg(\Big\{\delta_S : S\in 2^N\setminus\Delta_L\Big\}\Bigg) = V_N,$$

we conclude that there exists a subspace

$$W\subseteq \mathrm{Sp}\Bigg(\Big\{\delta_S : S\in 2^N\setminus\Delta_L\Big\}\Bigg) \text{ such that } V_N = (U+I)\oplus W.$$

Therefore, for each $v\in V_N$, there exist exactly one $v^1\in U+I$ and exactly one $w\in W$ such that:

$$v = v^1 + w.$$

Next, construct the solution Φ as follows:

$$\Phi(v, L) = \mathrm{AT}(v) + \Psi(w, L) \quad \text{where} \quad \exists w \in W \quad \text{such that} \quad \Psi(w, L) \neq \mathbf{0}_{\mathbb{R}^n}.$$

This solution satisfies Addition invariance on bi-partitions, the Inessential game axiom but violates Invariance to irrelevant coalitions due to the fact that there exists at least one

$$w \in W \subseteq \mathrm{Sp}\bigg(\Big\{\delta_S : S \in 2^N \setminus \Delta_L\Big\}\bigg) \quad \text{such that} \quad \Psi(w, L) \neq \mathbf{0}_{\mathbb{R}^n}.$$

• To prove that Addition invariance on bi-partitions is logically independent of the other two axioms, we proceed as follows. We know that each $v \in V_N$ admits a unique decomposition along the elements of $B_L \cup \{u_i : i \in N\}$:

$$v = \sum_{K \in \Delta_L^0, K \ni 1} \alpha_{K,K^c}(\delta_K^k + \delta_{K^c}^{k^c}) + \sum_{S \in 2^N \setminus \Delta_L} \alpha_S \delta_S + \sum_{i \in N} \beta_i u_i.$$

Define Φ as:

$$\Phi(v, L) = \mathrm{AT}(v, L) + \Psi\bigg(\sum_{K \in \Delta_L^0, K \ni 1} \alpha_{K,K^c}(\delta_K^k + \delta_{K^c}^{k^c}), L\bigg),$$

such that Ψ satisfies the following condition:

$$\exists w \in \mathrm{Sp}\bigg(\Big\{\delta_K^k + \delta_{K^c}^{k^c} : K \in \Delta_L^0, K \ni 1\Big\}\bigg) \quad \text{such that} \quad \Psi(w, L) \neq \mathbf{0}_{\mathbb{R}^n}.$$

The solution Φ satisfies Invariance to irrelevant coalitions and the Inessential game axiom, but violates Addition invariance on bi-partitions.

• To prove that the Inessential game axiom is logically independent of the two other axioms, consider the null solution $\Phi(v, L) = \mathbf{0}_{\mathbb{R}^n}$ for each $v \in V_N$. This solution satisfies Invariance to irrelevant coalitions, Addition invariance on bi-partitions but violates the Inessential game axiom.

We conclude that Invariance to irrelevant coalitions, Addition invariance on bi-partitions, and the Inessential game axiom are logically independent.

4. Conclusion

Since many allocation rules on graph TU-games are linear, it would be interesting to find relevant basis of their kernel with respect to the underlying graph in order to construct new axioms of invariance and/or covariance suitable to build new characterizations of these allocation rules. Though this approach has been recently applied to classical TU-games – see, for instance, Béal et al. (2013) and Yokote, (2014) – it has not been yet deeper investigated in the field of graph TU-games.

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