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April 2017

Working paper No. 2017 –8

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Coalitional desirability and the equal division value☆

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Abstract

We introduce three natural collective variants of the well-known axiom of Desirability (Maschler and Peleg, 1966), which require that if the (per capita) contributions of a first coalition are at least as large as the (per capita) contributions of a second coalition, then the (average) payoff in the first coalition should be as large as the (average) payoff in the second coalition. These axioms are called Coalitional desirability and Average coalitional desirability. The third variant, called Uniform coalitional desirability applies only to coalitions with the same size. We show that Coalitional desirability is very strong: no value satisfies simultaneously this axiom and Efficiency. To the contrary, the combination of either Average coalitional desirability or Uniform coalitional desirability with Efficiency and Additivity characterizes the Equal Division value.

Keywords: Desirability, Coalitional desirability, Average coalitional desirability, Uniform coalitional desirability, Equal Division value, Shapley value.

1. Introduction

van den Brink (2007) provides a clarifying axiomatic comparison between the Equal Division value and the Shapley value (Shapley, 1953) for cooperative games with transferable utility (simply games henceforth). He replaces the Null player axiom invoked in the classical characterization of the Shapley value, which imposes a null payoff to a player who contributes nothing to coalitions, by the Nullifying player axiom, which imposes a null payoff to a player whose coalitions have a null worth. The first axiom is defined from the marginal contributions of a player, while the second is not. The marginal contributions of players to coalitions have been at the heart of many concepts in cooperative game theory. They are the corner stone of the definitions of the Shapley value (Shapley, 1953) and the Banzhaf value (Banzhaf, 1965). Both values are obtained by averaging, in a certain sense, the players' marginal contributions. The marginal contributions are also used to

☆Financial support from research programs "DynaMITE: Dynamic Matching and Interactions: Theory and Experiments", contract ANR-13-BSHS1-0010, and "Mathématiques de la décision pour l'ingénierie physique et sociale" (MODMAD) is gratefully acknowledged.

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define popular axioms such as the axiom of Equal treatment of equals: two players with the same marginal contributions should end up with the same payoffs. Another stronger axiom is the axiom of Desirability introduced in Maschler and Peleg (1966). If a player enjoys marginal contributions that are at least as large as the marginal contributions of another player, then the first player should obtain a payoff at least as large as the second player's payoff. This desirability relation among the players originates from Isbell (1958) and has been studied extensively in order to evaluate the influence of voters on the class of simple games (see also Courtin and Tchantcho, 2015; Molinero et al., 2015, among others). The axiom of Desirability is often invoked in the characterization of classes of values such as the two classes of equal sharing values (van den Brink and Funaki, 2009; van den Brink et al., 2016), the procedural values (Malawski, 2013), the egalitarian Shapley values (Casajus and Huettner, 2013), a class of solidarity values (Béal et al., 2017) or to delimit subclasses of the linear, efficient and symmetric values (Levínský and Silársky, 2004; Radzik and Driessen, 2013).

This note aims to emphasize the difference between the Shapley value and the Equal Division value by taking a route alternative to van den Brink (2007), which is inspired by the desirability relation. We investigate to what extent the axiom of Equal treatment of equals has to be reinforced so as to single out the Equal Division value instead of the Shapley value among the values satisfying Efficiency and the classical axiom of Additivity. In particular, we would like to achieve this result without relying on an extra axiom such as the nullifying player axiom. More specifically, we exploit the extension of the desirability relation to coalitions as proposed by Lapidot (1968) for simple games (see also Peleg, 1980, 1981; Einy, 1985; Einy and Neyman, 1988; Einy and Lehrer, 1989; Carreras and Freixas, 1996) in order to construct three new axioms. These new axioms are stronger than Equal treatment of equals and Desirability: any value satisfying one of the new axioms also satisfies Equal treatment of equals and Desirability, while the converse is not true except in very specific cases. The first new axiom, called Coalitional desirability, imposes that if a first coalition has contributions at least as large as the contributions of a second coalition, then the total payoff in the first coalition should be at least as large as the total payoff in the second coalition. The second new axiom is similar except that it is based on the per-capita contributions and per-capita payoffs within the coalitions. We call it Average coalitional desirability. The third new axiom, called Uniform coalitional desirability, is weaker than the first two in that it only applies to coalitions with the same number of players.

Our results underline that the axioms of Equal treatment of equals and Desirability cannot be reinforced in any way. We show that the requirement imposed by Coalitional desirability is too strong in the sense that there exists no value satisfying this axiom and Efficiency at the same time whenever the game contains at least three players. This is no longer the case when Coalitional desirability is replaced by Average coalitional desirability: the latter axiom is compatible with Efficiency. More specifically, we show that the combination of Average coalitional desirability, Efficiency and the classical axiom of Additivity characterizes the Equal Division value. Therefore, one can move from the Shapley value to the Equal Division value by dropping the Null player axiom and reinforcing Equal treatment of equals into Average coalitional desirability. Finally, for classes of games with at least five players, this characterization of the Equal division value still holds when Average coalitional desirability is replaced by Uniform coalitional desirability. Our work continues a literature on the Equal Division value which has received a renewed interest in recent years, especially since van den Brink (2007). We refer to van den Brink and Funaki (2009), Béal et al. (2014, 2016) and Béal et al. (2015a,b) for recent contributions.

The rest of the note is organized as follows. Section 2 presents cooperative games with transferable utility, the axioms and the Equal Division value. The results are stated and proved in section 3. Section 4 provides concluding remarks.

2. Preliminaries

Let N be any finite set of n players. A **cooperative game with transferable utility** on N, or simply a **game**, is a function $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. We denote by V_N the set of all games on N. A game $v \in V_N$ is **symmetric** if $v(S) = v(T)$ whenever $|S| = |T|$. For any two games $v, w \in V_N$, the game $(v + w) \in V_N$ is defined as $(v + w)(S) = v(S) + w(S)$ for all $S \in 2^N$. For each nonempty $S \in 2^N$, the **Dirac game** induced by S on N is denoted by $1_S \in V_N$, and is defined by $1_S(T) = 1$ if $T = S$, and $1_S(T) = 0$ otherwise. It is obvious that each game $v \in V_N$ can be decomposed in a unique way as

$$v = \sum_{S \in 2^N \setminus \{\emptyset\}} v(S) \cdot 1_S.$$

Two players $i, j \in N$ are **equal** in a game $v \in V_N$ if for all $S \in 2^{N \setminus \{i,j\}}$, it holds that $v(S \cup \{i\}) = v(S \cup \{j\})$. A **value** on V_N is a function $f : V_N \longrightarrow \mathbb{R}^N$, which assigns to each game $v \in V_N$ and to each player $i \in N$ a payoff $f_i(v)$ for the participation of i to v. In this note, we investigate the **Equal Division value** ED, which assigns to each game $v \in V_N$ and to each player $i \in N$ a payoff

$$ED_i(v) = \frac{v(N)}{n}.$$

We also invoke the following axioms.

Equal treatment of equals. If $i, j \in N$ are equals in game $v \in V_N$, then $f_i(v) = f_j(v)$.

Efficiency. For any game $v \in V_N$, it holds that $\sum_{i \in N} f_i(v) = v(N)$.

Additivity. For any two games $v, w \in V_N$, it holds that $f(v + w) = f(v) + f(w)$.

Desirability. For each pair of distinct players $i, j \in N$, if $v(C \cup \{i\}) \geq v(C \cup \{j\})$ for all $C \subseteq N \setminus \{i, j\}$, then $f_i(v) \geq f_j(v)$.

The first three axioms are classical. Desirability (Maschler and Peleg, 1966) states that if a first player has marginal contributions to coalitions at least as large as the marginal contributions of a second player, then she should obtain a payoff at least as large as the payoff of the second player. The axiom is also known as Local monotonicity (Malawski, 2013; van den Brink et al.,

2013) and Fair treatment (Radzik and Driessen, 2013). We now introduce the new axioms, which can be considered as collective variants of Desirability.

The first one implements the same principle as Desirability, but for coalitions: if a first coalition has contributions to coalitions at least as large as the contributions of a second coalition, then the total payoff in the first coalition should be at least as large as the total payoff in the second coalition, where the two coalitions are nonempty and disjoint.

Coalitional Desirability. For all nonempty $R, T \in 2^N$ such that $R \cap T = \emptyset$, and all $C \subseteq N \backslash (R \cup T)$, if $v(C \cup R) \geq v(C \cup T)$ then it holds that

$$\sum_{j \in R} f_j(v) \geq \sum_{j \in T} f_j(v). \tag{1}$$

The second new axiom expresses these conditions in terms of average: if a first coalition has per-capita contributions to coalitions at least as large as the per-capita contributions of a second coalition, then the average payoff in the first coalition should be at least as large as the average payoff in the second coalition. Once again, the two chosen coalitions are supposed to be nonempty and disjoint.[1]

Average coalitional desirability. For all nonempty $R, T \in 2^N$ such that $R \cap T = \emptyset$, and all $C \subseteq N \backslash (R \cup T)$, if $v(C \cup R)/|R| \geq v(C \cup T)/|T|$ then it holds that

$$\frac{1}{|R|} \sum_{j \in R} f_j(v) \geq \frac{1}{|T|} \sum_{j \in T} f_j(v). \tag{2}$$

The third new axiom reuses the same principles, but only for pairs of coalitions with the same number of players.

Uniform coalitional desirability. For all nonempty $R, T \in 2^N$ such that $R \cap T = \emptyset$ and $|R| = |T|$, and all $C \subseteq N \backslash (R \cup T)$, if $v(C \cup R) \geq v(C \cup T)$ then it holds that

$$\sum_{j \in R} f_j(v) \geq \sum_{j \in T} f_j(v).$$

It easy to check that the three new axioms imply the axiom of Desirability since the requirement of latter axiom is obtained by choosing R and T satisfying $|R| = |T| = 1$. Since in addition it is well-known that Desirability implies Equal treatment of equals, we conclude that our new axioms are stronger than the axiom of Equal treatment of equals. Moreover, Uniform coalitional desirability can be considered as a weak compromise between Coalitional desirability and Average coalitional desirability. Since the two compared coalitions have the same size, the condition on their contributions is equivalent to the condition on their per-capita contributions, and similarly, the requirement on the total payoffs in the coalitions is equivalent to the average payoff in these coalitions.

[1] The results in this note are still valid if this assumption is relaxed in the definition of our new axioms.

3. Results

In this section, we show that Coalitional desirability is very demanding, while Average coalitional stability singles out the equal division value among the values satisfying the mild requirements of Efficiency and Additivity.

Proposition 1. *Let $n \geq 3$. There exist no value on V_N satisfying Coalitional desirability and Efficiency.*

Proof. Let (N, v) be a symmetric game such that $n \geq 3$, $v(N) \neq 0$ and for each $i \in N$,

$$v(\{i\}) > v(N \backslash \{i\}). \tag{3}$$

Since v is symmetric, all players are equal. By Efficiency and because Coalitional desirability implies Equal treatment of equals, we obtain $f_i(v) = v(N)/n \neq 0$ for all $i \in N$. Next, in the definition of Coalitional desirability, set $R = \{i\}$ and $T = N\backslash\{i\}$, so that it must be that $C = \emptyset$. By (3), the application of Coalitional desirability implies that

$$f_i(v) \geq \sum_{j \in N \backslash \{i\}} f_j(v),$$

which contradicts, since $n \geq 3$, the fact that $f_i(v) = v(N)/n \neq 0$ for all $i \in N$. ■

The condition that $n \geq 3$ in the statement of Proposition 1 cannot be weaken. In any two-player game on $N = \{i, j\}$, the only possible choice for the coalitions R and T in the definition of Coalitional desirability is $\{i\}$ and $\{j\}$, which means that Coalitional desirability is equivalent to Desirability on two-player games. Thus, it is enough to exhibit a value satisfying Efficiency and Desirability on two-player games. A well-known example is the standard solution (Hart and Mas-Colell, 1989), which assigns to game $v \in V_N$ and each player $i \in \{1, 2\}$, the payoff

$$v(\{i\}) + \frac{1}{2}\big(v(\{i,j\}) - v(\{i\}) - v(\{j\})\big).$$

The compatibility of Coalitional desirability and Efficiency can be restored on specific subclasses of games. An example is the class of data games introduced in Dehez and Tellone (2013).

Proposition 2. *Let $n \neq 2$. The Equal Division value ED is the unique value on V_N that satisfies Efficiency, Additivity and Average coalitional desirability.*

Proof. On the one hand, it is well-known that ED satisfies Efficiency and Additivity, and it satisfies Average coalitional desirability because (2) holds with equality for all pairs of coalitions, and not only those allowed in the definition of Average coalitional desirability. On the other, let f be a value on V_N, $n \neq 2$, that satisfies the three axioms. We show that there is at most one such value. If $n = 1$, f is uniquely determined by Efficiency. So assume that $n \geq 3$. Choose any $v \in V_N$. By Additivity, we know that

$$f(v) = \sum_{S \in 2^N \setminus \{\emptyset\}} f(v(S) \cdot 1_S).$$

Hence, it remains to show that f is uniquely determined in all games $(v(S) \cdot 1_S) \in V_N$, $S \in 2^N \setminus \{\emptyset\}$. We distinguish four cases depending on the number of players in S.

Case 1. Suppose that $|S| \in \{2, \cdots, n-2\}$. For any $i \in N$, let $R = \{i\}$ and $T = N \setminus \{i\}$ in the definition of Average coalitional desirability, so that it must be that $C = \emptyset$. We have $(v(S) \cdot 1_S)(\{i\}) = (v(S) \cdot 1_S)(N \setminus \{i\})/(n-1) = 0$. Applying Average coalitional desirability, Efficiency and using the fact that $(v(S) \cdot 1_S)(N) = 0$ yield that

$$f_i(v(S) \cdot 1_S) = \frac{1}{n-1} \sum_{j \in N \setminus \{i\}} f_j(v(S) \cdot 1_S) = \frac{1}{n-1}\big((v(S) \cdot 1_S)(N) - f_i(v(S) \cdot 1_S)\big) = -\frac{1}{n-1} f_i(v(S) \cdot 1_S),$$

which forces $f_i(v(S) \cdot 1_S) = 0$. Since $i \in N$ has been chosen arbitrarily, we get $f_i(v(S) \cdot 1_S) = 0$ for all $i \in N$.

Case 2. Suppose $|S| = 1$, which means that $v(S) \cdot 1_S = v(\{i\}) \cdot 1_{\{i\}}$ for some $i \in N$. Pick any $j \in N \setminus \{i\}$ and let $R = \{j\}$ and $T = N \setminus \{j\}$ in the definition of Average coalitional desirability. As in case 1, we get $f_j(v(\{i\}) \cdot 1_{\{i\}}) = 0$ for all $j \in N \setminus \{i\}$, so that by E, we also obtain $f_i(v(\{i\}) \cdot 1_{\{i\}}) = 0$.

Case 3. Suppose that $|S| = n-1$, which means that $v(S) \cdot 1_S = v(N \setminus \{i\}) \cdot 1_{N \setminus \{i\}}$ for some $i \in N$. As in case 2, pick any $j \in N \setminus \{i\}$ and let $R = \{j\}$ and $T = N \setminus \{j\}$ in the definition of Average coalitional desirability in order to obtain $f_j(v(N \setminus \{i\}) \cdot 1_{N \setminus \{i\}}) = 0$ for all $j \in N \setminus \{i\}$, and by E, $f_i(v(N \setminus \{i\}) \cdot 1_{N \setminus \{i\}}) = 0$.

Case 4. Suppose $|S| = n$, so that $v(S) \cdot 1_S = v(N) \cdot 1_N$. Recall that all players are equals in $v(N) \cdot 1_N$. Since Average coalitional desirability implies Equal treatment of equals, Average coalitional desirability and Efficiency imply that $f_i(v(N) \cdot 1_N) = v(N)/n$ for all $i \in N$.

The four cases together with Additivity ensure that there is at most one value f satisfying the three axioms, and the proof is complete. ■

The logical independence of the axioms invoked in Proposition 2 can be demonstrated as follows:

1. The Shapley value (Shapley, 1953) satisfies Efficiency and Additivity but violates Average coalitional desirability.
2. The null value, which assigns to each game $v \in V_N$ and to each player $i \in N$ and null payoff satisfies Additivity and Average coalitional desirability but violates Efficiency.
3. For each N and each $i \in N$, define the game $w^i \in V_N$ as

$$w^i = \sum_{S \subseteq N \setminus \{i\}, S \neq \emptyset} 1_S$$

and let $W_N = \{w^i : i \in N\}$. Construct the value f^* such that $f^*_i(w^i) = -1$ and $f^*_j(w^i) = 1/(n-1)$ if $j \in N\backslash\{i\}$ for each $w^i \in W_N$, and $f^*(v) = ED(v)$ if $v \in V_N\backslash W_N$. The value f^* satisfies Efficiency and Average coalitional desirability but violates Additivity.

The uniqueness of Proposition 2 does not hold in case $n = 2$. Similarly as the discussion following Proposition 1, it is easy to figure out that Average coalitional desirability is equivalent to Desirability on two-player games. Hence, the standard solution also satisfies the set of axioms in Proposition 2 when $n = 2$.

The statement of Proposition 2 remains valid when Uniform coalitional desirability replaces Average coalitional desirability, but only for classes of games with at least five players.

Proposition 3. *Let $n \notin \{2, 3, 4\}$. The Equal Division value ED is the unique value on V_N that satisfies Efficiency, Additivity and Uniform coalitional desirability.*

Proof. The Equal Division value clearly satisfies the three axioms. Now, let f be any value satisfying the three axioms. The case $n = 1$ is trivial. So assume $n \geq 5$. Consider any $v \in V_N$. As in the proof of Proposition 2, we use the decomposition of v into multiple of Dirac games. The uniqueness of f in $v(N)\cdot 1_N$ follows from Efficiency and the fact that Uniform coalitional desirability implies Equal treatment of equals. Next, consider any S such that $|S| \in \{1, \dots, n-1\}$ and the game $v(S) \cdot 1_S$. Pick $i \in S$ and $j \in N\backslash S$. In the definition of Uniform coalitional desirability, define $R = \{i, j\}$, and choose T such that $T \subseteq S$ (which is always possible if $|S| \geq 3$) or such that $T \subseteq N\backslash(S \cup \{j\})$ (which is always possible if $|S| \leq 2$ since $n \geq 5$). Let $T = \{i_1, i_2\}$. For each $C \subseteq N\backslash(R \cup T)$, observe that $(v(S) \cdot 1_S)(C \cup R) = (v(S) \cdot 1_S)(C \cup T)$. Thus, an application of Uniform coalitional desirability yields that

$$f_i(v(S) \cdot 1_S) + f_j(v(S) \cdot 1_S) = f_{i_1}(v(S) \cdot 1_S) + f_{i_2}(v(S) \cdot 1_S). \tag{4}$$

Furthermore, two players $k, l \in S$ are equal in $v(S) \cdot 1_S$, and similarly, two players $k, l \in N\backslash S$ are equal in $v(S) \cdot 1_S$. Since Uniform coalitional desirability implies Desirability, and in turn Equal treatment of equals, we also know that $f_k(v(S) \cdot 1_S) = f_l(v(S) \cdot 1_S)$ for each pair $k, l \in S$ and each pair $k, l \in N\backslash S$. As a consequence, if $T \subseteq S$, then (4) becomes

$$f_i(v(S) \cdot 1_S) + f_j(v(S) \cdot 1_S) = 2f_i(v(S) \cdot 1_S),$$

which implies $f_i(v(S) \cdot 1_S) = f_j(v(S) \cdot 1_S)$. If $T \subseteq N\backslash(S \cup \{j\})$, then (4) becomes

$$f_i(v(S) \cdot 1_S) + f_j(v(S) \cdot 1_S) = 2f_j(v(S) \cdot 1_S),$$

which implies $f_i(v(S) \cdot 1_S) = f_j(v(S) \cdot 1_S)$ as well. In both cases, we obtain that $f_i(v(S) \cdot 1_S) = f_j(v(S) \cdot 1_S)$ for all $i, j \in N$ since i and j were chosen arbitrarily in S and $N\backslash S$, respectively. By Efficiency, we get $f_i(v(S) \cdot 1_S) = 0$ for all $i \in N$. The proof that there is at most one value f satisfying the three axioms is complete after an application of Additivity. ■

The logical independence of the axioms in Proposition 3 can be proved with the three values exhibited after the proof of Proposition 2. Proposition 3 does not hold if $n \in \{2, 3, 4\}$. Firstly, note that if $n \in \{2, 3\}$, then in the definition of Uniform coalitional desirability, we must have $|R| = |T| = 1$, which means that Uniform coalitional desirability reduces to Desirability. Hence, any value satisfying Efficiency, Additivity and Desirability would satisfy the three axioms in Proposition 3. The Shapley value is one such value. Secondly, assume that $n = 4$. When choosing R and T in the definition of Uniform coalitional desirability, the case $|R| = |T| = 2$ becomes possible in addition to the case $|R| = |T| = 1$. As a consequence, Uniform coalitional desirability does not reduce to Desirability. Still, the value f^{**} on V_N, $|N| = 4$, which assigns to each game $v \in V_N$ and each player $i \in N$ the payoff

$$f_i^{**}(v) = \frac{v(N)}{4} + \sum_{S \subseteq N : |S|=2, S \ni i} \big(v(S) - v(N \backslash S)\big).$$

satisfies Efficiency, Additivity and Uniform coalitional desirability.

4. Concluding remark

We conclude this note a comparison with Theorem 3.1 in van den Brink (2007), who characterizes the Equal Division value by Efficiency, Additivity, Equal treatment of equals and the following Nullifying player axiom.

Nullifying player axiom. If player $i \in N$ is nullifying in game $v \in V_N$, *i.e.*, $v(S) = 0$ for all $S \ni i$, then $f_i(v) = 0$.

Since both Average coalitional desirability and Uniform coalitional desirability imply Equal treatment of equals, one way to prove Propositions 2 and 3 would be to invoke Theorem 3.1 in van den Brink (2007) and show that Average coalitional desirability or Uniform coalitional desirability implies the Nullifying player axiom. However, the latter implication does not hold: the above value f^* constructed after the proof of Proposition 2 satisfies Average coalitional desirability and Uniform coalitional desirability but violates the Nullifying player axiom on W_N since i is a nullifying player in game w^i but obtains $f_i^*(w^i) = -1$.

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