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Axiomatic characterizations under players nullification[1]

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Abstract

Many axiomatic characterizations of values for cooperative games invoke axioms which evaluate the consequences of removing an arbitrary player. Balanced contributions (Myerson, 1980) and balanced cycle contributions (Kamijo and Kongo, 2010) are two well-known examples of such axioms. We revisit these characterizations by nullifying a player instead of deleting her/him from a game. The nullification (Béal et al., 2014a) of a player is obtained by transforming a game into a new one in which this player is a null player, *i.e.* the worth of the coalitions containing this player is now identical to that of the same coalition without this player. The degree with which our results are close to the original results in the literature is connected to the fact that the targeted value satisfies the null player out axiom (Derks and Haller, 1999).

Keywords: Player nullification, balanced contributions, Shapley value, equal allocation of non-separable costs, potential.

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1. Introduction

This article studies cooperative games with transferable utility (denoted as TU-games). A TU-game is given by a set of players and a characteristic function which associates to any subset of players the worth created by the cooperation of its members. A value assigns to each TU-game and each player an individual payoff for participating to this TU-game. The axiomatic approach is adopted here, and following Thomson (2012), we sort the axioms in two kinds: punctual and relational axioms. A punctual axiom applies to each TU-games separately and a relational axiom

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relates payoff vectors of TU-games that are related in a certain way. This article introduce new relational axioms.

There exists several popular relational axioms among which balanced contributions (Myerson, 1980) and balanced cycle contributions (Kamijo and Kongo, 2010). The common characteristic of these two axioms is that they evaluate the consequences of removing a player from a TU-game on the payoff of some other players. For instance, balanced contributions requires, for any two players, equal allocation variation after the leave of the other player. As such the axiomatic study in Myerson (1980) operates on a class of TU-games with variable player sets. Together with the standard efficiency axiom, Myerson (1980) characterizes the Shapley value (Shapley, 1953).

An alternative to the approach of removing players has been introduced in Béal et al. (2014a) where the authors measure the influence of a complete loss of productivity of a player, in the sense that the worth of the coalitions containing this player is now identical to that of the same coalition without this player. This loss of productivity of a player is called his/her nullification in reference to the fact that he/she becomes a null player.

In both previous cases, whether a player has left the TU-game or has been nullified means somehow that the other players cannot expect anything from this player in terms of worths. In this article, we ask the following question: is the impact of deleting a player equivalent to keeping him nullified in the TU-game? In order to answer this question, we revisit the aforementioned relational axioms by nullifying a player instead of removing him from the TU-game. We extend this principle to the balanced collective contributions axiom (Béal et al., 2014b) and to the potential approach (Hart and Mas-Colell, 1989). We obtain the results below.

Firstly, we define the axiom of balanced contributions under nullification which requires, for any two players, equal allocation variation after the nullification of the other player. We prove that the combination of balanced contributions under nullification and efficiency characterizes a class of values: each such value is the sum of the Shapley value and an exogenous budget-balanced transfer scheme. In case the players are characterized by unequal initial exogenous endowments, such a transfer scheme can be interpreted as a solidarity mechanism, which serves to reduce inequalities resulting from these endowments. Our result possesses some similarities with the weighted Shapley values studied in Kalai and Samet (1987), where the weights have a multiplicative feature, while they are added to the Shapley value in our case. Adding the classical null game axiom, *i.e.* all players get a zero payoff if the worths of all coalitions are equal to zero, we get a characterization of the Shapley value as a corollary.

Secondly, in Béal et al. (2014b), the equal allocation of non-separable costs (see Moulin, 1987, for instance) is characterized by efficiency and balanced collective contributions. The latter axiom requires the identical average impact of the withdrawal of any player from a TU-game on the remaining population. This result is not valid anymore when efficiency is combined with our new axiom of balanced collective contributions under nullification. We prove that a new value is characterized by the later axiom in addition to equal treatment and efficiency. This value is linear and admits a closed form expression. It relies on a marginalistic principle which goes beyond the one expressed in the Shapley value by overpaying the productive players and taxing the unproductive ones, while the equal allocation of non-separable costs possesses a more egalitarian flavor. Replacing in this last result equal treatment by the null player axiom leads to an impossibility result.

Thirdly, we define the axiom of balanced cycle contributions under nullification as a variation of the axiom of balanced cycle contributions. The latter imposes, for all orderings of the players, that the sum of the impact on each player of removing his/her predecessor is balanced with the sum of the impact on each player of removing his/her successor. The former imposes the same requirement except that the removed players are nullified. Any linear and symmetric value satisfies both axioms. Kamijo and Kongo (2010) characterize the Shapley value by efficiency, null player out (Derks and Haller, 1999) and balanced cycle contributions. We prove that balanced cycle contributions under nullification, efficiency and the null player axiom characterize the Shapley value on the class of TU-games containing at least one null player. In order to recover a characterization on the full domain, we envisage two ways. On the one hand, adding linearity in the previous result yields a class of values: each such value is the sum of the Shapley value and a budget-balanced transfer scheme depending on the characteristic function in a simple manner. On the other hand, adding balanced collective contributions under nullification for TU-games possessing no null player characterizes a (non-additive) value which coincides with the Shapley value as soon as a null player is present in the TU-game, and coincides with the equal allocation of non-separable costs otherwise. In a sense, for monotonic TU-games, this value rules out any solidarity in an environment where there is no unproductive player. Solidarity can then emerge when every player has some positive contribution to at least one coalition.

Fourthly, we introduce a notion of nullified potential, similar to the original potential but based on the discrete derivative with respect to the nullification operation instead of the removal operation. These two potentials turn out to be equal. As a consequence, we obtain a characterization of the Shapley value analogous to the original one by Hart and Mas-Colell (1989): the Shapley value of a player in a TU-game is equal to the discrete derivative of the nullified potential of this TU-game with respect to this player's nullification. A recursive formula of the Shapley value relying on TU-games with nullified players is also provided in a similar way as the formula given by Hart and Mas-Colell (1989).

Let us mention two other facets of our approach. Firstly, our results are valid on classes of TU-games with fixed player sets, contrary to the corresponding original results in the literature. Secondly, the axiom of null player out is useful to deepen the relationship between the aforementioned relational axioms and their corresponding version with nullified players. In presence of null player out, we prove that the two versions of the previous relational axioms are equivalent. However the existence of a value satisfying these axioms is not always guaranteed.

The closest article in the literature is Béal et al. (2014a) in which the axiom of nullified solidarity is introduced. This axiom requires that all players weakly gain together or weakly lose together after one of them has been nullified. Together with efficiency, the null game axiom and a weak axiom of fairness, Béal et al. (2014a) characterize the equal division value.

The rest of the article is organized as follows. Section 2 provides definitions, notations and statements of the main existing results in the literature. Section 3 presents the nullification of a player, and contains the results invoking the axioms with players nullification. Section 4 revisits the potential approach. Section 5 concludes.

2. Basic definitions and notations

2.1. Cooperative games with transferable utility

Let $\mathcal{U} \subseteq \mathbb{N}$ be a fixed and infinite universe of players. Denote by U the set of all finite subsets of $\mathcal{U}$. A **TU-game** is a pair (N, v) where $N \in U$ and $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. A non-empty subset $S \subseteq N$ is a coalition, and $v(S)$ is the worth of the coalition. For any non-empty coalition S, let s be the cardinality of S. The **sub-game** of (N, v) induced by $S \subseteq N$ is denoted by $(S, v|_S)$, where $v|_S$ is the restriction of v to 2^S. For simplicity, we write the singleton $\{i\}$ as i. Define $\mathbb{V}$ and $\mathbb{V}(N)$ as the classes of all TU-games with a finite player set in $\mathcal{U}$ and of all TU-games with the fixed and finite player set $N \in U$.

Player $i \in N$ is **null** in $(N, v) \in \mathbb{V}$ if $v(S) = v(S\backslash\{i\})$ for all $S \subseteq N$ such that $S \ni i$. We denote by $K(N, v)$ the set of null players in (N, v). We often use the shortcuts $K(v)$ for $K(N, v)$ and $k(v)$ for $|K(v)|$. The subset $\mathbb{V}^0(N)$ of $\mathbb{V}(N)$ will denote the subset of TU-games with at least one null player in N. Two distinct players $i, j \in N$ are **equal** in $(N, v) \in \mathbb{V}$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N\backslash\{i, j\}$.

For $(N, v), (N, w) \in \mathbb{V}$ and $c \in \mathbb{R}$, the TU-games $(N, v+w)$ and (N, cv) are given by $(v+w)(S) = v(S)+w(S)$ and $(cv)(S) = cv(S)$ for all $S \subseteq N$. The **null game** on N is the TU-game $(N, \mathbf{0}) \in \mathbb{V}$ is given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. For each $N \in U$ and any nonempty $S \in 2^N$, the **unanimity TU-game** induced by S is the TU-game (N, u_S) such that $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise. It is well-known that any characteristic function $v : 2^N \longrightarrow \mathbb{R}$ admits a unique decomposition in terms of unanimity TU-games:

$$v = \sum_{S \in 2^N, S \neq \emptyset} \Delta_S(v) u_S,$$

where $\Delta_S(v) = \sum_{T \subseteq S} (-1)^{s-t} v(T)$ is called the **Harsanyi dividend** (Harsanyi, 1959) of S in TU-game (N, v).

2.2. *Values*

A **value** on $\mathbb{V}$ is a function φ that assigns a payoff vector $\varphi(N, v) \in \mathbb{R}^N$ to any $(N, v) \in \mathbb{V}$. The definition of a value on another class of TU-games is similar. We consider the following values.

The **Shapley value** (Shapley, 1953) is given by:

$$\mathrm{Sh}_i(N, v) = \sum_{S \subseteq N : S \ni i} \frac{(n-s)!\,(s-1)!}{n!} \big(v(S) - v(S\backslash i)\big) \qquad \text{for all } (N, v) \in \mathbb{V} \text{ and } i \in N.$$

The **Equal Division value** is the value ED given by:

$$\mathrm{ED}_i(N, v) = \frac{v(N)}{n} \qquad \text{for all } (N, v) \in \mathbb{V} \text{ and } i \in N.$$

The **Equal Allocation of Non-Separable Costs** is the value EANSC defined as:

$$\mathrm{EANSC}_i(N, v) = v(N) - v(N\backslash i) + \frac{1}{n}\Big(v(N) - \sum_{j \in N} \big(v(N) - v(N\backslash j)\big)\Big) \qquad \text{for all } (N, v) \in \mathbb{V} \text{ and } i \in N.$$

The equal allocation of non-separable costs first assigns to each player his/her marginal contribution to the grand coalition (his/her separable cost), and then splits equally the non-separable costs among the players.

2.3. *Some axioms and existing characterizations*

In this article, we invoke axioms which can be gathered into two categories according to whether they operate on a fixed player set or on variable player sets. The first category contains the following axioms.

Efficiency, E. For all $(N, v) \in \mathbb{V}$, $\sum_{i \in N} \varphi_i(N, v) = v(N)$.

Additivity, A. For all $(N, v), (N, w) \in \mathbb{V}$, $\varphi(N, v + w) = \varphi(N, v) + \varphi(N, w)$.
Linearity, L. For all $(N, v), (N, w) \in \mathbb{V}$ and $c \in \mathbb{R}$, $\varphi(cv + w) = c\varphi(v) + \varphi(w)$.
Null game axiom, NG. For all $(N, \mathbf{0}) \in \mathbb{V}$, all $i \in N$, $\varphi_i(N, \mathbf{0}) = 0$.
Equal treatment, ET. For all $(N, v) \in \mathbb{V}$, all $i, j \in N$ equal in (N, v), $\varphi_i(N, v) = \varphi_j(N, v)$.
Symmetry, S. For all $(N, v) \in \mathbb{V}$, all permutation $\sigma = (i_j)_{j \in N}$ on N, $\varphi_j(N, v) = \varphi_{i_j}(N, v_\sigma)$ where v_σ is defined by $v_\sigma(S) = v(\{k | i_k \in S\})$, $S \subseteq N$.
Null player, N. For all $(N, v) \in \mathbb{V}$, all $i \in N$ null in (N, v), $\varphi_i(N, v) = 0$.

It should be clear that all the aforementioned values and axioms operating on a fixed player set N can be defined/invoked on $\mathbb{V}$ as well. Below is a list of relational axioms operating on variable player sets.
Null player out axiom, NO. (Derks and Haller, 1999) For all $(N, v) \in \mathbb{V}$, all $i \in N$ null in (N, v), all $j \in N \backslash i$, $\varphi_j(N, v) = \varphi_j(N \backslash i, v|_{N \backslash i})$.
Balanced contributions, BC. (Myerson, 1980) For all $(N, v) \in \mathbb{V}$, for all $i, j \in N$,

$$\varphi_i(N, v) - \varphi_i(N, v|_{N \backslash j}) = \varphi_j(N, v) - \varphi_j(N, v|_{N \backslash i}).$$

Balanced cycle contributions, BCyC. (Kamijo and Kongo, 2010) For all $(N, v) \in \mathbb{V}$, all ordering $(i_1, \ldots, i_p, \ldots, i_n)$ on N,

$$\sum_{p=1}^{n} \Big(\varphi_{i_p}(N, v) - \varphi_{i_p}(N \backslash i_{p-1}, v|_{N \backslash i_{p-1}}) \Big) = \sum_{p=1}^{n} \Big(\varphi_{i_p}(N, v) - \varphi_{i_p}(N \backslash i_{p+1}, v|_{N \backslash i_{p+1}}) \Big),$$

where $i_0 = i_n$ and $i_{n+1} = i_1$.
Balanced collective contributions, BCoC. (Béal et al., 2014b) For all $(N, v) \in \mathbb{V}$, for all $i, j \in N$

$$\frac{1}{n-1} \sum_{k \in N \backslash i} \Big(\varphi_k(v) - \varphi_k(v|_{N \backslash i}) \Big) = \frac{1}{n-1} \sum_{k \in N \backslash j} \Big(\varphi_k(v) - \varphi_k(v|_{N \backslash j}) \Big).$$

We conclude this section by presenting some characterizations involving axioms operating on variable sets of players.

Proposition 1. *(Myerson, 1980) The Shapley value is the unique value on $\mathbb{V}$ that satisfies Efficiency (**E**) and Balanced contributions (**BC**).*

Proposition 2. *(Kamijo and Kongo, 2010) The Shapley value is the unique value on $\mathbb{V}$ that satisfies Efficiency (**E**), Null player out (**NO**) and Balanced cycle contributions (**BCyC**).*

Proposition 3. *(Béal et al., 2014b) The equal allocation of non-separable costs is the unique value on $\mathbb{V}$ that satisfies Efficiency (**E**), and Balanced collective contributions (**BCoC**).*

3. Player's nullification

The nullification of a player in a TU-game refers to the complete loss of productivity of this player. More specifically, a new TU-game is constructed from the original one, in which the worth of any coalition containing the nullified player is equal to the worth of the same coalition without the nullified player. Formally, for $(N, v) \in \mathbb{V}$ and $i \in N$, we denote by $(N, v^i) \in \mathbb{V}$ the TU-game obtained from (N, v) if player i is **nullified**: $v^i(S) = v(S\backslash i)$ for all $S \subseteq N$. As such, i is a null player in (N, v^i). Furthermore, it holds that $(v^i)^i = v^i$ and $(v^i)^j = v(^j)^i$ for $j \in N$. Thus, abusing notations, for any coalition $S \subseteq N$, we denote by (N, v^S) the TU-game obtained from (N, v) by the successive nullification of each player in S (in any order). For any $(N, v) \in \mathbb{V}$, note that $(N, v^N) = (N, \mathbf{0})$.

We continue this section by pointing out properties of the nullified TU-games. Firstly, there is a clear relationship between the Harsanyi dividends in (N, v) and (N, v^i): $\Delta_S(v^i) = 0$ if $S \ni i$ by definitions of a null player and of the Harsanyi dividends, and $\Delta_S(v^i) = \Delta_S(v)$ if $S \not\ni i$ since $v(T) = v^i(T)$ for all $T \subseteq S$ in such a case. Secondly, for any $(N, v) \in \mathbb{V}(N)$, the characteristic function v can be uniquely decomposed by means of nullified TU-games as follows:

$$v = \Delta_N(v)u_N + \sum_{\emptyset \subsetneq S \subsetneq N} (-1)^{s+1} v^S \tag{1}$$

In order to see this, observe that $u_N^S = \mathbf{0}$ for every $S \neq \emptyset$. So (1) is obvious for $v = u_N$. Next, for $S \subsetneq N$, we have $\Delta_N(u_S) = 0$ and, for every $T \neq \emptyset$,

$$u_S^T = \begin{cases} 0 & \text{if } S \cap T \neq \emptyset, \\ u_S & \text{if } S \cap T = \emptyset. \end{cases}$$

This implies that, for $v = u_S$, the right member in (1) simplifies to:

$$\sum_{\emptyset \subsetneq T \subseteq N\backslash S} (-1)^{t+1} u_S = -\left(\sum_{1 \leq t \leq n-s} (-1)^t \binom{n-s}{t} \right) u_S = u_S$$

proving formula (1) by linearity. Thirdly, remark that if (N, v) is a simple, monotone, superadditive, or convex TU-game, then so is (N, v^i) for all $i \in N$.

Next, we introduce variants of axioms operating on variable player sets and defined in section 2.3 (except Null player out) by using the TU-game in which a player is nullified instead of the subgame induced by the remaining players. In this sense, we rather keep this player in the TU-game, even though he/she is nullified, instead of removing him.

Balanced contributions under nullification, BCN. For all $(N, v) \in \mathbb{V}(N)$, all $i, j \in N$,

$$\varphi_i(N, v) - \varphi_i(N, v^j) = \varphi_j(N, v) - \varphi_j(N, v^i).$$

Balanced collective contributions under nullification, BCoCN. For all $(N, v) \in \mathbb{V}(N)$, all $i, j \in N$,

$$\frac{1}{n-1} \sum_{k \in N\backslash i} \Big(\varphi_k(N, v) - \varphi_k(N, v^i) \Big) = \frac{1}{n-1} \sum_{k \in N\backslash j} \Big(\varphi_k(N, v) - \varphi_k(N, v^j) \Big).$$

Balanced cycle contributions under nullification, BCyCN. For all $(N, v) \in \mathbb{V}(N)$, all ordering $(i_1, \ldots, i_n)$ of N,

$$\sum_{p=1}^{n} \Big(\varphi_{i_p}(N, v) - \varphi_{i_p}(N, v^{i_{p+1}}) \Big) = \sum_{p=1}^{n} \Big(\varphi_{i_p}(N, v) - \varphi_{i_p}(N, v^{i_{p-1}}) \Big)$$

where $i_0 = i_n$ and $i_{n+1} = i_1$.

Remark that the player set is now fixed in these axioms, so that we can consider the class of TU-games $\mathbb{V}(N)$ for some fixed player set $N \in U$.

Remark 1. *In our approach, the Null player out axiom plays a special role. To see this, note first that, for any $(N, v) \in \mathbb{V}(N)$ any $j \in N$, it holds that $(N\backslash j, v^j|_{N\backslash j}) = (N\backslash j, v|_{N\backslash j})$. So, for a value φ on $\mathbb{V}$ satisfying null player out and any $i \in N\backslash j$, it is easy to check that $\varphi_i(N, v^j) = \varphi_i(N\backslash j, v|_{N\backslash j})$. In a sense, this means that a player's nullification is equivalent to this player's removal on the payoff of the other players. More precisely, if a value φ defined on $\mathbb{V}$ satisfies Null player out, Balanced contributions is equivalent to Balanced contributions under nullification, Balanced cycle contributions is equivalent to Balanced cycle contributions under nullification, and Balanced collective contributions is equivalent to Balanced collective contributions under nullification. The latter remark assumes that such values exist (i.e. we assume that the considered value satisfies Null player out), and we show latter in this article that this is not always the case.*

Now, suppose that a value is characterized on $\mathbb{V}$ by one axiom relying on the removal of a player, denoted by A, in addition with other axioms which only involve TU-games with fixed player set. As a consequence of the previous remark, two questions can be addressed so far. If the value characterized on $\mathbb{V}$ satisfies Null player out, one may wonder whether replacing axiom A by its corresponding version under nullification will lead to an expanded set of values on $\mathbb{V}(N)$. Similarly, if the value characterized on $\mathbb{V}$ does not satisfy Null player out, one may wonder whether replacing axiom A by its corresponding version under nullification will give rise to new values on $\mathbb{V}(N)$.

Despite of Remark 1, the rest of this section will underline interesting differences with the existing literature, even for results in which the characterized value satisfies Null player out.

3.1. Balanced contributions under nullification

In order to state our first result, the following definition is needed. A **transfer scheme** on N is a vector $a \in \mathbb{R}^N$ such that $\sum_{i \in N} a_i \leq 0$. A transfer scheme a is **budget-balanced** if $\sum_{i \in N} a_i = 0$.

Proposition 4. *A value φ on $\mathbb{V}(N)$ satisfies Efficiency (**E**) and Balanced contributions under nullification (**BCN**) if and only if there exists a budget-balanced transfer scheme $a \in \mathbb{R}^N$ such that $\varphi_i = \mathrm{Sh}_i + a_i$, $i \in N$.*

Before proving Proposition 4, it makes sense to provide some comments. The family of values characterized by Proposition 4 is the sum of two parts. The first one is the classical Shapley value. It is endogenous in that it relies on the worths of the considered TU-game. The second one is a budget-balanced transfer scheme. It is exogenous in that it is independent of the characteristic functions (although it depends somehow on N). As such, this family belongs to the growing literature which tries to incorporates some solidarity aspects into cooperative TU-games. While the main stream of this literature studies (convex) combination between the Shapley value and a more egalitarian value, our family of values features another aspect of solidarity, which is purely exogenous. To see this, assume that each player $i \in N$ has an initial endowment $e_i \in \mathbb{R}_+$. For all $i \in N$, define a_i as the difference between the average endowment in the population and i's initial endowment, *i.e.* $a_i = \sum_{j \in N} e_j/n - e_i$. Then, transferring a_i to all players i is a mean to reduce inequalities in initial endowments. Many distributive issues involve situations of unequal endowments that call for such compensating transfers (see Fleurbaey and Maniquet, 2011, among others). Another variation on the Shapley value featuring weights is introduced by Kalai and Samet (1987). The so-called weighted Shapley values distribute to each player the dividends of the coalitions to which he/she belongs in proportion to his/her (positive) weight with respect to the total weight of the coalition. The main difference with our approach is that our weights a_i are simply added to the classical Shapley value, so that our value can be called an affine Shapley value.

Proof. (Proposition 4) For each budget-balanced transfer scheme $a \in \mathbb{R}^N$, define the value $\varphi^a = \mathrm{Sh} + a$.

Firstly, φ^a obviously satisfies **E** since $\sum_{i \in N} a_i = 0$. Since Sh satisfies **BC** and **NO**, by remark 1, it satisfies also **BCN** on $\mathbb{V}(N)$ for any $N \in U$. The constant value assigning the payoff vector a to each TU-game $(N, v) \in \mathbb{V}(N)$ trivially satisfies **BCN**, so that φ^a satisfies **BCN**.

Secondly, let $N \in U$ and $\varphi \in \mathbb{V}(N)$ satisfying **BCN** and **E**. The proof that $\varphi = \varphi^a$ for some $a \in \mathbb{R}^N$ is done by (descending) induction on the number of null players in a TU-game.

Initialization. If all players are null, *i.e.* in the null TU-game $(N, \mathbf{0})$, define $a := \varphi(N, \mathbf{0})$. Since φ satisfies **E**, we get $\sum_{i \in N} \varphi_i(N, \mathbf{0}) = 0$, and thus a is a budget-balanced transfer scheme. Furthermore, note that $\varphi^a(N, \mathbf{0}) = \mathrm{Sh}(N, \mathbf{0}) + a = a$ as Sh satisfies **NG**. Conclude that $\varphi(N, \mathbf{0}) = \varphi^a(N, \mathbf{0})$ for some budget-balanced transfer scheme $a \in \mathbb{R}^N$ as desired.

Induction hypothesis. Assume that $\varphi(N, v) = \varphi^a$ for some budget-balanced transfer scheme a for all TU-games $(N, v) \in \mathbb{V}(N)$ such that $k(v) \geq k$, $0 < k \leq n$.

Induction step. Choose any TU-game $(N, v) \in \mathbb{V}(N)$ such that $k(v) = k - 1$. Because $k(v) < n$, there exists $i \in N \backslash K(v)$ and $K(v^i) = K(v) \cup i$ so that $k(v^i) = k(v) + 1 = k$. For all $j \in K(v)$, as Sh satisfies **N**, $\mathrm{Sh}_j(N, v) = \mathrm{Sh}_j(N, v^i) = 0$. Moreover $v = v^j$, so that **BCN** and the induction hypothesis imply that

$$
\begin{aligned}
\varphi_j(N, v) &= \varphi_j(N, v^i) + \varphi_i(N, v) - \varphi_i(N, v^j) \\
&= \varphi_j(N, v^i) \\
&= \mathrm{Sh}_j(N, v^i) + a_j \\
&= \mathrm{Sh}_j(N, v) + a_j. \qquad (2)
\end{aligned}
$$

Conclude that the assertion is proved for null players in (N, v). Next, for $h \in N \backslash K(v^i)$, which may be an empty set in case $k(v) = n - 1$, using **BCN** for the Shapley value and the induction hypothesis, we can rewrite **BCN** as follows:

$$
\begin{aligned}
\varphi_h(N, v) &= \varphi_i(N, v) + \varphi_h(N, v^i) - \varphi_i(N, v^h) \\
&= \varphi_i(N, v) + \mathrm{Sh}_h(N, v^i) + a_h - \mathrm{Sh}_i(N, v^h) - a_i \\
&= \varphi_i(N, v) + \mathrm{Sh}_h(N, v) - \mathrm{Sh}_i(N, v) + a_h - a_i \qquad (3)
\end{aligned}
$$

Now **E** for φ gives:

$$
\varphi_i(N, v) + \sum_{j \in K(v)} \varphi_j(N, v) + \sum_{l \in N \backslash K(v^i)} \varphi_l(N, v) = v(N).
$$

Using (2) and (3) in the last equality yields:

$$\varphi_i(N,v) + \sum_{j \in K(v)} (a_j + \mathrm{Sh}_j(N,v)) + \sum_{l \in N \setminus K(v^i)} \Big(\varphi_i(N,v) + \mathrm{Sh}_l(N,v) - \mathrm{Sh}_i(N,v) + a_l - a_i \Big) = v(N).$$

Regrouping terms:

$$(n - k(v))(\varphi_i(N,v) - a_i - \mathrm{Sh}_i(N,v)) + \sum_{m \in N} (a_m + \mathrm{Sh}_m(N,v)) = v(N).$$

By **E** for Sh and $\sum_{i \in N} a_i = 0$ by the initial step, we obtain $(n-k(v))(\varphi_i(N,v)-a_i-\mathrm{Sh}_i(N,v)) = 0$. As $k(v) < n$ we have proved $\varphi_i(N,v) = \varphi_i^a$ for all non-null players $i \in (N,v)$ too. ■

The two axioms invoked in Proposition 4 are obviously logically independent. Adding the Null game axiom to them yields a characterization of the Shapley value.

Proposition 5. *A value φ on $\mathbb{V}(N)$ satisfies Efficiency (**E**), Balanced contributions under nullification (**BCN**), and the Null game axiom (**NG**) if and only if $\varphi = \mathrm{Sh}$.*

The proof is a corollary of Proposition 4. The axioms invoked in Proposition 5 are logically independent:

- The value φ^a, with $a_i \neq 0$ for some $i \in N$, defined in the proof of Proposition 4 satisfies all axioms except **NG**.
- The value φ such that $\varphi = 2\mathrm{Sh}$ satisfies all axioms except **E**.
- The ED-value satisfies all axioms except **BCN**.

Remark 2. *Balanced contributions under nullification is logically independent of Balanced contributions. On the one hand, Balanced contributions under nullification does not imply Balanced contributions by Propositions 1 and 4. On the other hand, in order to show that Balanced contributions does not imply Balanced contributions under nullification, consider the value φ on $\mathbb{V}$ such that*

$$\varphi_i(N,v) = \sum_{S \subseteq N : S \ni i} v(S) \qquad \text{for all } (N,v) \in \mathbb{V} \text{ and } i \in N.$$

For any two players $i, j \in N$, simple calculations show that

$$\varphi_i(N,v) - \varphi_i(N \setminus j, v|_{N \setminus j}) = \sum_{S \subseteq N : S \ni i,j} v(S),$$

and obviously equals $\varphi_j(N,v) - \varphi_j(N \setminus i, v|_{N \setminus i})$, while

$$\varphi_i(N,v) - \varphi_i(N,v^j) = \sum_{S \subseteq N : S \ni i,j} \Big(v(S) - v(S \setminus j) \Big),$$

which generically differs from $\varphi_j(N,v) - \varphi_j(N,v^i)$ unless, for instance, i and j are equals in (N,v).

3.2. Balanced collective contributions under nullification

The equal allocation of non-separable costs satisfies Balanced collective contributions but not its nullified version. In light of Remark 1, this also means that the equal allocation of non-separable costs violates Null player out. Furthermore, the next two results imply that this axiom is not strong enough to ensure the uniqueness of a value in presence of Efficiency. We start by an impossibility result if Null player is invoked in combination with Balanced collective contributions under nullification and Efficiency.

Proposition 6. *Fixed any $N \in U$ such that $n \geq 3$. There exists no value on $\mathbb{V}(N)$ satisfying Efficiency (**E**), Balanced collective contributions under nullification (**BCoCN**), and Null player (**N**).*

Proof. Consider any value φ on $\mathbb{V}(N)$ satisfying **E**, **BCoCN** and **N**. By **BCoCN**, for a given $(N, v) \in \mathbb{V}(N)$, the following sum does not depend on i and may be rewritten:

$$\sum_{k \in N \backslash i} \Big(\varphi_k(N, v) - \varphi_k(N, v^i) \Big) = \sum_{k \in N} \varphi_k(N, v) \; - \; \sum_{k \in N} \varphi_k(N, v^i) \; + \; \varphi_i(N, v^i) \; - \; \varphi_i(N, v).$$

By **N**, we have $\varphi_i(N, v^i) = 0$. Together with **E**, we can simplify the previous sum as $v(N) - v(N \backslash i) - \varphi_i(N, v)$ with $v^i(N) = v(N \backslash i)$. This last expression should not depend on i. Hence, it is equal to its average on N: $v(N) - (\sum_{j \in N}(v(N \backslash j)) + v(N))/n$. It follows that:

$$\varphi_i(N, v) = \frac{1}{n} \Big(v(N) + \sum_{j \in N} v(N \backslash j) \Big) - v(N \backslash i) = \mathrm{EANSC}_i(N, v).$$

Since EANSC does not satisfy **N**, we get the desired contradiction. ■

So, Null player is a too strong requirement in combination with Efficiency and Balanced collective contributions under nullification. In order to avoid this problem, we replace in Proposition 6 the Null player axiom by Equal treatment (*i.e* null player are still treated equally, but can obtain non-null payoffs). This leads to the characterization of a new value expressed in a closed form expression.

Proposition 7. *There is a unique value on $\mathbb{V}(N)$ that satisfies Efficiency (**E**), Equal treatment (**ET**) and Balanced collective contributions under nullification (**BCoCN**) which is given by*

$$\mathrm{SV}_i(N, v) = v(N) - \frac{n-1}{n} \Delta_N(v) - \sum_{S \not\ni i} \frac{n-1}{n-s} \Delta_S(v) \qquad \text{for all } (N, v) \in \mathbb{V}(N) \text{ and } i \in N. \tag{4}$$

Proof. Firstly, we prove uniqueness by (descending) induction on the number of null players in a TU-game. Let φ be a value on $\mathbb{V}(N)$ satisfying the three aforementioned axioms.
Initialization. If all players are null (and so equals), *i.e.* in the null TU-game $(N, \mathbf{0})$, **ET** and **E** implies $\varphi_i(N, \mathbf{0}) = 0$.
Induction hypothesis. Assume that $\varphi(N, v)$ is uniquely determined for all TU-games $(N, v) \in \mathbb{V}(N)$ such that $k(v) \geq k$, $0 < k \leq n$.

Induction step. Choose any TU-game $(N,v) \in \mathbb{V}(N)$ such that $k(v) = k-1$. For all $i \in N$, **E** gives:

$$\sum_{k \in N\backslash i} \Big(\varphi_k(N,v) - \varphi_k(N,v^i)\Big) = v(N) - \varphi_i(N,v) - v(N\backslash i) + \varphi_i(N,v^i).$$

Now **BCoCN** imposes that this last quantity should not depend on $i \in N$. If i is a null player in (N,v), then $v = v^i$ and $v(N) = v(N\backslash i)$, so that this quantity vanishes. Two cases are to be distinguished.

For $k(v) \geq 1$, there exists at least one null player $h \in K(v)$ and we get in particular for all non-null players $i \in N\backslash K(v)$ that $\varphi_i(N,v) = v(N) - v(N\backslash i) + \varphi_i(N,v^i)$ which is uniquely determined because $k(v^i) = k(v) + 1 = k$ and the induction hypothesis. Then **ET** applied to null players (which are equals) and **E** allows to complete the proof of uniqueness: for $h \in K(v)$, we have $\varphi_h(N,v) = (v(N) - \sum_{i \notin K(v)} \varphi_i(N,v))/k(v)$.

For $k(v) = 0$, **BCoCN** and **E** can be used to generate a system of n linearly independent equations involving $\varphi_i(N,v)$ as the unknown variables to be expressed in terms of v and also $\varphi_i(N,v^i)$ which are determined, by induction hypothesis and $k(v^i) = k(v) + 1 = k$ for all $i \in N$. For instance, for an ordered set N:

$$\begin{cases} \varphi_1(N,v) - \varphi_2(N,v) & & = v(N\backslash 2) - v(N\backslash 1) + \varphi_1(N,v^1) - \varphi_2(N,v^2) \\ \quad \vdots \qquad\qquad \ddots & & = \qquad \cdots \\ \varphi_1(N,v) & - \varphi_n(N,v) & = v(N\backslash n) - v(N\backslash 1) + \varphi_1(N,v^1) - \varphi_n(N,v^n) \\ \varphi_1(N,v) + \quad \cdots & + \varphi_n(N,v) & = v(N) \end{cases}$$

This implies that in this case too $\varphi_i(N,v)$ is uniquely determined (if it exists).

Secondly, we prove that SV satisfies **E**, **ET** and **BCoCN**. First notice that SV satisfies **L** so that we may use the unanimity TU-games basis for the proof. For each $N \in U$ and any nonempty $S \in 2^N$, denote by (N, δ_S) the Dirac TU-game induced by S, *i.e.* $\delta_S(T) = 1$ if $T = S$ and $\delta_S(T) = 0$ otherwise. For any nonempty $T \subseteq N$, we have:

$$\sum_{i \in N} \mathrm{SV}_i(N,u_T) = n - (n-1)\delta_N(T) - \sum_{S \subsetneq N} \sum_{i \in N\backslash S} \frac{n-1}{n-s}\delta_S(T) = n - (n-1) = 1,$$

so that **E** is proved by linearity. Next, if $i \in N$ and $j \in N$ are equal in a TU-game $(N,v) \in \mathbb{V}(N)$, recall that for all $S \subseteq N\backslash\{i,j\}$, we have $\Delta_{S\cup i}(v) = \Delta_{S \cup j}(v)$. A straight computation gives:

$$\begin{aligned} \mathrm{SV}_i(N,v) - \mathrm{SV}_j(N,v) &= \sum_{S \not\ni j} \frac{n-1}{n-s}\Delta_S(v) - \sum_{S \not\ni i} \frac{n-1}{n-s}\Delta_S(v) \\ &= \sum_{S \not\ni j, S \ni i} \frac{n-1}{n-s}\Delta_S(v) - \sum_{S \not\ni i, S \ni j} \frac{n-1}{n-s}\Delta_S(v) \\ &= \sum_{S \not\ni i,j} \frac{n-1}{n-(s+1)}\Delta_{S\cup i}(v) - \sum_{S \not\ni i,j} \frac{n-1}{n-(s+1)}\Delta_{S \cup j}(v) \\ &= 0. \end{aligned}$$

As in the proof of Proposition 6, the equality defining **BCoCN** is simplified by using **E**. Precisely, for a value φ on $\mathbb{V}(N)$ satisfying **E**, φ satisfies **BCoCN** if and only if for all $(N,v) \in \mathbb{V}(N)$,

$-v(N) + v(N\backslash i) + \varphi_i(N,v) - \varphi_i(N,v^i)$ does not depend on $i \in N$. Summing on all $i \in N$ and using **E** of φ in both (N,v) and (N,v^i), we can write that:

$$v^i(N) + \varphi_i(N,v) - \varphi_i(N,v^i) = \sum_{j \neq i} \Big(\varphi_j(N,v^i) - \varphi_j(N,v) \Big) + v(N).$$

Now we know that SV satisfies **E** so it remains to compute the left hand quantity. Since $\Delta_S(v^i) = 0$ for $i \in S$ and $\Delta_S(v^i) = \Delta_S(v)$ otherwise, we get:

$$\begin{aligned} v(N\backslash i) + \mathrm{SV}_i(N,v) - \mathrm{SV}_i(N,v^i) &= v(N) - \frac{n-1}{n}\Delta_N(v) - \sum_{S \not\ni i} \frac{n-1}{n-s}\Delta_S(v) \\ &\quad + \frac{n-1}{n}\Delta_N(v^i) + \sum_{S \not\ni i} \frac{n-1}{n-s}\Delta_S(v^i) \\ &= v(N) - \frac{n-1}{n}\Delta_N(v). \end{aligned}$$

This last expression does not depend on i so we proved that **BCoCN** is satisfied by SV. ■

The axioms in Proposition 7 are logically independent:

- The value given by $\varphi = 2\mathrm{SV}$ satisfies all axioms except **E**.
- The value given by $\varphi = \mathrm{Sh}$ satisfies all axioms except **BCoCN**.
- The value given by:

$$\widehat{\mathrm{SV}}_i(N,v) = v(N) - \frac{n-1}{n}\Delta_N(v) - \sum_{S \not\ni i} \frac{(n-1)i}{\sum_{j \notin S} j}\Delta_S(v)$$

  satisfies all axioms except **ET**. Indeed, $\widehat{\mathrm{SV}}$ is also linear so that the proof of **E** and **BCoCN** are the same. Note that for two different players $i, j \in N$, if $p \in N\backslash\{i,j\}$, i and j are equals (as null players) in u_p. Now we have:

$$\begin{aligned} \widehat{\mathrm{SV}}_i(N,u_p) - \widehat{\mathrm{SV}}_j(N,u_p) &= \sum_{S \not\ni j} \frac{(n-1)j}{\sum_{k \notin S} k}\delta_S(\{p\}) - \sum_{S \not\ni i} \frac{(n-1)i}{\sum_{k \notin S} k}\delta_S(\{p\}) \\ &= \frac{n-1}{\sum_{k \neq p} k}(j-i) \end{aligned} \tag{5}$$

Let us conclude this paragraph with four remarks. Firstly, we only needed Equal treatment for null players to prove the uniqueness of the proof of Proposition 7. Secondly, SV satisfies Efficiency, Balanced collective contributions under nullification and the Null game axiom, which enables a comparison with Proposition 6. Thirdly, SV can be interpreted, in particular in the unanimity TU-games. In (N,u_N), all players get $1/n$ as it is the case for all values satisfies Efficiency and Equal treatment. For any nonempty $S \subsetneq N$, SV assigns in (N,u_S) a payoff of 1 to each player in S and so each other player get a payoff of $(1-s)/(n-s)$. These payoffs can be obtained from a two-stage procedure. In the first step, each player receives a payoff equal to the worth generated by the cooperation of all members of N, *i.e.* one unit. Due to the efficiency constraint, this means that a total amount of $n-1$ units has to be funded. The principle of the value SV is that this amount is exclusively funded by the unproductive players, *i.e.* those in $N \backslash S$. Each of them eventually pays $-(s-1)/(n-s)$. Note that the fraction $(s-1)/(n-s)$ is increasing in s, which means that the less the number of unproductive players, the more each has to pay. This interpretation also implies that SV can be seen as an extremely marginalistic value: it amplifies/exaggerates the consideration of the contributions of the players to coalition. In other words, SV implements a kind of elitism since the productive players receive an even better treatment than in the Shapley value, at the expense of the unproductive players who receive a worse treatment than in the Shapley value. Fourthly, it is worth to note that Balanced collective contributions under nullification generates marginalistic effects through SV, which are the opposite of the more egalitarian results produced by Balanced collective contributions through the equal allocation of non-separable costs (see Béal et al., 2014b).

3.3. Balanced cycle contributions under nullification

Similarly as in Kamijo and Kongo (2012), we begin by underlying that any linear and symmetric value satisfies Balanced cycle contributions under nullification.

Proposition 8. *For $n \geq 3$, if a value φ defined on $\mathbb{V}(N)$ satisfies Linearity (**L**) and Symmetry (**S**), then it also satisfies Balanced cycle contributions under nullification (**BCyCN**).*

The proof is similar to those in Kamijo and Kongo (2012) and is omitted. By remark 1, the characterization of the Shapley value in Kamijo and Kongo (2010, Proposition 1) by Efficiency, Null player out and Balanced cycle contributions is still valid if the latter axiom is replaced by its nullified counterpart. However, such a result is not in the spirit of our article where we work on a class of TU-game with a fixed player set. As a consequence, in order to cope with this constraint, a first attempt is to replace null player out by null player. We obtain the following result, which makes use of the class of all TU-games with player set N containing at least one null player.

Proposition 9. *For $n \geq 3$, a value φ on $\mathbb{V}^0(N)$ satisfies Efficiency (**E**), Balanced cycle contributions under nullification (**BCyCN**), and the Null player axiom (**N**) if and only if $\varphi = \mathrm{Sh}$.*

Proof. First Sh satisfies **E** and **N** on $\mathbb{V}^0(N)$. Since Sh also satisfies **S** and **L**, it satisfies **BCyCN** by Proposition 8.

Now, let φ be a value on $\mathbb{V}^0(N)$ satisfying **E**, **BCyCN**, and **N**. The proof that $\varphi = \mathrm{Sh}$ is done by (descending) induction on the number $k(v)$ of null players in a TU-game $(N, v) \in \mathbb{V}^0(N)$.
Initialization If all players are null, *i.e.* in the null TU-game $(N, \mathbf{0})$, we directly get $\varphi_i(N, \mathbf{0}) = 0 = \mathrm{Sh}_i(N, \mathbf{0})$ by **N**. If all players except one are null in v, *i.e.* if $K(v) = N \backslash i$, **N** and **E** directly lead to $\varphi_j(N, v) = 0 = \mathrm{Sh}_j(N, v)$ for all $j \in K(v)$ and $\varphi_i(N, v) = v(N) = \mathrm{Sh}_i(N, v)$.
Induction hypothesis. Assume that $\varphi(N, v) = \mathrm{Sh}(N, v)$ for all TU-games $(N, v) \in \mathbb{V}(N)$ such that $k(v) \geq k$, $1 < k \leq n-1$.
Induction step. Choose any TU-game $(N, v) \in \mathbb{V}^0(N)$ such that $k(v) = k-1 > 0$. Because $k(v) < n-1$, there exists at least two different non null players $i, j \in N \backslash K(v)$ and $k(v^i) = k(v^j) = k(v) + 1 = k$. By induction hypothesis, $\varphi(N, v^i) = \mathrm{Sh}(N, v^i)$ and $\varphi(N, v^j) = \mathrm{Sh}(N, v^j)$. Since $(N, v) \in \mathbb{V}^0(N)$, there exists $h \in K(v)$. Similarly as in Kamijo and Kongo (2010), **BCyCN** is equivalent to the axiom of Balanced 3-cycle contributions under nullification (*i.e.* when only cycles of length 3 are considered).[2] Therefore, $\varphi_i(N, v^j) + \varphi_j(N, v^h) + \varphi_h(N, v^i) = \varphi_i(N, v^h) +$

[2] The proof of this statement and the proof of Proposition 8 are available upon request.

$\varphi_j(N, v^i)+\varphi_h(N, v^j)$ which simplifies to $\varphi_j(N, v)-\varphi_i(N, v) = \mathrm{Sh}_j(N, v^i)-\mathrm{Sh}_i(N, v^j)$ by noting that $(N, v^h) = (N, v)$ since h is a null player. Now Sh satisfies **BCN** and we get $\varphi_j(N, v) - \varphi_i(N, v) = \mathrm{Sh}_j(N, v) - \mathrm{Sh}_i(N, v)$. Summing this equality for all $i \in N\backslash K(v)$ leads to:

$$\begin{aligned}(n-k(v))\Big(\varphi_j(N,v)-\mathrm{Sh}_j(N,v)\Big) &= \sum_{i\in N\backslash K(v)}\Big(\varphi_i(N,v)-\mathrm{Sh}_i(N,v)\Big)\\ &\overset{\mathbf{N}}{=} \sum_{i\in N}\Big(\varphi_i(N,v)-\mathrm{Sh}_i(N,v)\Big)\\ &\overset{\mathbf{E}}{=} 0. \qquad (6)\end{aligned}$$

So we have $\varphi_j(N, v) = \mathrm{Sh}_j(N, v)$ for non null players and, by **N** for null players too. ■

The axioms in Proposition 9 are logically independent:

- The Banzhaf value (Banzhaf, 1965) satisfies all axioms except **E**.
- The ED-value satisfies all axioms except **N**.
- The value given by $\varphi_i(N, v) = v(\{1, \ldots, i\}) - v(\{1, \ldots, i-1\})$ for all $(N, v) \in \mathbb{V}^0(N)$ and $i \in N$ satisfies all axioms except **BCyCN**.

This result in Proposition 9 is partial since it does not deal with TU-games having no null players. There are two reasons for that. Firstly, Balanced cycle contributions under nullification and null player axiom have no implication when applied to such TU-games. For a TU-game (N, v) without null players, the payoffs in (N, v) cancel since they appear in both sides of the formula of Balanced cycle contributions under nullification. So, for such a TU-game and all ordering $(i_1, \ldots, i_n)$ of N, the axiom reduces to:

$$\sum_{p=1}^{n} \varphi_{i_p}(N, v^{i_{p+1}}) = \sum_{p=1}^{n} \varphi_{i_p}(N, v^{i_{p-1}}).$$

where $i_0 = i_n$ and $i_{n+1} = i_1$. Since all involved TU-games contain one null player, the axiom is silent on the original TU-game (N, v). This is no longer the case when (N, v) possesses a null player $i \in N$ since $v = v^i$ enables to retain the original TU-game (N, v) in some part of the axiom. Secondly, Kamijo and Kongo (2010, 2012) use the elevator principle: starting from a TU-game with n players, they construct a TU-game with $n + 1$ players by adding a new null player, and then they come back to TU-games with n players by removing a player through the operation in Balanced cycle contributions. We cannot proceed in this fashion since the class of TU-games under consideration in this article has a fixed player set. More complete characterizations can be obtained at the cost of adding extra axioms. We present below two ways to do so. It is also interesting to remark that the set $\mathbb{V}^0(N)$ has an empty interior with respect to the natural topology on $\mathbb{R}^{2^n-1}$ and so is a measure-zero set for any density measure on $\mathbb{V}(N)$.

Proposition 10. *For $n \geq 3$, a value φ on $\mathbb{V}(N)$ satisfies Efficiency (**E**), Balanced cycle contributions under nullification (**BCyCN**), the Null player axiom (**N**), and Linearity (**L**) if and only if there exists a budget-balanced transfer scheme $a \in \mathbb{R}^N$ such that for $(N, v) \in \mathbb{V}(N), \varphi_i(N, v) = \mathrm{Sh}_i(N, v) + a_i\Delta_N(v)$, $i \in N$.*

Proof. Firstly, for every budget-balanced transfer scheme $a \in \mathbb{R}^N$, the value $\mathrm{Sh} + a\Delta_N$ obviously satisfies **L**. Since it coincides with Sh on $\mathbb{V}^0(N)$, it also satisfies **N**, and **BCyCN** on $\mathbb{V}^0(N)$ by Proposition 9 and the fact that $\Delta_N(v) = 0$ for all $(N, v) \in \mathbb{V}^0(N)$. Furthermore, by the remark preceding Proposition 10, **BCyCN** has no implication on $\mathbb{V}(N)\backslash\mathbb{V}^0(N)$, which means that $\mathrm{Sh} + a\Delta_N$ satisfies **BCyCN** on $\mathbb{V}(N)$. Finally, the value satisfies **E** since $\Delta_N(v)\sum_{i\in N} a_i = 0$.

Secondly, let φ be a value on $\mathbb{V}(N)$ satisfying the four aforementioned axioms. By Proposition 9, φ coincides with Sh on $\mathbb{V}^0(N)$. Define $a_i = \varphi_i(N, u_N) - \mathrm{Sh}_i(N, u_N)$. By **E**, we get $\sum_{i\in N} a_i = 0$ and, with the help of (1), for any $(N, v) \in \mathbb{V}(N)$, it holds that:

$$\begin{aligned}
\varphi_i(N,v) &\overset{(1)}{=} \varphi_i\Bigg(N, \Delta_N(v)u_N + \sum_{\emptyset\subsetneq S\subsetneq N} (-1)^{s+1} v^S\Bigg) \\
&\overset{\mathbf{L}}{=} \Delta_N(v)\varphi_i(N,u_N) + \sum_{\emptyset\subsetneq S\subsetneq N} (-1)^{s+1}\varphi_i\big(N, v^S\big) \\
&\overset{\text{Prop. 9}}{=} \Delta_N(v)\varphi_i(N,u_N) + \sum_{\emptyset\subsetneq S\subsetneq N} (-1)^{s+1}\mathrm{Sh}_i\big(N, v^S\big) \\
&\overset{\mathbf{L}}{=} \Delta_N(v)\big(\varphi_i(N,u_N) - \mathrm{Sh}_i(N,u_N)\big) + \mathrm{Sh}_i\Bigg(N, \Delta_N(v)u_N + \sum_{\emptyset\subsetneq S\subsetneq N} (-1)^{s+1} v^S\Bigg) \\
&\overset{(1)}{=} a_i\Delta_N(v) + \mathrm{Sh}_i(N,v),
\end{aligned}$$

which completes the proof. ∎

The axioms in Proposition 10 are logically independent:

- The value given by $\varphi = 2\mathrm{Sh}$ satisfies all axioms except **E**.
- The ED-value satisfies all axioms except **N**.
- The value characterized in Proposition 11 satisfies all axioms except **L**.
- The value given by $\varphi_i(N, v) = v(\{1, \ldots, i\}) - v(\{1, \ldots, i-1\})$ for all $(N, v) \in \mathbb{V}(N)$ and $i \in N$ satisfies all axioms except **BCyCN**.

Proposition 10 relies on the fact that any characteristic function can be decomposed into unanimity TU-games, all of which contain null players except the unanimity TU-game on the grand coalition. This result is comparable to Proposition 4, with the notable difference that the part including exogenous coefficients is independent of v in Proposition 4, while it depends on v through the Harsanyi dividend of the grand coalition in Proposition 10. In relevant classes of TU-games (see Maniquet, 2003, for instance), the Harsanyi dividend of the grand coalition is null, which means that Proposition 10 characterizes the Shapley value, provided that the other axioms are valid on the class under consideration.

Proposition 11. *For $n \geq 3$, a value φ on $\mathbb{V}(N)$ satisfies Efficiency (**E**), Balanced cycle contributions under nullification (**BCyCN**), the Null player axiom (**N**), and Balanced collective contributions under nullification on TU-games without null players (**BCoCN***) if and only if $\varphi(N, v) = \mathrm{EANSC}(N, v)$ if $(N, v) \in \mathbb{V}(N)\backslash\mathbb{V}^0(N)$ and $\varphi(N, v) = \mathrm{Sh}(N, v)$ if $(N, v) \in \mathbb{V}^0(N)$.*

Proof. Firstly, the aforementioned value coincides with Sh on $\mathbb{V}^0(N)$, inherits Efficiency on $\mathbb{V}(N)$ from Sh and EANSC by Propositions 1 and 3. **BCyCN** and **N** are satisfied on $\mathbb{V}(N)\backslash\mathbb{V}^0(N)$ by the remark preceding Proposition 10, so that it only remains to prove that it satisfies **BCoCN***. Consider any $(N, v) \in \mathbb{V}(N)\backslash\mathbb{V}^0(N)$, and any $i \in N$, we have:

$$\begin{aligned}\sum_{k\neq i}\Big(\mathrm{EANSC}_k(N,v) - \mathrm{Sh}_k(N,v^i)\Big) &\overset{\mathbf{E}}{=} v(N) - \mathrm{EANSC}_i(N,v) - v(N\backslash i) + \mathrm{Sh}_i(N,v^i)\\ &\overset{\mathbf{N}}{=} -\frac{1}{n}\Big(v(N) - \sum_{j\in N}\Big(v(N) - v(N\backslash j)\Big)\Big)\end{aligned}$$

This last quantity does not depend on $i \in N$ so **BCoCN*** is fulfilled.

Secondly, let φ be a value on $\mathbb{V}(N)$ satisfying the four aforementioned axioms. By Proposition 9, φ coincides with Sh on $\mathbb{V}^0(N)$. Next, for any $(N, v) \in \mathbb{V}(N)\backslash\mathbb{V}^0(N)$, **BCoCN*** imposes that the following quantity is independent of $i \in N$, and in turn equal to its average on N:

$$\begin{aligned}\sum_{k\neq i}\Big(\varphi_k(N,v) - \varphi_k(N,v^i)\Big) &\overset{\mathbf{E}}{=} v(N) - \varphi_i(N,v) - v(N\backslash i) + \mathrm{Sh}_i(N,v^i)\\ &\overset{\mathbf{N}}{=} v(N) - \varphi_i(N,v) - v(N\backslash i)\\ &\overset{\textbf{average}}{=} v(N) - \frac{v(N)}{n} - \frac{1}{n}\Big(\sum_{j\in N} v(N\backslash j)\Big)\end{aligned}$$

This last two equalities yield:

$$\varphi_i(N,v) = \frac{1}{n}\Big(v(N) + \sum_{j\in N} v(N\backslash j)\Big) - v(N\backslash i) = \mathrm{EANSC}_i(N,v),$$

as desired. ■

The axioms in Proposition 11 are logically independent:

- The null value satisfies all axioms except **E**.
- The linear and symmetric value given by $\varphi = \mathrm{SV}$ satisfies all axioms except **N** by Propositions 7 and 6.
- The Shapley value satisfies all axioms except **BCoCN***.
- For $i \in N$, the value given by $\varphi_i(N, v) = v(\{1, \ldots, i\}) - v(\{1, \ldots, i-1\})$ for all $(N, v) \in \mathbb{V}^0(N)$ and $\varphi(N, v) = \mathrm{EANSC}(N, v)$ if $(N, v) \in \mathbb{V}(N)\backslash\mathbb{V}^0(N)$ satisfies all axioms except **BCyCN**.

Proposition 11 calls upon several comments. Firstly, it enables a comparison with the impossibility result in Proposition 6. Indeed, there is no value satisfying Efficiency, Null player and Balanced collective contributions under nullification. By Proposition 11, this is no longer the case if Balanced collective contributions under nullification is only required on the class of TU-games containing no null players. Secondly, the value SV characterized in Proposition 7 by Efficiency, Equal treatment and Balanced collective contributions under nullification does not coincide with the equal allocation of non-separable costs obtained in Béal et al. (2014b) if Balanced collective contributions under nullification is replaced by Balanced collective contributions. This difference is reduced if Balanced collective contributions under nullification is required only on the class of TU-games containing no null players since the equal allocation of non-separable costs takes part of the value characterized in Proposition 11. Thirdly, Proposition 11 illustrates the fact that our relational axioms do not automatically lead to linear values. In particular, this result highlights a non continuous switch in the allocation process depending on the composition of the player set.

Regarding the last two propositions, which extend the Shapley value differently on $\mathbb{V}(N)\backslash\mathbb{V}^0(N)$, it is interesting to note that both characterized values satisfy Efficiency and Null player but differ on the two remaining axioms involved in the standard Shapley's characterization: the value in Proposition 10 satisfies Linearity but not Equal treatment, while the value in Proposition 11 satisfies Equal treatment but not Linearity. Another rather trivial extension of Proposition 9 is to impose Balanced contributions under nullification on TU-games without null players (**BCN***) which, together with Efficiency, Balanced cycle contributions under nullification and Null player, characterizes the Shapley value on $\mathbb{V}(N)$.

4. Revisiting the potential approach

Following Hart and Mas-Colell (1989), a function $P : \mathbb{V} \longrightarrow \mathbb{R}$ is called a **potential** if $P(\emptyset, v) = 0$ and, for all $(N, v) \in \mathbb{V}$,

$$\sum_{i\in N} \big(P(N, v) - P(N\backslash i, v|_{N\backslash i})\big) = v(N).$$

This condition means that the sum of the marginal contributions of the players in N with respect to P add up to the worth of grand coalition.

Proposition 12. *(Hart and Mas-Colell, 1989) There exists a unique potential function P. For all $(N, v) \in \mathbb{V}$ and $i \in N$, it is given by:*

$$P(N, v) - P(N\backslash i, v|_{N\backslash i}) = \mathrm{Sh}_i(N, v), \tag{7}$$

and thus

$$P(N, v) = \sum_{S\subseteq N} \frac{(s-1)!(n-s)!}{n!} v(S).$$

As in section 3, we substitute the TU-game in which a player is nullified for the subgame induced by the leave of this player. Formally, a **nullified potential** on N is a function $Q : \mathbb{V}(N) \longrightarrow \mathbb{R}$ such that $Q(N, \mathbf{0}) = 0$ and

$$\sum_{i\in N} \Big(Q(N, v) - Q(N, v^i)\Big) = v(N).$$

Proposition 13. *There exists a unique nullified potential function Q. For all $(N, v) \in \mathbb{V}(N)$ and $i \in N$, it holds that $Q(N, v) - Q(N, v^i) = \mathrm{Sh}_i(N, v)$. Furthermore, $Q(N, v) = P(N, v)$.*

Proof. Firstly, we recall that the potential P satisfies:

$$P(N, v^i) = P(N\backslash i, v^i|_{N\backslash i}) + \mathrm{Sh}_i(v^i) = P(N\backslash i, v^i|_{N\backslash i}) = P(N\backslash i, v|_{N\backslash i}). \tag{8}$$

Again, $Q(N,v) = P(N,v)$ will be proved by descending induction on the number $k(v)$ of null players in (N,v).
INITIALIZATION. For $k(v) = n$, clearly $Q(N,\mathbf{0}) = 0 = P(N,\mathbf{0})$.
INDUCTION HYPOTHESIS. Assume that the result holds for all TU-games $(N,v) \in \mathbb{V}(N)$ such that $k(v) \geq k$, for any $0 < k \leq n$.
INDUCTION STEP. Consider any $(N,v) \in \mathbb{V}(N)$ such that $k(v) = k-1$. It holds that $k(v) < n$, so there exists at least one player $i \in N \backslash K(v)$. For any such non null player i, by the induction hypothesis we have $Q(N,v^i) = P(N,v^i)$. For any null player $j \in K(v)$, $v = v^j$ so $Q(N,v^j) = Q(N,v)$. The definition of Q and the two previous remarks then imply:

$$nQ(N,v) = v(N) + \sum_{i \in N \backslash K(v)} P(N,v^i) + \sum_{j \in K(v)} Q(N,v).$$

Hence, using **E** and **N** for Sh, we have:

$$\begin{aligned}
(n-k(v))Q(N,v) &= v(N) + \sum_{i \in N \backslash K(v)} P(N,v^i) \\
&\overset{\mathbf{E}}{=} \sum_{i \in N} \mathrm{Sh}_i(N,v) + \sum_{i \in N \backslash K(v)} P(N,v^i) \\
&\overset{\mathbf{N}}{=} \sum_{i \in N \backslash K(v)} \mathrm{Sh}_i(N,v) + \sum_{i \in N \backslash K(v)} P(N,v^i) \\
&= \sum_{i \in N \backslash K(v)} \big(\mathrm{Sh}_i(N,v) + P(N,v^i)\big) \\
&\overset{(8)}{=} \sum_{i \in N \backslash K(v)} \big(\mathrm{Sh}_i(N,v) + P(N \backslash i, v|_{N \backslash i})\big) \\
&\overset{(7)}{=} \sum_{i \in N \backslash K(v)} P(N,v) \\
&= (n-k(v))P(N,v)
\end{aligned}$$

This completes the proof. ■

Proposition 13 was expected because of Proposition 5 and section 3 in Hart and Mas-Colell (1989) in which an equivalence between the potential approach and the so-called notion of preservation of differences is established, and linked to the axiom of Balanced contributions. In our framework the only novelty is that the Null game axiom is required in addition to Balanced contribution under nullification and Efficiency is order to single out the Shapley value. This axiom somehow appears in the condition that $Q(N,\mathbf{0}) = 0$, albeit in a different form.

Our variation on the potential approach is also useful to provide a recursive formula of the Shapley value on a class of TU-games with a fixed player set. More specifically, from Hart and Mas-Colell (1989), we now that for any $(N,v) \in \mathbb{V}$ and any $i \in N$,

$$\mathrm{Sh}_i(N,v) = \frac{1}{n}\big(v(N) - v(N \backslash i)\big) + \frac{1}{n} \sum_{j \in N \backslash i} \mathrm{Sh}_i(N \backslash j, v|_{N \backslash j}).$$

Since the Shapley value satisfies Null player out, by Remark 1, for all $(N,v) \in \mathbb{V}$, all $j \in N$ and all $i \in N\backslash j$, it holds that $\mathrm{Sh}_i(N,v^j) = \mathrm{Sh}_i(N\backslash j, v|_{N\backslash j})$. Thus, the previous expression can be rewritten as

$$\mathrm{Sh}_i(N,v) = \frac{1}{n}\big(v(N) - v^i(N)\big) + \frac{1}{n}\sum_{j\in N\backslash i} \mathrm{Sh}_i(N,v^j).$$

5. Conclusion

Our article opens the ground for an extension of the nullification approach to the class of TU-games augmented by a graph. For such TU-games, many axioms are based on deleting a link from a graph instead of removing a player. The axioms of fairness (Myerson, 1977) and component fairness (Herings et al., 2008) are two well-known examples. Therefore, it would make sense to explore the nullification of a link in a similar way as the nullification of a player studied in this article. This extension is left for a future work.

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