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Characterizations of three linear values for TU games by associated consistency: simple proofs using the Jordan normal form

Sylvain Béal*, Eric Rémila†, Philippe Solal‡

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Abstract

This article studies values for cooperative games with transferable utility. Numerous such values can be characterized by axioms of Ψ_ε-associated consistency, which require that a value is invariant under some parametrized linear transformation Ψ_ε on the vector space of cooperative games with transferable utility. Xu et al. (2008, 2009, *Linear Algebra Appl.*), Xu et al. (2013, *Linear Algebra Appl.*), Hamiache (2010, *Int. Game Theory Rev.*) and more recently Xu et al. (2015, *Linear Algebra Appl.*) follow this approach by using a matrix analysis. The main drawback of these articles is the heaviness of the proofs to show that the matrix expression of the linear transformations is diagonalizable. By contrast, we provide quick proofs by relying on the Jordan normal form of the previous matrix.

1 Introduction

A transferable utility (TU henceforth) game on a finite player set is a function which assigns a real number, called the worth, to each subset of players. A (single-valued) solution, often called a value, assigns a single payoff to each player in each TU game. The payoff received by a player in a TU game represents an assessment by this player of his or her gains for participating in this TU game. In the axiomatic characterization of values for TU games, consistency is an important principle. Roughly speaking, it indicates that the value is invariant under some plausible transformation of the TU game.

Hamiache (2001) introduces the parameterized linear transformation H_ε, $\varepsilon \in [0,1]$, on the vector space of TU games, and shows that the Shapley value (1953), the most popular linear value for transferable utility games is invariant under H_ε. He then characterizes the Shapley value by three axioms: H_ε-associated consistency, continuity and the inessential game axiom. The axiom of H_ε-associated consistency indicates that the value is invariant under H_ε; continuity stipulates that the value is continuous on the vector space of TU games; and the inessential game axiom indicates that the value assigns to each player his or her stand-alone worth in case the game is inessential / additive.

Hwang (2006) introduces an alternative parameterized linear transformation Hw_ε, $\varepsilon \in [0,1]$, on the vector space of TU games, and shows that the Equal Allocation of Non-Separable Contributions, a well-known value popularized by Moulin (1985), satisfies Hw_ε-associated consistency. He then characterizes this value by using Hw_ε-associated consistency, continuity, and three other standard axioms for TU games. Xu et al. (2013) define another parameterized linear transformation K_ε, $\varepsilon \in [0,1]$, and show that replacing Hw_ε-associated consistency by K_ε-associated consistency in Hwang's axiomatic system yields the Center of gravity of Imputation Set, a value popularized by Driessen and Funaki (1991).

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Recently, Xu et al. (2015) define two other parameterized linear transformations X_ε and L_ε, $\varepsilon \in [0,1]$, in order to characterize the Equal Allocation of Non-Separable Contributions and the Center of gravity of Imputation Set, respectively. The Equal Allocation of Non-Separable Contributions is characterized by X_ε-associated consistency, continuity and the inessential game axiom. Replacing X_ε-associated consistency by L_ε-associated consistency yields the characterization of the Center of gravity of Imputation Set.

The key point behind the proof of these characterizations is to show that the sequences $(H_\varepsilon^{(t)})_{t\in\mathbb{N}}$, $(Hw_\varepsilon^{(t)})_{t\in\mathbb{N}}$, $(K_\varepsilon^{(t)})_{t\in\mathbb{N}}$, $(X_\varepsilon^{(t)})_{t\in\mathbb{N}}$ and $(L_\varepsilon^{(t)})_{t\in\mathbb{N}}$ are convergent sequences for sufficiently small ε. To do this, Xu et al. (2008, 2009, 2013), Hamiache (2010) and more recently Xu et al. (2015) use a matrix analysis. More specifically, they show that the matrix expression of each linear transformation is diagonalizable. From this point, it is easy to find the conditions under which each of the aforementioned sequences is convergent. Nevertheless, showing that the matrix expression of each of these linear transformations is diagonalizable is computationally demanding and not necessary at all. First, it is computationally demanding because the authors express the matrix with respect to the standard basis, which is not the most suitable basis to examine H_ε, Hw_ε, K_ε, X_ε or L_ε. Second, it is unnecessary to prove that the matrix is diagonalizable to study the corresponding sequences $(H_\varepsilon^{(t)})_{t\in\mathbb{N}}$, $(Hw_\varepsilon^{(t)})_{t\in\mathbb{N}}$, $(K_\varepsilon^{(t)})_{t\in\mathbb{N}}$, $(X_\varepsilon^{(t)})_{t\in\mathbb{N}}$ and $(L_\varepsilon^{(t)})_{t\in\mathbb{N}}$. Indeed, it suffices to prove that (a) each eigenvalue $\lambda \neq 1$ of the linear transformation under consideration is such that $|\lambda| < 1$; (b) the multiplicity of the eigenvalue $\lambda = 1$, when the latter exists, is equal to the dimension of its eigenspace. These two properties can simultaneously be proved by exhibiting a basis in which the matrix expression of the linear transformation is triangular, which is a weaker condition than exhibiting a diagonal basis. At this step, it remains to put the matrix of the linear transformation in the Jordan normal form to obtain the desired convergence result.

In this article, we show that the Jordan normal form of each matrix associated with the four linear transformations can be quickly computed. As a byproduct, the proofs of convergence as well as the axiomatic characterizations can be dramatically shortened. In fact, these proofs become very simple and short as soon as a suitable basis for the vector space of TU games is chosen. Moreover, the choice of these bases appears quite natural.

The rest of the article is organized as follows. In section 2, notations, definitions and related notions are introduced. In section 3, we define the three different types of linear transformations on the space of transferable utility games, and state the axiomatic characterization results obtained by Hamiache (2001), Hwang (2006) and Xu et al. (2015). In section 4, we present the proof strategy behind the three above-mentioned axiomatic characterization results. In section 5, we introduce what we call the Jordan normal form approach as opposed to the "matrix analysis to associated consistency for linear values" followed by Xu et al. (2008, 2009, 2013), Hamiache (2010) and Xu et al. (2015). In section 6, we use the Jordan normal form approach to construct a short and simple proof of each axiomatic characterization obtained by Hamiache (2001), Hwang (2006), Xu et al. (2013), and Xu et al. (2015), respectively.

2 Notations and definitions

Given a vector space V, its additive identity element will be denoted by 0_V. If a vector space V is the direct sum of the subspaces V^1 and V^2, i.e. $V = V^1 + V^2$ and $V^1 \cap V^2 = 0_V$, we write $V = V^1 \oplus V^2$. If $\Psi : V \longrightarrow U$ is a linear transformation, then denote by $Ker(\Psi)$ its kernel, i.e. the set of vectors $v \in V$ such that $\Psi(v) = 0_U$.

Let $N = \{1, 2, \ldots, n\}$ be a finite and fixed set of size n. An element $i \in N$ and a subset $S \subseteq N$ are called a *player* and *coalition* (of players), respectively. For each nonempty coalition $S \subseteq N$, its cardinality will be denoted by s. Denote by Ω_N the collection of all nonempty coalitions of N. Throughout this article, we assume that Ω_N is ordered according to some linear extension of the partial order $\subseteq$. Let σ be a permutation on N and let Σ_N be the set of $n!$ permutations on N. Throughout this article, we assume that Ω_N is ordered according to some linear extension of the partial order $\subseteq$. Let V_N be the set all coalition functions or *transferable utility (TU) games* on N, $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. The real number $v(S)$ is interpreted as the worth that coalition S can reach when all its members cooperate. A coalition function

$v \in V_N$ is *inessential* or additive if, for each $S \subseteq N$, it holds that $v(S) = \sum_{i \in S} v(\{i\})$. Denote by I_N the set of inessential TU games. A TU game $v \in V_N$ is *constant* if there is $\alpha \in \mathbb{R}$ such that, for each $S \in \Omega_N$, $v(S) = \alpha$. Denote by C_N the subset of constant TU games. For any $v \in V_N$ and any $\sigma \in \Sigma_N$ define $\sigma v \in V_N$ as: for each $S \in \Omega_N$, $\sigma v(\sigma(S)) = v(S)$, where $\sigma(S)$ stands for $\cup_{i \in S} \sigma(i)$.

Let $v \in V_N$, $w \in V_N$ and $\alpha \in \mathbb{R}$. The sum game $v + w \in V_N$ is defined for each $S \subseteq N$ as $(v + w)(S) = v(S) + w(S)$. The scalar product game $\alpha v \in V_N$, where $\alpha \in \mathbb{R}$, is defined, for each $S \subseteq N$, as $(\alpha v)(S) = \alpha v(S)$. The set V_N together with the operations defined above has a (real) vector space structure of dimension $2^n - 1$. Denote by 0_{V_N} its identity element, i.e. the null game. The *standard basis* for V_N is the ordered collection of games $\mathcal{E} = (e_R)_{R \in \Omega_N}$, where each standard TU game $e_R \in \mathcal{E}$ is defined as: for each $S \subseteq N$, $e_R(S) = 1$ if $S = R$, and $e_R(S) = 0$ if $S \neq R$. Another well-used basis for V_N is the ordered collection *unanimity games* $\mathcal{U} = (u_R)_{R \in \Omega_N}$, where each unanimity TU game $u_R \in \mathcal{U}$ is defined as: for each $S \subseteq N$, $u_R(S) = 1$ if $S \supseteq R$, and $u_R(S) = 0$ if $S \not\supseteq R$. Notice that I_N is a n-dimensional subspace of V_N, the ordered collection of unanimity TU games $\mathcal{I} = (u_{\{1\}}, u_{\{2\}}, \ldots, u_{\{n\}})$ is a basis for I_N; C_N is a one-dimensional subspace of V_N, and the TU game $c \in C_N$ defined as, for each $S \in \Omega_N$, $c(S) = 1$, constitutes a basis for C_N. Finally, D_N denotes the one-dimensional subspace of V_N spanned by the unanimity TU game u_N. Thus, TU games in D_N assign a zero worth to each coalition different from N.

A *value* on V_N is a function $\Phi : V_N \longrightarrow \mathbb{R}^n$ that assigns a single real payoff vector $\Phi(v) \in \mathbb{R}$ to each $v \in V_N$. The payoff $\Phi_i(v)$ represents an assessment by i of his or her gains for participating in the TU game v.

The best-known value for TU games is the value Sh, known as the *Shapley value* (Shapley, 1953), and defined as:

$$\forall i \in N, \quad Sh_i(v) \;=\; \sum_{S \in \Omega_N : S \not\ni i} \frac{(n-s-1)!s!}{n!} \big(v(S \cup \{i\}) - v(S)\big). \tag{1}$$

In particular, it is well known that, for each unanimity TU game $u_R \in \mathcal{U}$, it holds that:

$$\forall i \in R, \quad Sh_i(u_R) = \frac{1}{r} \quad \text{and} \quad \forall i \in N \setminus R, \quad Sh_i(u_R) = 0. \tag{2}$$

The *Equal Allocation of Non-Separable Contributions* ($EANSC$) is the value introduced by Moulin (1985) and defined as:

$$\forall i \in N, \quad EANSC_i(v) \;=\; SC_i(v) + \frac{1}{n}\left(v(N) - \sum_{j \in N} SC_j(v)\right), \tag{3}$$

where,

$$\forall i \in N, \quad SC_i(v) = v(N) - v(N \setminus \{i\})$$

is the marginal contribution of player $i \in N$ to the coalition N.
The *Center of gravity of Imputation Set* (CIS), introduced in Driessen and Funaki (1991), is the dual of the EANSC value where the dual v^D of a TU game $v \in V_N$ is defined as:

$$\forall S \in \Omega_N, \quad v^D(S) = v(N) - v(S).$$

Therefore, the CIS value is given by:

$$\begin{aligned} \forall i \in N, \quad CIS_i(v) &= EANSC(v^D) \\ &= v(\{i\}) + \frac{1}{n}\left(v(N) - \sum_{j \in N} v(\{j\})\right). \end{aligned} \tag{4}$$

A value Φ on V_N satisfies:
Linearity if Φ is linear.
Continuity if Φ is continuous.

Efficiency if, for each $v \in V_N$, it holds that:

$$\sum_{i \in N} \Phi_i(v) = v(N).$$

Inessential game axiom if, for each inessential game $v \in I_N$, it holds that:

$$\forall i \in N, \quad \Phi_i(v) = v(\{i\}).$$

Translation covariance if, for each $v \in V_N$ and each $w \in I_N$, it holds that:

$$\Phi(v + w) = \Phi(v) + (w(\{1\}), \ldots, w(\{n\})).$$

Anonymity if, for each $v \in V_N$ and each $\sigma \in \Sigma_N$, it holds that:

$$\forall i \in N, \quad \Phi_{\sigma(i)}(\sigma v) = \Phi_i(v).$$

Fact 1: the values Sh, EANSC and CIS satisfy all the above axioms. Note that Linearity implies Continuity, and the combination of Linearity and the Inessential game axiom implies Translation covariance.

3 Associated consistency and characterization of Sh, EANSC and CIS

Hamiache (2001) constructs a consistency axiom with respect to an associated TU game. Let $\Psi : V_N \longrightarrow V_N$ be a linear transformation which transforms any $v \in V_N$ into $\Psi(v) \in V_N$. The axiom of Ψ-associated consistency indicates that a value Φ is invariant with respect to the transformation Ψ. Formally:

Ψ-associated consistency Given a linear transformation Ψ, a value Φ on V_N satisfies Ψ-associated consistency if, for all $v \in V_N$, it holds that $\Phi(v) = \Phi(\Psi(v))$.

In order to characterize Sh, Hamiache (2001) introduces, for each parameter $\varepsilon \in [0, 1]$, the linear transformation H_ε defined as:

$$\forall S \in \Omega_N, \quad H_\varepsilon(v)(S) = v(S) + \varepsilon \sum_{j \in N \setminus S} \big(v(S \cup \{j\}) - v(S) - v(\{j\})\big). \tag{5}$$

Theorem 1 *(Hamiache, 2001)*
Assume that $0 < \varepsilon < 2/n$. Then, the Shapley value Sh defined as in (1) is the unique value on V_N satisfying H_ε-associated consistency, Continuity and the Inessential game axiom.

In order to characterize the value EANSC, Hwang (2006) introduces, for each parameter $\varepsilon \in [0, 1]$, the linear transformation Hw_ε defined as:

$$\forall S \in \Omega_N, \quad Hw_\varepsilon(v)(S) = v(S) + \varepsilon \sum_{j \in N \setminus S} \big(v(S \cup \{j\}) - v(S) - SC_j(v)\big). \tag{6}$$

Theorem 2 *(Hwang, 2006)*
Assume that $0 < \varepsilon < 2/(n-1)$. Then, the EANSC value defined as in (3) is the unique value on V_N satisfying Hw_ε-associated consistency, Continuity, Efficiency, Anonymity and Translation covariance.

Xu et al. (2013) suggest another parameterized linear transformation. For each $\varepsilon \in [0, 1]$, they define the linear transformation K_ε as follows:

$$K_\varepsilon(v)(S) = \begin{cases} v(S) - \varepsilon \sum_{j \in S} \big(v(S) - v(S \setminus \{j\}) - v(\{j\})\big) & \text{if } S \in \Omega_N, S \neq N, \\ v(N) & \text{if } S = N. \end{cases} \tag{7}$$

They obtain, within "Hwang's framework", i.e. by using Ψ_ε-associated consistency, Continuity, Efficiency, Anonymity and Translation covariance, the following characterization result.

Theorem 3 *(Xu et al. 2013)*
Assume that $0 < \varepsilon < 2/(n-1)$. Then, the CIS value defined as in (4) is the unique value on V_N satisfying K_ε-associated consistency, Continuity, Efficiency, Anonymity and Translation covariance.

Recently, Xu et al. (2015) characterize the EANSC and the CIS value within "Hamiache's framework" i.e. by using Ψ-associated consistency, Continuity and the Inessential game axiom. To this end, the authors introduce, for each parameter $\varepsilon \in [0,1]$, two linear transformations X_ε and L_ε defined respectively as:

$$\forall S \in \Omega_N, \quad X_\varepsilon(v)(S) = v(S) + \varepsilon\Big(\frac{s}{n}\big(v(N) - \sum_{j\in N} SC_j(v)\big) - \big(v(S) - \sum_{j\in S} SC_j(v)\big)\Big), \tag{8}$$

and

$$\forall S \in \Omega_N, \quad L_\varepsilon(v)(S) = v(S) + \varepsilon\Big(\frac{s}{n}\big(v(N) - \sum_{j\in N} v(\{j\})\big) - \big(v(S) - \sum_{j\in S} v(\{j\})\big)\Big). \tag{9}$$

Theorem 4 *(Xu et al., 2015)*
Let ε be such that $0 < \varepsilon < 1$.

1. *The EANSC value defined as in (3) is the unique value on V_N satisfying X_ε-associated consistency, Continuity and the Inessential game axiom.*
2. *The CIS value defined as in (4) is the unique value on V_N satisfying L_ε-associated consistency, Continuity and the Inessential game axiom.*

4 Proof strategy

The proof strategy of each of the aforementioned results is based, for some fixed $\varepsilon \in [0,1]$, on the study of the infinite sequences $(\Psi_\varepsilon^{(t)}(v))_{t\in\mathbb{N}} \subseteq V_N$, where $\Psi_\varepsilon^{(1)}(v) = \Psi_\varepsilon(v)$, and for each $t \geq 2$, $\Psi_\varepsilon^{(t)}(v) = \Psi_\varepsilon(\Psi_\varepsilon^{(t-1)}(v))$. For each result, the main step of the proof reduces to prove that each sequence converges to a TU game belonging to a certain subspace E_{Ψ_ε} of V_N, provided that ε is sufficiently small, i.e. for sufficiently small ε,

$$\forall v \in V_N, \quad (\Psi_\varepsilon^{(t)}(v))_{t\in\mathbb{N}} \xrightarrow{t\to+\infty} v^* \in E_{\Psi_\varepsilon}. \tag{10}$$

Then, by Ψ_ε-associated consistency and Continuity, we obtain:

$$\Phi(v) = \Phi(\Psi_\varepsilon^{(t)}(v)), \text{ and } \lim_{t\to+\infty} \Phi\big(\Psi_\varepsilon^{(t)}(v)\big) = \Phi\big(\lim_{t\to+\infty} \Psi_\varepsilon^{(t)}(v)\big) = \Phi(v^*),$$

and so

$$\Phi(v) = \Phi(v^*).$$

The second step consists in computing $\Phi(v^*)$ with the other axioms, using the fact that $v^* \in E_{\Psi_\varepsilon}$. There are some variations according to E_{Ψ_ε}.

- Regarding Theorems 1 and 4, we have $E_{\Psi_\varepsilon} = I_N$, where $\Psi_\varepsilon \in \{H_\varepsilon, X_\varepsilon, L_\varepsilon\}$, and, by the Inessential game axiom, we conclude that Φ is uniquely determined:

$$\Phi(v) = \Phi(v^*) = (v^*(\{1\}), \ldots, v^*(\{n\}).$$

- Regarding Theorem 2, we have $E_{Hw_\varepsilon} = C_N \oplus I_N$. After some elementary computations, we obtain:

$$\forall S \in \Omega_N, \quad v^*(S) = \underbrace{\sum_{j\in S} SC_j(v^*)}_{\text{Inessential TU game}} + \underbrace{v^*(N) - \sum_{j\in N} SC_j(v^*)}_{\text{Constant TU game}}. \tag{11}$$

By Translation covariance, we obtain:

$$\Phi(v^*) = (SC_1(v^*), \ldots, SC_n(v^*)) + \Phi(c_v^*), \tag{12}$$

where c_v^* stands for the constant TU game in (11). By Anonymity and Efficiency apply to $\Phi(c_v^*)$, we obtain:

$$\Phi(c_v^*) = \left(\frac{c_v^*(N)}{n}, \ldots, \frac{c_v^*(N)}{n}\right). \tag{13}$$

By (12)-(13), $\Phi(v^*)$ is uniquely determined, so is $\Phi(v)$.

- Regarding Theorem 3, we have $E_{K_\varepsilon} = D_N \oplus I_N$. In particular, elementary computations give:

$$\forall S \in \Omega_N, \quad v^*(S) = \underbrace{\sum_{j \in S} v^*(\{j\})}_{\text{Inessential TU game}} + \underbrace{\left(v^*(N) - \sum_{j \in N} v^*(\{j\})\right) u_N(S)}_{\text{TU game of } D_N}. \tag{14}$$

By applying Translation covariance, Efficiency, and Anonymity, we obtain:

$$\Phi(v^*) = (v^*(\{1\}), \ldots, v^*(\{n\})) + n^{-1}\left(v^*(N) - \sum_{j \in N} v^*(\{j\}), \ldots, v^*(N) - \sum_{j \in N} v^*(\{j\})\right). \tag{15}$$

By (15), $\Phi(v^*)$ is uniquely determined, so is $\Phi(v)$.

In any case, to complete the proof, it remains to apply **Fact 1** and to verify that the value under consideration satisfies the Ψ_ε-associated consistency axiom, which is a routine exercise.

In Xu et al. (2008) and Hamiache (2010) a matrix approach has been applied to study the linear transformation H_ε. Xu et al. (2013) and Xu et al. (2015) also apply this matrix approach to study the linear transformations K_ε, X_ε and L_ε. See also Hamiache (2013) for axiomatic characterizations of other values by using this approach.

The so-called matrix approach consists in studying the matrix expression $M_{\Psi_\varepsilon}^{\mathcal{E}}$ with respect to the standard basis $\mathcal{E} = (e_R)_{R \in \Omega_N}$ of the linear transformation Ψ_ε defining the Ψ_ε-consistency axiom. The authors prove that $M_{\Psi_\varepsilon}^{\mathcal{E}}$ is diagonalizable by exhibiting its eigenvalues and by showing that the dimension of the eigenspace of each eigenvalue equals the multiplicities of the eigenvalue. As underlined in the introduction, a main drawback of this approach is that it is unnecessarily long and computationally demanding, mainly because the matrix of the linear transformation is expressed with respect to the standard basis. In section 6, we show that the proof of convergence of (10) can be dramatically shortened by focusing the analysis on the linear transformation Ψ_ε. Considering alternative bases to $\mathcal{E}$ for which the matrix expression of Ψ_ε is upper triangular, we directly conclude by using the Jordan normal form. This approach is detailed in section 5. Section 6 applies this approach in order to provide a shorter proof of Theorems 1, 2 and 4.

5 Jordan normal form approach

In order to prove Theorems 1, 2, 3 and 4, we will use the Jordan normal form of a matrix (see Horn and Johnson (1990, chapter 3) for instance).

Theorem 5 *(Jordan normal form) Let V be a finite dimensional vector space (over an algebraically closed field) of dimension n and let Ψ be a linear transformation $\Psi : V \longrightarrow V$. Then, there exists a basis $\mathcal{B}$ for V such that the matrix expression $M_\Psi^{\mathcal{B}}$ of Ψ with respect to the basis $\mathcal{B}$ has the following form:*

$$M_{\Psi}^{\mathcal{B}} = \begin{pmatrix} J_1 & O & \dots & O \\ O & J_2 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & \dots & O & J_k, \end{pmatrix}$$

where O is a zero matrix, and, for each $p \in \{1, \dots, k\}$, the matrix J_p, called a Jordan block, is $d_p \times d_p$ for some $d_p \in \mathbb{N}$ and has the upper triangular matrix form

$$J_p = \lambda_p I_{d_p} + N_{d_p},$$

where λ_p is an eigenvalue of Ψ, N_{d_p} is the upper triangular $d_p \times d_p$ matrix with 0's on the main diagonal, 1's on the super diagonal and 0's above the super diagonal ; and I_{d_p} is the $d_p \times d_p$ identity matrix with 1's on the main diagonal and 0's elsewhere. Furthermore, if B_λ denotes the set of Jordan blocks associated with the eigenvalue λ, then the cardinality of B_λ is equal to the dimension of the eigenspace associated with λ, and the sum of d_p for $J_p \in B_\lambda$ is equal to the multiplicity of λ.

>From Theorem 5, we obtain the following corollary

Corollary 1 *Let V be a finite dimensional real vector space of dimension n and let Ψ be a linear transformation $\Psi : V \longrightarrow V$. Assume that the following conditions hold.*

1. *1 is an eigenvalue of Ψ and the multiplicity of 1 is equal to the dimension of its eigenspace.*
2. *Each other eigenvalue λ of Ψ is such that $|\lambda| < 1$.*

Then,

$$\forall v \in V_N, \quad (\Psi^{(t)}(v))_{t\in\mathbb{N}} \xrightarrow{t\to+\infty} v^*,$$

where v^* belongs to the eigenspace associated with the eigenvalue 1.

Proof: Consider the blocks $J_p \in B_1$ (see Theorem 5). On the one hand, because the multiplicity of 1 is equal to the dimension of its eigenspace, it follows that each $J_p \in B_1$ is a 1×1 matrix. All these blocks induce a submatrix $J_1^{'}$ of the Jordan normal form $M_{\Psi}^{\mathcal{B}}$ such that $J_1^{'} = I_m$, where m denotes the multiplicity of 1. Therefore,

$$J_1^{'(t)} \xrightarrow{t\to+\infty} I_m. \tag{16}$$

On the other hand, for each Jordan block $J_p \in B_\lambda$, $\lambda \neq 1$, we have:

$$J_p = \lambda I_{d_p} + N_{d_p}.$$

Because $|\lambda| < 1$, from standard computations, we obtain:

$$J_p^{(t)} \xrightarrow{t\to+\infty} O. \tag{17}$$

The result follows from (16) and (17). □

What we call the Jordan normal form approach by opposition to the matrix approach can be exposed as follows.

1. We choose a basis $\mathcal{B}' = (b'_1, \dots, b'_{2^n-1})$ for V_N.
2. For each linear transformation Ψ and for each $p \in \{1, \dots, 2^n - 1\}$, we compute the image $\Psi(b'_p)$.
3. From step 2, we express the matrix of Ψ with respect to $\mathcal{B}'$ and conclude that it is upper triangular on the main diagonal. It follows that the entries on the main diagonal are the eigenvalues of Ψ.
4. We verify that the conditions 1 and 2 in the statement of Corollary 1 (of Theorem 5) are satisfied, which ensures the convergence result of the linear transformation Ψ.
5. We apply the proof strategy described in section 4.

6 Applications

6.1 Proof of Theorem 1

To prove Theorem 1, we have to show that (10) holds for $\Psi_\varepsilon = H_\varepsilon$ and $E_{H_\varepsilon} = I_N$. To this end, we choose the unanimity basis $\mathcal{U} = (u_T)_{T\in\Omega_N}$, mainly because it is well known that $\mathcal{U}$ is suitable for the study of the Shapley value. By the way, the seminal paper by Shapley (1953) uses this basis. Furthermore, given the expression of H_ε given in (5), it is readily verify that $H_\varepsilon(u_R) = u_R$ in case $r = 1$, meaning that 1 is an eigenvalue of H_ε, and $\mathcal{I} = (u_{\{1\}}, \ldots, u_{\{n\}})$, a basis for I_N, contains (all) the eigenvectors associated with the eigenvalue 1. It remains to show that each other eigenvalue λ is such that $|\lambda| < 1$ for sufficiently small ε. This fact is a consequence of the statement of the next proposition.

Proposition 1 *For $R \in \Omega_N$, the image $H_\varepsilon(u_R)$ is such that:*

1. *If $r = 1$, $H_\varepsilon(u_R) = u_R$.*
2. *If $r \geq 2$, $H_\varepsilon(u_R) = (1 - \varepsilon r)u_R + \varepsilon \sum_{i\in R} u_{R\setminus\{i\}}$.*

We relegate the proof of Proposition 1 to the end of this section.

>From Proposition 1, we see that the matrix expression $M^{\mathcal{U}}_{H_\varepsilon}$ of H_ε with respect to the basis $\mathcal{U}$ is upper triangular. This allows us to quickly compute the eigenvalues of H_ε as well as their multiplicities: 1 is an eigenvalue of multiplicity n, and, for $r \in \{2, \ldots, n\}$, $1 - \varepsilon r$ is an eigenvalue with multiplicity $\binom{n}{r}$. There is no other eigenvalue. At this step, we verify that conditions 1 and 2 in the statement of Corollary 1 hold.

(a) On the one hand $\mathcal{I} = (u_1, \ldots, u_n)$ is a basis for I_N. Thus, the dimension of the eigenspace associated with the eigenvalue 1 is at least equal to n. On the other hand, the multiplicity of the eigenvalue 1 is n. This enforces that n is the dimension of the eigenspace associated with the eigenvalue 1. In other words I_N is the eigenspace associated with the eigenvalue 1. So, condition 1 in the statement of Corollary 1 holds.

(b) For sufficiently small ε, actually $0 < \varepsilon < 2/n$, for each $r \in \{2, \ldots, n\}$, it holds that $|1 - \varepsilon r| < 1$. So, condition 2 in the statement of Corollary 1 holds.

By (a)-(b) and Corollary 1, the convergence result (10) holds for $\Psi_\varepsilon = H_\varepsilon$. It remains to apply the proof strategy as described in section 4 and Theorem 1 is proved.

To establish Proposition 1, define the linear transform $\Upsilon : V_N \longrightarrow V_N$ as:

$$\forall S \in \Omega_N, \quad \Upsilon(v)(S) = \sum_{j\in N\setminus S} \big(v(S \cup \{j\}) - v(S) - v(\{j\})\big), \tag{18}$$

so that H_ε is defined as the sum of the identity transformation and $\varepsilon\Upsilon$. Proposition 1 is a direct consequence of the following result.

Proposition 2 *Let $R \in \Omega_N$.*

1. *If $r = 1$, then $\Upsilon(u_R) = 0_{V_N}$.*
2. *If $r \geq 2$, then*
$$\forall S \in \Omega_N, \quad \Upsilon(u_R)(S) = \begin{cases} 1 & \text{if } |S \cap R| = r - 1, \\ 0 & \text{otherwise.} \end{cases}$$
3. *If $r \geq 2$, it holds that:*
$$\Upsilon(u_R) = \sum_{i\in R} u_{R\setminus\{i\}} - r u_R. \tag{19}$$

Proof: Point 1. Pick any $R \in \Omega_N$ such that $r = 1$. Given the expression (18) of Υ and the fact that $u_R \in I_N$, the results holds trivially.

Point 2. Pick any $R \in \Omega_N$ such that $r \geq 2$. For each $S \in \Omega_N$, we have to compute:

$$\sum_{j \in N \setminus S} \Big(u_R(S \cup \{j\}) - u_R(S) - u_R(\{j\}) \Big). \tag{20}$$

Because $r \geq 2$, for each $j \in N \setminus S$, $u_R(\{j\}) = 0$. Therefore, (20) rewrites:

$$\sum_{j \in N \setminus S} \Big(u_R(S \cup \{j\}) - u_R(S) \Big). \tag{21}$$

Two exclusive cases arise depending on the choice of S.
(a) If S is such that $|S \cap R| = r - 1$. Denote by j^* the unique player belonging to $R \cap N \setminus S$. We have $u_R(S) = 0$, $S \cup \{j^*\} \supseteq R$ and so $u_R(S \cup \{j^*\}) = 1$. Therefore, (21) is equal to 1. (b) If S is such that $|S \cap R| \neq r - 1$, then it is obvious that $u_R(S) = u_R(S \cup \{j\}) = 0$ for each $j \in N \setminus S$ so that(21) is equal to 0.

Point 2 follows from (a) and (b).

Point 3. It follows from point 2. Indeed, pick any $S \in \Omega_N$. If $|S \cap R| = r - 1$, we have $u_{R \setminus \{i\}}(S) = 1$ for exactly one $i \in R$, i.e. for the unique $i \in R \setminus S$, and thus:

$$\sum_{i \in R} u_{R \setminus \{i\}}(S) - r u_R(S) = 1 - 0 = \Upsilon(u_R)(S).$$

If $|S \cap R| = r$, then we have:

$$\sum_{i \in R} u_{R \setminus \{i\}}(S) - r u_R(S) = r - r = \Upsilon(u_R)(S).$$

If $|S \cap R| < r - 1$, then we obviously have:

$$\sum_{i \in R} u_{R \setminus \{i\}}(S) - r u_R(S) = 0 - 0 = \Upsilon(u_R)(S).$$

This completes the proof of point 3. □

Proof: (of Proposition 1). Recall that H_ε is the sum of the identity transformation and $\varepsilon\Upsilon$. Pick any $R \in \Omega_N$. If $r = 1$, then by point 1 of Proposition 2, we have $\Upsilon(u_R) = 0_{V_N}$, so that $H_\varepsilon(u_R) = u_R$. If $r \geq 2$, then by point 3 of Proposition 2, we have:

$$H_\varepsilon(u_R) = u_R + \varepsilon \Big(\sum_{i \in R} u_{R \setminus \{i\}} - r u_R \Big) = (1 - \varepsilon r) u_R + \varepsilon \sum_{i \in R} u_{R \setminus \{i\}},$$

as asserted. □

For the sake of completeness, it is easy to verify that $(\Upsilon(u_R))_{R \in \Omega_N, r \geq 2} \subseteq V_N$ is a basis for $Ker(Sh)$. Indeed, since Sh satisfies the Inessential game property, it is onto and so the dimension of $Ker(Sh)$ is equal to $2^n - 1 - n$. The above collection of TU games contains exactly $2^n - 1 - n$, and these TU games are linearly independent. At last, using the expression (19) of Υ, the linearity of Sh, and the expression (2) of Sh for unanimity TU games, it is immediate to verify that $Sh(\Upsilon(u_R)) = 0_{V_N}$ for $r \geq 2$. >From this fact, it follows that Sh satisfies H_ε-associated for each $\varepsilon \in [0, 1]$.

6.2 Proof of Theorem 2

To prove Theorem 2, we have to show that (10) holds for $\Psi_\varepsilon = Hw_\varepsilon$ and $E_{Hw_\varepsilon} = I_N \oplus C_N$. By expression (6) of Hw_ε, we see that $Hw_\varepsilon(c) = c$ for the constant TU game c, and $Hw_\varepsilon(u_{\{i\}}) = u_{\{i\}}$ for each $i \in N$, meaning that that the unique element of the basis (c) for C_N and the elements of the basis $\mathcal{I} = (u_{\{1\}}, \ldots, u_{\{n\}})$ for I_N are eigenvectors associated with the eigenvalue 1. Furthermore, consider the subspace of TU games $V_{n-2} \subseteq V_N$ such that each $v \in V_{n-2}$ is defined as:

$$\forall i \in N, \quad v(N) = v(N \setminus \{i\}) = 0.$$

We see that $Hw_\varepsilon(v)(N) = Hw_\varepsilon(v)(N \setminus \{i\}) = 0$. Therefore, C_N, I_N and V_{n-2} are invariant subspaces under Hw_ε. The standard basis for V_{n-2} is $\mathcal{E}_{n-2} = (\delta_R)_{R \in \Omega_N, r \leq n-2}$. It turns out that that the ordered collection $\mathcal{B}^{(1)} = ((c), \mathcal{I}, \mathcal{E}_{n-2})$ constitutes a basis for V_N, so that V_N is the direct sum of C_N, I_N and V_{n-2}. All these facts provide a strong incentive to choose $\mathcal{B}^{(1)}$ to study the convergence of the linear transformation Hw_ε. As a start, we prove that $\mathcal{B}^{(1)}$ is a basis for V_N.

Proposition 3 *The ordered collection $\mathcal{B}^{(1)}$ is a basis for V_N. Thus, $V_N = C_N \oplus I_N \oplus V_{n-2}$.*

Proof: The collection $\mathcal{B}^{(1)}$ contains exactly $2^n - 1$ vectors, i.e. the dimension of V_N. It remains to prove that these vectors are linearly independent. So, pick any any linear combination

$$\alpha_c c + \sum_{i \in N} \alpha_i u_i + \sum_{R \in \Omega_N : r \leq n-2} \alpha_R \delta_R = 0_{V_N}.$$

On the one hand, we have:

$$\begin{aligned} \alpha_c c(N) + \sum_{i \in N} \alpha_i u_i(N) + \sum_{R \in \Omega_N : r \leq n-2} \alpha_R \delta_R(N) &= \alpha_c + \sum_{i \in N} \alpha_i \\ &= 0, \end{aligned}$$

and, for each $j \in N$,

$$\begin{aligned} \alpha_c c(N \setminus \{j\}) + \sum_{i \in N} \alpha_i u_i(N \setminus \{j\}) + \sum_{R \in \Omega_N : r \leq n-2} \alpha_R \delta_R(N \setminus \{j\}) &= \alpha_c + \sum_{i \in N \setminus \{j\}} \alpha_i \\ &= 0. \end{aligned}$$

Thus,

$$\forall j \in N, \quad \sum_{i \in N} \alpha_i = \sum_{i \in N \setminus \{j\}} \alpha_i$$

which implies, for each $j \in N$, $\alpha_j = 0$, and so $\alpha_c = 0$.

On the other hand, and taking into account that, for each $i \in N$, $\alpha_i = \alpha_c = 0$, we have, for each $S \in \Omega_N$, $s \leq n - 2$:

$$\begin{aligned} \alpha_c c(S) + \sum_{i \in N} \alpha_i u_i(S) + \sum_{R \in \Omega_N : r \leq n-2} \alpha_R \delta_R(S) &= \alpha_S \\ &= 0, \end{aligned}$$

which proves that the elements of $\mathcal{B}^{(1)}$ are linearly independent. □

Proposition 4 *Consider the basis $\mathcal{B}^{(1)}$ for V_N. The following facts hold.*

1. *$Hw_\varepsilon(c) = c$.*
2. *For each $i \in N$, $Hw_\varepsilon(u_{\{i\}}) = u_{\{i\}}$.*

3. For each $R \in \Omega_N$ such that $r \leq n-2$,

$$Hw_\varepsilon(\delta_R) = (1-(n-r)\varepsilon)\delta_R + \varepsilon \sum_{i \in R} \delta_{R\setminus\{i\}}.$$

Proof: Points 1 and 2 are obvious given the expression (6) of Hw_ε.

Point 3. Pick any $R \in \Omega_N$ such that $r \leq n-2$. By (6) and definition of δ_R and SC_j, we have:

$$\begin{aligned} \forall S \in \Omega_N, \quad Hw_\varepsilon(\delta_R)(S) &= \delta_R(S) + \varepsilon \sum_{j \in N\setminus S} \big(\delta_R(S\cup\{j\}) - \delta_R(S) - SC_j(\delta_R)\big) \\ &= \delta_R(S) + \varepsilon \sum_{j \in N\setminus S} \big(\delta_R(S\cup\{j\}) - \delta_R(S)\big) \\ &= (1-(n-s)\varepsilon)\delta_R(S) + \varepsilon \sum_{j \in N\setminus S} \big(\delta_R(S\cup\{j\}). \end{aligned} \tag{22}$$

If $S = R$, then (22) is equal to:

$$Hw_\varepsilon(\delta_R)(R) = 1-\varepsilon(n-r).$$

If $S = R\setminus\{i\}$ for some $i \in R$, then (22) is equal to:

$$Hw_\varepsilon(\delta_R)(R\setminus\{i\}) = \varepsilon.$$

In each other case, (22) is equal to:

$$Hw_\varepsilon(\delta_R)(R) = 0.$$

Thus, for each $R \in \Omega_N$ such that $r \leq n-2$, it holds that:

$$Hw_\varepsilon(\delta_R) = (1-(n-r)\varepsilon)\delta_R + \varepsilon \sum_{i \in R} \delta_{R\setminus\{i\}},$$

as asserted. □

>From Proposition 4, it is easy to conclude that EANSC satisfies Hw_ε-associated consistency. >From Proposition 4, we also see that the matrix expression $M^{\mathcal{B}^{(1)}}_{Hw_\varepsilon}$ of Hw_ε with respect to the basis $\mathcal{B}^{(1)}$ is upper triangular. This allows us to quickly compute the eigenvalues of Hw_ε as well as their multiplicities: 1 is an eigenvalue of multiplicity $n+1$, and, for $r \leq n-2$, $1-(n-r)\varepsilon$ is an eigenvalue with multiplicity $\binom{n}{r}$. There is no other eigenvalue. At this step, we verify that conditions 1 and 2 in the statement of Corollary 1 hold.

(a) On the one hand, $(c, u_1, \ldots, u_n)$ is a basis for $C_N \oplus I_N$. Thus, the dimension of the eigenspace associated with the eigenvalue 1 is at least equal to $n+1$. On the other hand, the multiplicity of the eigenvalue 1 is $n+1$. This enforces that $n+1$ is the dimension of the eigenspace associated with the eigenvalue 1. In other words, $C_N \oplus I_N$ is the eigenspace associated with the eigenvalue 1. So, condition 1 in the statement of Corollary 1 holds.

(b) For sufficiently small ε, actually $0 < \varepsilon < 2/(n-1)$, it holds that, for each $r \leq n-2$, $|1-(n-r)\varepsilon| < 1$. So, condition 2 in the statement of Corollary 1 holds.

By (a)-(b) and Corollary 1, the convergence result (10) holds for $\Psi_\varepsilon = Hw_\varepsilon$ and $E_{Hw_\varepsilon} = I_N \oplus C_N$. It remains to apply the proof strategy as described in section 4, and Theorem 2 is proved.

6.3 Proof of Theorem 3

To prove Theorem 3, we have to show that (10) holds for $\Psi_\varepsilon = K_\varepsilon\ E_{K_\varepsilon} = D_N \oplus I_N$. By expression (7) of K_ε, we immediately see that $K_\varepsilon(u_{\{i\}}) = u_{\{i\}}$ for each unanimity TU game $u_{\{i\}}$, $i \in N$, and $K_\varepsilon(u_N) = u_N$ for the unanimity TU game u_N, meaning that these $n+1$ unanimity TU games are eigenvectors associated with the eigenvalue 1. So, let us choose the collection of unanimity TU games as a basis for V_N. For convenience, we order the unanimity TU games as follows $(u_N, (u_i)_{i\in N}, (u_R)_{R\in\Omega_N, r\geq 2, r\neq n})$. Denote this basis by $\mathcal{U}^{(1)}$.

Proposition 5 *Consider the basis $\mathcal{U}^{(1)}$ for V_N. The following facts hold.*

1. *$K_\varepsilon(u_N) = u_N$.*
2. *For each $i \in N$, $K_\varepsilon(u_{\{i\}}) = u_{\{i\}}$.*
3. *For each $R \in \Omega_N$ such that $r \in \{2, \ldots, n-1\}$, $K_\varepsilon(u_R) = (1 - r\varepsilon)u_R + r\varepsilon u_N$.*

Proof: Points 1 and 2 are immediate by expression (7).

Point 3 Pick any $R \in \Omega_N$ such that $r \in \{2, \ldots, n-1\}$. By (7) and $r \geq 2$, we have:

$$\begin{aligned} \forall S \in \Omega_N, S \neq N, \quad K_\varepsilon(u_R)(S) &= u_R(S) - \varepsilon \sum_{j\in S} \big(u_R(S) - u_R(S\setminus\{j\}) - u_R(\{j\})\big) \\ &= u_R(S) - \varepsilon \sum_{j\in S} \big(u_R(S) - u_R(S\setminus\{j\})\big), \end{aligned} \tag{23}$$

Two cases arise:

(a) Consider any coalition $S \supseteq R$, $S \neq N$. For each $j \in R \cap S$, we have:

$$u_R(S) - u_R(S\setminus\{j\}) = 1 - 0 = 1.$$

For each For each $j \in S \setminus R$, we have:

$$u_R(S) - u_R(S\setminus\{j\}) = 1 - 1 = 0.$$

Therefore, for $S \supseteq R$, $S \neq N$, (23) writes:

$$K_\varepsilon(u_R)(S) = (1 - r\varepsilon)u_R(S) + r\varepsilon u_N(S) \tag{24}$$

(b) Consider any coalition $S \not\supseteq R$. We obviously have:

$$\forall j \in S, \quad u_R(S) = u_R(S\setminus\{j\}) = 0.$$

It follows that (23) writes:

$$\begin{aligned} K_\varepsilon(u_R)(S) &= 0 \\ &= (1 - r\varepsilon)u_R(S) + \varepsilon r u_N(S). \end{aligned} \tag{25}$$

It remains to consider the case $S = N$. By (7), we have:

$$\begin{aligned} K_\varepsilon(u_R)(N) &= u_R(N) \\ &= 1 \\ &= (1 - r\varepsilon)u_R(N) + r\varepsilon u_N(N). \end{aligned} \tag{26}$$

Point 3 follows from (24), (25) and (26). □

>From Proposition 5, it is easy to verify that CIS satisfies K_ε-associated consistency. Because CIS is a linear value, it is sufficient to verify that CIS satisfies K_ε-associated consistency on the elements of the basis. In particular, for $r \in \{2, \ldots, n-1\}$, by linearity of the CIS value, we have:

$$\forall i \in N, \quad CIS_i(K_\varepsilon(u_R)) = (1 - r\varepsilon)CIS_i(u_R) + r\varepsilon CIS_i(u_N).$$

But by (4), $CIS_i(u_R) = CIS_i(u_N) = n^{-1}$, so that

$$CIS_i(K_\varepsilon(u_R)) = (1 - r\varepsilon)n^{-1} + r\varepsilon n^{-1} = n^{-1} = CIS_i(u_R),$$

as desired. Next, from Proposition 5, we also see that the matrix expression $M_{K_\varepsilon}^{\mathcal{U}^{(1)}}$ of K_ε with respect to the basis $\mathcal{U}^{(2)}$ is upper triangular. This allows us to quickly compute the eigenvalues of K_ε as well as their multiplicities: 1 is an eigenvalue of multiplicity $n+1$, and, for $r \in \{2, \ldots, n-1\}$, $1 - r\varepsilon$ is an eigenvalue with multiplicity $\binom{n}{r}$. There is no other eigenvalue. At this step, we verify that conditions 1 and 2 in the statement of Corollary 1 hold.

(a) On the one hand, $(u_N, u_1, \ldots, u_n)$ is a basis for $D_N \oplus I_N$. Thus, the dimension of the eigenspace associated with the eigenvalue 1 is at least equal to $n+1$. On the other hand, the multiplicity of the eigenvalue 1 is $n+1$. This enforces that $n+1$ is the dimension of the eigenspace associated with the eigenvalue 1. In other words, $D_N \oplus I_N$ is the eigenspace associated with the eigenvalue 1. So, condition 1 in the statement of Corollary 1 holds.

(b) For sufficiently small ε, actually $0 < \varepsilon < 2/(n-1)$, it holds that, for each $r \in \{2, \ldots, n-1\}$, $|1 - r\varepsilon| < 1$. So, condition 2 in the statement of Corollary 1 holds.

By (a)-(b) and Corollary 1, the convergence result (10) holds for $\Psi_\varepsilon = K_\varepsilon$ and $E_{K_\varepsilon} = D_N \oplus I_N$. It remains to apply the proof strategy as described in section 4, and Theorem 5 is proved.

6.4 Proof of Theorem 4

To prove point 1 of Theorem 4, we have to show that (10) holds for $\Psi_\varepsilon = X_\varepsilon$ and $E_{X_\varepsilon} = I_N$. For similar reasons to Theorem 2, we choose the same basis for V_N, with the difference that here $X_\varepsilon(c) \neq c$. For this reason, we order the elements of this basis as follows: $(\mathcal{I}, c, \mathcal{E}_{n-2})$. This change can be made without loss of generality, and it is more convenient to express the matrix of X_ε as an upper triangular matrix. We denote this basis by $\mathcal{B}^{(2)}$.

Proposition 6 *Consider the basis $\mathcal{B}^{(2)}$ for V_N. The following facts hold.*

1. *For each $i \in N$, $X_\varepsilon(u_{\{i\}}) = u_{\{i\}}$.*
2. *$X_\varepsilon(c) = (1-\varepsilon)c + \varepsilon n^{-1} \sum_{i \in N} u_i$.*
3. *For each $R \in \Omega_N$ such that $r \leq n-2$, $X_\varepsilon(\delta_R) = (1-\varepsilon)\delta_R$.*

Proof: Point 1. For each $i \in N$ and each $j \in N$, $SC_j(u_{\{i\}}) = u_i(\{j\})$. By definition of X_ε as given by (8), it follows that:

$$\begin{aligned} \forall S \in \Omega_N, \quad X_\varepsilon(u_{\{i\}})(S) &= u_{\{i\}}(S) + \varepsilon\left(\frac{s}{n}\big(u_{\{i\}}(N) - \sum_{j \in N} u_i(\{j\})\big) - \big(u_{\{i\}}(S) - \sum_{j \in S} u_i(\{j\})\big)\right) \\ &= u_{\{i\}}(S), \end{aligned}$$

which proves point 1.

Point 2. Consider the constant TU game c. For each $j \in N$, $SC_j(c) = 0$. We have:

$$\begin{aligned} \forall S \in \Omega_N, \quad X_\varepsilon(c)(S) &= c(S) + \varepsilon\left(\frac{s}{n}c(N) - c(S)\right) \\ &= (1-\varepsilon) + \varepsilon\frac{s}{n}. \end{aligned}$$

Therefore,

$$X_\varepsilon(c) = (1-\varepsilon)c + \frac{\varepsilon}{n}\sum_{i\in N} u_i,$$

as asserted.

Point 3. Pick any $R \in \Omega_N$ such that $r \leq n-2$. Because $r \leq n-2$, for each $j \in N$, $SC_j(\delta_R) = 0$ and $\delta_R(N) = 0$. Thus, we have:

$$\begin{aligned} \forall S \in \Omega_N, \quad X_\varepsilon(\delta_R)(S) &= \delta_R(S) + \varepsilon(-\delta_R(S)) \\ &= (1-\varepsilon)\delta_R(S). \end{aligned}$$

which proves point 3. □

>From Proposition 6, it is easy to conclude that EANSC satisfies X_ε-associated consistency. >From Proposition 6, we also see that the matrix expression $M_{X_\varepsilon}^{\mathcal{B}^{(2)}}$ of X_ε with respect to the basis $\mathcal{B}^{(2)}$ is upper triangular with non-zero entries. This allows us to compute the eigenvalues of X_ε as well as their multiplicities: 1 is an eigenvalue of multiplicity n, and, for $1-\varepsilon$ is an eigenvalue with multiplicity $2^n - 1 - n$. There is no other eigenvalue. At this step of the proof, we verify that conditions 1 and 2 in the statement of Corollary 1 hold.

(a) As for the proof of Theorem 1, we conclude that I_N is the eigenspace associated with the eigenvalue 1. So, condition 1 in the statement of Corollary 1 holds.

(b) For each ε such that $0 < \varepsilon < 1$, it holds that $|1-\varepsilon| < 1$. So, condition 2 in the statement of Corollary 1 holds.

>From (a)-(b), the convergence result (10) holds for $\Psi_\varepsilon = X_\varepsilon$ and $E_{X_\varepsilon} = I_N$. It remains to apply the proof strategy as described in section 4 and point 1 of Theorem 4 is proved.

Remark. By Proposition 6, each vector of the basis $\mathcal{B}^{(2)}$ is an eigenvector associated with an eigenvalue except the vector c. To obtain a basis of eigenvectors, it suffices to choose the basis $((c - n^{-1}\sum_{i\in N} u_i), \mathcal{I}, \mathcal{E}_{n-2})$ instead of $\mathcal{B}'$. In that case, by linearity of X_ε, we have:

$$\begin{aligned} X_\varepsilon\left(c - n^{-1}\sum_{i\in N} u_i\right) &= X_\varepsilon(c) - n^{-1}\sum_{i\in N} X_\varepsilon(u_i) \\ &= (1-\varepsilon)c + \varepsilon n^{-1}\sum_{i\in N} u_i - n^{-1}\sum_{i\in N} u_i \\ &= (1-\varepsilon)\left(c - n^{-1}\sum_{i\in N} u_i\right), \end{aligned}$$

which proves that $c - n^{-1}\sum_{i\in N} u_i$ is an eigenvector associated with the eigenvalue $1-\varepsilon$.

To prove point 2 of Theorem 4, we have to show that (10) holds for $\Psi_\varepsilon = L_\varepsilon$ and $E_{L_\varepsilon} = I_N$. By expression (9) of L_ε, we see that, for each $i \in N$, $L_\varepsilon(u_{\{i\}}) = u_{\{i\}}$ meaning that that the elements of the basis $\mathcal{I} = (u_{\{1\}}, \ldots, u_{\{n\}})$ for I_N are eigenvectors associated with the eigenvalue 1. Furthermore, consider the subspace of TU games $V_{2,n-1} \subseteq V_N$ such that each $v \in V_{2,n-1}$ is such that:

$$\forall i \in N, \quad v(\{i\}) = v(N) = 0.$$

By using (9), we see that $L_\varepsilon(v)(N) = v(N) = 0$ and $L_\varepsilon(v)(\{i\}) = v(\{i\}) = 0$. Therefore, I_N and $V_{2,n-1}$ are invariant subspaces under L_ε. The standard basis for $V_{2,n-1}$ is $\mathcal{E}_{2,n-1} = (\delta_R)_{R\in\Omega_N, 2\leq r\leq n-1}$. By similar arguments than those used to prove Proposition 3, we obtain $I_N \oplus V_{2,n-1}$. To obtain a basis for V_N, we complete the ordered collection $(\mathcal{I}, \mathcal{E}_{2,n-1})$ by adding the constant TU game c. For the rest of this section, we thus consider the basis $\mathcal{C} = (\mathcal{I}, (c), \mathcal{E}_{2,n-1})$. This fact is summarized in the following proposition. We omit the proof since it is similar to the proof of Proposition 3.

Proposition 7 *The ordered collection $\mathcal{C}$ is a basis for V_N. Thus, $V_N = I_N \oplus C_N \oplus V_{2,n-1}$.*

Proposition 8 *Consider the basis $\mathcal{C}$ for V_N. The following facts hold.*

1. *For each $i \in N$, $L_\varepsilon(u_{\{i\}}) = u_{\{i\}}$.*
2. *$L_\varepsilon(c) = (1-\varepsilon)c + \varepsilon n^{-1} \sum_{i\in N} u_i$.*
3. *For each $R \in \Omega_N$ such that $r \in \{2, \ldots, n-1\}$, $L_\varepsilon(\delta_R) = (1-\varepsilon)\delta_R$.*

The proof of Proposition 8 is similar to the proof of Proposition 6 and so it is omitted. The rest of the proof of point 2 of Theorem 4 follows the same arguments as the proof of point 1 of this theorem.

References

Driessen T.S.H., Funaki Y., Coincidence of and collinearity between game theoretic solutions, *OR Spektrum* 13 (1991) 15-30.

Hamiache G., Associated consistency and Shapley value, *Internat. J. Game Theory* 30 (2001) 279-289.

Hamiache G., A matrix approach to the associated consistency with an application to the Shapley value, *Int. Game Theory Rev.* 12 (2010) 175-187.

Hamiache G., A matrix approach to TU games with coalition and communication structures, *Soc. Choice Welf.* 38 (2012) 85-100.

Horn R., Johnson C.R., *Matrix Analysis*, Cambridge University Press, 1990.

Hwang Y., Associated consistency and equal allocation of nonseparable costs, *Econom. Theory* 28 (2006) 709-719.

Moulin H., The separability axiom and equal sharing method, *J. Econom. Theory* 36 (1985) 120-148.

Shapley L.S., A value for n-person games. In: Contributions to the Theory of Games II, Ann. Math. Stud., Kuhn HW, Tucker AW (eds.), Princeton University Press, Princeton, 1953, 307-317.

Xu G., Driessen T.S.H., Sun H., Matrix analysis for associated consistency in cooperative game theory, *Linear Algebra Appl.* 428 (2008) 1571-1586.

Xu G., Driessen T.S.H., Sun H., Matrix approach to dual similar associated consistency for the Shapley value, *Linear Algebra Appl.* 430 (2009) 2896-2897.

Xu G., van den. Brink R., van der Laan G., Associated consistency characterization of two linear values for TU games by matrix apporach, *Linear Algebra Appl.* 471 (2015) 224-240.

Xu G., Wang W., Dong H., Axiomatization for the center-of-gravity of imputation set value, *Linear Algebra Appl.* 439 (2013) 2205-2215.