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The Priority Value for Cooperative Games with a Priority Structure

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Abstract

We study cooperative games with a priority structure modeled by a poset on the agent set. We introduce the Priority value, which splits the Harsanyi dividend of each coalition among the set of its priority agents, *i.e.* the members of the coalition over which no other coalition member has priority. This allocation shares many desirable properties with the classical Shapley value: it is efficient, additive and satisfies the null agent axiom, which assigns a null payoff to any agent with null contributions to coalitions. We provide two axiomatic characterizations of the Priority value which invoke both classical axioms and new axioms describing various effects that the priority structure can impose on the payoff allocation. Applications to queueing and bankruptcy problems are discussed.

Keywords: Priority structure, Shapley value, Priority value, necessary agent, Harsanyi solution, queueing problems, bankruptcy problems.

JEL code: C71

1. Introduction

The theory of cooperative games is a remarkable set of tools to analyze many problems of resource allocation resulting from the cooperation of several agents. Among the numerous allocation rules proposed in this literature, the Shapley value (Shapley, 1953) is by far the most studied and applied(see Algaba et al., 2019, for instance). The Shapley value allocates to each agent an average of its contributions to the coalitions of agents. Equivalently, the Shapley value can be described as the allocation rule which splits equally the so-called Harsanyi dividend (Harsanyi, 1959) of each coalition among its members.

In this classical model, it is assumed that the agents only differ with respect to their ability to contribute to the worths of the coalitions. However, there are many situations in which the allocation of resources can also be influenced by economical, hierarchical or communicational structures between the agents. Such structures can impose communicational constraints modeled by a graph

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in Myerson (1977), coalitional constraints modeled by a partition in Aumann and Dreze (1974) and Owen (1977) or hierarchical constraints modeled by a permission structure in Gilles et al. (1992). In all these approaches, the Shapley value is adapted to account for the position of the agents in the associated structure.

In this article, we enrich the classical model of cooperative games by a priority structure. The aim of this structure is to reflect the fact that some agents may have priority over other agents in the allocation process. There are natural examples in which a priority structure is relevant. For instance, in the context of bankruptcy problems, Thomson (2003) argues that "some claims (for instance, secured claims) have higher priority than others (such as unsecured claims).". He insists that "the objective would be to give priority to agents who have risked relatively greater amounts." (see also Flores-Szwagrzak et al., 2019). When selling tickets to sporting events such as Roland Garros, licensees often have priority over other buyers. Allocating the research budget of a university also systematically gives priority to certain projects rather than others. In private procurement auctions, a buyer may give priority to an incumbent supplier through the Right-of-First-Refusal: the incumbent supplier can win the auction by simply matching the best offer made by its rivals (Brisset et al., 2015). Mitchell (2001) explains that such a preemptive right can also exist in real estate, corporate securities, franchise agreements, oil and gas leases, employment contracts, among others. In the well-known "school choice" problem, each school has a priority order over the students (Bu, 2014). Finally, the allocation of resources in cloud computing often necessitates to determine priority among the user requests (Ghanbaria and Othman, 2012).

We model the priority structure by a partially ordered set (poset) $\succcurlyeq$ on the set of participating agents: an agent i has priority over an agent j if $i \succcurlyeq j$. The objective is then to design an allocation rule which takes into account both the economic possibilities resulting from the cooperation of the agents and modeled by a cooperative game with transferable utility and the priority structure given by this poset. From a mathematical point of view, our model is identical to the acyclic permission structures introduced in Gilles et al. (1992) and the structures modeling precedence constraints introduced in Faigle and Kern (1992), even if these structures are sometimes defined by means of digraphs. These two models have been extensively studied in the literature (van den Brink, 2017; Algaba et al., 2017, survey some results) and have in common that the associated structure imposes some restriction on the formation of coalitions or on the subset of coalitions to which an allocation rule is sensitive. In van den Brink and Gilles (1996), van den Brink (1997) and van den Brink and Dietz (2014), the so-called conjunctive, disjunctive and local permission values are computed as the Shapley value of a restricted game in which only the "feasible" part of a coalition is productive, where the "feasible" part is the largest subset of the coalition that contains all the hierarchical superiors deemed necessary for the worth generation (these vary depending on the model: conjunctive, disjunctive or local). In the Shapley value for games with precedence constraints studied by Faigle and Kern (1992), the principle of the marginal vectors underlying the classical Shapley value is restricted to those orderings in which each agent must appear after its hierarchical superiors, preventing de facto the formation of some coalitions.

In this article, we adopt a completely new point of view which does not reduce our contribution to a simple reinterpretation of the poset in terms of priority. More specifically, we consider that the priority structure has no impact on the formation of coalitions as in Faigle and Kern (1992) or on the result that these coalitions can produce as in Gilles et al. (1992). This important feature of our model is consistent with the aforementioned applications: in a bankruptcy problem, it makes little sense to prevent a coalition from forming on the grounds that a creditor has priority over its members or to prevent some of its members from being productive simply because other agents have priority over them. Obviously, the priority structure should influence the allocation of the results of cooperation between agents. In order to stay as close as possible to the spirit of the Shapley value, we introduce the Priority value, which shares the Harsanyi dividend of each coalition equally among the subset of its members over whom no other agent in the coalition has priority. As the Shapley value, the Priority value can be written as an Harsanyi solution, unlike the conjunctive, disjunctive and local permission values. As a consequence, a difference between of the Priority value and these allocation rules is that it satisfies the Null agent axiom: an agent with null contribution to coalitions receives a null payoff. This means that unproductive agents should not be rewarded even if they have priority over the other agents or, equivalently, that the productivities take over the priorities. The Shapley value for games with precedence constraints satisfies the null agent axiom, but an advantage of the Priority value over this value is that it is sensitive to the worth of all coalitions: if the result of the cooperation between any coalition of agents increases, then some of its members should be better off while some of the members of the complementary coalition should be worse off.

We conduct an axiomatic study on the class of cooperative games with a priority structure whose purpose is to provide two characterizations of the Priority value. This analysis invokes classical axioms as well as new axioms. Classical axioms are Efficiency (the sum of distributed payoffs equals the worth of the grand coalition), the already mentioned Null agent axiom, the Null game axiom (if all worths are null, then everyone gets a null payoff), Additivity (the sum of payoffs in two games equals the payoffs in the sum of the two games) and the Null agent out axiom (removing null agents does not alter the payoffs of the remaining agents) adapted from Derks and Haller (1999) to our framework in the most natural way. The new axioms describe how the priority structure can affect the agents' payoffs. Two of them rely on necessary agents, a specific type of agents often used in the design of axioms, particularly for games with a permission structure (see Béal and Navarro, 2020, for a list of references). An agent is necessary if the worth of any coalition without it is equal to zero. The first new axiom imposes that two necessary agents get the same payoff if the set of agents with priority over each of them is the same (A1). The rationale is that two necessary agents are also equals. Hence axiom (A1) deals with two agents that cannot be distinguished by their contributions and which are also indistinguishable from the point of view of the agents having priority over each of them. The second axiom involving necessary agents requires that an agent receives a null payoff if a necessary agent has priority over it (A2). This agent receives a null payoff since it is in a sense doubly blocked by the fact that there is another agent which both has priority over it and is necessary to any worth creation. The third new axiom is an invariance axiom which states that removing any agent does not alter the payoffs of the agents over which it has priority (A3). The explanation of this axiom is that removing any agent j which has priority over an agent i has a double effect: on the one hand, the situation of agent's i improves since it has fewer agents having priority over it but, on the other hand, i's situation is also deteriorating because agent j takes with it some possibilities of cooperation. Axiom (A3) requires that these effects neutralize each other. The fourth and final new axiom is another invariance axiom. If an agent and all the agents who have priority over it are null and if a new priority is created for this agent over another agent, then all payoffs are unchanged (A4). An interpretation of this axiom is that an agent does not care that a group of unproductive agents take priority over it, and that this local modification of the priority structure should not impact the other agents. In a sense here, the absence of productivity takes precedence over priorities.

The results are as follows. The combination of Efficiency, Additivity A3 and A4 implies the Null agent axiom (Proposition 1). The combination of the Null game axiom and A3 implies A2 and the combination of the Null agent out axiom and Efficiency implies A4 (Proposition 3). We also show that the Priority value is the unique allocation rule satisfying Efficiency, Additivity, A1, A3 and A4 (Proposition 2). Our second characterization of the Priority value invokes Efficiency, Additivity, the Null agent out axiom, A1 and A2 (Proposition 4). The last part of the article is devoted to a comparison between the Priority value and other allocation rules and to two applications to queueing and bankruptcy problems. In a queueing problem where agents are also associated with a priority structure, the Priority value applied to the queueing game suggested by Maniquet (2003) is natural and can be interpreted, if the priority structure is linear, as a gradual regime change from the optimal queue to the priority ordering. Moreover, the Priority value of a natural bankruptcy game constructed from a linear priority structure belongs to the family of priority rules characterized by Moulin (2000): the individual claims are satisfied sequentially according to the priority ordering of the agents until the estate is exhausted; the remaining agents getting nothing.

The plan of the article is the following. Section 2 provides definitions. Section 3 introduces and motivates the axioms. Section 4 presents and proves the main results. The comparison between the Priority value and other allocation rules is provided in section 5. Section 6 discusses two applications. Section 7 concludes.

2. Basic definitions and notation

2.1. Cooperative games with transferable utility and the Harsanyi solutions

A situation in which a finite set of agents can generate certain monetary payoffs by agreeing to cooperate can be described by a cooperative game with transferable utility (or simply a TU-game). A **TU-game** is a pair (N, v) where $N \subseteq \mathbb{N}$ is a finite set of $n \in \mathbb{N}$ agents, and $v : 2^N \longrightarrow \mathbb{R}$ is a coalition function on N such that $v(\varnothing) = 0$. A subset S of N is a **coalition** and $v(S) \in \mathbb{R}$ is the worth that the members of S can obtain when they agree to cooperate. In the sequel, the singleton $\{i\}$ is denoted by i, and, for any non-empty coalition S, we will often use notation s to denote its cardinality $|S|$.

The **subgame** (S, v^S) of (N, v) induced by coalition $S \neq \varnothing$ is such that v^S is the restriction of v to 2^S. When there is no risk of confusion, (S, v^S) will be denoted by (S, v).

An agent $i \in N$ is a **null agent** in (N, v) if, for each $S \subseteq N$ such that $S \ni i$, $v(S) = v(S \backslash i)$. Two distinct agents i and j in N are **equal agents** in (N, v) if, for each $S \subseteq N \backslash \{i, j\}$, it holds that $v(S \cup i) = v(S \cup j)$. An agent i is a **necessary agent** (sometimes called veto agent) in (N, v)

if, for each $S \subseteq N\backslash i$, $v(S) = 0$. If two distinct agents i and j are necessary in (N, v), then they are indeed equal agents because $v(S \cup i) = v(S \cup j) = 0$ for each $S \subseteq N\backslash\{i, j\}$.

For two TU-games (N, v) and (N, w) defined on the same agent set N and for each $c \in \mathbb{R}$, the TU-games $(N, v+w)$ and (N, cv) are defined as follows: for each $S \subseteq N$, $(v+w)(S) = v(S)+w(S)$, and $(cv)(S) = cv(S)$. The **null TU-game** on N is the TU-game $(N, \mathbf{0})$ such that $\mathbf{0}(S) = 0$ whatever $S \subseteq N$. A TU-game is **monotone** if $v(S) \geqslant v(T)$ whenever $S \supseteq T$. For each nonempty coalition $S \subseteq N$, the **unanimity TU-game** induced by S is the TU-game (N, u_S) defined as: $u_S(T) = 1$ if $T \supseteq S$, and $u_S(T) = 0$ otherwise. Each unanimity TU-game is monotone. It is well-known that any TU-game (N, v) admits a unique linear decomposition in terms of unanimity TU-games:

$$v = \sum_{\varnothing \subsetneq S \subseteq N} \Delta_S(v) u_S, \tag{1}$$

where each coordinate $\Delta_S(v) \in \mathbb{R}$ is called the **Harsanyi dividend** (Harsanyi, 1959) of S in (N, v), and is computed from the following recursive formula:

$$\Delta_S(v) = v(S) - \sum_{T \subsetneq S} \Delta_T(v).$$

Therefore, the dividend of a singleton is the worth of that singleton, and the dividends of all other coalitions represent the additional contribution that such a coalition earns more than the sum of the dividends of all its subcoalitions. These dividends can thus be seen as the inner contribution of cooperation.

Remark 1. *From the above recursive formula, the following properties hold:*

1. *if i is a null agent in (N, v), then $\Delta_S(v) = 0$ whenever $S \ni i$;*
2. *if i is a necessary agent in (N, v), then $\Delta_S(v) = 0$ whenever $i \in N\backslash S$;*
3. *given a TU-game (N, v) and its subgame (S, v^S), each Harsanyi dividend $\Delta_T(v^S)$ in (S, v^S) coincides with the Harsanyi dividend $\Delta_T(v)$ in (N, v) for each $T \subseteq S$, $T \neq \varnothing$.*

Pick any finite and nonempty set of agents $N' \subseteq \mathbb{N}$. Denote by $G_{N'}$ the set of all TU-games (N, v) where N is a nonempty subset of N' and v is a coalition function $v : 2^N \longrightarrow \mathbb{R}$. A **payoff vector** for a TU-game (N, v) is a vector $x = (x_i)_{i \in N}$ assigning a payoff $x_i \in \mathbb{R}$ to each agent i. An **allocation rule** on $G_{N'}$ is a function f that assigns a payoff vector $f(N, v) \in \mathbb{R}^N$ to any $(N, v) \in G_{N'}$.

The class of allocation rules proposed by Vasil'ev (1978), Hammer et al. (1977) and studied by Derks et al. (2000), and more recently by Besner (2019), distributes the Harsanyi dividends through a sharing system. A **sharing system** on the agent set N' is a system $p = (p^S)_{S \subseteq N', S \neq \varnothing}$, where p^S is a vector of real numbers $(p_i^S)_{i \in S}$ assigning a nonnegative share $p_i^S \geqslant 0$ to each $i \in S$ and such that $\sum_{i \in S} p_i^S = 1$. Given a sharing system p, the **Harsanyi solution** h^p on $G_{N'}$ is the allocation rule that assigns to each agent in a TU-game $(N, v) \in G_{N'}$ a payoff equal to the sum, over all coalitions S containing i, of the share $p_i^S \Delta_S(v)$ of agent i in the Harsanyi dividend of coalition S. Formally,

$$\forall i \in N, \quad h_i^p(N, v) = \sum_{S \subseteq N : S \ni i} p_i^S \Delta_S(v).$$

One of the most famous Harsanyi solutions for TU-games is the Shapley value (Shapley, 1953) defined as the average, over all orderings on the agent set, of the marginal contribution vectors. More precisely, given an agent set N of size n, an ordering $\sigma : N \longrightarrow \{1, \ldots, n\}$ is a bijective map assigning to each agent $i \in N$ a rank $\sigma(i) \in \{1, \ldots n\}$. Such an ordering represents the situation where the agents enter in a room one by one according to σ. Let $O(N)$ be the set of all orderings on N. Given an ordering $\sigma \in O(N)$ of N and a TU-game $(N, v) \in G_{N'}$, one defines the corresponding **marginal vector** $m^\sigma(N, v)$ as

$$\forall i \in N, \quad m_i^\sigma(N, v) = v\big(\{j \in N : \sigma(j) \leqslant \sigma(i)\}\big) - v\big(\{j \in N : \sigma(j) < \sigma(i)\}\big).$$

Thus m^σ, viewed as an allocation rule on $G_{N'}$, distributes to each agent its contribution to the coalition formed by its entrance according to the ordering $\sigma \in O(N)$.

The **Shapley value** is the allocation rule Sh defined as the average of all these marginal vectors:

$$\forall i \in N, \quad \mathrm{Sh}_i(N, v) = \frac{1}{n!} \sum_{\sigma \in O(N)} m_i^\sigma(N, v).$$

It is well-known that the Shapley value is the Harsanyi solution which distributes equally the dividend of a coalition among its members, that is, the Shapley can also be written as:

$$\forall i \in N, \quad \mathrm{Sh}_i(N, v) = \sum_{S \subseteq N : S \ni i} \frac{\Delta_S(v)}{s}.$$

2.2. TU-games with a priority structure

A TU-game with a priority structure describes a situation where some agents in the TU-game have priority over some other agents. Formally, a **priority structure** on N is a **partially ordered set**, also called a **poset**, $\succcurlyeq$ on the agent set N. Recall that a poset $(N, \succcurlyeq)$ is a reflexive, antisymmetric and transitive binary relation. The relation $i \succcurlyeq j$ means that i has priority over j. The poset $(N, \succcurlyeq^0)$ containing no priority relations among pair of distinct agents is called the **trivial poset**. A poset $(N, \succcurlyeq)$ is a **linear order** if, for any pair of agents $\{i, j\} \subseteq N$, either $i \succcurlyeq j$ or $j \succcurlyeq i$, that is, if $(N, \succcurlyeq)$ is complete. The poset $(N, \succcurlyeq')$ **contains** the poset $(N, \succcurlyeq)$ if for each $i, j \in N$, $i \succcurlyeq j$ implies $i \succcurlyeq' j$. In this case, we say that $(N, \succcurlyeq')$ is an **extension** of the poset $(N, \succcurlyeq)$. Furthermore, it is the **elementary extension** of $(N, \succcurlyeq)$ with respect to the pair of distinct agents $\{i, j\} \subseteq N$ if it is the smallest extension of $(N, \succcurlyeq)$ such that $i \succcurlyeq' j$. Faigle and Kern (1992) and Gilles et al. (1992) also consider a poset to model precedence constraints and hierarchical constraints. We will come back later on the differences between our model and their model.

A poset $(N, \succcurlyeq)$ gives rise to the asymmetric binary relation $(N, \succ)$: $i \succ j$ if $i \succcurlyeq j$ and $i \neq j$. For an agent $i \in N$, define the **priority group on** i, denoted by $\uparrow_{\succcurlyeq} i$, as the set of agents having priority over i in $(N, \succcurlyeq)$:

$$\uparrow_{\succcurlyeq} i = \big\{j \in N : j \succ i\big\},$$

and the **set of agents over whom** i **has priority** in $(N, \succcurlyeq)$ as

$$\downarrow_{\succcurlyeq} i = \big\{j \in N : i \succ j\big\}.$$

Two distinct agents i and j are **incomparable** in $(N, \succcurlyeq)$ if neither $i \succcurlyeq j$ nor $j \succcurlyeq i$. For each nonempty coalition S, the **subposet** $(S, \succcurlyeq^S)$ of $(N, \succcurlyeq)$ induced by S is defined as follows: for each $i \in S$ and $j \in S$, $i \succcurlyeq^S j$ if $i \succcurlyeq j$. We will also use the notation $(S, \succcurlyeq)$ instead of $(S, \succcurlyeq^S)$. An agent i is a **priority agent** in $(S, \succcurlyeq)$ if, for $j \in S$, the relation $j \succcurlyeq i$ implies $i = j$. Denote by $M(S, \succcurlyeq)$ the nonempty subset of priority agents in $(S, \succcurlyeq)$. Denote by $P_{N'}$ the set of all posets $(N, \succcurlyeq)$, where N is a nonempty subset of a finite set of agents $N' \subseteq \mathbb{N}$ and $\succcurlyeq$ is a poset on N.

Example 1. Consider the Hasse diagram of the poset $(N, \succcurlyeq)$, where $N = \{6, \ldots, 10\}$, represented in Figure 1. For instance, $\uparrow_{\succcurlyeq} 9 = \{6, 7, 8\}$ and $\downarrow_{\succcurlyeq} 9 = \varnothing$, and agent 9 and agent 10 are incomparable with respect to $(N, \succcurlyeq)$. Consider the subposet $(S, \succcurlyeq)$ induced by $S = \{7, 8, 9\}$. Then, $M(S, \succcurlyeq) = \{7, 8\}$ whereas $M(N, \succcurlyeq) = \{6\}$. $\square$

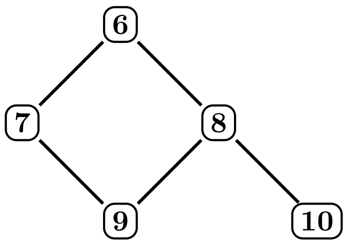

Figure 1: Hasse diagram of the priority structure $(N, \succcurlyeq)$.

Given a finite and nonempty set of agents $N' \subseteq \mathbb{N}$ and a nonempty coalition $N \subseteq N'$, the triple $(N, v, \succcurlyeq)$ where $(N, v) \in G_{N'}$ and $(N, \succcurlyeq) \in P_{N'}$ is called a **TU-game with a priority structure** on N. Denote by $GP_{N'}$ the class of TU-games with a priority structure that we can construct from $G_{N'}$ and $P_{N'}$. In this article, we consider Harsanyi solutions on $GP_{N'}$. The main difference with the Harsanyi solutions on $G_{N'}$ is that the sharing system depends on the priority structure. That is, the priority structure may affect the distribution of the dividends among the agents. Given a priority structure $(N', \succcurlyeq)$, the sharing system $p = \big(p^{(S,\succcurlyeq)}\big)_{S \subseteq N', S \neq \varnothing}$ is such that $p^{(S,\succcurlyeq)}$ is a vector of real numbers $\big(p_i^{(S,\succcurlyeq)}\big)_{i \in S}$ assigning a nonnegative share $p_i^{(S,\succcurlyeq)} \geqslant 0$ to each $i \in S$ in the subposet $(S, \succcurlyeq)$ of $(N', \succcurlyeq)$ and $\sum_{i \in S} p_i^{(S,\succcurlyeq)} = 1$. That is, the Harsanyi solution h^p on $GP_{N'}$ with respect to the sharing system p is given by:

$$\forall i \in N, \quad h_i^p(N, v, \succcurlyeq) = \sum_{S \subseteq N : S \ni i} p_i^{(S,\succcurlyeq)} \Delta_S(v) \tag{2}$$

3. Axioms for allocation rules on games with a priority structure

First, we list a set of axioms for an allocation rule for TU-games on a priority structure, that are straightforward generalizations of axioms for allocation rules on TU-games. Consider an allocation rule f on $GP_{N'}$.

Efficiency (E). For each $(N, v, \succcurlyeq) \in GP_{N'}$, it holds that:

$$\sum_{i \in N} f_i(N, v, \succcurlyeq) = v(N).$$

Null agent axiom (N). For each $(N, v, \succcurlyeq) \in GP_{N'}$, and each null agent $i \in N$ in (N, v), it holds that:

$$f_i(N, v, \succcurlyeq) = 0.$$

Equal treatment of equals (ET). For each $(N, v, \succcurlyeq) \in GP_{N'}$, each pair $\{i, j\} \subseteq N$ of distinct equal agents in (N, v), it holds that:

$$f_i(N, v, \succcurlyeq) = f_j(N, v, \succcurlyeq).$$

Null game axiom (NG). For each $(N, \mathbf{0}, \succcurlyeq) \in GP_{N'}$, it holds that:

$$f(N, \mathbf{0}, \succcurlyeq) = (0, \ldots, 0).$$

Additivity (A). For each $(N, v, \succcurlyeq)$ and each $(N, w, \succcurlyeq) \in GP_{N'}$, it holds that:

$$f(N, v + w, \succcurlyeq) = f(N, v, \succcurlyeq) + f(N, w, \succcurlyeq).$$

Null agent out axiom (NAO). For each $(N, v, \succcurlyeq) \in GP_{N'}$, each null agent $j \in N$ in (N, v), it holds that:

$$\forall i \in N\backslash j, \quad f_i(N, v, \succcurlyeq) = f_i(N\backslash j, v, \succcurlyeq).$$

It is well-known that Efficiency Additivity, the Null agent axiom and Equal treatment of equals characterize the Shapley value (Shapley, 1953; Shubik, 1962). Additivity implies the Null game axiom, and the combination of Efficiency and Null agent out axiom implies the Null agent axiom. Each Harsanyi solution satisfies Additivity, Efficiency and the Null agent out axiom. Finally, note that the above axioms are independent of the priority structure. Below, we introduce new axioms which take into account the priority structure. The first axiom is a weak version of Equal treatment of equals. It imposes that two necessary agents with the same priority group on them are treated equally, no matter over which set of agents they have priority.

Equal treatment for necessary agents with equal priority group (A1). For each $(N, v, \succcurlyeq) \in GP_{N'}$ and each pair $\{i, j\} \subseteq N$ of distinct necessary agents in (N, v) such that $\uparrow_{\succcurlyeq} i = \uparrow_{\succcurlyeq} j$, it holds that:

$$f_i(N, v, \succcurlyeq) = f_j(N, v, \succcurlyeq).$$

Because two necessary agents with the same priority group on them are equal agents, Equal treatment of equals implies Equal treatment for necessary agents with equal priority, while the converse does not hold. The second new axiom indicates that if a necessary agent has priority over another agent, then the later obtains a null payoff.

Necessary and priority agent axiom (A2). For each $(N, v, \succcurlyeq) \in GP_{N'}$ and each necessary agent j in (N, v), it holds that:

$$\forall i \in \downarrow_{\succcurlyeq} j, \quad f_i(N, v, \succcurlyeq) = 0.$$

The rationale behind this axiom is that the payoff possibilities for an agent $i \in \downarrow_{\succcurlyeq} j$ are doubly blocked by the fact that agent j both has priority over it and is necessary to any worth creation. In this case, nothing accrues to agent i. The third new axiom expresses in another way the fact that one agent has priority over another agent. It states that removing an agent, necessary or not, does not affect the payoffs of the agents over which it has priority.

Priority agent out (A3). For each $(N, v, \succcurlyeq) \in GP_{N'}$ and each agent $j \in N$, it holds that:

$$\forall i \in \downarrow_{\succcurlyeq} j \quad f_i(N, v, \succcurlyeq) = f_i(N \backslash j, v, \succcurlyeq).$$

For the last axiom, we need a definition. Given a poset $(N, \succcurlyeq)$ and two distinct agents $i, j \in N$ such that $j \in N \backslash \uparrow_{\succcurlyeq} i$, define the poset $(N, \succcurlyeq_{i \to j})$ as follows:

$$\forall \ell, m \in N, \quad \ell \succcurlyeq_{i \to j} m :\Longleftrightarrow \begin{cases} (a) & \ell \in i \cup (\uparrow_{\succcurlyeq} i) \text{ and } m \in j \cup (\downarrow_{\succcurlyeq} j), \\ (b) & \ell \succcurlyeq m \text{ otherwise.} \end{cases} \tag{3}$$

Note that the poset $(N, \succcurlyeq_{i \to j})$ is the **elementary extension** of $(N, \succcurlyeq)$ with respect to the pair of agents $\{i, j\}$.

Remark 2. *Note that $(N, \succcurlyeq_{i \to j}) = (N, \succcurlyeq)$ whenever $i \succcurlyeq j$. Furthermore, for $m \in (i \cup \uparrow_{\succcurlyeq} i)$ and $\ell \in (j \cup \downarrow_{\succcurlyeq} j)$, it cannot be the case that $\ell \succcurlyeq m$, otherwise $j \succcurlyeq i$ which contradicts the assumption $j \in N \backslash \uparrow_{\succcurlyeq} i$.*

Next, the binary relation $(N, \succcurlyeq_{i \to j})$ is indeed a poset such that $\succcurlyeq_{i \to j}$ contains $\succcurlyeq$. First, by construction $(N, \succcurlyeq_{i \to j})$ inherits reflexivity from $(N, \succcurlyeq)$. Regarding antisymmetry, note that $j \in N \backslash \uparrow_{\succcurlyeq} i$ implies

$$\big(i \cup (\uparrow_{\succcurlyeq} i)\big) \cap \big(j \cup (\downarrow_{\succcurlyeq} j)\big) = \varnothing.$$

Assume $m \succcurlyeq_{i \to j} \ell$ and $\ell \succcurlyeq_{i \to j} m$. It cannot be the case that $m \succcurlyeq_{i \to j} \ell$ from (a). Otherwise, $m \in (i \cup \uparrow_{\succcurlyeq} i)$, $\ell \in (j \cup \downarrow_{\succcurlyeq} j)$, and $m \notin (j \cup \downarrow_{\succcurlyeq} j)$, force $\ell \succcurlyeq m$ from (b), which is impossible as noticed above. For a similar reason, it cannot be the case that $\ell \succcurlyeq_{i \to j} m$ from (a). Thus, $m \succcurlyeq_{i \to j} \ell$ and $\ell \succcurlyeq_{i \to j} m$ occur from (b), that is, $m \succcurlyeq \ell$ and $\ell \succcurlyeq m$, and so $\ell = m$ by antisymmetry of $(N, \succcurlyeq)$. Regarding transitivity, assume that $k \succcurlyeq_{i \to j} \ell$ and $\ell \succcurlyeq_{i \to j} m$. We have three cases: if $k \succcurlyeq \ell$ and $\ell \succcurlyeq m$ from (b), then $k \succcurlyeq m$ by transitivity of $(N, \succcurlyeq)$, and so $k \succcurlyeq_{i \to j} m$ as desired; if $k \in (i \cup \uparrow_{\succcurlyeq} i)$ and $\ell \in (j \cup \downarrow_{\succcurlyeq} j)$, then, by (b) $\ell \succcurlyeq m$ so that $m \in (j \cup \downarrow_{\succcurlyeq} j)$; and, by (a), $k \succcurlyeq_{i \to j} m$ as desired; if $k \succcurlyeq \ell$ from (b), $\ell \in (i \cup \uparrow_{\succcurlyeq} i)$ and $m \in (j \cup \downarrow_{\succcurlyeq} j)$, the same reasoning applies because $k \in (i \cup \uparrow_{\succcurlyeq} i)$.

The last axiom indicates that adding a priority relation between two incomparable agents i and j, in the sense that i has now priority over j, does not affect the agents' payoffs, including j, if agent i and the agents in the priority group on i are null agents. This means that an agent does not care that a group of unproductive agents take priority over it, and that this local modification of the priority structure should not impact the other agents. In a sense here, the absence of productivity takes precedence over priorities.

Invariance to unproductive priority extension (A4). For each $(N, v, \succcurlyeq) \in GP_{N'}$ and each pair of incomparable agents $\{i, j\} \subseteq N$ such that i and each $k \in \uparrow_{\succcurlyeq} i$ are null agents in (N, v), it holds that:

$$f(N, v, \succcurlyeq) = f(N, v, \succcurlyeq_{i \to j}).$$

Remark 3. *By Remark 2, if $i \succcurlyeq j$, then $(N, \succcurlyeq_{i \to j})$ and $(N, \succcurlyeq)$ coincide so that Invariance to unproductive priority extension applies trivially.*

4. Axiomatic study

Our first result states that the combination of Efficiency, Additivity, Invariance to deletion of superiors and Structural invariance for null agents implies the Null agent axiom. This result is a consequence of the following lemma.

Lemma 1. *If an allocation rule f on $GP_{N'}$ satisfies Efficiency* **(E)**, *Null game* **(NG)**, *Priority agent out* **(A3)** *and Invariance to unproductive priority extension* **(A4)**, *then, for each $(N, cu_S, \succcurlyeq) \in GP_{N'}$, where $c \in \mathbb{R}$ and $S \subseteq N$, it holds that:*

$$\forall i \in N \backslash S, \quad f_i(N, cu_S, \succcurlyeq) = 0.$$

Proof. Pick any allocation rule f as hypothesized, and consider the unanimity TU-game with a priority structure $(N, u_S, \succcurlyeq) \in GP_{N'}$. Note that each agent $i \in N \backslash S$ are null agents in (N, cu_S), and that, for each agent $j \in S$, $(N \backslash j, cu_S) = (N \backslash j, \mathbf{0})$. We partition $N \backslash S$ in two subsets in the following way: for $i \in N \backslash S$, either there exists $j \in S$ such that $j \succcurlyeq i$, and we write $i \in \downarrow_{\succcurlyeq} S$, or, for each $j \in S$, we have $j \in N \backslash (\uparrow_{\succcurlyeq} i)$. We prove the statement of Lemma 1 in three steps. In a first step, we extend the subposet $(N \backslash (S \cup \downarrow_{\succcurlyeq} S), \succcurlyeq)$ of $(N, \succcurlyeq)$ to a linear order $(N \backslash (S \cup \downarrow_{\succcurlyeq} S), \succcurlyeq')$ containing $(N \backslash (S \cup \downarrow_{\succcurlyeq} S), \succcurlyeq)$. Then, we define $(N, \succcurlyeq^{(1)})$ containing $(N, \succcurlyeq)$ and such that $(N \backslash (S \cup \downarrow_{\succcurlyeq} S), \succcurlyeq^{(1)}) = (N \backslash (S \cup \downarrow_{\succcurlyeq} S), \succcurlyeq')$. In a second step, we construct a new poset $(N, \succcurlyeq^{(2)})$ containing $(N, \succcurlyeq^{(1)})$ through elementary extensions which connects the lowest priority agent in $(N \backslash (S \cup \downarrow_{\succcurlyeq} S), \succcurlyeq^{(1)})$ to all priority agents of S. In this way, for each $i \in N \backslash (S \cup \downarrow_{\succcurlyeq} S)$ and each $j \in M(S, \succcurlyeq)$, $i \succcurlyeq^{(2)} j$. Recall that $M(S, \succcurlyeq)$ stands for the set of priority agents in the subposet $(S, \succcurlyeq)$ of $(N, \succcurlyeq)$. The constructions involved in these two steps are illustrated below in Example 2. In a third step, we conclude by using axioms **(E)**, **(NG)**, **(A3)** and **(A4)**.

Step 1. Consider the subposet $(N \backslash (S \cup \downarrow_{\succcurlyeq} S), \succcurlyeq)$ of $(N, \succcurlyeq)$. By the Szpilrajn extension theorem (Szpilrajn, 1930), there is a linear order $(N \backslash (S \cup \downarrow_{\succcurlyeq} S), \succcurlyeq')$ containing $(N \backslash (S \cup \downarrow_{\succcurlyeq} S), \succcurlyeq)$. This extension only involves elementary extensions as in (3). Finally, define the poset $(N, \succcurlyeq^{(1)})$ as, for each $\ell, m \in N$:

$$\ell \succcurlyeq^{(1)} m :\Longleftrightarrow \begin{cases} \ell \succcurlyeq' m & \text{if } \ell, m \in N \backslash (S \cup \downarrow_{\succcurlyeq} S), \\ \ell \succcurlyeq m & \text{if } \ell, m \in S \cup \downarrow_{\succcurlyeq} S, \\ \exists \ell' \in N \backslash (S \cup \downarrow_{\succcurlyeq} S) \text{ such that} & \\ \quad \ell' \succcurlyeq m \text{ and } \ell \succcurlyeq' \ell' & \text{if } \ \ell \in N \backslash (S \cup \downarrow_{\succcurlyeq} S) \text{ and } m \in S \cup \downarrow_{\succcurlyeq} S. \end{cases}$$

Note that there exists a lowest priority agent i_m in the linear order $(N\backslash(S\cup\downarrow_{\succcurlyeq} S), \succcurlyeq^{(1)})$ and that $M(S, \succcurlyeq^{(1)}) = M(S, \succcurlyeq)$.

Step 2. We construct sequentially the poset $(N, \succcurlyeq^{(2)})$ as follows: consider the lowest priority agent i_m in $(N\backslash(S\cup\downarrow_{\succcurlyeq} S), \succcurlyeq^{(1)})$ and any $j \in M(S, \succcurlyeq^{(1)})$. Consider the elementary extension $(N, \succcurlyeq^{(1)}_{i_m\to j})$ as defined in (3). By Remark 2, the binary relation $(N, \succcurlyeq^{(1)}_{i_m\to j})$ is a poset containing $(N, \succcurlyeq^{(1)})$. From i_m, $(N, \succcurlyeq^{(1)}_{i_m\to j})$ and $k \in M(S, \succcurlyeq^{(1)})\backslash j$, construct the elementary extension $(N, (\succcurlyeq^{(1)}_{i_m\to j})_{i_m\to k})$. Then, continue in this fashion until all agents in $M(S, \succcurlyeq^{(1)})$ have been exhausted. When the procedure stops, we obtain the poset $(N, \succcurlyeq^{(2)})$. By construction, for each $i \in N\backslash(S\cup\downarrow_{\succcurlyeq} S)$, each $j \in S\cup\downarrow_{\succcurlyeq} S$, we have $i \succcurlyeq^{(2)} j$; and, for each $i, j \in N$ such that $i \succcurlyeq j$, we have $i \succcurlyeq^{(2)} j$. Furthermore, $M(S, \succcurlyeq^{(2)}) = M(S, \succcurlyeq)$ and all priority agents in $M(S, \succcurlyeq^{(2)})$ have now the same priority group on them $N\backslash(S\cup\downarrow_{\succcurlyeq} S)$.

Step 3. First, consider a given agent $i \in\downarrow_{\succcurlyeq} S$. On the one hand, by definition of $\downarrow_{\succcurlyeq} S$, there is $j \in S$ such that $j \succcurlyeq i$. By **(A3)**, $f_i(N, cu_S, \succcurlyeq) = f_i(N\backslash j, cu_S, \succcurlyeq)$. On the other hand, $(N\backslash j, cu_S, \succcurlyeq)$ coincides with $(N\backslash j, \mathbf{0}, \succcurlyeq)$, and so, by **(NG)**, $f_i(N\backslash j, \mathbf{0}, \succcurlyeq) = 0$. Therefore, we conclude that:

$$\forall i \in\downarrow_{\succcurlyeq} S, \quad f_i(N, cu_S, \succcurlyeq) = 0.$$

Next, for $i \in N\backslash(S\cup\downarrow_{\succcurlyeq} S)$, consider the priority group $\uparrow_{\succcurlyeq^{(2)}} i$ on i and the set $\downarrow_{\succcurlyeq^{(2)}} i$ of agents over whom i has priority. By construction, $(S\cup\downarrow_{\succcurlyeq} S) \subseteq \big(\downarrow_{\succcurlyeq^{(2)}} i\big)$ and $N\backslash i = (\uparrow_{\succcurlyeq^{(2)}} i) \cup (\downarrow_{\succcurlyeq^{(2)}} i)$. Recall also that each agent in $N\backslash(S\cup\downarrow_{\succcurlyeq} S)$ is a null agent in (N, cu_S). We obtain:

$$\begin{aligned}
f_i(N, cu_S, \succcurlyeq) \quad & \overset{\textbf{(E)}}{=} c - \sum_{j\in(\uparrow_{\succcurlyeq^{(2)}} i)} f_j(N, cu_S, \succcurlyeq) - \sum_{j\in(\downarrow_{\succcurlyeq^{(2)}} i)} f_j(N, cu_S, \succcurlyeq) \\
& \overset{\textbf{(A4)}}{=} c - \sum_{j\in(\uparrow_{\succcurlyeq^{(2)}} i)} f_j(N, cu_S, \succcurlyeq) - \sum_{j\in(\downarrow_{\succcurlyeq^{(2)}} i)} f_j(N, cu_S, \succcurlyeq^{(2)}) \\
& \overset{\textbf{(A3)}}{=} c - \sum_{j\in(\uparrow_{\succcurlyeq^{(2)}} i)} f_j(N, cu_S, \succcurlyeq) - \sum_{j\in(\downarrow_{\succcurlyeq^{(2)}} i)} f_j((\downarrow_{\succcurlyeq^{(2)}} i), cu_S, \succcurlyeq^{(2)}) \\
& \overset{\textbf{(E)}}{=} - \sum_{j\in(\uparrow_{\succcurlyeq^{(2)}} i)} f_j(N, cu_S, \succcurlyeq)
\end{aligned} \tag{4}$$

By construction, there exists a unique priority agent in $(N, \succcurlyeq^{(2)})$, say i_0. Therefore, $\uparrow_{\succcurlyeq^{(2)}} i_0 = \varnothing$. Set $i = i_0$ in (4) and we immediately get

$$f_{i_0}(N, cu_S, \succcurlyeq) = 0.$$

If $N\backslash\big((S\cup\downarrow_{\succcurlyeq} S) \cup i_0\big)$ is empty, then we are done. Otherwise, consider the subposet $(N\backslash i_0, \succcurlyeq^{(2)})$. Once again, it contains a unique priority agent, say i_1, and $\uparrow_{\succcurlyeq^{(2)}} i_1 = \{i_0\}$ in $(N, \succcurlyeq^{(2)})$. By (4) and the previous step,

$$f_{i_1}(N, cu_S, \succcurlyeq) = -f_{i_0}(N, cu_S, \succcurlyeq) = 0.$$

Continuing in this fashion by taking the remaining agents in $N\backslash\big((S\cup \downarrow_{\succcurlyeq} S) \cup \{i_0,\ldots,i_{k-1}\}\big)$ in order and starting with the priority agent i_k of $(N\backslash\big((S\cup \downarrow_{\succcurlyeq} S) \cup \{i_0,\ldots,i_{k-1}\}\big), \succcurlyeq^{(2)})$, we get the desired result. ■

Example 2. Consider the Hasse diagram of the poset $(N, \succcurlyeq)$, where $N = \{1,\ldots,9\}$, represented in the left part of Figure 2. Consider the coalition $S = \{3,4,7\}$. Then, $M(S,\succcurlyeq) = \{3,4\}$ and $\downarrow_{\succcurlyeq} S = \{6,8,9\}$. In **Step 1** of the proof of Lemma 1, the subposet $(\{1,2,5\},\succcurlyeq)$ is extended to a linear order. Then, the poset $(N,\succcurlyeq^{(1)})$ containing $(N,\succcurlyeq)$ is constructed. In **Step 2** of the proof of Lemma 1, the lowest priority agent 5 of $(N\backslash(S\cup \downarrow_{\succcurlyeq} S), \succcurlyeq^{(1)})$ is connected to the priority agents 3 and 4 of $S = \{3,4,7\}$. These two steps are represented in the central and right part of Figure 2, respectively. □

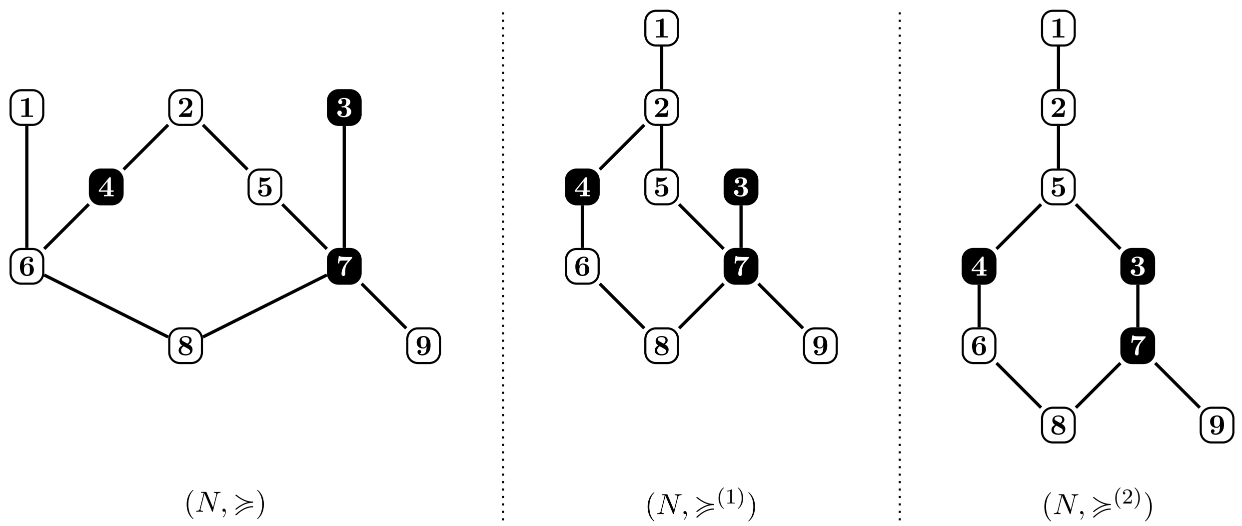

Figure 2: Hasse diagrams of $(N, \succcurlyeq)$, $(N, \succcurlyeq^{(1)})$ after **Step** 1 and $(N, \succcurlyeq^{(2)})$ after **Step** 2.

We have the material to prove our first result.

Proposition 1. *Efficiency* **(E)**, *Additivity* **(A)**, *Priority agent out* **(A3)** *and Invariance to unproductive priority extension* **(A4)** *on* $GP_{N'}$ *implies the Null agent axiom* **(N)**.

Proof. First, **(A)** implies **(NG)**. Next, assume that $i \in N$ is a null agent in $(N, v, \succcurlyeq) \in GP_{N'}$. By point 1 of Remark 1, $\Delta_S(v) = 0$ for each $S \ni i$, so that v can be expressed as follows:

$$v = \sum_{S\subseteq N: i \in N\backslash S} \Delta_S(v) u_S.$$

By **(A)** and Lemma 1, conclude that i obtains a null payoff. ■

Adding Equal treatment for necessary agents with equal priority group in the statement of Proposition 1 yields a characterization of the Harsanyi solution as in (2), which distributes the dividend of each coalition equally among its priority agents. For this reason, this Harsanyi solution is named the Priority value. Formally, the **Priority value** P on $GP_{N'}$ is the Harsanyi solution given by:

$$\forall i \in N, \quad P_i(N, v, \succcurlyeq) = \sum_{S \subseteq N : M(S, \succcurlyeq) \ni i} \frac{\Delta_S(v)}{|M(S, \succcurlyeq)|}. \tag{5}$$

Proposition 2. *The Priority value P is the unique allocation rule on $GP_{N'}$ satisfying Efficiency* **(E)**, *Additivity* **(A)**, *Priority agent out* **(A3)**, *Invariance to unproductive priority extension* **(A4)** *and Equal treatment for necessary agents with equal priority group* **(A1)**.

Proof. We first show that P satisfies all the axioms of the statement of Proposition 2.

(E) and **(A)**: follow from the fact that P is a Harsanyi solution.

(A3): pick any j in a TU-game with a priority structure $(N, v, \succcurlyeq) \in GP_{N'}$, and consider any $i \in \downarrow_{\succcurlyeq} j$. For $S \subseteq N$, if $i \in M(S, \succcurlyeq)$, then $j \in N \backslash S$, so that $S \subseteq N \backslash j$. Hence, we have:

$$P_i(N, v, \succcurlyeq) = \sum_{S \subseteq N : M(S, \succcurlyeq) \ni i} \frac{\Delta_S(v)}{|M(S, \succcurlyeq)|} = \sum_{S \subseteq N \backslash j : M(S, \succcurlyeq) \ni i} \frac{\Delta_S(v)}{|M(S, \succcurlyeq)|} = P_i(N \backslash j, v, \succcurlyeq),$$

which shows that P satisfies **(A3)**.

(A4): consider any $(N, v, \succcurlyeq) \in GP_{N'}$ and two incomparable agents i and j in $(N, \succcurlyeq)$ such that i and all the agents $k \in \uparrow_{\succcurlyeq} i$ having priority over i are null agents in (N, v). By definition of an elementary extension, $M(S, \succcurlyeq_{i->j}) \subseteq M(S, \succcurlyeq)$, and $M(S, \succcurlyeq) \neq M(S, \succcurlyeq_{i \to j})$ if and only if there exist $k, q \in M(S, \succcurlyeq)$ where $k \in (i \cup \uparrow_{\succcurlyeq} i)$ and $q \in (j \cup \downarrow_{\succcurlyeq} j)$. In this case, $M(S, \succcurlyeq_{i \to j}) = M(S, \succcurlyeq) \backslash (j \cup \downarrow_{\succcurlyeq} j)$, and because S contains a null agent k in $i \cup \uparrow_{\succcurlyeq} i$, $\Delta_S(v) = 0$. Hence, we can write:

$$\begin{aligned}
\forall \ell \in N, \quad P_\ell(N, v, \succcurlyeq) &= \sum_{S \subseteq N : M(S, \succcurlyeq) \ni \ell} \frac{\Delta_S(v)}{|M(S, \succcurlyeq)|} \\
&= \sum_{\substack{S \subseteq N : M(S, \succcurlyeq) \ni \ell \\ M(S, \succcurlyeq) = M(S, \succcurlyeq_{i \to j})}} \frac{\Delta_S(v)}{|M(S, \succcurlyeq)|} + \sum_{\substack{S \subseteq N : M(S, \succcurlyeq) \ni \ell \\ M(S, \succcurlyeq) \neq M(S, \succcurlyeq_{i \to j})}} \frac{\Delta_S(v)}{|M(S, \succcurlyeq)|} \\
&= \sum_{S \subseteq N : M(S, \succcurlyeq_{i \to j}) \ni \ell} \frac{\Delta_S(v)}{|M(S, \succcurlyeq_{i \to j})|} \\
&= P_\ell(N, v, \succcurlyeq_{i \to j}),
\end{aligned}$$

which shows that P satisfies **(A4)**.

(A1): consider any $(N, v, \succcurlyeq) \in GP_{N'}$ and any two necessary agents i and j in (N, v) such that $\uparrow_{\succcurlyeq} i = \uparrow_{\succcurlyeq} j$. By Remark 1, $\Delta_S(v) = 0$ for each $S \subseteq ((N \backslash i) \cup (N \backslash j))$. In addition, for a coalition S containing both i and j, $\uparrow_{\succcurlyeq} i = \uparrow_{\succcurlyeq} j$ implies that $i \in M(S, \succcurlyeq)$ if and only if $j \in M(S, \succcurlyeq)$. Hence, we have:

$$
\begin{aligned}
P_i(N,v,\succcurlyeq) &= \sum_{S\subseteq N: M(S,\succcurlyeq)\ni i} \frac{\Delta_S(v)}{|M(S,\succcurlyeq)|} \\
&= \sum_{\substack{S\subseteq N: S\supseteq\{i,j\}\\ i\in M(S,\succcurlyeq)}} \frac{\Delta_S(v)}{|M(S,\succcurlyeq)|} \\
&= \sum_{\substack{S\subseteq N: S\supseteq\{i,j\}\\ j\in M(S,\succcurlyeq)}} \frac{\Delta_S(v)}{|M(S,\succcurlyeq)|} \\
&= \sum_{S\subseteq N: M(S,\succcurlyeq)\ni j} \frac{\Delta_S(v)}{|M(S,\succcurlyeq)|} \\
&= P_j(N,v,\succcurlyeq),
\end{aligned}
$$

as desired.

For the uniqueness part, consider any allocation rule f on $GP_{N'}$ satisfying **(E)**, **(A)**, **(A3)**, **(A4)** and **(A1)**. To show: $f = P$. Choose any $(N,v,\succcurlyeq)\in GP_{N'}$. We have:

$$f(N,v,\succcurlyeq) \overset{\textbf{(A)}}{=} \sum_{\varnothing\subsetneq S\subseteq N} f(N,\Delta_S(v)u_S,\succcurlyeq).$$

Thus, it is enough to show that, for each nonempty $S\subseteq N$, $f(N,\Delta_S(v)u_S,\succcurlyeq) = P(N,\Delta_S(v)u_S,\succcurlyeq)$. Consider any such coalition S. Each $i\in N\backslash S$ is null in $(N,\Delta_S(v)u_S)$, so that $f_i(N,\Delta_S(v)u_S,\succcurlyeq) = 0$ by Proposition 1. Next, if $i\in S\backslash M(S,\succcurlyeq)$, then there exists $j\in S$ such that $j\succ i$. We have:

$$f_i(N,\Delta_S(v)u_S,\succcurlyeq) \overset{\textbf{(A3)}}{=} f_i(N\backslash j,\Delta_S(v)u_S,\succcurlyeq) = f_i(N\backslash j,\mathbf{0},\succcurlyeq) \overset{\textbf{(A)}}{=} 0,$$

where the last equality comes from the fact that **(A)** implies **(NG)**.

It remains to prove that all agents in $M(S,\succcurlyeq)$ receive the same payoff in $(N,\Delta_S(v)u_S,\succcurlyeq)$, and we will conclude by **(E)**. Notice that each member of $M(S,\succcurlyeq)\subseteq S$ is a necessary agent in $(N,\Delta_S(v)u_S)$. As in **Step** 1 and **Step** 2 of the proof of Lemma 1, we construct an extension $(N,\succcurlyeq^{(2)})$ of $(N,\succcurlyeq)$, using only elementary extensions and such that $M(S,\succcurlyeq) = M(S,\succcurlyeq^{(2)})$ and all agents in $M(S,\succcurlyeq)$ now have the same priority group on them in $(N,\succcurlyeq^{(2)})$. So, for any two agents i and j in $M(S,\succcurlyeq)$, we get:

$$f_i(N,\Delta_S(v)u_S,\succcurlyeq) \overset{\textbf{(A4)}}{=} f_i\big(N,\Delta_S(v)u_S,\succcurlyeq^{(2)}\big) \overset{\textbf{(A1)}}{=} f_j\big(N,\Delta_S(v)u_S,\succcurlyeq^{(2)}\big) \overset{\textbf{(A4)}}{=} f_j(N,\Delta_S(v)u_S,\succcurlyeq),$$

as desired. Together with **(E)**, conclude that:

$$f_i(N,\Delta_S(v)u_S,\succcurlyeq) = \begin{cases} \dfrac{\Delta_S(v)}{|M(S,\succcurlyeq)|} & \text{if } i\in M(S,\succcurlyeq),\\ 0 & \text{otherwise.}\end{cases}$$

which is precisely $P_i(N,\Delta_S(v)u_S,\succcurlyeq)$ and completes the proof. ■

The logical independence of the axioms can be demonstrated as follows:

- The null allocation rule satisfies all axioms except **(E)**;
- The allocation rule f such that, for each $(N, v, \succcurlyeq)$ and each $i \in N$,

$$f_i(N, v, \succcurlyeq) = \sum_{S \subseteq N : M(S, \succcurlyeq) \ni i} \frac{v(i)^2 + 1}{\sum_{j \in M(S, \succcurlyeq)} \big(v(j)^2 + 1\big)} \Delta_S(v)$$

  otherwise satisfies all axioms except **(A)**;
- The allocation rule f such that, for each $(N, v, \succcurlyeq)$ and each $i \in N$, $f_i(N, v, \succcurlyeq) = v(N)/n$ satisfies all axioms except **(A3)**;
- The allocation rule f such that, for each $(N, v, \succcurlyeq)$ and each $i \in N$,

$$f_i(N, v, \succcurlyeq) = \sum_{S \subseteq N : M(S, \succcurlyeq) \ni i} \frac{|\uparrow_{\succcurlyeq} i| + 1}{\sum_{j \in M(S, \succcurlyeq)} \big(|\uparrow_{\succcurlyeq} j| + 1\big)} \Delta_S(v)$$

  satisfies all axioms except **(A4)**. To see this, consider the game with priority structure $(N, u_{\{2,3,4\}}, \succcurlyeq)$ where $N = \{1, 2, 3, 4\}$ and where $\succcurlyeq$ is described by $2 \succcurlyeq 4$ and $3 \succcurlyeq 4$. Hence, the dividends of all coalitions are null except for coalition $\{2, 3, 4\}$ for which it is 6. In this initial situation, it is straightforward to compute that $f(N, v, \succcurlyeq) = (0, 3, 3, 0)$. Now consider the game with priority structure $(N, u_{\{2,3,4\}}, \succcurlyeq_{1 \to 2})$. Since $\uparrow_{\succcurlyeq} 1 = \varnothing$ and 1 is a null agent in $(N, u_{\{2,3,4\}})$, the conditions required in **(A4)** are met. However, $f(N, v, \succcurlyeq_{1 \to 2}) = (0, 4, 2, 0) \neq f(N, v, \succcurlyeq)$, as desired ;
- The allocation rule f such that, for each $(N, v, \succcurlyeq)$ and each $i \in N$,

$$f_i(N, v, \succcurlyeq) = \sum_{S \subseteq N : M(S, \succcurlyeq) \ni i} \frac{\omega_i}{\sum_{j \in M(S, \succcurlyeq)} \omega_j} \Delta_S(v)$$

  for some weight vector $\omega = (\omega_i)_{i \in N} \in \mathbb{R}^N_{++}$ in which all coordinates are different satisfies all axioms except **(A1)**;

Another characterization of the Priority value can be obtained by substituting in the statement of Proposition 2 the two axioms involving natural operations on the poset structure, namely Priority agent out and Invariance to unproductive priority extension, by the Null agent out axiom and the Necessary and priority agent axiom. The Null agent out axiom does not refer to the priority structure whereas the Necessary and priority agent axiom indicates that a necessary agent imposes a null payoff on the agents over which it has priority. Interestingly, the combination of Priority agent out and Null Game implies the Necessary and priority agent axiom, whereas the combination of Null agent out axiom and Efficiency implies Invariance to unproductive priority extension. These results are collected in the following propositions.

Proposition 3. *On the class of TU-games with a priority structure $GP_{N'}$,*

1. *Priority agent out* **(A3)** *and Null game* **(NG)** *imply Necessary and priority agent axiom* **(A2)***;*
2. *Null agent out axiom* **(NAO)** *and Efficiency* **(E)** *imply Invariance to unproductive priority extension* **(A4)***.*

Proof. Point 1. Consider any $(N, v, \succcurlyeq) \in GP_{N'}$ and any necessary agent i in (N, v) and any agent $j \in \downarrow_{\succcurlyeq} i$ over which i has priority. Because i is a necessary agent in (N, v), the subgame $(N \backslash i, v)$ is the null game. We have

$$f_j(N, v, \succcurlyeq) \overset{\textbf{(A3)}}{=} f_j(N \backslash i, v, \succcurlyeq) = f_j(N \backslash i, \mathbf{0}, \succcurlyeq) \overset{\textbf{(NG)}}{=} 0,$$

which shows that f satisfies the **(A2)**.

Point 2. Consider any $(N, v, \succcurlyeq) \in GP_{N'}$ and two incomparable agents i and j in $(N, \succcurlyeq)$ such that i and each agent $k \in \uparrow_{\succcurlyeq} i$ in the priority group on i are null agents in (N, v). Note that $(N \backslash (i \cup \uparrow_{\succcurlyeq} i), \succcurlyeq)$ coincides with $(N \backslash (i \cup \uparrow_{\succcurlyeq} i), \succcurlyeq_{i \to j})$. Therefore, for any $\ell \in N \backslash (i \cup \uparrow_{\succcurlyeq} i)$, it holds that:

$$f_\ell(N, v, \succcurlyeq) \overset{\textbf{(NAO)}}{=} f_\ell((N \backslash (i \cup \uparrow_{\succcurlyeq} i), v, \succcurlyeq)) = f_\ell((N \backslash (i \cup \uparrow_{\succcurlyeq} i), v, \succcurlyeq_{i \to j})) \overset{\textbf{(NAO)}}{=} f_\ell(N, v, \succcurlyeq_{i \to j}).$$

Furthermore, **(NAO)** and **(E)** imply **(N)**, so that:

$$\forall k \in (i \cup \uparrow_{\succcurlyeq} i), \quad f_k(N, v, \succcurlyeq) = 0 = f_k(N, v, \succcurlyeq_{i \to j}).$$

Conclude that $f(N, v, \succcurlyeq) = f(N, v, \succcurlyeq_{i \to j})$, which shows that f satisfies **(A4)**. ∎

Proposition 4. *The Priority value P is the unique allocation rule on $GP_{N'}$ satisfying Efficiency* **(E)***, Additivity* **(A)***, the Null agent out axiom* **(NAO)***, the Necessary and priority agent axiom* **(A2)***, and Equal treatment for necessary agents with equal priority group* **(A1)***.*

Proof. By Proposition 2, we already know that P satisfies **(E)**, **(A)** and **(A1)**. Because P is a Harsanyi solution, it also satisfies **(NAO)**. It remains to show that P satisfies **(A2)**. To this end, consider any $(N, v \succcurlyeq) \in GP_{N'}$ and any two distinct agents $i, j \in N$ such that $j \in \uparrow_{\succcurlyeq} i$ and j is necessary in (N, v). Thus, if a coalition S contains both i and j, agent i cannot belong to $M(S, \succcurlyeq)$ since $j \succ i$. It follows that:

$$P_i(N, v, \succcurlyeq) = \sum_{S \subseteq N : M(S, \succcurlyeq) \ni i} \frac{\Delta_S(v)}{|M(S, \succcurlyeq)|} = \sum_{S \subseteq N \backslash j : M(S, \succcurlyeq) \ni i} \frac{\Delta_S(v)}{|M(S, \succcurlyeq)|}.$$

Because j is a necessary agent in (N, v), it also holds that $\Delta_S(v) = 0$ whenever $j \in N \backslash S$ (see Remark 1). This implies that:

$$\sum_{S \subseteq N \backslash j : M(S, \succcurlyeq) \ni i} \frac{\Delta_S(v)}{|M(S, \succcurlyeq)|} = 0,$$

and so $P_i(N, v, \succcurlyeq) = 0$, proving that P satisfies **(A2)**.

For the uniqueness part, consider any allocation rule f on $GP_{N'}$ satisfying **(E)**, **(A)**, **(NAO)**, **(A2)** and **(A1)**. To show: $f = P$. Choose any $(N, v, \succcurlyeq) \in GP_{N'}$. As in the proof of Proposition 2, **(A)** implies that it is enough to prove that, for each nonempty coalition S, $f(N, \Delta_S(v)u_S, \succcurlyeq) = P(N, \Delta_S(v)u_S, \succcurlyeq)$. Consider any such S. Each $i \in N \backslash S$ is null in $(N, \Delta_S(v)u_S)$. The combination of **(NAO)** and **(E)** implies **(N)**, which rewards 0 each null agent. Because each $i \in N \backslash S$ is null in $(N, \Delta_S(v)u_S)$, we get $f_i(N, \Delta_S(v)u_S, \succcurlyeq) = 0$ for each $i \in N \backslash S$. We also have:

$$\forall i \in S, \quad f_i(N, \Delta_S(v)u_S, \succcurlyeq) \overset{(NAO)}{=} f_i(S, \Delta_S(v)u_S, \succcurlyeq).$$

In the TU-game $(S, \Delta_S(v)u_S)$, all agents are necessary. In case $M(S, \succcurlyeq) \neq S$, then for each $i \in S \backslash M(S, \succcurlyeq)$, there is $j \in M(S, \succcurlyeq)$ such that $j \in \uparrow_{\succcurlyeq} i$. Therefore:

$$\forall i \in S \backslash M(S, \succcurlyeq), \quad f_i(S, \Delta_S(v)u_S, \succcurlyeq) \overset{\textbf{(A2)}}{=} 0.$$

Finally, consider the priority agents in $M(S, \succcurlyeq)$. In $(S, \Delta_S(v)u_S, \succcurlyeq)$, they obviously have the same (empty) priority group on them. Thus, by **(A1)** and **(E)**, we get:

$$\forall i \in M(S, \succcurlyeq), \quad f_i(S, \Delta_S(v)u_S, \succcurlyeq) = \frac{\Delta_S(v)}{|M(S, \succcurlyeq)|}.$$

All in all, we have reached the desired result $f = P$, which completes the proof. ■

The logical independence of the axioms can be demonstrated as follows:

- The null allocation rule satisfies all axioms except **(E)**;
- The allocation rule f such that $f(N, v, \succcurlyeq) = P(N, v, \succcurlyeq)$ if (N, v) contains at least one necessary agent and $f(N, v, S) = Sh(N, v)$ otherwise satisfies all axioms except **(A)**;
- The allocation rule f such that, for each $(N, v, \succcurlyeq)$, $f_i(N, v, \succcurlyeq) = v(N)/|M(N, \succcurlyeq)|$ if $i \in M(N, \succcurlyeq)$ and 0 otherwise satisfies all axioms except **(NAO)**;
- The allocation rule f such that $f(N, v, \succcurlyeq) = Sh(N, v)$ satisfies all axioms except **(A2)**;
- The allocation rule f such that, for each $(N, v, \succcurlyeq)$ and each $i \in N$,

  $$f_i(N, v, \succcurlyeq) = \sum_{\substack{S \subseteq N: \\ \min_{j \in M(S, \succcurlyeq)} j = i}} \Delta_v(S).$$

  satisfies all axioms except **(A1)**.

We now detail the application of the Priority value to three particular structures.

The trivial poset

In case the priority structure is the trivial poset $(N, \succcurlyeq^0) \in P_N$, then, for each nonempty $S \subseteq N$, $M(S, \succcurlyeq^0) = S$. It follows that the Priority value coincides with the Shapley value: for each $i \in N$, we have

$$P_i(N, v, \succcurlyeq^0) = \sum_{S \subseteq N: M(S, \succcurlyeq^0) \ni i} \frac{\Delta_S(v)}{|M(S, \succcurlyeq^0)|} = \sum_{S \subseteq N: S \ni i} \frac{\Delta_S(v)}{s} = \mathrm{Sh}_i(N, v).$$

The Priority value can thus be viewed as a generalization of the Shapley value.

The outward pointing partial order

Consider now the case where the Hasse diagram of the priority structure $(N, \succcurlyeq^*) \in P_N$, $n \geqslant 2$, is shaped as a star. Precisely, there is an agent $r \in N$ such that $i \succ^* j$ if and only if $i = r$. For each $S \ni r$, $M(S, \succcurlyeq^*) = \{r\}$. Note that the subposet $(N \backslash r \succcurlyeq^*)$ is the trivial order so that for each $S \subseteq N \backslash r$, $M(S, \succcurlyeq^*) = S$. Therefore, by definition of the Priority value,

$$\begin{aligned}
\forall i \in N \backslash r, \quad P_i(N, v, \succcurlyeq^*) &= \sum_{S \subseteq N: M(S, \succcurlyeq^*) \ni i} \frac{\Delta_S(v)}{|M(S, \succcurlyeq^*)|} \\
&= \sum_{S \subseteq N \backslash r: M(S, \succcurlyeq^*) \ni i} \frac{\Delta_S(v)}{|M(S, \succcurlyeq^*)|} \\
&= \sum_{S \subseteq N \backslash r: S \ni i} \frac{\Delta_S(v)}{s} \\
&= \mathrm{Sh}_i(N \backslash r, v).
\end{aligned}$$

By Efficiency of the Shapley value, we have:

$$\sum_{i \in N \backslash r} \mathrm{Sh}_i(N \backslash r, v) = v(N \backslash r),$$

so that, by Efficiency of the Priority value, the unique top agent r gets its contribution to the grand coalition N, that is,

$$P_r(N, v, \succcurlyeq^*) = v(N) - v(N \backslash r).$$

The linear order

Consider the situation where the priority structure $(N, \succcurlyeq) \in P_N$ is a linear order. Without loss of generality, set $N = \{1, \ldots, n\}$, $n \geqslant 2$, and for each $i \in \{1, \ldots, n-1\}$, $i \succ i+1$. The Hasse diagram of $(N, \succcurlyeq)$ is shaped like a line where agent 1 is the unique priority agent. Consider first the lowest priority agent n. By using repeatedly Priority agent out, we get:

$$P_n(N, v, \succcurlyeq) = P_n(N \backslash 1, v, \succcurlyeq) = P_n(N \backslash \{1, 2\}, v, \succcurlyeq) = \cdots = P_n(\{n, n-1\}, v, \succcurlyeq) = P_n(\{n\}, v, \succcurlyeq),$$

By Efficiency applied to $(\{n\}, v, \succcurlyeq)$, we deduce that $P_n(N, v, \succcurlyeq) = v(n)$. Next, consider agent $n-1$. Proceeding in a similar way, we have:

$$P_{n-1}(N, v, \succcurlyeq) = P_{n-1}(N \backslash 1, v, \succcurlyeq) = P_n(N \backslash \{1, 2\}, v, \succcurlyeq) = \cdots = P_{n-1}(\{n, n-1\}, v, \succcurlyeq).$$

By Efficiency of the Priority value, we also have:

$$P_{n-1}(\{n,n-1\},v,\succcurlyeq)+P_n(\{n,n-1\},v,\succcurlyeq)=v(\{n,n-1\}),$$

and, by the previous step,

$$P_n(\{n,n-1\},v,\succcurlyeq)=P_n(\{n\},v,\succcurlyeq)=v(n).$$

Therefore, we get:

$$P_{n-1}(N,v,\succcurlyeq)=v(\{n,n-1\})-v(n).$$

Proceeding inductively from n to 1, we conclude that the Priority value applied to a linear order coincides withe the so-called **downward marginal vector** m^{σ^d} where $\sigma^d(i)=n-i+1$ for $i\in\{1,\ldots,n-1\}$. Furthermore, given the above reasoning, we obtain that the Priority value is characterized by Efficiency and Priority agent out on the subdomain where the priority structures are linear orders. This result is summarized below.

Proposition 5. *On the subdomain of $GP_{N'}$ where the priority structures $(N,\succcurlyeq)$ are linear orders, the Priority value P is the unique allocation rule satisfying Efficiency* **(E)** *and Priority agent out* **(A3)**, *and it coincides with the downward marginal vector m^{σ^d}:*

$$P(N,v,\succcurlyeq)=m^{\sigma^d}(N,v).$$

The downward marginal vector as been successfully applied to river games and sequencing games by van den Brink et al. (2007). Nevertheless, it should be stressed that van den Brink et al. (2007) apply the downward marginal vector to the Myerson restricted-game (Myerson, 1977) on situations where the Hasse diagram of the priority structure is partitioned into several components $\{C_p : p=1,\ldots,c\}$, where each subposet $(C_p,\succcurlyeq)$ of $(N,\succcurlyeq)$ is a linear order.[1] Thus, in case the priority structure has several components, that is $c>1$, their allocation rule is different from ours because it computes the downward marginal vector on each (sub)linear order $(C_p,\succcurlyeq)$ of the priority structure, meaning that their allocation rule is Component efficient whereas the Priority value is Efficient. They characterize their allocation rule by using Component efficiency and Lower equivalence. The latter axiom indicates that deleting the priority relation between agent i and agent $i+1$ does not affect the payoff of $i+1$ and all the agents over which it has priority in the component. Lower equivalence and Priority agent out have the same flavor. The difference between both axioms is that Lower equivalence means that the payoff of an agent does not depend on the presence of "upward" priority relations while Priority agent out indicates that the payoff of an agent does not depend on the presence of "upward" agents. However, the combination of Component efficiency and Priority agent out yields a characterization of the allocation rule proposed by van den Brink et al. (2007) on the corresponding domain with a variable agent set.

[1] van den Brink et al. (2007) use a different formalism: the structure is represented by a collection of directed graphs, where each of these graphs is shaped as a directed line.

5. Comparison with other models and allocation rules

Two other prominent models use a poset to model the relations among the agents: cooperative games with a permission structure introduced by Gilles et al. (1992) and cooperative games under precedence constraints introduced by Faigle and Kern (1992). In this section, we compare the Priority value with the Permission value for cooperative games with a permission structure and with the Precedence Shapley value for cooperative games under precedence constraints.

5.1. *Cooperative games with a permission structure*

Gilles et al. (1992) consider situations where cooperation between agents in a cooperative TU-game $(N, v) \in G_N$ is influenced by a hierarchical structure represented by a directed graph on N. In case the directed graph is acyclic these situations can be represented by a poset $(N, \succcurlyeq) \in P_N$, called a **permission structure**, and the triple $(N, v, \succcurlyeq)$ is a TU-game with a permission structure. In this model, the permission structure affects the possibilities of cooperation between the agents in the sense that each agent needs permission from all its **superiors** to cooperate, where the set of superiors of agent i corresponds to the priority group on i in our model. The relation $\succcurlyeq$ determines how coalitions are **evaluated**: a coalition is called **feasible** if for each member of this coalition all its superiors are also in the coalition. Therefore, a coalition S is feasible if it satisfies the following permission constraints: for each agent $i \in S$, $j \succcurlyeq i$ implies $j \in S$. For each coalition S, the interior $\sigma_{\succcurlyeq}(S)$ of S, also called the sovereign part of S, is the greatest – with respect to set inclusion – feasible subcoalition of S:

$$\sigma_{\succcurlyeq}(S) = S \backslash \Big(\bigcup_{j \in N \backslash S} \downarrow_{\succcurlyeq} j \Big).$$

The closure $\alpha_{\succcurlyeq}(S)$ of a coalition S, also called the authorizing part of S – is the smallest feasible coalition containing S:

$$\alpha_{\succcurlyeq}(S) = S \cup \Big(\bigcup_{i \in S} \uparrow_{\succcurlyeq} i \Big).$$

Thus, a coalition S is feasible if and only if coincides with its interior $\sigma_{\succcurlyeq}(S)$ and its closure $\alpha_{\succcurlyeq}(S)$.

Gilles et al. (1992) introduce a **restricted game** $(N, v_{\succcurlyeq}) \in G_N$ which assigns to each coalition the worth of its interior:

$$\forall S \subseteq N, \quad v_{\succcurlyeq}(S) = v(\sigma_{\succcurlyeq}(S)).$$

van den Brink and Gilles (1996) define the **Permission value** ρ as the allocation rule that assigns to each TU-game with a permission structure $(N, v, \succcurlyeq)$ the Shapley value of its restricted game $(N, v_{\succcurlyeq})$:

$$\rho(N, v, \succcurlyeq) = \mathrm{Sh}(N, v_{\succcurlyeq}).$$

By definition of the closure and the interior of a coalition S, we have:

$$\forall S, T \subseteq N, \quad \big[S \subseteq \sigma_{\succcurlyeq}(T)\big] \Longleftrightarrow \big[\alpha_{\succcurlyeq}(S) \subseteq T\big], \text{ and so } u_S(\sigma_{\succcurlyeq}(T)) = u_{\alpha_{\succcurlyeq}(S)}(T).$$

From this and the definition of the Shapley value, the Permission value can be rewritten as:

$$\forall i \in N, \quad \rho_i(N, v, \succcurlyeq) = \sum_{S \in 2^N : \alpha_{\succcurlyeq}(S) \ni i} \frac{\Delta_S(v)}{|\alpha_{\succcurlyeq}(S)|}.$$

There are two major differences with our model. First, in our model all coalitions are feasible, the priority structure only comes into play in the process of allocating the Harsanyi dividend of each coalition and not in the evaluation of a coalition. Second, the Priority value is a Harsanyi solution, as defined in (2), while the Permission value is not. The Permission value distributes the Harsanyi dividend of each coalition equally within its authorizing set and so possibly outside the coalition itself.

van den Brink and Gilles (1996, Theorem 4.4) provide an axiomatic characterization of the permission value in terms of Efficiency, Additivity, the Strongly Inessential agent axiom, Strongly inessential relational axiom, Structural monotonicity, and the Necessary agent axiom. Strongly Inessential agent axiom indicates if a null agent has no subordinates (the agents over whom it has priority in our model), then it obtains a null payoff. Strongly inessential relational axiom indicates that the deletion of a null agent with no subordinates does not affect the payoff of the remaining agents. Structural monotonicity states that in a monotone TU-game, an agent earns a least as much as their successors.[2] Finally, the Necessary agent axiom indicates that any necessary agent earns as least as much as any other agents in a monotone TU-game.

The Priority value satisfies the Strongly Inessential agent axiom because it satisfies the stronger Null agent axiom by Proposition 1 and Proposition 2. The Priority value also satisfies the Strongly inessential relational axiom because, by Proposition 4, it satisfies the stronger Null agent out axiom. On the other hand, the Priority value satisfies neither the Necessary agent axiom nor Structural monotonicity. It does not satisfy the Necessary agent axiom because the later implies that two necessary agents obtain the same payoff in a monotone TU-game. Under the Priority value, two necessary agents do not necessary earn the same payoff in a monotone TU-game, unless they share the same priority group on them by the Necessary and priority agent axiom and Proposition 4. To see that the Priority value violates Structural monotonicity, consider the following example.

Example 3. Assume that the members of $N = \{1, 2, 3\}$ are linearly ordered as follows: $1 \succ 2 \succ 3$; and that the monotone TU-game (N, v) is given by $v(N) = 3$, $v(S) = 2$ if $|S| = 2$, and $v(S) = 0$ if $|S| = 1$. Because $(N, \succcurlyeq)$ is a linear order, by Proposition 5, the Priority value coincides with the downward marginal vector m^{σ^d}. Even if $1 \succ 2$, we have $m_1^{\sigma^d}(N, v, \succcurlyeq) = v(N) - v(\{2, 3\}) = 1$ which is strictly less that $m_2^{\sigma^d}(N, v, \succcurlyeq) = v(\{1, 2\}) - v(\{1\}) = 2$. □

For a detailed survey on the Permission value and its applications, we refer the reader to van den Brink (2017) who presents the alternative disjunctive and local restricted games.

5.2. Cooperative games under precedence constraints

Faigle and Kern (1992) introduce TU-games under precedence constraints represented by a triple $(N, v, \succcurlyeq)$ where $(N, \succcurlyeq)$ is a poset. They consider that the relation $i \succcurlyeq j$ means that the presence of i enforces the presence of j in a coalition, and that the poset determines how coalitions **form**. Thus, and unlike in TU-games with permission structures, certain coalitions are not allowed to form in TU-games under precedence constraints. In this model, the coalition function v is defined

[2] An agent j is a successor of i in $(N, \succcurlyeq)$ if $i \succ j$ and there is not other $\ell \in N \backslash \{i, j\}$ such that $i \succ \ell \succ j$.

on the **restricted domain** $\mathcal{D}_{\succcurlyeq} \subseteq 2^N$ which is the set of coalitions such that for any agent in the coalition all its successors with respect to $(N, \succcurlyeq)$ also belong to this coalition. That is, $S \in \mathcal{D}_{\succcurlyeq}$ if for each $i \in S$, $i \succcurlyeq j$ implies $j \in S$. In a more formal way, $\mathcal{D}_{\succcurlyeq}$ is the collection of **downsets** of $(N, \succcurlyeq)$ defined as

$$\mathcal{D}_{\succcurlyeq} = \big\{S \subseteq N : \forall i \in S, \downarrow_{\succcurlyeq} i \subseteq S\big\},$$

and forms a lattice of sets endowed with set inclusion, meaning that $\mathcal{D}_{\succcurlyeq}$ is closed under union and intersection.[3]

Example 4. Consider the poset $(N \succcurlyeq)$ given in Example 1. The collection of downsets is $\mathcal{D}_{\succcurlyeq} = \big\{\{9\}, \{10\}, \{8, 9, 10\}, \{7, 9\}, \{6, 7, 8, 9, 10\}\big\}$ and the corresponding Hasse diagram of $(\mathcal{D}_{\succcurlyeq}, \subseteq)$ is represented by the Figure 3. □

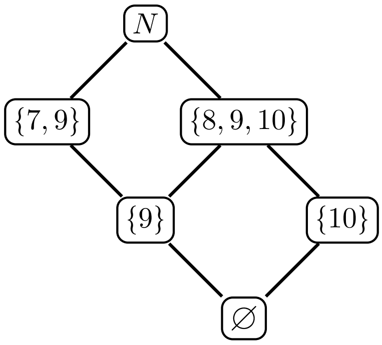

Figure 3: The lattice $(\mathcal{D}_{\succcurlyeq}, \subseteq)$.

In short a **TU-game under precedence constraints** is a triple $(N, v, \succcurlyeq)$ where $(N, \succcurlyeq)$ is a poset and (N, v) a TU-game such that $v : \mathcal{D}_{\succcurlyeq} \longrightarrow \mathbb{R}$. Faigle and Kern (1992) introduce a Shapley-like value for TU-game under precedence constraints by restricting the set of orderings $O(N)$ to the subset of orderings $O_{\succcurlyeq}(N)$ compatible with $(N, \succcurlyeq)$:

$$O_{\succcurlyeq}(N) = \big\{\sigma \in O(N) : \sigma(i) > \sigma(j) \text{ if } i \succ j\big\}.$$

Given a TU-game under precedence constraints $(N, v, \succcurlyeq)$, the **precedence Shapley value** ϕ is defined as:

$$\forall i \in N, \quad \phi_i(N, v \succcurlyeq) = \frac{1}{|O_{\succcurlyeq}(N)|} \sum_{\sigma \in O_{\succcurlyeq}(N)} m_i^\sigma(N, v) = \sum_{S \in \mathcal{D}_{\succcurlyeq} : S \ni i} \frac{h_{S,i}(N, \succcurlyeq)}{\sum_{j \in S} h_{S,j}(N \succcurlyeq)} \Delta_S(v),$$

where $h_{S,i}(N, \succcurlyeq)$ is called the **hierarchical strength** of i in S an is given by the number of orderings in $O_{\succcurlyeq}(N)$ where agent $i \in S$ enters after the agents in $S \backslash i$.

Once again, there are two major differences with our model. The first one is that here certain coalitions are not allowed to form. Second, if we concentrate the analysis on a downset S, the

[3]In a dual way, this is also true for the set of feasible coalitions in the model of Gilles et al. (1992).

allocation rule ϕ distributes the Harsanyi dividend $\Delta_S(v)$ among the priority agents in $M(S, \succcurlyeq)$ but proportionally to their hierarchical strength. Indeed, if there j such that $j \succ i$, then $h_{S,j}(N, \succcurlyeq) = 0$. In contrast, the Priority value P distributes $\Delta_S(v)$ equally among the members of $M(S, \succcurlyeq)$. Of course, in case $(N, \succcurlyeq)$ is a linear order, the only one ordering compatible with $(N, \succcurlyeq)$ is σ^d so that

$$\phi_i(N, v \succcurlyeq) = m^{\sigma^d}(N, v) = P(N, w, \succcurlyeq)$$

for any $(N, w) \in G_N$ such that the subgame $w^{\mathcal{D}_\succcurlyeq}$ induced by $\mathcal{D}_\succcurlyeq$ coincides with v.

Faigle and Kern (1992) provide an axiomatization of the precedence Shapley value, using Efficiency, Linearity (Additivity plus Homogeneity of degree 1), the Null agent axiom and the Hierarchical strength axiom. The latter stipulates that, in a unanimity game on a downset S, agents in S are rewarded according to their relative hierarchical strengths. As noted above, the Priority value does not satisfies this principle, but satisfies Linearity and the Null agent axiom. For more information on cooperative games under precedence constraints and their solutions, see Algaba et al. (2017).

Below is a numerical example that captures the differences between the three allocation rules discussed in this section.

Example 5. For the poset given in Example 1, the set of feasible orderings is

$$O_\succcurlyeq(N) = \big\{(10, 9, 8, 7, 6), (9, 10, 8, 7, 6), (10, 9, 7, 8, 6), (9, 10, 7, 8, 6), (9, 7, 10, 8, 6)\big\}$$

where, for the sake of simplicity, $(10, 9, 8, 7, 6)$ means that agent 10 is ranked first, and so on. Consider the unanimity TU-game on the downset $S = \{7, 8, 9, 10\} \in \mathcal{D}_\succcurlyeq$, the hierarchical strengths are $h_{S,9}(N, \succcurlyeq) = h_{S,10}(N, \succcurlyeq) = 0$ and $h_{S,7}(N, \succcurlyeq) = 2$ and $h_{S,8}(N, \succcurlyeq) = 3$. We also have $\sigma_\succcurlyeq(S) = \varnothing$ and $\alpha_\succcurlyeq(S) = N$ so that $(u_S)_\succcurlyeq = u_N$. Finally, $M(S, \succcurlyeq) = \{7, 8\}$. Therefore, with a slight abuse of notation,

$$P(N, u_S, \succcurlyeq) = \left(0, \frac{1}{2}, \frac{1}{2}, 0, 0\right), \quad \phi(N, u_S, \succcurlyeq) = \left(0, \frac{2}{5}, \frac{3}{5}, 0, 0\right), \text{ and } \rho(N, u_S, \succcurlyeq) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right).$$

$\square$

Remark 4. *If $(N, \succcurlyeq)$ is the trivial poset, then $\mathcal{D}_\succcurlyeq = 2^N$, $\sigma(S) = S$ and $M(S, \succcurlyeq) = S$ for any coalition S, so that:*

$$P(N, v, \succcurlyeq) = \rho(N, v, \succcurlyeq) = \phi(N, v, \succcurlyeq) = \mathrm{Sh}(N, v).$$

This means that the Priority value P, the Permission value ρ and the Precedence Shapley value ϕ are generalizations of the Shapley value.

6. Applications

6.1. Queueing problems

A finite set of agents $N \subseteq \mathbb{N}$ of size n stands to receive a service from a facility that can handle only one agent at time. Each agent $i \in N$ is characterized by its unitary waiting cost $\theta_i \geqslant 0$.

Each agent is assigned a position $\sigma(i) \in \{1, \dots, n\}$ in a queue and receives a transfer (positive or negative) $t_i \in \mathbb{R}$. If agent i is served in the $\sigma(i)$th position, then its waiting cost is $-(\sigma(i) - 1)\theta_i$. Utility is linear in position and transfer, that is:

$$u_i(\sigma_i, t_i \,|\, \theta_i) = -(\sigma(i) - 1)\theta_i + t_i,$$

where $t_i > 0$ means that i receives a compensation from the other agents, and $t_i < 0$ means that i has to pay that amount as compensation to other agents.

A **queueing problem** Q is a pair (N, θ), where N is a finite set of agents and $\theta = (\theta_i)_{i \in N}$ is a list of waiting costs. Given $Q = (N, \theta)$, a **feasible allocation** is a pair (σ, t) where $\sigma \in O(N)$ is an ordering, representing a queue, on N and $t = (t_i)_{i \in N}$ is a list of transfers such that $\sum_{i \in N} t_i \leqslant 0$. An allocation is **optimal** if σ minimizes the total waiting costs and t is budget balanced, that is:

$$\sigma \in \arg \min_{\sigma' \in O(N)} \sum_{i \in N} (\sigma'(i) - 1)\theta_i, \quad \text{and} \quad \sum_{i \in N} t_i = 0.$$

It is known that an optimal queue σ serves agents in the non-increasing order of waiting costs and that any queue with this property is optimal. In particular, if all unitary waiting costs are different, there is a unique optimal queue. If two agents have identical unitary waiting costs, these agents are served consecutively but in any order. An **allocation rule** f associates to each queueing problem $Q = (N, \theta)$ a non-empty subset $f(Q)$ of optimal allocations.

Maniquet (2003) solves queueing problems by treating them as TU-games. Given a queueing problem $Q = (N, \theta)$, he introduces the corresponding TU-game (N, v_Q) where the worth of each coalition S is given by its optimal total waiting costs under the optimistic assumption that they are served before agents in $N \backslash S$:

$$\forall S \subseteq N, \quad v_Q(S) = -\sum_{i \in S} (\sigma_S(i) - 1)\theta_i,$$

where $\sigma_S \in O(S)$ is an optimal queue for the subproblem $(S, (\theta_i)_{i \in S})$.

Maniquet (2003, Lemma 1) provides an expression of the Shapley value applied to (N, v_Q). To compute the Shapley value of the optimistic queueing TU-game (N, v_Q), it should be noted that the Harsanyi dividend of any coalition S of size $s \neq 2$ is equal to zero and the Harsanyi dividend of a coalition S of size 2 is equal to the minimum between the unitary waiting costs of its members. Formally,

$$\Delta_S(v) = \begin{cases} -\min\{\theta_i : i \in S\} & \text{if } |S| = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

From this, it easy to compute the Shapley value viewed as the Harsanyi solution which distributes equally the dividend of each coalition to its members:

$$\forall i \in N, \quad \mathrm{Sh}_i(N, v_Q) = -(\sigma(i) - 1)\frac{\theta_i}{2} \quad - \sum_{j \in N : \sigma(j) > \sigma(i)} \frac{\theta_j}{2}.$$

Therefore, each agent pays half of its waiting cost in an optimal queue plus half of the unit waiting cost of each agent placed behind it in that queue. Now, assume that the allocation (σ, t) for the queueing problem $Q = (N, \theta)$ is such that each agent's utility is equal to its Shapley value in (N, v_Q). Then, the transfers are given by:

$$\begin{aligned}\forall i \in N, \quad t_i &= u(\sigma_i, t_i \,|\, \theta_i) + (\sigma(i) - 1)\theta_i \\ &= \mathrm{Sh}_i(N, v_Q) + (\sigma(i) - 1)\theta_i \\ &= (\sigma(i) - 1)\frac{\theta_i}{2} - \sum_{j \in N : \sigma(j) > \sigma(i)} \frac{\theta_j}{2}. \end{aligned} \tag{7}$$

Moreover, the allocation (σ, t) is an optimal allocation. In particular, the fact that t is budged balanced comes from the Efficiency of the Shapley value. One can remark that each agent's transfer does not depend on the unitary waiting costs of agents preceding it in an optimal queue. Therefore, an increase in agent's unitary waiting cost should not affect the agents following it in an optimal queue. Maniquet (2003, Theorem 1, see also Chun, 2016) uses this property together with a principle of efficiency, a principle of equal treatment of equals and a principle of Pareto indifference to characterize the allocation rule f which selects all the allocations assigning to the agents utilities the Shapley value of the corresponding TU-game.

Now, we consider **queueing problems with priority structure** as a triplet $(N, \theta, \succcurlyeq)$, where (N, θ) is a queueing problem and $(N, \succcurlyeq)$ is a poset. The relation $i \succcurlyeq j$ still means that agent i has priority over agent j. However, in the context of a cost sharing problem, this priority should translate into a larger share of the total cost to i than to j, all other things being equal. In other words, the total waiting costs must be allocated as a priority to i rather than j.

For a given $i \in N$, denote by $\mathcal{I}_{\succcurlyeq}(i)$ the set of incomparable agents with i in $(N, \succcurlyeq)$. Consider the optimistic queueing TU-game endowed with the corresponding priority structure $(N, v_Q, \succcurlyeq)$. The following proposition shows that the Priority value is a very natural solution which can be easily compared to the Shapley value. For each pair of agents, the Shapley value distributes equally the minimal unit waiting cost among them whereas the Priority value takes into account their relative priority, *i.e.* which has priority in assuming this minimal unit waiting cost. Precisely, if one agent has priority over another agent, then it fully pays the minimal unit waiting cost. Otherwise, the distribution is not impacted. Furthermore, each agent's transfer computed with respect to the Priority value does not depend on the unitary waiting costs of agents preceding it in an optimal queue nor of agents having priority on it (whether they are preceding it in an optimal queue or not).

Proposition 6. *Consider the queueing problem with a priority structure $(N, \theta, \succcurlyeq)$ and the corresponding optimistic queueing TU-game with priority structure $(N, v_Q, \succcurlyeq)$. Then, the Priority value is given by: for each $i \in N$,*

$$\begin{aligned} P_i(N, v_Q, \succcurlyeq) = & - \theta_i \left(\left| \{ j \in N : \sigma(j) < \sigma(i) \,\&\, i \succcurlyeq j \} \right| + \frac{\left| \{ j \in N : \sigma(j) < \sigma(i) \,\&\, j \in \mathcal{I}_{\succcurlyeq}(i) \} \right|}{2} \right) \\ & - \sum_{\substack{j \in N : \sigma(j) > \sigma(i) \\ i \succcurlyeq j}} \theta_j - \sum_{\substack{j \in N : \sigma(j) > \sigma(i) \\ j \in \mathcal{I}_{\succcurlyeq}(i)}} \frac{\theta_j}{2} \end{aligned} \tag{8}$$

Consequently, the associated transfer is equal to

$$t_i = \theta_i\left(\left|\{j \in N : \sigma(j) < \sigma(i) \,\&\, j \succcurlyeq i\}\right| + \frac{\left|\{j \in N : \sigma(j) < \sigma(i) \,\&\, j \in \mathcal{I}_{\succcurlyeq}(i)\}\right|}{2}\right) - \sum_{\substack{j\in N:\sigma(j)>\sigma(i)\\ i\succcurlyeq j}} \theta_j - \sum_{\substack{j\in N:\sigma(j)>\sigma(i)\\ j\in\mathcal{I}_{\succcurlyeq}(i)}} \frac{\theta_j}{2} \tag{9}$$

Proof. Pick an agent $i \in N$. The Harsanyi dividends shown in (6) only involve coalitions of size 2. Thus, to compute the Priority value for each agent $i \in N$, we only have to consider coalitions $S = \{i, j\}$ for $j \in N\backslash i$ in (5). Now, we have four cases: either $M(\{i, j\}, \succcurlyeq) = \{i\}$, that is $i \succcurlyeq j$, or $M(\{i, j\}, \succcurlyeq) = \{i, j\}$, that is $j \in \mathcal{I}_{\succcurlyeq}(i)$; and either $\min\{\theta_i, \theta_j\} = \theta_i$, that is $\sigma(j) < \sigma(i)$, or $\min\{\theta_i, \theta_j\} = \theta_j$, that is $\sigma(j) > \sigma(i)$. Each case gives rise to the four parts of the right hand side of (8). ■

Example 6. Consider the queueing problem with priority structure $(N, \theta, \succcurlyeq)$ such that $\succcurlyeq$ is a linear order. Without loss of generality, suppose $1 \succcurlyeq 2 \succcurlyeq \ldots \succcurlyeq n$, and take an optimal queue σ. Because there is no pair of incomparable agents, the transfers corresponding to the Priority value (8) simplify to:

$$t_i = \theta_i\left|\{j \in N : \sigma(j) < \sigma(i) \,\&\, j \succcurlyeq i\}\right| - \sum_{\substack{j\in N:\sigma(j)>\sigma(i)\\ i\succcurlyeq j}} \theta_j. \tag{10}$$

This formula can be implemented through the following transfer mechanism. From the optimal queue $(\sigma(1), \ldots, \sigma(n))$, we sort the agents according to the bubble sort algorithm that runs as follows. Starting from the first pair of consecutive agents in the queue, it compares and swaps them if they are in the wrong order with respect to the linear priority structure $(N, \succcurlyeq)$. The swap gives rise to a transfer between them: the agent which moves to the front position obtains the other agent's unit waiting cost. This operation is repeatedly applied to the newly obtained queue up to the last pair of consecutive agents. Then, the procedure restarts from the first pair of consecutive agents along the newly created queue and goes on until it coincides with the priority structure $(N, \succcurlyeq)$. The overall transfers paid or received by an agent are equal to the transfer given by (10).

For instance, set $n = 4$, so that $1 \succcurlyeq 2 \succcurlyeq 3 \succcurlyeq 4$, and $\sigma = (3, 1, 4, 2)$. The resulting optimal queue and the bubble sort mechanism are represented below in Figure 4. In a situation where the optimal queue and the priority ordering are very different and necessitate important transfers between the agents, the bubble sort algorithm can be considered as a gradual regime change. In the first step, agent 3 and 1 swap positions since agent 1 has priority over 3 in assuming a larger share of the total waiting costs. A first transfer from 1 to 3 corrects the gap between the position of 1 in the queue and the position it occupies in the order of payment priorities. The new ordering that is obtained is one step closer to the priority ordering. Two additional swaps with similar interpretations are further needed to match the priority ordering. □

6.2. Bankruptcy problems

A **bankruptcy problem** $B = (N, E, c)$ is described by a set of creditors N, an estate $E > 0$ and a vector of nonnegative claims $c = (c_i)_{i\in N}$ such that $E < \sum_{i\in N} c_i$. Bankruptcy problems can

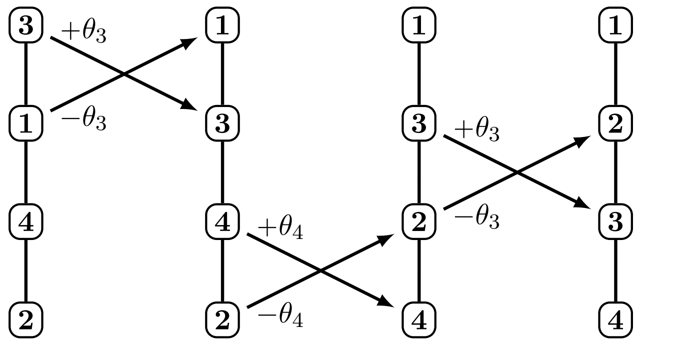

Figure 4: Computation of transfers using the bubble sort mechanism

be apprehended by cooperative games. O'Neill (1982) introduces the classical **bankruptcy game** (N, v_B) associated with B such that, for each $S \subseteq N$,

$$v_B(S) = \max\left\{0; E - \sum_{i \in N \setminus S} c_i\right\},$$

in which each coalition S gets either zero or what remains of the estate once the other creditors have obtained their claims. Alternatively, a bankruptcy problem can be studied without introducing cooperative games. As an example, Moulin (2000) characterizes the so-called priority rules, in which the individual claims are satisfied according to an exogenous ordering of the agents until the estate is exhausted; the remaining agents getting nothing. Formally, for an ordering $(\sigma(1), \ldots, \sigma(n))$, the interpretation is that agent i has priority over agent j if $\sigma(i) < \sigma(j)$. The **priority rule** associated with σ is the payoff vector $y^\sigma(B)$ defined as follows. Beforehand, compute the unique rank $r^* \in \{1, \ldots, n\}$ such that

$$\sum_{k=1}^{r^*-1} c_{\sigma^{-1}(k)} \leqslant E < \sum_{k=1}^{r^*} c_{\sigma^{-1}(k)}.$$

Then,

$$y_i^\sigma(B) = \begin{cases} c_i & \text{if } \sigma(i) \in \{1, \ldots, r^* - 1\}, \\ E - \sum_{k=1}^{r^*} c_{\sigma^{-1}(k)} & \text{if } \sigma(i) = r^*, \\ 0 & \text{if } \sigma(i) \in \{r^* + 1, \ldots, n\}. \end{cases}$$

Each agent with a rank lower than r^* gets its full claim, the agent at rank r^* gets what remains of E and the other agents get nothing.

Now, we consider **bankruptcy problems with priority structure** as a pair $(B, \succcurlyeq)$, where $\succcurlyeq$ is a priority structure on N. In this framework, we would like to construct a cooperative game which takes the priority structure into account in the most natural way. In order to do so, for each coalition $S \subseteq N$, define the priority group on S as

$$P_{\succcurlyeq}(S) = \left(\bigcup_{i \in S} \uparrow_{\succcurlyeq} i\right) \Big\backslash \left(S \cup \left(\bigcup_{i \in S} \downarrow_{\succcurlyeq} i\right)\right).$$

These are the agents outside S which have priority over a member of S but on which no-one in S has priority. The **bankruptcy game with priority structure** $(N, v_B^{\succcurlyeq})$ associated with $(B, \succcurlyeq)$ is such that, for each $S \subseteq N$,

$$v_B^{\succcurlyeq}(S) = \max\left\{0; E - \sum_{i \in P_{\succcurlyeq}(S)} c_i\right\}.$$

In words, $v_B^{\succcurlyeq}(S)$ is either zero or what remains of the estate once the agents having priority over S have obtained their claims. Since $P_{\succcurlyeq}(S) \subseteq N \backslash S$, note that $v_B^{\succcurlyeq}(S) \geqslant v_B(S)$, and of course $v_B^{\succcurlyeq}(N) = v_B(N) = E$. We show that if the Priority value is applied to the game $(N, v_B^{\succcurlyeq})$ associated with a linear order $\succcurlyeq$, then it coincides with a certain priority rule characterized in Moulin (2000): claims are distributed in accordance with the linear priority order $\succcurlyeq$ until the estate is exhausted.

Proposition 7. *Consider the bankruptcy problems with priority claims $(B, \succcurlyeq)$ such that $\succcurlyeq$ is a linear order. Then $P(N, v_B^{\succcurlyeq}) = y^{\sigma^{\succcurlyeq}}(B)$ where $\sigma^{\succcurlyeq}(i) = n - |\downarrow_{\succcurlyeq} i|$ for each $i \in N$.*

Before proving this result, note that an equivalent definition of $\sigma^{\succcurlyeq}$ is that $\sigma^{\succcurlyeq}(i) < \sigma^{\succcurlyeq}(j)$ if and only if $i \succcurlyeq j$. In words, the rank of agent i is lower than the rank of agent j in the order $\sigma^{\succcurlyeq}$ if and only if i has priority over j according to $\succcurlyeq$.

Proof. Assume that $\succcurlyeq$ is a linear order. From Proposition 4, for each $i \in N$, we already know that $P_i(N, v_B^{\succcurlyeq})$ can be written as $v_B^{\succcurlyeq}((\downarrow_{\succcurlyeq} i) \cup i) - v_B^{\succcurlyeq}(\downarrow_{\succcurlyeq} i)$, so that it is enough to prove that

$$v_B^{\succcurlyeq}((\downarrow_{\succcurlyeq} i) \cup i) - v_B^{\succcurlyeq}(\downarrow_{\succcurlyeq} i) = y_i^{\sigma^{\succcurlyeq}}(B).$$

Since $\succcurlyeq$ is a linear order, remark that $P_{\succcurlyeq}((\downarrow_{\succcurlyeq} i) \cup i) = (\uparrow_{\succcurlyeq} i)$ for each $i \in N$. Hence, the definition of $v_B^{\succcurlyeq}$ yields that $v_B^{\succcurlyeq}((\downarrow_{\succcurlyeq} i) \cup i) = \max\{0; E - \sum_{j \in \uparrow_{\succcurlyeq} i} c_j\}$ and $v_B^{\succcurlyeq}(\downarrow_{\succcurlyeq} i) = \max\{0; E - \sum_{j \in \uparrow_{\succcurlyeq} i} c_j - c_i\}$. Now, we distinguish three cases depending on the position of i with respect to the rank r^* used to define $y^{\sigma^{\succcurlyeq}}(B)$.

Firstly, suppose that $\sigma^{\succcurlyeq}(i) \in \{1, \ldots, r^* - 1\}$. Then $v_B^{\succcurlyeq}((\downarrow_{\succcurlyeq} i) \cup i) = E - \sum_{j \in \uparrow_{\succcurlyeq} i} c_j$ and $v_B^{\succcurlyeq}(\downarrow_{\succcurlyeq} i) = E - \sum_{j \in \uparrow_{\succcurlyeq} i} c_j - c_i$, and so $v_B^{\succcurlyeq}((\downarrow_{\succcurlyeq} i) \cup i) - v_B^{\succcurlyeq}(\downarrow_{\succcurlyeq} i) = c_i = y_i^{\sigma^{\succcurlyeq}}(B)$.

Secondly, suppose that $\sigma^{\succcurlyeq}(i) = r^*$. Then $v_B^{\succcurlyeq}((\downarrow_{\succcurlyeq} i) \cup i) = E - \sum_{j \in \uparrow_{\succcurlyeq} i} c_j$ and $v_B^{\succcurlyeq}(\downarrow_{\succcurlyeq} i) = 0$, and so $v_B^{\succcurlyeq}((\downarrow_{\succcurlyeq} i) \cup i) - v_B^{\succcurlyeq}(\downarrow_{\succcurlyeq} i) = E - \sum_{j \in \uparrow_{\succcurlyeq} i} c_j = y_i^{\sigma^{\succcurlyeq}}(B)$. In this case, by definition of r^*, note that $0 \leqslant y_i^{\sigma}(B) < c_i$.

Thirdly, suppose that $\sigma^{\succcurlyeq}(i) \in \{r^* + 1, \ldots, n\}$. Then $v_B^{\succcurlyeq}((\downarrow_{\succcurlyeq} i) \cup i) = v_B^{\succcurlyeq}(\downarrow_{\succcurlyeq} i) = 0$, and so $v_B^{\succcurlyeq}((\downarrow_{\succcurlyeq} i) \cup i) - v_B^{\succcurlyeq}(\downarrow_{\succcurlyeq} i) = 0 = y_i^{\sigma^{\succcurlyeq}}(B)$. This completes the proof. ■

7. Conclusion

The approach developed in this article can be extended potentially in two directions.

Firstly, other sharing systems that the one used in the Priority value can be design from the priority structure for distributing the Harsanyi dividends. As an example, the dividend of a coalition S can be shared in proportion to the number of agents over which each member of S has priority in S, including it. That is, for each $(N, v, \succcurlyeq)$ and each $i \in N$,

$$f_i(N, v, \succcurlyeq) = \sum_{S \subseteq N : S \ni i} \frac{|(\downarrow_{\succcurlyeq^S} i) \cup i|}{\sum_{j \in S} |(\downarrow_{\succcurlyeq^S} j) \cup j|} \Delta_v(S).$$

Secondly, the absence of constraint imposed on both the formation and the evaluation of coalition can be replicated to the class of TU-games enriched by a graph as proposed by Myerson (1977). In that context, the Harsanyi dividend of each coalition S would be distributed to its members by using a sharing system determined from the subgraph induced by S. Such a system can rely on the number of neighbors in the subgraph induced by S or any popular centrality measure applied to this subgraph (see Bloch et al., 2017, for instance).

These lines of research are left for future works.

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