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Efficient extensions of communication values[☆]

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Abstract

We study values for transferable utility games enriched by a communication graph. The most well-known such values are component-efficient and characterized by some link-deletion property. We study efficient extensions of such values: for a given component-efficient value, we look for a value that *(i)* satisfies efficiency, *(ii)* satisfies the link-deletion property underlying the original component-efficient value, and *(iii)* coincides with the original component-efficient value whenever the underlying graph is connected. Béal et al. (2015) prove that the Myerson value (Myerson, 1977) admits a unique efficient extension, which has been introduced by van den Brink et al. (2012). We pursue this line of research by showing that the average tree solution (Herings et al., 2008) and the compensation solution (Béal et al., 2012a) admit similar unique efficient extensions, and that there exists no efficient extension of the position Value (Meessen, 1988; Borm et al., 1992). As byproducts, we obtain new characterizations of the average tree solution and the compensation solution, and of their efficient extensions.

Keywords: Efficient extension, average tree solution, compensation solution, position value, component fairness, relative fairness, balanced total threats, Myerson value, component-wise egalitarian solution
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1. Introduction

Cooperative games with transferable utility (henceforth TU-games) describe the worth that each coalition of players can generate by cooperating. The objective is to find a value, which rewards the players for participating in the TU-game with a certain payoff. In this classical model, such a value can only depend (possibly) on the worths of the coalitions of players. The

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Shapley value (Shapley, 1953) is the most well-known value for TU-games. However, in many situations the exogenous affinities among players are represented by some social, hierarchical, economical, communicational, or technical structure. Two prominent examples are the games with a coalition structure introduced by Aumann and Dreze (1974) and Owen (1977) and the games with a communication structure (called CO-games hereafter) proposed by Myerson (1977). In the first model, the players are organized in a priori unions. In the second model, the communication among the players are is modeled by the links of an undirected graph. In both models, it is crucial to evaluate the influence of the exogenous structure on the payoff allocation, and in each of the above-mentioned articles, the Shapley value is generalized in a specific way. For games with a coalition structure, two interpretations coexist since the beginning. On the one hand, Aumann and Dreze (1974) propose a value that is component-efficient: the players only share the worth of their own a priori union. On the other hand, Owen (1977) introduces an efficient value, which is motivated by Hart and Kurz (1983, p.1048) as follows:

> "(...) outcomes are "overall efficient", no matter the players are organized. Thus, we assume as a postulate that society as a whole operates efficiently; the problem we address here is how are the benefits distributed among the participants. With this view in mind, coalitions do not form in order to obtain their "worth" and then "leave" the game. But rather, they "stay" in the game and bargain as a unit with all the other players."

The Myerson value (Myerson, 1977) can be considered as the counterpart for CO-games of the value suggested by Aumann and Dreze (1974) for games with a coalition structure. It is component-efficient: the players only share the worth of their own component of the graph. This property is also satisfied by numerous values for CO-games (henceforth CO-values) that appeared consequently in the literature such as the position value (Meessen, 1988; Borm et al., 1992), the average tree solution (Herings et al., 2008) and the compensation solution (Béal et al., 2012a). All such CO-values are characterized by component efficiency and an appealing link deletion property which reflects the payoff variation of some players when some links are removed from the graph. Surprisingly, for CO-games, counterparts of the value proposed by Owen (1977) have been less popular so far and only appeared recently (see Casajus, 2007; Hamiache, 2012; Béal et al., 2012b; van den Brink et al., 2012).

In this article, we introduce new efficient CO-values that are *efficient extensions* of well-known component-efficient CO-values. More specifically, for a given component-efficient CO-value, an efficient extension is any CO-value which *(i)* is efficient, *(ii)* satisfies the link deletion property characterizing the original component-efficient CO-value, and *(iii)* coincides with the original component-efficient CO-value whenever the underlying graph is connected. This approach has been initiated by Béal et al. (2015), in which it is shown that the Myerson value admits a unique efficient extension: the CO-value studied by van den Brink et al. (2012). We show that the average tree solution and the Compensation admit unique efficient extensions (Theorems 4 and 6, respectively). Moreover, these efficient extensions can be constructed similarly to the efficient extension of the Myerson value: each player receives his payoff according to the corresponding component-efficient CO-value plus an equal share of the surplus of worth generated by the grand coalition compared to the total worth achieved by the components of the graph.

These results seem to suggest that an efficient extension always exists, is unique, and is always built by means of the previous construction. We show that these assertions are not all valid. Firstly, we prove that there does not exist any efficient extension of the position value (Theorem

8). Secondly, we conclude the article by showing that the component-wise egalitarian solution characterized by Slikker (2007) admits a unique efficient extension, but that it is not constructed by evenly splitting surplus produced by the grand coalition compared to the total worth achieved by the components of the graph in addition to the component-wise egalitarian solution.

Our (possibility) results are obtained by means of new axiomatic characterizations of the average tree solution and the compensation solution on the domain of connected cycle-free graphs (Propositions 1 and 2, respectively). These characterizations are also adapted to provide axiomatic characterizations of their efficient extensions (parts *(ii)* in Theorems 4 and 6).

The rest of the article is organized as follows. Section 2 contains preliminaries, the definition of an efficient extension, and exposes a result from Béal et al. (2015). Sections 3 and 4 provide all the material on the average tree solution and the compensation solution, respectively. Section 5 proves the impossibility result about the position value. Section 6 concludes by a discussion on the unique efficient extension of the component-wise egalitarian solution.

2. Cooperative games and graphs

Fix an infinite set $\mathfrak{U}$, the universe of players, and let $\mathcal{N}$ denote the set of non-empty and finite subsets of $\mathfrak{U}$.

2.1. Cooperative games with transferable utilities

A **TU-game** is a pair (N, v) consisting of a set of players $N \in \mathcal{N}$ and a **coalition function** $v \in \{f : 2^N \longrightarrow \mathbb{R} \mid f(\varnothing) = 0\}$, where 2^N denotes the power set of N. Subsets of N are called **coalitions**, and $v(S)$ is called the worth of coalition S. For any TU-game (N, v) and any $S \subseteq N$, the sub-game of (N, v) induced by S is denoted by $(S, v|_S)$, where $v|_S$ is the restriction of v to 2^S. A TU-game (N, v) is **zero-normalized** if $v(\{i\}) = 0$ for all $i \in N$.

A **value** on $\mathcal{N}$ is an operator φ that assigns a payoff vector $\varphi(N, v) \in \mathbb{R}^N$ to any TU-game (N, v). The **Shapley value** (Shapley, 1953) is the value given by

$$\mathrm{SH}_i(N, v) = \sum_{S \subseteq N \backslash \{i\}} \frac{1}{|N|} \cdot \binom{|N|-1}{|S|}^{-1} \cdot (v(S \cup \{i\}) - v(S))$$

for all TU-games (N, v) and $i \in N$.

2.2. Graphs

A **communication graph** for $N \in \mathcal{N}$ is an undirected graph (N, L), where $L \subseteq \mathcal{L}^N := \{\{i,j\} \,| i,j \in N, i \neq j\}$. A typical element (**link**) of L is written as $ij := \{i, j\}$. Player's $i, j \in N$ are called connected in (N, L) if there is a sequence of players $(i_1, i_2, \ldots, i_k)$, $k \in \mathbb{N}$, $k > 1$ from N such that $i_1 = i$, $i_k = j$, and $i_\ell i_{\ell+1} \in L$ for all $\ell \in \{1, \ldots, k-1\}$. It is clear that connectedness is an equivalence relation. Hence, it induces a partition $\mathcal{C}(N, L)$ of N, the set of **components** of (N, L), such that $C \in \mathcal{C}(N, L)$, $i, j \in C$, $k \in N \backslash C$, $i \neq j$ implies that i and j are connected and that i and k are not connected in (N, L). The component of (N, L) containing $i \in N$ is denoted by $C_i(N, L)$. The graph (N, L) is called **connected** if $\mathcal{C}(N, L) = \{N\}$. A link $ij \in L$ is called a **bridge** in (N, L) if $\mathcal{C}(N, L) \neq \mathcal{C}(N, L \backslash \{ij\})$. For $S \subseteq N$ and $L \subseteq \mathcal{L}^N$, set $L|_S := \{ij \in L | i, j \in S\}$. A graph (N, L) is **cycle-free** if each $ij \in L$ is a bridge, and each of its components is called a **tree**.

A **directed graph** for $N \in \mathcal{N}$ is pair (N, D), $D \subseteq \mathcal{D}^N := (N \times N) \backslash \{(i, i) \mid i \in N\}$; $(i, j) \in \mathcal{D}^N$ is called a **directed link** from i to j. For any cycle-free graph (N, L) and any $C \in \mathcal{C}(N, L)$, each player $r \in C$ induces a **rooted spanning tree** on C, *i.e.*, a directed graph that arises from the tree $(C, L|_C)$ by directing all links away from the root r. If a spanning tree rooted at r contains a directed link (i, j), then j is a called a **successor** of i. Denote by $s_r(i)$ the possibly empty set of successors of player $i \in C$ in the spanning tree rooted at r. A player j is a **subordinate** of i if there is a directed path from i to j, *i.e.*, if there is a sequence of distinct players $(i_1, \ldots, i_k)$, $k \in \mathbb{N}$, $k > 1$ from N such that $i_1 = i$, $i_k = j$, and, for each $\ell = \{1, \ldots, k-1\}$, $i_{\ell+1} \in s_r(i_\ell)$. The set $S_r(i)$ denotes the union of all subordinates of i in the spanning tree rooted at r and i.

2.3. Communication games

A **CO-game** is a triple (N, v, L), where (N, v) is a TU-game and $L \subseteq \mathcal{L}^N$. We denote by $\mathcal{G}$ the set of all such CO-games. A CO-game is called connected if the associated graph is connected, and cycle-free is the associated graph is cycle-free. We denote by $\mathcal{G}_C \subseteq \mathcal{G}$, $\mathcal{G}_{CF} \subseteq \mathcal{G}$ and $\mathcal{G}_0 \subset \mathcal{G}$ the classes of all **connected CO-games**, of all **cycle-free CO-games**, and of all **zero-normalized CO-games** respectively. A **CO-value** on some class of CO-games $\mathcal{G}^* \subseteq \mathcal{G}$ is an operator φ that assigns a payoff vector $\varphi(N, v, L) \in \mathbb{R}^N$ to every CO-game $(N, v, L) \in \mathcal{G}^*$.

The **Myerson value** (Myerson, 1977) is the CO-value on $\mathcal{G}$ given by

$$\mathrm{MY}\,(N, v, L) := \mathrm{SH}\left(N, v^L\right), \qquad v^L\,(S) := \sum_{T \in \mathcal{C}(S, L|_S)} v\,(T)\,, \quad S \subseteq N$$

It is characterized by component efficiency and fairness. Throughout this article, we sometimes invoke axioms on different subclasses of CO-games indicated by "$|_{\mathcal{G}^*}$" in their definition. For any such subclass, all the CO-games used in the axiom belong to the subclass. If an axiom is invoked on a unique class of CO-games or if $\mathcal{G}^* = \mathcal{G}$, we omit this indicator.

Component efficiency, CE$|_{\mathcal{G}^*}$. For all $(N, v, L) \in \mathcal{G}^*$ and $C \in \mathcal{C}(N, L)$,

$$\sum_{i \in C} \varphi_i(N, v, L) = v(C).$$

Fairness, F. For all $(N, v, L) \in \mathcal{G}$, and $ij \in L$,

$$\varphi_i\,(N, v, L) - \varphi_i\,(N, v, L\backslash\{ij\}) = \varphi_j\,(N, v, L) - \varphi_j\,(N, v, L\backslash\{ij\})\,.$$

Component efficiency states that the worth of each component of the graph is distributed among its members. Fairness requires that removing a link from the graph changes the payoffs of the players forming this link by the same amount.

Theorem 1. (Myerson, 1977) *The Myerson value is the unique CO-value on $\mathcal{G}$ that satisfies component efficiency (**CE**) and fairness (**F**).*

2.4. Efficient extensions

The axiom of component efficiency is natural if the communication among the players in the communication graph is interpreted as a necessity to generate worth. Another plausible interpretation is that the generation of worth is not constrained by the communication graph, which is simply used to evaluate the a priori affinities among the players in order to provide a payoff allocation. Under this alternative interpretation, the axiom of efficiency below is natural.

Efficiency, $\mathbf{E}|_{\mathcal{G}^*}$. For all $(N, v, L) \in \mathcal{G}^*$,

$$\sum_{i \in N} \varphi_i(N, v, L) = v(N).$$

Let ψ be any CO-value characterized on some class of CO-games $\mathcal{G}^*$ by component efficiency and some link deletion property denoted by $\mathbf{LDP}^{\psi}$. The efficient extension of ψ is a CO-value φ on $\mathcal{G}^*$ such that

(i) φ satisfies efficiency ($\mathbf{E}|_{\mathcal{G}^*}$),

(ii) φ satisfies the link deletion property $\mathbf{LDP}^{\psi}|_{\mathcal{G}^*}$,

(iii) φ coincides with ψ on $\mathcal{G}^* \cap \mathcal{G}_C$.

Point *(i)* means that φ is efficient while ψ is component efficient. Point *(ii)* means that φ and ψ both satisfy the link deletion property which is characteristic of ψ. Point *(iii)* means that φ and ψ prescribe the same payoff vector in all CO-games in $\mathcal{G}^*$ where the graph is connected, *i.e.*, where efficiency and component efficiency are the same condition. All in all, this means that φ is very close to ψ, somehow the CO-value closest to ψ among the efficient CO-values as will be explained later.

The first efficient extension has been proposed by van den Brink et al. (2012) for the Myerson value. We call it the **efficient egalitarian Myerson value** EEMY, given by

$$\mathrm{EEMY}_i\,(N, v, L) := \mathrm{MY}_i\,(N, v, L) + \frac{v\,(N) - v^L(N)}{|N|}$$

for all CO-games $(N, v, L) \in \mathcal{G}$, and $i \in N$.

Béal et al. (2015) show that EEMY is the unique efficient extension of the Myerson value.

Theorem 2. (Béal et al., 2015) *The efficient egalitarian Myerson value* EEMY *is the unique efficient extension of the Myerson value, i.e.,* φ *satisfies* $\varphi = \mathrm{MY}$ *on* $\mathcal{G}_C$ *and meets efficiency (**E**) and fairness (**F**), if and only if* $\varphi = \mathrm{EEMY}$ *on* $\mathcal{G}$.

In the next three sections, we study the efficient extensions of three other CO-values: the average tree solution, the compensation solution, and the position value.

3. The average tree solution: characterizations and a unique efficient extension

For all $(N, v, L) \in \mathcal{G}_{CF}$, all $C \in \mathcal{C}(N, L)$, and $r \in C$, Demange (2004) defines the **hierarchical outcome** for the spanning tree on C rooted at r as:

$$m_i^r(N, v, L) = v(S_r(i)) - \sum_{j \in s_r(i)} v(S_r(j))$$

for each $i \in C$. The **average tree solution** AT introduced by Herings et al. (2008) is the CO-value on $\mathcal{G}_{CF}$ that assigns to each cycle-free CO-game and to each player the average of his hierarchical outcomes:

$$\mathrm{AT}_i(N, v, L) = \frac{1}{|C_i(N, L)|} \sum_{r \in C_i(N,L)} m_i^r(N, v, L)$$

for all $(N, v, L) \in \mathcal{G}_{CF}$, and $i \in N$. They use component efficiency and component fairness below in order to characterize the average tree solution for cycle-free CO-games.

Component fairness, CF. For all $(N, v, L) \in \mathcal{G}_{CF}$, and $ij \in L$,

$$\sum_{k \in C_i(N, L\backslash\{ij\})} \frac{\varphi_k(N, v, L) - \varphi_k(N, v, L\backslash\{ij\})}{|C_i(N, L\backslash\{ij\})|} = \sum_{k \in C_j(N, L\backslash\{ij\})} \frac{\varphi_k(N, v, L) - \varphi_k(N, v, L\backslash\{ij\})}{|C_j(N, L\backslash\{ij\})|}.$$

Component fairness states that deleting a link between two players yields for both resulting new components the same per-capita change in payoffs.

Theorem 3. (Herings et al., 2008) *The average tree solution is the unique CO-value on $\mathcal{G}_{CF}$ that satisfies component efficiency (**CE**$|_{\mathcal{G}_{CF}}$) and component fairness (**CF**).*

In this section, we show that the average tree solution admits a unique efficient extension. This result will be the consequence of other results, which are analogous of those in Béal et al. (2015) for the Myerson value. In order to understand the average tree solution as an efficient CO-value for connected and cycle-free CO-games, we begin by invoking the following property.

Connected component fairness, CCF. For all $(N, v, L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$ and $ij \in L$,

$$\begin{aligned} &\sum_{k \in C_i(N, L\backslash\{ij\})} \frac{\varphi_k(N, v, L) - \varphi_k(C_i(N, L\backslash\{ij\}), v|_{C_i(N, L\backslash\{ij\})}, L\backslash\{ij\}|_{C_i(N, L\backslash\{ij\})})}{|C_i(N, L\backslash\{ij\})|} \\ = &\sum_{k \in C_j(N, L\backslash\{ij\})} \frac{\varphi_k(N, v, L) - \varphi_k(C_j(N, L\backslash\{ij\}), v|_{C_j(N, L\backslash\{ij\})}, L\backslash\{ij\}|_{C_j(N, L\backslash\{ij\})})}{|C_j(N, L\backslash\{ij\})|}. \end{aligned}$$

Similarly to component fairness, connected component fairness considers the average change of the payoffs of the components of two players i and j if the link ij is removed. Connected component fairness compares the original payoffs with the payoffs obtained if the CO-game is restricted to each player's component, respectively, and imposes an equal average payoff variation. Note that all graphs involved in this axiom are connected and cycle-free. We suggest the following characterization of the average tree solution on the class of connected and cycle-free CO-games.

Proposition 1. *A CO-value φ on $\mathcal{G}_{CF} \cap \mathcal{G}_C$ satisfies efficiency (**E**$|_{\mathcal{G}_{CF} \cap \mathcal{G}_C}$) and connected component fairness (**CCF**) if and only if $\varphi = \mathrm{AT}$ on $\mathcal{G}_{CF} \cap \mathcal{G}_C$.*

Proof. By construction, AT satisfies $\mathbf{E}|_{\mathcal{G}_{CF} \cap \mathcal{G}_C}$. Concerning **CCF**, recall first that for a CO-game $(N, v, L) \in \mathcal{G}_{CF}$, a component $C \in \mathcal{C}(N, L)$, and a player $i \in C$, the payoff assigned by AT to player i only relies on the worths of some coalitions in 2^C by the definition of all hierarchical outcomes. This means that AT satisfies the axiom of component decomposability[1] (van den Nouweland, 1993, pp. 28-29). As a consequence, for any link $ij \in L$, it holds that both

$$\mathrm{AT}_k(N, v, L\backslash\{ij\}) = \mathrm{AT}_k(C_i(N, L\backslash\{ij\}), v|_{C_i(N, L\backslash\{ij\})}, L\backslash\{ij\}|_{C_i(N, L\backslash\{ij\})})$$

[1] **Component decomposability, CD**$|_{\mathcal{G}^*}$**.** For all $(N, v, L) \in \mathcal{G}^*$, $C \in \mathcal{C}(N, L)$, and $i \in C$ such that $(C, v|_C, L|_C) \in \mathcal{G}^*$, we have $\varphi_i(N, v, L) = \varphi_i(C, v|_C, L|_C)$.

for all $k \in C_i(N, L\backslash\{ij\})$ and

$$\mathrm{AT}_k(N, v, L\backslash\{ij\}) = \mathrm{AT}_k(C_j(N, L\backslash\{ij\}), v|_{C_j(N,L\backslash\{ij\})}, L\backslash\{ij\}|_{C_j(N,L\backslash\{ij\})})$$

for all $k \in C_j(N, L\backslash\{ij\})$. Using these equalities and the fact that AT satisfies **CF**, we obtain

$$\begin{aligned} &\sum_{k \in C_i(N,L\backslash\{ij\})} \frac{\mathrm{AT}_k(N, v, L) - \mathrm{AT}_k(C_i(N, L\backslash\{ij\}), v|_{C_i(N,L\backslash\{ij\})}, L\backslash\{ij\}|_{C_i(N,L\backslash\{ij\})})}{|C_i(N, L\backslash\{ij\})|} \\ = &\sum_{k \in C_j(N,L\backslash\{ij\})} \frac{\mathrm{AT}_k(N, v, L) - \mathrm{AT}_k(C_j(N, L\backslash\{ij\}), v|_{C_j(N,L\backslash\{ij\})}, L\backslash\{ij\}|_{C_j(N,L\backslash\{ij\})})}{|C_j(N, L\backslash\{ij\})|}. \end{aligned}$$

for all $(N, v, L) \in \mathcal{G}_{CF}$ and $ij \in L$. Since this condition holds for all $(N, v, L) \in \mathcal{G}_{CF}$, it obviously holds for all $(N, v, L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$, which means that AT satisfies **CCF**.

It remains to show that if a CO-value φ on $\mathcal{G}_{CF} \cap \mathcal{G}_C$ satisfies the two axioms, then it is uniquely determined. This part of the proof is similar to the proof of Theorem 3.4 of Herings et al. (2008). Thus, we only sketch the proof by induction on the cardinality of the player set. For a CO-game $(N, v, L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$ such that $|N| = 1$, $\mathbf{E}|_{\mathcal{G}_{CF} \cap \mathcal{G}_C}$ uniquely determines φ. Now, suppose that φ is uniquely determined for all $(N, v, L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$ such that $|N| < n$ and consider a CO-game $(N, v, L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$ such that $|N| = n$. Applying **CCF**, $\mathbf{E}|_{\mathcal{G}_{CF} \cap \mathcal{G}_C}$, and the induction hypothesis as in proof of Theorem 3.4 of Herings et al. (2008), we get a system of linearly independent equations as desired. □

In the next Theorem, we prove that there exists a unique efficient extension of the average tree solution for cycle-free CO-games. In other words, there exists a unique CO-value that satisfies efficiency and component fairness and that coincides with the average tree solution for connected cycle-free CO-games. We call this value the **efficient egalitarian average tree solution** EEAT that is defined by

$$\mathrm{EEAT}_i(N, v, L) := \mathrm{AT}_i(N, v, L) + \frac{v(N) - v^L(N)}{|N|}$$

for all $(N, v, L) \in \mathcal{G}_{CF}$, and $i \in N$. Moreover, we give a concise characterization of the efficient egalitarian average tree solution.

Theorem 4. *(i) A CO-value φ on $\mathcal{G}_{CF}$ satisfies efficiency ($\boldsymbol{E}|_{\mathcal{G}_{CF}}$), component fairness (**CF**), and $\varphi = \mathrm{AT}$ on $\mathcal{G}_{CF} \cap \mathcal{G}_C$ if and only if $\varphi = \mathrm{EEAT}$ on $\mathcal{G}_{CF}$.*

*(ii) A CO-value φ on $\mathcal{G}_{CF}$ satisfies efficiency ($\boldsymbol{E}|_{\mathcal{G}_{CF}}$), component fairness (**CF**), and connected component fairness (**CCF**) if and only if $\varphi = \mathrm{EEAT}$.*

The proof of Theorem 4 is based on the following lemma, which states that if two CO-values for cycle-free CO-games satisfy efficiency, component fairness, and agree on connected cycle-free CO-games, then they must assign the same total payoff to all components of a graph in all cycle-free CO-games.

Lemma 1. *Let φ and ψ be two CO-values on $\mathcal{G}_{CF}$ that satisfy efficiency ($\boldsymbol{E}|_{\mathcal{G}_{CF}}$) and component fairness (**CF**). If $\varphi = \psi$ on $\mathcal{G}_{CF} \cap \mathcal{G}_C$, then $\sum_{i \in C} \varphi_i(N, v, L) = \sum_{i \in C} \psi_i(N, v, L)$ for all $(N, v, L) \in \mathcal{G}_{CF}$ and all $C \in \mathcal{C}(N, L)$.*

Proof. Let φ and ψ be two CO-values for cycle-free CO-games that satisfy $\mathbf{E}|_{\mathcal{G}_{CF}}$ and **CF** and suppose that $\varphi(N, v, L) = \psi(N, v, L)$ for all $(N, v, L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$. The result follows by $\mathbf{E}|_{\mathcal{G}_{CF}}$ on connected cycle-free CO-games. So, consider any cycle-free CO-game $(N, v, L) \in \mathcal{G}_{CF} \backslash \mathcal{G}_C$, which implies $|\mathcal{C}(N, L)| > 1$. Suppose that there is a cycle-free CO-game in which φ and ψ do not assign the same total payoff to some component. More specifically, consider $(N, v, L) \in \mathcal{G}_{CF} \backslash \mathcal{G}_C$ with a maximal $L \subseteq \mathcal{L}^N$ such that $\sum_{i \in C} \varphi_i(N, v, L) \neq \sum_{i \in C} \psi_i(N, v, L)$ for some $C \in \mathcal{C}(N, L)$. Since $|\mathcal{C}(N, L)| > 1$, we can consider distinct players $i, j \in N$ such that $i \in C$ and $j \in N \backslash C$. By **CF**, the maximality of L, and the initial assumption, we get

$$\begin{aligned} & \frac{\sum_{k \in C} \varphi_k(N, v, L)}{|C|} - \frac{\sum_{k \in C_j(N,L)} \varphi_k(N, v, L)}{|C_j(N, L)|} \\ = \quad & \frac{\sum_{k \in C} \varphi_k(N, v, L \cup \{ij\})}{|C|} - \frac{\sum_{k \in C_j(N,L)} \varphi_k(N, v, L \cup \{ij\})}{|C_j(N, L)|} \\ = \quad & \frac{\sum_{k \in C} \psi_k(N, v, L \cup \{ij\})}{|C|} - \frac{\sum_{k \in C_j(N,L)} \psi_k(N, v, L \cup \{ij\})}{|C_j(N, L)|} \\ = \quad & \frac{\sum_{k \in C} \varphi_k(N, v, L)}{|C|} - \frac{\sum_{k \in C_j(N,L)} \varphi_k(N, v, L)}{|C_j(N, L)|}. \end{aligned}$$

Equivalently, the latter equality can be written as

$$\frac{|C_j(N, L)|}{|C|} \sum_{k \in C} (\varphi_k(N, v, L) - \psi_k(N, v, L)) = \sum_{k \in C_j(N,L)} (\varphi_k(N, v, L) - \psi_k(N, v, L)).$$

Summing the last expression on all $C_j(N, L)$ in $\mathcal{C}(N, L)$ and using $\mathbf{E}|_{\mathcal{G}_{CF}}$ yields

$$\begin{aligned} & \sum_{C_j(N,L) \in \mathcal{C}(N,L)} \frac{|C_j(N, L)|}{|C|} \sum_{k \in C} (\varphi_k(N, v, L) - \psi_k(N, v, L)) \\ = \quad & \sum_{C_j(N,L) \in \mathcal{C}(N,L)} \sum_{k \in C_j(N,L)} (\varphi_k(N, v, L) - \psi_k(N, v, L)) \\ \iff \quad & \frac{|N|}{|C|} \sum_{k \in C} (\varphi_k(N, v, L) - \psi_k(N, v, L)) = \sum_{k \in N} (\varphi_k(N, v, L) - \psi_k(N, v, L)) \\ \iff \quad & \sum_{k \in C} (\varphi_k(N, v, L) - \psi_k(N, v, L)) \overset{\mathbf{E}|_{\mathcal{G}_{CF}}}{=} 0, \end{aligned}$$

a contradiction that proves the result. □

We are now ready to prove Theorem 4.

Proof. (Theorem 4) (i) EEAT satisfies $\mathbf{E}|_{\mathcal{G}_{CF}}$ and coincides with AT on connected cycle-free CO-games by construction, as well as inherits **CF** from AT. For the uniqueness part, consider any CO-value φ on $\mathcal{G}_{CF}$ that satisfies the two axioms and that coincides with AT for connected cycle-free CO-games. By definition of EEAT, for any $(N, v, L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$, it holds that $\varphi(N, v, L) = \mathrm{AT}(N, v, L) = \mathrm{EEAT}(N, v, L)$. Therefore, Lemma 1 implies that for all $(N, v, L) \in \mathcal{G}_{CF}$ and all $C \in \mathcal{C}(N, L)$, we have $\sum_{i \in C} \varphi_i(N, v, L) = \sum_{i \in C} \mathrm{EEAT}_i(N, v, L)$. In particular, if $L = \varnothing$, then $\mathcal{C}(N, L) = \{\{i\}, i \in N\}$ and thus $\varphi(N, v, L) = \mathrm{EEAT}(N, v, L)$. This proves that φ is uniquely determined for all cycle-free CO-games with an empty graph. It remains to consider cycle-free CO-games with a non-empty graph. So pick any $(N, v, L) \in \mathcal{G}_{CF}$, $L \neq \varnothing$, and any $ij \in L$. By **CF**, it holds that

$$\sum_{k \in C_i(N, L\backslash\{ij\})} \frac{\varphi_k(N, v, L) - \varphi_k(N, v, L\backslash\{ij\})}{|C_i(N, L\backslash\{ij\})|} = \sum_{k \in C_j(N, L\backslash\{ij\})} \frac{\varphi_k(N, v, L) - \varphi_k(N, v, L\backslash\{ij\})}{|C_j(N, L\backslash\{ij\})|}. \quad (1)$$

Since $\{C_i(N, L\backslash\{ij\}), C_j(N, L\backslash\{ij\})\} \subseteq \mathcal{C}(N, L\backslash\{ij\})$, Lemma 1 implies that both

$$\sum_{k \in C_i(N, L\backslash\{ij\})} \varphi_k(N, v, L\backslash\{ij\}) = \sum_{k \in C_i(N, L\backslash\{ij\})} \mathrm{EEAT}_k(N, v, L\backslash\{ij\})$$

and

$$\sum_{k \in C_j(N, L\backslash\{ij\})} \varphi_k(N, v, L\backslash\{ij\}) = \sum_{k \in C_j(N, L\backslash\{ij\})} \mathrm{EEAT}_k(N, v, L\backslash\{ij\}).$$

Therefore, (1) becomes:

$$\sum_{k \in C_i(N, L\backslash\{ij\})} \frac{\varphi_k(N, v, L) - \mathrm{EEAT}_k(N, v, L\backslash\{ij\})}{|C_i(N, L\backslash\{ij\})|} = \sum_{k \in C_j(N, L\backslash\{ij\})} \frac{\varphi_k(N, v, L) - \mathrm{EEAT}_k(N, v, L\backslash\{ij\})}{|C_j(N, L\backslash\{ij\})|}.$$

It is useful to express this equality as:

$$\begin{aligned} & \frac{\sum_{k \in C_j(N, L\backslash\{ij\})} \varphi_k(N, v, L)}{|C_i(N, L\backslash\{ij\})|} - \frac{\sum_{k \in C_j(N, L\backslash\{ij\})} \varphi_k(N, v, L)}{|C_j(N, L\backslash\{ij\})|} \\ = \; & \frac{\sum_{k \in C_j(N, L\backslash\{ij\})} \mathrm{EEAT}_k(N, v, L\backslash\{ij\})}{|C_i(N, L\backslash\{ij\})|} - \frac{\sum_{k \in C_j(N, L\backslash\{ij\})} \mathrm{EEAT}_k(N, v, L\backslash\{ij\})}{|C_j(N, L\backslash\{ij\})|} \end{aligned} \quad (2)$$

There are $|L| = |N| - |\mathcal{C}(N, L)|$ equations of type (2). Furthermore, for each $C \in \mathcal{C}(N, L)$, we also know from Lemma 1 that

$$\sum_{i \in C} \varphi_i(N, v, L) = \sum_{i \in C} \mathrm{EEAT}_i(N, v, L). \quad (3)$$

There are $|\mathcal{C}(N, L)|$ equations of type (3). All in all, we obtain a system of $|N|$ equations. The left-hand sides of this system and of the system with a unique solution obtained by Herings et al. (2008, proof of Theorem 3.4) are identical. Since only the right-hand side is different in our system, it also has a unique solution.

(ii) One easily checks that EEAT obeys **CCF**. Uniqueness follows from Lemma 1, Theorem 4 (i), and Proposition 1. □

Remark 1. *The characterization of the efficient egalitarian average tree solution in Theorem 4 (ii) is non-redundant. The Null value assigning a null payoff to all players in all CO-games satisfies component fairness and connected component fairness but not efficiency. The* **egalitarian value** EV *defined as*

$$\mathrm{EV}_i(N, v, L) = \frac{v(N)}{|N|}$$

for all $(N, v, L) \in \mathcal{G}$ and $i \in N$ satisfies efficiency and component fairness but not connected component fairness. The CO-value $\varphi^{✠}$ given by

$$\varphi_i^{✠}(N, v, L) = \mathrm{AT}_i(v) + \frac{v(N) - v^L(N)}{|\mathcal{C}(N, L)| \cdot |C_i(N, L)|}$$

for all $(N,v,L) \in \mathcal{G}$ and $i \in N$ satisfies efficiency and connected component fairness but not component fairness.

4. The compensation solution: characterizations and a unique efficient extension

The compensation solution has been introduced in Béal et al. (2012a) for cycle-free CO-games. It is build from the compensation vectors defined as follows. For each CO-game $(N,v,L) \in \mathcal{G}_{CF}$, each component $C \in \mathcal{C}(N,L)$, and each spanning tree rooted at r on C, define the compensation vector as

$$c_i^r(N,v,L) = \sum_{j \in C: i \in S_j^r} \frac{v(S_j^r)}{|S_j^r|} - \sum_{j \in C: i \in C \backslash S_j^r} \frac{v(S_j^r)}{|C \backslash S_j^r|} \tag{4}$$

for all $i \in N$. Firstly, the contribution of player $i \in C$ consists in sharing equally the worth $v(C)$ with the other members of component C. Then, for each coalition S_j^r, $j \in C\backslash\{r\}$, formed according to the partial order of the spanning tree rooted at r, player i receives a share $v(S_j^r)/|S_j^r|$ if he belongs to this coalition or pays $v(S_j^r)/|C\backslash S_j^r|$ otherwise. On $\mathcal{G}_{CF}$, the **compensation solution** CS is defined as the average over all rooted spanning trees of the contribution vector (4). Formally,

$$\mathrm{CS}_i(N,v,L) = \frac{1}{|C_i(N,L)|} \sum_{r \in C_i(N,L)} c_i^r(N,v,L) \tag{5}$$

for all $(N,v,L) \in \mathcal{G}_{CF}$ and $i \in N$. Béal et al. (2012a) characterize the compensation solution by component efficiency and relative fairness.

Relative fairness, RF. For all $(N,v,L) \in \mathcal{G}_{CF}$, and $ij \in L$,

$$\varphi_i(N,v,L) - \sum_{k \in C_i(N,L\backslash\{ij\})} \frac{\varphi_k(N,v,L\backslash\{ij\})}{|C_i(N,L\backslash\{ij\})|} = \varphi_j(N,v,L) - \sum_{k \in C_j(N,L\backslash\{ij\})} \frac{\varphi_k(N,v,L\backslash\{ij\})}{|C_j(N,L\backslash\{ij\})|}.$$

Relative fairness can be interpreted in the context of the merging of components $C_i(N,L\backslash\{ij\})$ and $C_j(N,L\backslash\{ij\})$ through a new link ij. The axiom says that the payoff variation of players i and j with respect to the per-capita payoff in their pre-existing components $C_i(N,L\backslash\{ij\})$ and $C_i(N,L\backslash\{ij\})$ should be the same.

Theorem 5. (Béal et al., 2012a) *The compensation solution* CS *is the unique CO-value that satisfies component efficiency (**CE**$|_{\mathcal{G}_{CF}}$) and relative fairness (**RF**).*

The method developed in Béal et al. (2015) for the Myerson value and in the previous section for the average tree solution also works to construct the unique efficient extension of the compensation solution and to provide new characterizations of the compensation solution. Since the proof of the results in this section exploits similar arguments, we omit them and make them available upon request. We start by a characterization of the compensation solution on connected cycle-free CO-games by relying on the following axiom.

Connected relative fairness, CRF. For all $(N,v,L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$ and $ij \in L$,

$$\begin{aligned} &\varphi_i(N,v,L) - \sum_{k \in C_i(N,L\backslash\{ij\})} \frac{\varphi_k(C_i(N,L\backslash\{ij\}, v|_{C_i(N,L\backslash\{ij\}}, L\backslash\{ij\}|_{C_i(N,L\backslash\{ij\}})}{|C_i(N,L\backslash\{ij\})|} \\ = \;& \varphi_j(N,v,L) - \sum_{k \in C_j(N,L\backslash\{ij\})} \frac{\varphi_k(C_j(N,L\backslash\{ij\}, v|_{C_j(N,L\backslash\{ij\}}, L\backslash\{ij\}|_{C_j(N,L\backslash\{ij\}})}{|C_j(N,L\backslash\{ij\})|}. \end{aligned}$$

Connected relative fairness is similar to relative fairness except that after deleting link ij, the CO-game is restricted to each player's component, respectively. Thus, all graphs involved in this axiom are connected and cycle-free.

Proposition 2. *A CO-value φ on $\mathcal{G}_{CF} \cap \mathcal{G}_C$ satisfies efficiency ($\boldsymbol{E}|_{\mathcal{G}_{CF} \cap \mathcal{G}_C}$) and connected relative fairness (**CRF**) if and only if $\varphi = \mathrm{CS}$ on $\mathcal{G}_{CF} \cap \mathcal{G}_C$.*

Next, we show that the compensation solution admits a unique efficient extension, which we call the efficient egalitarian compensation solution. As for the Myerson value and the average tree solution, this CO-value is obtained by adding an equal share of the surplus created between the components, $\left(v\left(N\right) - v^L\left(N\right)\right) / |N|$, to the compensation solution. More specifically, the **efficient egalitarian compensation solution** EECS is defined by

$$\mathrm{EECS}_i(N, v, L) := \mathrm{CS}_i(N, v, L) + \frac{v(N) - v^L(N)}{|N|}$$

for all $(N, v, L) \in \mathcal{G}_{CF}$ and $i \in N$. The next theorem also provides a characterization of the efficient egalitarian compensation solution.[2]

Theorem 6. *(i) A CO-value φ on $\mathcal{G}_{CF}$ satisfies efficiency ($\boldsymbol{E}|_{\mathcal{G}_{CF}}$), relative fairness (**RF**), and $\varphi = \mathrm{CS}$ on $\mathcal{G}_{CF} \cap \mathcal{G}_C$ if and only if $\varphi = \mathrm{EECS}$ on $\mathcal{G}_{CF}$.*

*(ii) A CO-value φ on $\mathcal{G}_{CF}$ satisfies efficiency ($\boldsymbol{E}|_{\mathcal{G}_{CF}}$), relative fairness (**RF**), and connected relative fairness (**CRF**) if and only if $\varphi = \mathrm{EECS}$.*

Remark 2. *The characterization of the efficient egalitarian compensation solution in Theorem 6 (ii) is non-redundant. The null value assigning a null payoff to all player in all CO-games satisfies relative fairness and connected relative fairness but not efficiency. The egalitarian value* EV *defined in Remark 1 satisfies efficiency and relative fairness but not connected relative fairness. The CO-value $\varphi^{\diamond}$ given by*

$$\varphi_i^{\diamond}\left(N, v, L\right) = \mathrm{CS}_i\left(v\right) + \frac{v\left(N\right) - v^L\left(N\right)}{|\mathcal{C}(N, L)| \cdot |C_i\left(N, L\right)|}$$

for all $(N, v, L) \in \mathcal{G}$ and $i \in N$ satisfies efficiency and connected component fairness but not relative fairness.

Remark 3. *Among all the efficient CO-values, the efficient extensions of the average tree solution and of the compensation solution uniquely minimize the euclidean distance to the average tree solution and the compensation solution, respectively. More specifically, it is straightforward to show that*

$$\mathrm{EEAT}(N, v, L) = \underset{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N)}{\mathit{argmin}} d\left(x, \mathrm{AT}(N, v, L)\right)$$

and

$$\mathrm{EECS}(N, v, L) = \underset{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N)}{\mathit{argmin}} d\left(x, \mathrm{CS}(N, v, L)\right)$$

[2] The proof of the Theorem also makes use of a lemma, which proves that if two CO-values for cycle-free CO-games satisfy efficiency, relative fairness, and agree on connected cycle-free CO-games, then they must assign the same total payoff to all components of a graph in all cycle-free CO-games.

for all $(N, v, L) \in \mathcal{G}$, where $d(x, y) := \sqrt{\sum_{i \in N}(x_i - y_i)^2}$ denotes the Euclidean distance between $x \in \mathbb{R}^N$ and $y \in \mathbb{R}^N$. This property provides a supplementary evidence that these two efficient extensions can be considered as very close to their original component-efficient CO-values.

5. The position value: an impossibility result

The position value is introduced by Meessen (1988) and Borm et al. (1992). For any $(N, v, L) \in \mathcal{G}_0$, the associated **link game** is the TU-game (L, r^v) such that

$$r^v(A) := v^A(N) = \sum_{S \in \mathcal{C}(N,A)} v(S), \quad A \subseteq L.$$

This link game is a TU-game in which the players can be identified with the links in the original CO-game. The worth of a coalition of players A in the link game is the total worth generated by the components of graph (N, A). The link game is well-defined: $r^v(\varnothing) = 0$ because (N, v) is zero-normalized. The **position value** is the CO-value PV that assigns to each player half of the Shapley value of each of its links in the link game, that is,

$$\mathrm{PV}_i(N, v, L) := \sum_{k \in N : ik \in L} \frac{1}{2} \mathrm{SH}_{ik}(L, r^v)$$

for all $(N, v, L) \in \mathcal{G}_0$ and $i \in N$. Slikker (2005) characterizes the position value by component efficiency and balanced total threats.

Balanced total threats, BTT. (Slikker, 2005) For all $(N, v, L) \in \mathcal{G}_0$ and $i, j \in N$,

$$\sum_{k \in N : jk \in L} (\varphi_i(N, v, L) - \varphi_i(N, v, L \backslash \{jk\})) = \sum_{k \in N : ik \in L} (\varphi_j(N, v, L) - \varphi_j(N, v, L \backslash \{ik\}))\,.$$

Balanced total threats says that the total threat of any player towards another player equals to the total threat of that player towards the first player, where the total threat of a player towards another player is the sum over all links of the first player of payoff differences the second player experiences if such a link is broken.

Theorem 7. (Slikker, 2005) *The position value* PV *is the unique CO-value that satisfies component efficiency (**CE**$|_{\mathcal{G}_0}$) and balanced total threats (**BTT**).*

It is easy to check that the CO-value defined by assigning every player $(v(N) - v^L(N))/|N|$ plus his/her payoff according to the position value satisfies efficiency and coincides with the position value whenever the underlying graph is connected. However, it does not satisfy balanced total threats. This means that the method used to construct the unique efficient extension of the Myerson value, the average tree solution and the compensation Solution does not yield the efficient extension of the position Value. Below, a stronger result is demonstrated: there does not exist any efficient extension of the position Value.

Theorem 8. *There does not exists any efficient extension of the position value, i.e., no value φ on $\mathcal{G}^0$ satisfies $\varphi = \mathrm{PV}$ on $\mathcal{G}^0 \cap \mathcal{GC}$ and meets efficiency (**E**$|_{\mathcal{G}^0}$) and balanced total threats (**BTT**).*

Proof. We proceed by contradiction. So assume that there is a CO-value φ on $\mathcal{G}^0$ that satisfies $\varphi = \text{PV}$ on $\mathcal{G}^0 \cap \mathcal{G}_C$, $\mathbf{E}|_{\mathcal{G}^0}$, and **BTT**. Consider any player set N such that $|N| \geqslant 4$, and without any loss of generality, assume that $\{1, 2, 3\} \subset N$. Pick player $1 \in N$ in order to show that φ is not well-defined in the CO-game (N, v, L) where $L = \mathcal{L}^{N\backslash\{1\}}$. In graph (N, L), player 1 has no link and the subgraph $(N\backslash\{1\}, L(N\backslash\{1\}))$ induced by (N, L) on $N\backslash\{1\}$ is complete. First, note that any graph (N, L') such that $|L'| = |L| + 1$ and $L' \supset L$ is connected, *i.e.*, adding any missing link to the unconnected graph (N, L) yields a connected graph. Of course, any such added link is of the form $1i$ for some $i \in N\backslash\{1\}$.

Next, we consider the CO-game $(N, v, L \cup 12)$ in order to compute $\varphi(N, v, L)$. According to the previous remark, $(N, L \cup 12)$ is a connected graph, or equivalently, $(N, v, L \cup 12) \in \mathcal{G}^0 \cap \mathcal{G}_C$. Another important property is that, for all links $ij \in L$, the graph $(N, (L \cup 12)\backslash ij)$ remains connected since $|N\backslash\{1\}| \geqslant 3$ and $(N\backslash\{1\}, L(N\backslash\{1\}))$ is a complete graph (in other words, no link in $(N\backslash\{1\}, L(N\backslash\{1\}))$ is a bridge). Therefore, $(N, v, (L \cup 12)\backslash ij) \in \mathcal{G}^0 \cap \mathcal{G}_C$ as well. Now, let us apply **BTT** to all pairs of players $\{1, i\}$, $i \in N\backslash\{1\}$. Two cases are possible. For $i = 2$, we get

$$\sum_{j \in N\backslash\{2\}} (\varphi_1(N, v, L \cup 12) - \varphi_1(N, v, (L \cup 12)\backslash 2j)) = \varphi_2(N, v, L \cup 12) - \varphi_2(N, v, L).$$

Since $\varphi = \text{PV}$ on $\mathcal{G}^0 \cap \mathcal{G}_C$, the previous expression can be rewritten as:

$$(|N|-1)\text{PV}_1(N, v, L \cup 12) - \varphi_1(N, v, L) - \sum_{j \in N\backslash\{1,2\}} \text{PV}_1(N, v, (L \cup 12)\backslash 2j) = \text{PV}_2(N, v, L \cup 12) - \varphi_2(N, v, L),$$

or equivalently,

$$\varphi_1(N, v, L) - \varphi_2(N, v, L) = (|N|-1)\text{PV}_1(N, v, L \cup 12) - \sum_{j \in N\backslash\{1,2\}} \text{PV}_1(N, v, (L \cup 12)\backslash 2j) - \text{PV}_2(N, v, L \cup 12).$$

Observe that the right-hand side of this expression is uniquely determined, and more importantly, that it has been obtained by invoking **BTT** and the fact that $\varphi = \text{PV}$ on $\mathcal{G}^0 \cap \mathcal{G}_C$ only. This means that this equality holds for every CO-value satisfying **BTT** and that coincides with PV on $\mathcal{G}^0 \cap \mathcal{G}_C$. Since the position value is obviously one such CO-value, it must be that

$$\varphi_1(N, v, L) - \varphi_2(N, v, L) = \text{PV}_1(N, v, L) - \text{PV}_2(N, v, L). \tag{6}$$

Similarly, for all $i \in N\backslash\{1, 2\}$, the application of **BTT** to the pair of players $\{1, i\}$ yields

$$\sum_{j \in N\backslash\{i\}} (\varphi_1(N, v, L \cup 12) - \varphi(N, v, (L \cup 12)\backslash ij) = \varphi_i(N, v, L \cup 12) - \varphi_i(N, v, L).$$

Using once again the fact that $\varphi = \text{PV}$ on $\mathcal{G}^0 \cap \mathcal{G}_C$, we can rewrite the previous expression as

$$\varphi_i(N, v, L) = \text{PV}_i(N, v, L \cup 12) + \sum_{j \in N\backslash\{i\}} \text{PV}_1(N, v, (L \cup 12)\backslash ij) - (|N| - 1)\text{PV}_1(N, v, L \cup 12).$$

For the same reason as above (*i.e.*, the case where $i = 2$), this equality is also true for the position value, so that we can write

$$\varphi_i(N, v, L) = \text{PV}_i(N, v, L) \tag{7}$$

for all $i \in N\backslash\{1,2\}$. Furthermore, since φ satisfies $\mathbf{E}|_{\mathcal{G}^0}$, it holds that $\sum_{i\in N} \varphi_i(N,v,L) = v(N)$, and by using both (7) and the fact that PV is component-efficient in the CO-game (N,v,L), the previous efficiency condition is equivalent to

$$\varphi_1(N,v,L) + \varphi_2(N,v,L) = v(N) - v^L(N) + \mathrm{PV}_1(N,v,L) + \mathrm{PV}_2(N,v,L). \tag{8}$$

It is easy to see that the unique solution of the system of two equations formed by (6) and (8) is

$$\varphi_i(N,v,L) = \mathrm{PV}_i(N,v,L) + \frac{1}{2}\left(v(N) - v^L(N)\right) \tag{9}$$

for all $i \in \{1,2\}$.

For the final step of the proof, repeat the above procedure in the CO-game $(N,v,L \cup 13)$ instead of the CO-game $(N,v,L \cup 12)$, and obtain $\varphi_i(N,v,L) = \mathrm{PV}_i(N,v,L)$ for all $i \in N\backslash\{1,3\}$ and $\varphi_i(N,v,L) = \mathrm{PV}_i(N,v,L)+(v(N)-v^L(N))/2$ for each $i \in \{1,3\}$. Provided that $v(N) \neq v^L(N)$, note that $\varphi_2(N,v,L)$ has two different values according to whether the original CO-game to which **BTT** is applied is $(N,v,L \cup 12)$ or $(N,v,L \cup 13)$, a contradiction. This completes the proof. ▫

6. Concluding remarks

This article has extended the approach initiated by Béal et al. (2015) on the efficient extension of communication values. Combined with the results in Béal et al. (2015), our findings suggest that whenever an efficient extension of a component-efficient CO-value exists, it is unique and assigns to each player an equal share of the surplus created by the grand coalition in addition to its payoff according to the component-efficient CO-value. This is not always the case. To see this, consider the **component-wise egalitarian solution** CW defined on $\mathcal{G}$ as

$$\mathrm{CW}_i(N,v,L) = \frac{v(C_i(N,L))}{|C_i(N,L)|}$$

for all $(N,v,L) \in \mathcal{G}$ and $i \in N$. Slikker (2007) characterizes component-wise egalitarian solution by component efficiency and balanced component contributions.[3]

Balanced component contributions, BCC. For all $(N,v,L) \in \mathcal{G}$ and $i,j \in N$,

$$\varphi_i(N,v,L) - \varphi_i(N,v,L\backslash L(C_j(N,L))) = \varphi_j(N,v,L) - \varphi_j(N,v,L\backslash L(C_i(N,L)))$$

This axiom states that the payoff variation experienced by a first player when all links in the component of a second player are severed is identical to payoff variation of the second player when all links in the component of the first player are deleted. Now, consider the egalitarian value EV defined in Remark 1. This CO-value is an efficient extension of the component-wise egalitarian solution. Indeed, it trivially coincides with CW for connected CO-games, satisfies efficiency and balanced component contributions. The latter property drops from the fact that EV is not sensitive to L. Nonetheless, EV is not obtained by adding to CW an equal share of the surplus $v(N)-v^L(N)$. Furthermore, it is not difficult to show that there is no other efficient extension of the component-wise egalitarian solution. As a consequence, if an efficient extension exists, the question of whether it is always unique remains open.

[3] In Slikker (2007), this result is given for the larger class of network games. We adapt it here to the class of all CO-games.

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