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Inconsistent weighting in weighted voting games*

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Abstract

In a weighted voting game, each voter has a given weight and a coalition of voters is successful if the sum of its weights exceeds a given quota. Such voting systems translate the idea that voters are not all equal by assigning them different weights. In such a situation, two voters are symmetric in a game if interchanging the two voters leaves the outcome of the game unchanged. Two voters with the same weight are naturally symmetric in every weighted voting game, but the converse statement is not necessarily true. We call this latter type of scenario **inconsistent weighting**. We investigate the conditions that give rise to such a phenomenon within the class of weighted voting games. We also study how the choice of the quota and the total weight can affect the probability of observing inconsistent weighting. Finally, we investigate various applications where inconsistent weighting is observed.

JEL classification: C7, D7
Keywords: Weighted voting games, symmetric voters, inconsistent weighting, probability.

1 Introduction

The class of weighted voting games is an important class of cooperative games, widely used for many important real-world social choice problems in political and economic life. They are particularly relevant for representing and studying electoral bodies in which the voters have different weights. A typical example is parliamentary voting, where voters are parties, and the weight of each voter is the number of votes it controls. More precisely, in weighted voting games, each voter is assigned a non-negative weight and makes a binary yes/no decision on some particular issue; a voting procedure among the members of the electoral body is then used to either accept or reject the resolution: the decision is carried if the sum of weights of voters in favor of it meets or exceeds some specific given threshold, called the quota.

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In weighted voting games, the use of weights is intended to capture the relative importance of each voter in the electoral body under consideration. In other words, it is often recognized that if voters are given different amounts of weights, then a greater voting weight should translate into greater influence in the voting process. For instance, the distribution of seats of each member state within the European Parliament has been established and modified according to the various European Union treaties following the adherence of new members (or withdrawal of a member such as the UK's recent withdrawal) and demographic changes. The distribution of seats is not proportional to each state's population; however, it is stated that it should take into account the population of member states and follows the principle of decreasing proportionality, which means that states that are bigger in terms of population should have more seats than smaller states, with a minimum level of representation for those states. This principle can be compared to the composition of the Electoral College that is formed for the purpose of electing the President of the United States of America.

It is well known that two voters with the same weight are symmetric in the resulting game in the sense that interchanging the two voters in any coalition of voters does not alter the ability of the coalition to pass a decision. However, the weights of the voters do not always indicate the influence the voter has in affecting decisions. In other words, the importance of voters is not necessarily proportional to their weights, and this may lead to a situation in which two voters with different weights are symmetric. As an example, consider an electoral body with three voters, 1, 2, and 3 having respectively 50, 30, and 20 seats and suppose that a decision is carried if a coalition of voters in favor of it controls a simple majority of the votes, i.e., it should have at least 51 votes. Such a coalition is called a winning coalition. Voters 2 and 3 are clearly symmetric despite the fact that they have different weights: each of these voters does not form a winning coalition alone and forms a winning coalition together with voter 1. Based on this observation, we investigate this kind of scenario that we refer to as **inconsistent weighting** in the class of weighted voting games.

We obtain analytical, numerical, and empirical results. Proposition 1 characterizes symmetric voters with unequal weights while Proposition 3 shows that a weighted voting game is not altered by swapping the possibly unequal weights of two symmetric voters. Proposition 6 reveals that the presence of symmetric voters in a weighted voting game is always compatible with both consistent and inconsistent weighting. In Propositions 7 and 9, we provide sufficient conditions for consistent weighting and inconsistent weighting, respectively. In particular, weighted voting games with three or four voters are always inconsistent if weights are all different, a frequent situation in some applications, as we underline with the case of French public inter-municipality cooperation establishments. Our numerical results calculate the probability of inconsistent weighting as a function of the choice of the quota and the total weight of voters. These results highlight the following phenomena: the probability of inconsistent weighting is increasing in the total weight and tends towards 1 very quickly, it is symmetric around a quota corresponding to simple majority, and it seems to increase when the number of voters increases. Propositions 11 and 12 deal with games with two and three voters when the total weight is finite, while Propositions 14 to 16 analyze games with 2 to 5 voters when the total weight tends to infinity. On top of the French public inter-municipality cooperation establishments, we emphasize inconsistent weighting in other applications such as the French association "Union Technique de l'Electricité" and the Ankara international agreement signed in 1963 between the European Community and Turkey.

Numerous research articles analyze weighted voting games in various theoretical settings (e.g., de Keijzer et al., 2010, Freixas and Molinero, 2010, 2009, Freixas and Zwicker, 2003, Kurz, 2012, Taylor and Zwicker, 1992). Weighted voting games have also been considered in many empirical applications, including for instance the Council of the European Union (e.g., Algaba et al., 2007, Bilbao et al., 2002, Felsenthal and Machover, 2004, Lane and Maeland, 2000, Laruelle and Widgren, 1998, Leech, 2002a), the Electoral College of the United States of America (e.g., Leech, 1992, Mann and Shapley, 1960), the International Monetary Fund (e.g., Alonso-Meijide, 2005, Leech, 2002b), the U.N. Security Council (e.g., O'Neill, 1996, Strand and Rapkin, 2010), joint stock companies where each shareholder gets votes in proportion to the ownership (e.g., Arcaini and Gambarelli, 1986, Gambarelli, 1994), community of municipalities in France (e.g., Barthélemy and Martin, 2007, Bison et al., 2004, Bonnet and Lepelley, 2001, Dia and Kamwa, 2019), as well as different national parliaments (e.g., Diss and Steffen, 2018, Diss and Zouache, 2015, Koki and Leonardos, 2019, Van Deemen and Rusinowska, 2003). The above list of references is incomplete, but demonstrates the relevance of weighted voting games and its applications. However, it is worth noting that our approach is close in spirit to the one of Barthélemy et al. (2020, 2013), and Barthélemy and Martin (2020). In their framework, the authors consider the class of weighted voting games and investigate the conditions as well as the probability of having at least one dummy voter in a game, that is a voter with no effect on the outcome of the game in spite of its non-zero individual weight.

The rest of the article is organized as follows. After presenting the necessary definitions in Section 2, we present some applications from several real institutional contexts in Section 3. In Section 4 we establish some analytical results regarding symmetric voters as well as inconsistent weighting. Section 5 describes our numerical results for the probability of observing inconsistent weighting for various values of the number of voters, the total weight, and the established quota. Section 6 concludes.

2 Preliminary definitions

Let $N = \{1, 2, \ldots, n\}$ be a finite non-empty set of **voters**. A **coalition** of voters is any non-empty subset of N. A **voting game** is a pair $G = (N, W)$ where W is a set of coalitions such that $W \neq \emptyset$; and for all $T \subseteq N$, $T \in W$ whenever $T \supseteq S$ for some $S \in W$. Each coalition in W is called a **winning coalition** in the game $G = (N, W)$. The **dual game** of the game $G = (N, W)$ is the game $\widehat{G} = \left(N, \widehat{W}\right)$ such that $\widehat{W} = \{N \backslash S \mid S \notin W\}$.

A voting game G is **weighted** if there are non-negative integers $w_1, w_2, \ldots, w_n$ and $q \geq 1$ such that $\sum_{i \in N} w_i \geq q$, and for all $S \subseteq N$, $S \in W$ if and only if $w_S := \sum_{i \in S} w_i \geq q$. For simplicity, we write w instead of w_N. In this case, the game is denoted by $[q; w_1, w_2, \ldots, w_n]$ and the total weight w_N. For all $i \in N$, w_i is the **weight** or the number of votes of voter i, q is the **quota** needed to form a winning coalition and $[q; w_1, w_2, \ldots, w_n]$ is called a **weighted form** of G. A weighted form of the dual game $\widehat{G}$ of G is $[\widehat{q}; w_1, w_2, \ldots, w_n]$ where $\widehat{q} = w - q + 1$. A particular class of weighted voting games is the class of **weighted majority games** $[q; w_1, w_2, \ldots, w_n]$ with $q = \left\lceil \frac{w+1}{2} \right\rceil$. When $q = w$, we get the **unanimity rule** where the unique winning coalition is N. We denote by $\mathcal{WG}_n$ the set of all weighted voting games with n voters. We assume that $w_1 \geq w_2 \geq \cdots \geq w_n$ up to a relabeling of voters; otherwise, we write $[q; w_1, w_2, \ldots, w_n]^*$. Given a permutation σ of N, define the game $[q; \sigma w_1, \sigma w_2, \ldots, \sigma w_n]^*$ by $\sigma w_i = w_{\sigma(i)}$ for all $i \in N$ (voter i is now endowed with the weight of voter $\sigma(i)$).

Two voters i and j are **symmetric** in a voting game $G = (N, W)$ if for all $T \subseteq N \backslash \{i, j\}$, $S \cup \{i\} \in W \iff S \cup \{j\} \in W$. As noted in the introduction, two voters with the same weight are symmetric in every weighted voting game, but the converse statement is not necessarily true. Inconsistent weighting arises when two symmetric voters have distinct weights. It is obvious that the set of voters can be partitioned into equivalent classes containing voters that are symmetrical to each other. If, furthermore, the voters are listed in descending order of weight, then one obtains an ordered partition denoted by $\mathcal{SYM}(G) = (N_1, N_2, \ldots, N_s)$ for a voting game G. Hence, the voters in each N_t, $t \in \{1, \ldots, s\}$, are symmetric and for all $i \in N_t$ and $j \in N_{t+1}$, $w_i > w_j$.[1] The sequence $\mathcal{SYM}(G)$ will be called the **symmetry class** of the game G. We denote by $\mathcal{WG}_n^*$ the set of all weighted voting games G with n voters that admit no pair of symmetric voters. In other words, $\mathcal{WG}_n^*$ contains all weighted voting games such that $\mathcal{SYM}(G) = (\{1\}, \{2\}, \ldots, \{n\})$.

The binary relation $\succeq_G$ over subsets of N is defined for all $S, T \subseteq N$, by $T \succeq_G S$ if $w_T \geq w_S$, where $w_\emptyset = 0$. We label all subsets of T in such a way that

$$T = T_p \succeq_G T_{p-1} \succeq_G \cdots \succeq_G T_2 \succeq_G T_1 = \emptyset$$

with $p = 2^{|T|}$. The sequence $(T_p, T_{p-1}, \ldots, T_2, T_1)$ will be called a **weight ranking** over subsets of T induced by the game $[q; w_1, w_2, \ldots, w_n]$. In the example of three voters 1, 2, and 3, presented in the introduction, it is clear that for $T = N \backslash \{1\}$ the sequence $(T_p, T_{p-1}, \ldots, T_2, T_1)$ is such that $T_p = T_4 = \{2, 3\}$, $T_3 = \{2\}$, $T_2 = \{3\}$, and $T_1 = \emptyset$.

3 Motivation and applications

In this section, we show that inconsistent weighting is not just a theoretical phenomenon but occurs in several real institutional contexts. To illustrate, without claiming to be exhaustive, we propose three distinct real cases concerning, (i) the voting rules employed in French associations, (ii) the voting rules related to French public inter-municipality cooperation establishments, and (iii) the 1963 Ankara international agreement between the European Community and Turkey. Those examples are illustrations of cases where two or several voters are symmetric despite the fact they have distinct weights. Beyond the quota used in reality, we can also provide a complete map of consistent and inconsistent weighted forms when the quota varies.

Example 1: Voting rules within French associations

Among French non-profit organizations, associations occupy a singular place: first, because this is the most common legal form, there being currently nearly 1.5 million of them,[2] and, second, because this is one of the most malleable legal forms. Indeed, civil law, commercial law, the association law of 1901 as well as jurisprudence authorize the creation of a huge diversity of forms of associations, e.g., charitable associations, sports associations, cultural associations, professional associations.

Regarding voting rules, French law allows associations to: (i) choose their quorum, (ii) distinguish between different categories of members, and most importantly (iii) allocate different weights to them. In this respect, since freedom is the rule, it seems to us quite possible that some associations are indeed confronted with the problem of inconsistent weighting.

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[1] All the infinitely many distinct weighted forms $[q; w_1, \ldots, w_n]$ for a weighted voting game $G = (N, W)$ yield the same ordered partition $\mathcal{SYM}(G)$. If a weighted form $[q; w_1, \ldots, w_n]$ is consistent, then for all $t \in \{1, \ldots, s\}$ and all $i, j \in N_t$, $w_i = w_j$, while the latter equality does not necessarily hold if $[q; w_1, \ldots, w_n]$ is inconsistent. This is the reason why we can write $\mathcal{SYM}(G)$ instead of $\mathcal{SYM}(q; w_1, \ldots, w_n)$.

[2] See, for instance, INJEP, 2019, p. 4: https://injep.fr/wp-content/uploads/2019/07/Chiffres-cles-Vie-associative-2019.pdf.

As an example, we take **l'Union Technique de l'Electricité** (UTE), an association created as a professional union in 1907 under the name **Union des Syndicats de l'Electricité** which became UTE in 2006. This association brings together stakeholders who represent French interests in the field of electrotechnical standardization.[3] A review of their statutes adopted on January 21, 2014 is particularly instructive. Indeed, Article 2.2 describes the college of contributing members and their respective voting weights, which can be summarized as follows:

Table 1: Contributing members of UTE and their weights

College	Relative weightings in %
Manufacturers in the electrotechnical field	35
Users/integrators of infrastructures, equipment, products, services, technologies in the electrotechnical field	27
Producers and operators of electrical networks	22
Installers and service providers in the electro-technical field	8
Technical bodies in the electrotechnical field	8
Total	100

Article 2.1 of the UTE's statutes indicates that the vote is carried out by an absolute majority of the members. Therefore, it is easy to see here that a college with 22% of the votes and one with 27% of the votes are symmetric in the voting game even though they do not have the same weight.

This example can be written as the weighted form $[q; w_1, w_2, w_3, w_4, w_5] = [51; 35, 27, 22, 8, 8]$. Now, if we assume that the quota can be any integer between 1 and 100, then we can summarize the different possibilities in Table 2, in which the bold line highlights the quota interval corresponding to the real case where $q = 51$ under absolute majority.

Table 2: The map of consistent and inconsistent weighted forms

Quota q	Weighted form
$\{1, \ldots, 38\}$	Inconsistent
$\{39, \ldots, 43\}$	Consistent
$\{\mathbf{44}, \ldots, \mathbf{57}\}$	**Inconsistent**
$\{58, \ldots, 62\}$	Consistent
$\{63, \ldots, 100\}$	Inconsistent

Table 2 points out that there can be multiple changes in the status of a given weighted form when the quota increases. It should be noted that within a given quota interval in which the weighted form is inconsistent, the symmetric voters with different weights may not always be the same. As an example, consider the interval where $q \in [63, 100]$. If $q = 70$, voters with weights 27 and 22 are symmetric but not with the voter with weight 35. If $q = 75$, voters with weights 35 and 27 are symmetric but not with the voter with weight 22. If $q = 80$, these three voters are symmetric.

[3] For more information on their duties, we refer the reader to http://ute-asso.fr/index-2.html.

Example 2: EPCI

In order to better manage some facilities or spaces (e.g., university, port, airport), public services (e.g., school transport, sanitation, garbage collection) or to jointly develop projects, French municipalities can create public inter-municipality cooperation establishments known under the name of EPCI (**Établissements Public de Coopération Intercommunale**).[4] As public institutions, these EPCIs are governed by the general code of local authorities[5] and, in particular, by the principle of specialization. More precisely, they are competent only in the fields and matters that the law assigns to them, or for those delegated by the member municipalities.

Each EPCI is administered by a deliberative council whose members are drawn from each of the member municipalities with the condition that each municipality has to be represented by at least one member. Concomitantly or following municipal elections (which take place every six years), the members of the deliberative council for each EPCI are elected depending on the type of EPCI. The electoral code and the general code of local government detail the modalities of elections and the governance by the deliberative council. The law requires EPCIs to have the following properties: (i) the number of councilors elected to represent a municipality depends on its population, (ii) decisions are taken according to the majority quota, that is, the smallest integer strictly greater than half of the total weight, and (iii) no municipality may have more than 50% of the total number of seats of the EPCI.

Despite their precision, these rules do not guarantee that the weightings correspond to the symmetrical character of the municipalities in the voting game. By way of illustration, it is possible to show this inconsistency using the distribution of seats in the EPCI of Grand Pontarlier,[6] thanks to the following table:

Table 3: The number of seats in the EPCI of Grand Pontarlier (2020)

Municipalities	Number of seats (%)
Pontarlier	16 (47.06)
Doubs	3 (8.82)
Chaffois, Cluse et Mijoux, Dommartin, Les Grandes Narboz, Houtaud, Sainte Colombe, Les Verrières De Joux	2 each (5.88)
Vuillecin	1 (2.94)
Total	34 (100)

This table shows that, although the Doubs municipality and all the other municipalities with two seats each have a different number of seats, they are symmetric in the corresponding weighted voting game.

Let us consider the following former EPCI called Communauté des bords de Vire in the French Basse-Normandie region, which existed between 1992 and 2005 (Bonnet and Lepelley, 2001).

[4]For more details on cooperative administrative structures in France, especially EPCI, we refer the reader to `https://www.collectivites-locales.gouv.fr/intercommunalite-1`.

[5]**Code général des collectivités territoriales**.

[6]See, for instance, `https://sig.ville.gouv.fr/Territoire/242500338`. Notice that based on the data and analysis of Blancard et al. (2020), it is also possible to show this type of inconsistency in some EPCIs of French Reunion Island.

Table 4: The number of seats in the EPCI Communauté des bords de Vire

Municipalities	Number of seats (%)
La Meauffe	5 (33.33)
Pont-Hébert	7 (46.67)
Rampan	3 (20.00)
Total	15 (100)

This new example is interesting in that the corresponding weighted form is always inconsistent, whatever the chosen quota. The table below specifies which pairs of municipalities are symmetric as a function of the quota.

Table 5: The map of symmetric municipalities

Quota q	Symmetric municipalities
$\{1, 2, 3\}$	La Meauffe, Pont-Hébert, Rampan
$\{4, 5\}$	La Meauffe, Pont-Hébert
$\{6, 7\}$	La Meauffe, Rampan
$\{8\}$	La Meauffe, Pont-Hébert, Rampan
$\{9, 10\}$	La Meauffe, Rampan
$\{11, 12\}$	La Meauffe, Pont-Hébert
$\{13, 14, 15\}$	La Meauffe, Pont-Hébert, Rampan

Example 3 : Ankara international agreement

On September 12, 1963, Agreement 64/733/EEC was signed between the European Community (EC) and Turkey creating an association with the aim "to promote the continuous and balanced strengthening of trade and economic relations between the parties, while taking full account of the need to ensure an accelerated development of the Turkish economy and to improve the level of employment and the living conditions of the Turkish people" (Art. 2, 1). The essential purpose of this agreement is to both gradually establish a customs union and to approximate economic policies. It provided for the future possibility of Turkey's accession to the EC (Article 28). The implementation of this association was planned to take place in three phases: (i) a preparatory phase, (ii) a transitional phase, and (iii) a final phase. In order to define the modalities of the first phase, which lasted for five years, in particular Community aid, the agreement includes an appendix providing a provisional protocol and a financial protocol.

The Financial Protocol provides for "the financing of investment projects which will serve to increase the productivity of the Turkish economy and further the objectives of the Agreement of Association, and which are part of the Turkish development plan" (Art. 1), via the possibility of loans from the European Investment Bank for a total amount of 175 million US $ over five years (Art. 2). The report of French Senator Roger Carcassonne on December 6, 1963 informs us that this sum is distributed among the member states of the EC as follows:[7]

[7] See, for instance, https://www.senat.fr/rap/1963-1964/i1963_1964_0063.pdf.

Table 6: Contributions following Ankara international agreement

Countries	Amount in millions of US $ (%)
France	58.5 (33.42)
Germany (FRG)	58.5 (33.42)
Italy	32 (18.28)
Belgium	13 (7.42)
The Netherlands	12.7 (7.25)
Luxembourg	0.3 (0.17)
Total	175 (100)

Loan applications must be approved by the Turkish government. The application is then forwarded by the European Investment Bank to the member States and the European Commission. Without objection, the loan is granted. However, if a member state so requires, a committee consisting of a representative of each member state and a representative of the Commission examines the admissibility of the application and votes, by a qualified majority, according to a system of proportional voting in proportion to the financial participation of each state. According to the Internal Agreement (Art. 10),[8] the qualified majority is 67 votes according to the following distribution.

Table 7: Distribution of votes following Ankara international agreement

Countries	Votes in %
France	33
Germany (FRG)	33
Italy	17
Belgium	8
The Netherlands	8
Luxembourg	1
Total	100

Except France and Germany, we can easily show that the other four countries are symmetric in the voting game even though they do not have the same weight at all. Denote by $[q; 33, 33, 17, 8, 8, 1]$ the weighted form in which the quota q is an integer between 1 and 100. The table below shows that there are eight successive switches in the status of the weighted form when the quota increases.

[8]The Internal Agreement can be found, for instance, in https://eur-lex.europa.eu/legal-content/EN/TXT/PDF/?uri=CELEX:01964A1229(01)-20040501&from=EN.

Table 8: The map of consistent and inconsistent weighted forms

Quota q	Weighted form
$\{1,\ldots,18\}$	Inconsistent
$\{19,\ldots,25\}$	Consistent
$\{26,\ldots,42\}$	Inconsistent
$\{43,\ldots,49\}$	Consistent
$\{50,\ldots,51\}$	Inconsistent
$\{52,\ldots,58\}$	Consistent
$\{\mathbf{59},\ldots,\mathbf{75}\}$	**Inconsistent**
$\{76,\ldots,82\}$	Consistent
$\{83,\ldots,100\}$	Inconsistent

The examples presented in this section naturally raise the following question. Can we characterize inconsistent weighted forms? Can the choice of suitable weights rule out inconsistent forms? If a weighted voting game is obtained from an inconsistent weighted form, is there always a consistent weighted form inducing the same weighted voting game? What is the likelihood of an inconsistent weighted form? What are the most frequent regime changes?

4 Some analytical results

4.1 About symmetric voters

The first result of our article gives the three conditions under which two voters i and j are symmetric in a game. More exactly, two voters i and j, with $i < j$, are symmetric if (a) the smallest voter j alone constitutes a winning coalition, or (b) the coalition containing all but the smallest voter j is not a winning coalition, or (c) the coalition containing the smallest voter j together with a given subset S_k of other voters is a winning coalition whereas the coalition containing the largest voter i together with any subset not having a total weight greater than S_k is a losing coalition.

Proposition 1 *Let i and j be two voters (with $i < j$) in a weighted voting game $[q; w_1, w_2, \ldots, w_n]$ and let $(S_p, S_{p-1}, \ldots, S_2, S_1)$ be a weight ranking over all subsets of $S = N\setminus\{i,j\}$ induced by the game. The following assertions are equivalent:*

(i) i and j are symmetric voters.

(ii) $(a) : w_j \geq q$; or
$(b) : w_i + w_{S_p} < q$; or
$(c) : w_j + w_{S_k} \geq q$ and $w_i + w_{S_{k-1}} < q$ for some $k \in \{1, \ldots, p\}$.

Proof. Assume that i and j are symmetric voters in $[q; w_1, w_2, \ldots, w_n]$. Assume that (a) and (b) do not hold. Let $I = \{t \in \{1, 2, \ldots, p\} : w_j + w_{S_t} \geq q\}$. By assumption, $w_j < q$ and $w_i + w_{S_p} \geq q$. That is $\{j\} \notin W$ and $S_p \cup \{i\} \in W$. Since i and j are symmetric voters, we also have $\{i\} \notin W$ and $S_p \cup \{j\} \in W$. This implies that $w_j = w_j + w_{S_1} < q$ and $w_j + w_{S_p} \geq q$. It follows that $p \in I \neq \emptyset$ and $1 \notin I$. This proves that I has a minimum denoted by k such that $k \geq 2$. By definition of k, $w_j + w_{S_{k-1}} < q$. By recalling that i and j are symmetric voters, it follows that $w_i + w_{S_{k-1}} < q$. Therefore (c) holds.

Conversely, assume that (a), (b), or (c) hold and let S be a subset of $N\backslash\{i,j\}$. By assumption, $S = S_t$ for some $t \in \{1,2,\ldots,p\}$. First, suppose that (a) holds. That is $w_j \geq q$. Consider $t \in \{1,2,\ldots,p\}$. It follows that $w_i + w_{S_t} \geq w_j + w_{S_t} \geq w_j \geq q$. Therefore $S_t \cup \{j\} \in W$ and $S_t \cup \{i\} \in W$. Hence i and j are symmetric voters. Now suppose that (b) holds. That is $w_i + w_{S_p} < q$. Consider $t \in \{1,2,\ldots,p\}$. It follows that $w_j + w_{S_t} \leq w_i + w_{S_t} \leq w_i + w_{S_p} < q$. Therefore $S_t \cup \{j\} \notin W$ and $S_t \cup \{i\} \notin W$. Hence i and j are symmetric voters. Finally, suppose that (c) holds. That is $w_j + w_{S_k} \geq q$ and $w_i + w_{S_{k-1}} < q$ for some $k \in \{1,\ldots,p\}$. Consider $t \in \{1,2,\ldots,p\}$. Two possible cases arise. Suppose that $t \leq k-1$. Then by the definition of $(S_p, S_{p-1},\ldots,S_2,S_1)$, $w_j + w_{S_t} \leq w_i + w_{S_t} \leq w_i + w_{S_{k-1}} < q$. Therefore $S_t \cup \{j\} \notin W$ and $S_t \cup \{i\} \notin W$. Now suppose that $t \geq k$. Then by the definition of $(S_p, S_{p-1},\ldots,S_2,S_1)$, $w_i + w_{S_t} \geq w_j + w_{S_t} \geq w_j + w_{S_k} \geq q$. Therefore $S_t \cup \{j\} \in W$ and $S_t \cup \{i\} \in W$. In both cases, $S \cup \{i\} \in W \Longleftrightarrow S \cup \{j\} \in W$, meaning that the two voters i and j are symmetric in the game $[q; w_1, w_2, \ldots, w_n]$. ∎

Proposition 1 can be illustrated with example 1. In the weighted voting game $[q; w_1,\ldots,w_5] = [51; 35, 27, 22, 8, 8]$, voters 2 and 3 are symmetric even if $w_2 = 27 > w_3 = 22$. Point (a) does not hold since $w_3 = 22 < 51 = q$. Point (b) does not hold since $w_{N\backslash\{3\}} = 78 > 51$. Hence, it must be that point (c) holds. As a start, we construct the weight ranking $(S_1,\ldots,S_8)$ over all subsets of $S = \{1,4,5\}$:

$$\big(S_8, S_7, S_6, S_5, S_4, S_3, S_2, S_1\big) = \big(\{1,4,5\}, \{1,4\}, \{1,5\}, \{1\}, \{4,5\}, \{4\}, \{5\}, \emptyset\big),$$

which corresponds to the sequence of weights $(51, 43, 43, 35, 16, 8, 8, 0)$. Point (c) holds by considering $S_5 = \{1\}$ and $S_4 = \{4,5\}$ since $w_{\{1,3\}} = 57 > 51$ and $w_{\{2,4,5\}} = 43 < 51$.

The following proposition tells us that all voters whose weights are between the weights of two symmetric voters in a game are also symmetric.

Proposition 2 *Let i and j be two symmetric voters (with $i < j$) in a weighted voting game $[q; w_1, w_2, \ldots, w_n]$. Then all voters i' and j' such that $j \geq j' > i' \geq i$ are symmetric voters.*

Proof. Consider two voters i' and j' such that $j \geq j' > i' \geq i$. Since i and j are symmetric voters, (a), (b), or (c) hold. First suppose that (a) holds. It follows that $w_{j'} \geq w_j \geq q$. Therefore i' and j' are symmetric voters by Proposition 1. Now suppose that (b) holds. It follows that $w_{i'} + w_{S_p} \leq w_i + w_{S_p} < q$ which means that i' and j' are symmetric voters by Proposition 1. Finally, suppose that (c) holds for some $k \in \{1,\ldots,p\}$. Note that $w_{j'} + w_{S_k} \geq w_j + w_{S_k} \geq q$ and $w_{i'} + w_{S_{k-1}} \leq w_i + w_{S_{k-1}} < q$. Therefore i' and j' are symmetric voters by Proposition 1. ∎

4.2 About inconsistent weighting of a game

Note that each weighted voting game G with no pair of symmetric voters only admits consistent weighted forms. Indeed, since an inconsistent weighting is observed when two symmetric voters have distinct weights, G admits no inconsistent weighted form. In the next result, it is shown that the set of winning coalitions in a weighted voting game does not change by permuting the weights of two symmetric voters.

Proposition 3 *Let G be a weighted voting game and $[q; w_1, w_2, \ldots, w_n]$ a representation of G. Suppose that $(w'_1, w'_2, \ldots, w'_n)$ is obtained from $(w_1, w_2, \ldots, w_n)$ by permuting the* *weights of two symmetric voters. Then* $[q; w'_1, w'_2, \ldots, w'_n]^*$ *is also a representation of* G*; that is, both* $[q; w_1, w_2, \ldots, w_n]$ *and* $[q; w'_1, w'_2, \ldots, w'_n]^*$ *lead to the same set of winning coalitions.*

Proof. Suppose that $(w'_1, w'_2, \ldots, w'_n)$ is obtained from $(w_1, w_2, \ldots, w_n)$ by permuting the weights of two symmetric voters, say i and j. Let $G' = (N, W')$ be the weighted voting game a representation of which is $[q; w'_1, w'_2, \ldots, w'_n]^*$. Consider $S \in 2^N$. There are two possible cases. Case (a) : $S \cap \{i, j\} = \emptyset$ or $S \cap \{i, j\} = \{i, j\}$. Then $w'_S = w_S$. This implies that $w'_S \geq q \iff w_S \geq q$; that is $S \in W' \iff S \in W$. Case (b) : $S \cap \{i, j\} = \{i\}$. Then $S = T \cup \{j\}$ and $w'_S = w_T + w_i = w_{T\cup\{i\}}$ where $T = S \backslash \{i\} \subseteq N \backslash \{i, j\}$. Since i and j are symmetric, then $T \cup \{j\} \in W \iff T \cup \{i\} \in W$. That is $w'_S \geq q \iff w_S \geq q$. Equivalently, $S \in W' \iff S \in W$. In both cases, $S \in W' \iff S \in W$. We deduce that $W' = W$. Hence $G = G'$. ■

The previous proposition still holds for all other permutations that preserve all symmetry classes in the game. This is the objective of the following result.

Corollary 4 *Let* $[q; w_1, w_2, \ldots, w_n]$ *and* $\mathcal{SYM}(G) = (N_1, N_2, \ldots, N_s)$ *be a representation of a weighted voting game* G*. Then for all permutations* σ *of* N *such that* $\sigma(N_j) = N_j$ *for all* $j \in \{1, 2, \ldots, s\}$*,* $[q; \sigma w_1, \sigma w_2, \ldots, \sigma w_n]^*$ *is also a representation of* G*.*

Proof. Noting that all permutations σ of N such that $\sigma(N_j) = N_j$ for all $j \in \{1, 2, \ldots, s\}$ can be obtained from a finite number of steps, each consisting in a transposition of the weights of two symmetric voters, the result holds from Proposition 3. ■

Now, let us denote by G_0 the unanimity game such that $W = \{N\}$. Note that $\mathcal{SYM}(G_0) = (\{1, 2, \ldots, n\})$, and that both $[n; 1, 1, \ldots, 1]$ and $[n(n+1); 2n, 2n-2, \ldots, 2]$ are distinct representations of G_0. Therefore, the representation $[n(n+1); 2n, 2n-2, \ldots, 2]$ is an inconsistent weighted form of G_0 while the representation $[n; 1, 1, \ldots, 1]$ is not. This observation can be generalized to all weighted voting games that admit a pair of symmetric voters. But first we provide a way of modifying the quota and the weights in a weighted voting game without changing the set of winning coalitions. This helps us in proving the generalization just announced.

Proposition 5 *Let* G *be a weighted voting game. If* $[q; w_1, w_2, \ldots, w_n]^*$ *and* $[q; w'_1, w'_2, \ldots, w'_n]^*$ *are two weighted forms of* G*, then so is* $[2q; w_1 + w'_1, \ldots, w_n + w'_n]^*$*.*

Proof. Consider $[q; w_1, w_2, \ldots, w_n]^*$ and $[q; w'_1, w'_2, \ldots, w'_n]^*$ two weighted forms of a given weighted voting game G. Denote by $G' = (N, W')$ the weighted voting game represented by $[2q; w_1 + w'_1, \ldots, w_n + w'_n]^*$. Let $S \in 2^N$. Suppose that $S \in W$. Then $w_S + w'_S \geq 2q$. Thus $S \in W'$ and $W \subseteq W'$. Now suppose that $S \notin W$. Then $w_S + w'_S < q$. This necessary implies that $w_S < q$ or $w'_S < q$. Therefore, $S \notin W'$ and $W' \subseteq W$. It follows that $W = W'$ and then $G = G'$. ■

Now, the question raised below is: for a weighted voting game that admits at least one pair of symmetric voters, is it possible to find both consistent and inconsistent weighted forms? The next result answers yes.

Proposition 6 *Each weighted voting game that admits a pair of symmetric voters has some consistent weighted forms as well as some inconsistent weighted forms.*

Proof. Let $G = (N, W)$ be a weighted voting game with n voters. Then G admits a weighted form $[q; w_1, w_2, \ldots, w_n]$. Assume that G admits some pairs of symmetric voters. Then $\mathcal{SYM}(G) = (N_1, N_2, \ldots, N_s)$ with $|N_k| \geq 2$ for some $k \in \{1, 2, \ldots, s\}$.
First suppose that $[q; w_1, w_2, \ldots, w_n]$ is a consistent form of G. There are two possible cases. Case $(a) : W = 2^N$. In this case, all non-empty subsets of N are winning coalitions. Thus $[1; n, n-1, \ldots, 2, 1]$ is an inconsistent form of G. Case $(b) : W \neq 2^N$. Then for all $S \in 2^N \backslash W$, $w_S < q$. Let $\alpha = \frac{1}{n(n+1)} \min_{S \in 2^N \backslash W} (q - w_S) > 0$, $w'_i = w_i + i\alpha$ for all $i \in N$ and denote by G' the weighted voting game represented by $[dq; dw'_1, dw'_2, \ldots, dw'_n]$ with $d = n(n+1)$. We prove that $G = G'$. For this purpose, let W' be the set of all winning coalitions in G'. Suppose that $T \in W$. Then $dw'_T = dw_T + \sum_{i \in T} id\alpha > dw_T \geq dq$. Thus $T \in W'$. Therefore $W \subseteq W'$. Suppose that $T \notin W$. Then

$$\begin{aligned} dw'_T &= dw_T + \sum_{i \in T} id\alpha \leq dw_T + \sum_{i \in N} id\alpha \text{ since } T \subseteq N \\ &\leq dw_T + \frac{n(n+1)}{2} d\alpha \text{ since } \sum_{i \in N} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \\ &= dw_T + \frac{d}{2} \min_{S \in 2^N \backslash W} (q - w_S) \text{ by the definition of } \alpha \\ &\leq d\left(w_T + \frac{1}{2}(q - w_T)\right) < dq \text{ since } T \notin W \end{aligned}$$

Hence $T \notin W'$. Therefore $W' \subseteq W$. We conclude that $W = W'$. Moreover $w_i \geq w_{i+1}$ for $1 \leq i < n$. This implies that $dw'_i = dw_i + (n - i + 1) d\alpha > dw_{i+1} + (n - i) d\alpha = dw'_{i+1}$ for $1 \leq i < n$. Therefore, $[dq; dw'_1, dw'_2, \ldots, dw'_n]$ is an inconsistent weighted form of G since G admits a pair of symmetric voters.
Now suppose that $[q; w_1, w_2, \ldots, w_n]$ is an inconsistent form of G. There are two possible cases. Let $\mathcal{SYM}(G) = (N_1, N_2, \ldots, N_s)$ and denote by E the set of all permutations σ of N such that $\sigma(N_j) = N_j$ for all $j \in \{1, 2, \ldots, s\}$. Pose $w'_i = \sum_{\sigma \in E} \sigma w_i$. By Proposition 5, $[|E| q; w'_1, w'_2, \ldots, w'_n]^*$ is also a weighted form of G. Consider two voters i and j. Assume that i and j are symmetric. Then $i, j \in N_k$ for some $k \in \{1, 2, \ldots, s\}$. When σ describes E, $\sigma(i)$ and $\sigma(j)$ take each value in N_k exactly $\frac{|E|}{|N_k|!} = \frac{|N_1|! * |N_2|! * \cdots * |N_s|!}{|N_k|!}$ times. Therefore

$$w'_i = \frac{1}{|E|} \sum_{\sigma \in E} \sigma w_i = \frac{1}{|N_k|!} \sum_{v \in N_k} w_v = w'_j.$$

Therefore two symmetric voters necessary have the same weight with respect to $[q; w'_1, w'_2, \ldots, w'_n]^*$. Thus $[q; w'_1, w'_2, \ldots, w'_n]^*$ is a consistent weighted form of G. ■

Let us now take into consideration a particular case among weighted voting games. A voting game $G = (N, W)$ is called proper if $S \in W$ implies $N \backslash S \notin W$. A weighted voting game is called proper if its induced voting game is proper. A representation $[q; w_1, \ldots, w_n]$ of a weighted voting game has a large minority if $|\{i \in N : w_i = 1\}| \geq q$. Below, we provide a sufficient condition for consistent weighting.

Proposition 7 *If a weighted voting game G admits a representation $[q; w_1, \ldots, w_n]$ having a large minority and such that $w_1 < q$, then this representation is a consistent weighted form of G.*

Proof. Consider any weighted voting game G that admits a representation $[q; w_1, \ldots, w_n]$ having a large minority and such that $w_1 < q$. Denote by M its large minority, i.e.,

$$M = \{i \in N : w_i = 1\}.$$

It holds that $n \geq q$ since there are at least q voters i satisfying $w_i = 1$.
Next, we have to prove that $[q; w_1, \dots, w_n]$ only admits consistent weighted forms. In other words, we have to show that whenever two voters i and j are symmetric, it holds that $w_i = w_j$. We proceed by contradiction. So assume that i and j are symmetric but that $w_i > w_j$. By assumption $w_1 < q$, we have $w_i < q$. As a consequence and since $[q; w_1, \dots, w_n]$ has a large minority, we can choose $S \subseteq M$ such that $|S| = q - w_i$. Then

$$w_{S \cup \{i\}} = (q - w_i) + w_i = q > (q - w_i) + w_j = w_{S \cup \{j\}},$$

which implies that $(S \cup \{i\}) \in W$ but $(S \cup \{j\}) \notin W$. This contradicts the fact that i and j are symmetric and completes the proof. ■

It should be noted that none of the voters (even those in the large minority) is null in the weighted voting games considered in Proposition 7. This result also naturally raises the question of whether there are relevant classes of weighting voting games satisfying the two conditions in the statement of Proposition 7. The corollary below provides an answer.

Corollary 8 *Any weighted form of a proper weighted voting game having a large minority is always consistent.*

Proof. It is enough to show that $w_1 < q$ for any representation $[q; w_1, \dots, w_n]$ of a proper weighted voting game having a large minority. By contradiction, assume $[q; w_1, \dots, w_n]$ is such that $w_1 \geq q$, which implies $\{1\} \in W$. For the coalition M, we also have $w_M \geq q$ and so $M \in W$. Hence $M \subseteq N\backslash\{1\}$ entails $N\backslash\{1\} \in W$. The fact that both $\{1\} \in W$ and $N\backslash\{1\} \in W$ contradicts the fact that the weighting game $[q; w_1, \dots, w_n]$ is proper. ■

As an example, consider the weighted voting game $[7; w_1, w_2, 1, 1, 1, 1, 1, 1, 1]$. Such a game belongs to the class of games in Proposition 7 if $w_1 \leq 7$ but is not necessarily proper (as in Corollary 8). For instance, if $w_1 = 6$ and $w_2 = 5$, then both $\{1, 3\}$ and $\{2, 4, \dots, 9\}$ are winning coalitions. In all cases, voters 1 and 2 are symmetric in this game if and only if $w_1 = w_2$.

We end this section with a negative or disappointing result. Any weighted voting games with three our four voters having all distinct weights must be inconsistent, that is, there is no value for the quota such that no pair of voters is symmetric.

Proposition 9 *If a weighted voting game $G = (N, W)$ with $n \in \{3, 4\}$ admits a representation $[q; w_1, \dots, w_n]$ such that $w_1 > \dots > w_n$, then this representation is an inconsistent weighted form of G whatever the choice of the quota.*

Proof. We start by considering the 4-voter situation. The assumption that $w_1 > w_2 > w_3 > w_4$ implies that for each $i, j \in \{1, 2, 3, 4\}$, $i < j$, and each $S \subseteq N\backslash\{i, j\}$,

$$\big[(S \cup \{j\}) \in W\big] \Longrightarrow \big[(S \cup \{i\}) \in W\big].$$

Hence, the consistency of $[q; w_1, w_2, w_3, w_4]$ results in

(i) the existence of at least one (possibly empty) coalition $S \subseteq \{3, 4\}$ such that $S \cup \{1\} \in W$ but $S \cup \{2\} \notin W$;

(ii) the existence of at least one (possibly empty) coalition $S \subseteq \{1, 4\}$ such that $S \cup \{2\} \in W$ but $S \cup \{3\} \notin W$;

(iii) the existence of at least one (possibly empty) coalition $S \subseteq \{1,2\}$ such that $S \cup \{3\} \in W$ but $S \cup \{4\} \notin W$.

Case (i) means that $\{1\} \in W$ but $\{2\} \notin W$, or $\{1,3\} \in W$ but $\{2,3\} \notin W$, or $\{1,4\} \in W$ but $\{2,4\} \notin W$ or $\{1,3,4\} \in W$ but $\{2,3,4\} \notin W$. These possibilities give rise to the following four inequalities:

$$w_1 \geq q > w_2, \tag{1}$$
$$w_1 + w_3 \geq q > w_2 + w_3, \tag{2}$$
$$w_1 + w_4 \geq q > w_2 + w_4, \tag{3}$$
$$w_1 + w_3 + w_4 \geq q > w_2 + w_3 + w_4. \tag{4}$$

Similarly, case (ii) means that $\{2\} \in W$ but $\{3\} \notin W$, or $\{1,2\} \in W$ but $\{1,3\} \notin W$, or $\{2,4\} \in W$ but $\{3,4\} \notin W$ or $\{1,2,4\} \in W$ but $\{1,3,4\} \notin W$. These possibilities also give rise to four new inequalities:

$$w_2 \geq q > w_3, \tag{5}$$
$$w_1 + w_2 \geq q > w_1 + w_3, \tag{6}$$
$$w_2 + w_4 \geq q > w_3 + w_4, \tag{7}$$
$$w_1 + w_2 + w_4 \geq q > w_1 + w_3 + w_4. \tag{8}$$

Finally, case (ii) means that $\{3\} \in W$ but $\{4\} \notin W$, or $\{1,3\} \in W$ but $\{1,4\} \notin W$, or $\{2,3\} \in W$ but $\{2,4\} \notin W$ or $\{1,2,3\} \in W$ but $\{1,2,4\} \notin W$. These possibilities also give rise to four more inequalities:

$$w_3 \geq q > w_4, \tag{9}$$
$$w_1 + w_3 \geq q > w_1 + w_4, \tag{10}$$
$$w_2 + w_3 \geq q > w_2 + w_4, \tag{11}$$
$$w_1 + w_2 + w_3 \geq q > w_1 + w_2 + w_4. \tag{12}$$

Let us prove that there is no compatible triple of inequalities, one belonging to the system (1)–(4) to ensure case (i), one belonging to the system (5)–(8) to ensure case (ii) and one belonging to the system (9)–(12) to ensure case (iii).

Firstly, note that (1) is incompatible with (5), (6), (8), (9), (10) and (12). Hence, if (1) holds, then (7) and (11) must hold as well. However, these last two equalities are incompatible with each other, proving that (1) is not possible under a consistent weighted form.

Secondly, remark that (2) is incompatible with (5), (6), (8), (9), (11) and (12). Hence, if (2) holds, then (7) and (11) must hold as well. But these last two inequalities lead to the impossibility:

$$q \overset{(10)}{>} w_1 + w_4 > w_2 + w_4 \overset{(7)}{\geq} q.$$

This implies that (2) is not possible either under a consistent weighted form.

Thirdly, note that (3) is incompatible with (5), (6), (7) and (8), which directly makes case (ii) impossible and proves that (3) cannot hold under a consistent weighted form.

Fourthly, exactly as for (1), remark that (4) is incompatible with (5), (6), (8), (9), (10) and (12), which yields the same conclusion as for (1). Since the quota q was arbitrarily chosen, this completes the proof that a representation $[q; w_1, w_2, w_3, w_4]$ such that $w_1 > w_2 > w_3 > w_4$ is always inconsistent.

Now, let us deal with the 3-voter situation. Instead of replicating the same reasoning as that developed above, we can consider again the 4-voter situation and assume that $w_4 = 0$. This assumption implies that (3) reduces to (1) and that (4) reduces to (2), which means that the system (1)–(4) describing case (i) reduces to the system of two inequalities needed for case (i) in the absence of voter 4. Similarly, the assumption $w_4 = 0$ also implies that (7) reduces to (5) and that (8) reduces to (6), which means that the system (5)–(8) describing case (ii) reduces to the system of two inequalities needed for case (ii) in the absence of voter 4. As a consequence, the demonstration for the 4-voter situation suffices to show that a representation $[q; w_1, w_2, w_3]$ such that $w_1 > w_2 > w_3$ is always inconsistent whatever the choice of the quota. ■

The EPCI Communaué des bords de Vire introduced in section 3 (see tables 4 and 5) is a particular instance of Proposition 9. The impossibility pointed out in Proposition 9 disappears in the 2-voter and 5-voter cases: for $[q; w_1, w_2]$ with $w_1 \geq q > w_2$, voters 1 and 2 are not symmetric, and for $[9; 5, 4, 3, 2, 1]$, there is no pair of symmetric voters.

Finally, it is useful to note that Proposition 9 can be seen as a consequence of data from Kurz (2012), which emphasizes that no weighted voting game with three or four voters admits a minimum sum representation in which weights are all different.

5 Probabilities of inconsistent weighting

5.1 Overview

Let us denote by $\mathcal{G}_{n,w,q}$ the set of all representations $[q; w_1, w_2, \ldots, w_n]$ of weighted voting games G with n voters having a total weight w and a quota q. Recall that every possible representation $[q; w_1, w_2, \ldots, w_n]$ in $\mathcal{G}_{n,w,q}$ can be described by the following system of (in)equalities:

$$(\mathcal{G}_{n,w,q}) : \begin{cases} w - q \geq 0 & \\ q \geq 1 & \\ w_i \geq w_{i+1} & i = 1, 2, \ldots, n-1 \\ w - \sum_{i \in N} w_i = 0 & \end{cases} \tag{13}$$

where q, w_1, w_2, …, w_n are non-negative integers. We assume that all representations of games in $\mathcal{G}_{n,w,q}$ are equally likely. This assumption is called IAC (Impartial Anonymous Culture) and it is widely employed in voting theory when computing the theoretical likelihood of electoral events. For more details on this and other probabilistic assumptions and their use in social choice theory, the reader may refer to Gehrlein and Lepelley (2011, 2017). Furthermore, we denote by $\mathcal{C}_{n,w,q}$ the set of all representations in $\mathcal{G}_{n,w,q}$ with consistent weights and by $\mathcal{I}_{n,w,q}$ the set of all representations in $\mathcal{G}_{n,w,q}$ with inconsistent weights. The frequency to be evaluated is that of inconsistent weighting of all games in $\mathcal{G}_{n,w,q}$ given n, w, and q. This amounts to evaluating the following frequencies as functions of both q and w for some values of n.

$$I(n, w, q) = \frac{|\mathcal{I}_{n,w,q}|}{|\mathcal{G}_{n,w,q}|} = 1 - \frac{|\mathcal{C}_{n,w,q}|}{|\mathcal{G}_{n,w,q}|}. \tag{14}$$

As shown in (13), each representation $[q; w_1, w_2, \ldots, w_n]$ of games in $\mathcal{G}_{n,w,q}$ is given by a set of linear constraints on q, w_1, w_2, …, w_n and w and, in order to calculate our probabilities, we need the list of constraints corresponding to $\mathcal{C}_{n,w,q}$ or $\mathcal{I}_{n,w,q}$. For this, let us first mention that it is possible to describe the structure of the game $G = (N, W)$ by enumerating all its winning coalitions or all its losing coalitions. However, this generally includes redundant constraints. Instead, Freixas and Molinero (2009) introduce the notion of a winning coalition that is shift-minimal and a losing coalition that is shift-maximal. A winning coalition S is shift-minimal if by replacing any voter i in S with a voter j out of S, one obtains a coalition T which is losing whenever $w_i > w_j$. The set of all winning coalitions in $G = (N, W)$ that are shift-minimal is denoted by W^*. A losing coalition S is shift-maximal if by replacing any voter j in S with a voter i out of S, one obtains a coalition T which is winning whenever $w_i > w_j$. The set of all losing coalitions in $G = (N, W)$ that are shift-maximal is denoted by L^*. More formally, for all coalitions S,

$$S \in W^* \iff S \in W \text{ and for all } i \in S,\, j \in N\backslash S : w_j < w_i \Longrightarrow (S\backslash \{i\}) \cup \{j\} \notin W \quad (15)$$

and

$$S \in L^* \iff S \notin W \text{ and for all } j \in S,\, i \in N\backslash S : w_j < w_i \Longrightarrow (S\backslash \{j\}) \cup \{i\} \in W \quad (16)$$

Freixas and Molinero (2009) prove that $[q; w_1, w_2, \ldots, w_n]$ is a weighted form of the game $G = (N, W)$ if

$$\begin{cases} w_S \geq q & \text{for all } S \in W^* \\ q - w_S \geq 1 & \text{for all } S \in L^* \end{cases} \quad (17)$$

Now, recall that $\mathcal{SYM}(G) = (N_1, N_2, \ldots, N_s)$ is the ordered partition of symmetric voters in the game G; that is voters in each N_t are symmetric and for all $i \in N_t$ and $j \in N_{t+1}$, $w_i > w_j$. It follows that $\mathcal{C}_{n,w,q}(G)$, the set of consistent weighted forms $[q; w_1, w_2, \ldots, w_n]$ of the weighted game $G = (N, W)$, can be described by the following system of (in)equalities:

$$(\mathcal{C}_{n,w,q}(G)) : \begin{cases} w_S - q \geq 0 & \text{for all } S \in W^* \\ q - w_S \geq 1 & \text{for all } S \in L^* \\ w_i - w_j \geq 1 & \text{for all } i \in N_t,\, j \in N_{t+1},\, t = 1, 2, \ldots, s-1 \\ w_i - w_j = 0 & \text{for all } i, j \in N_t,\, t = 1, 2, \ldots, s \\ w - q \geq 0 & \\ q \geq 1 & \\ w - \sum_{i \in N} w_i = 0 & \end{cases} \quad (18)$$

The two first inequalities of (18) come from (17); the next inequality comes from the definition of $\mathcal{SYM}(G)$; the next equality is due to the fact that $\mathcal{C}_{n,w,q}$ represents consistent weighted forms, that is voters belonging to the same N_t, for all $t = 1, 2, \ldots, s$, have the same weight; finally, the last three (in)equalities come from (13) when the inequality $w_i \geq w_{i+1}$, for all $i = 1, 2, \ldots, n-1$, is redundant.

In order to establish (18), we need the complete list $\mathcal{WG}_n$ of all weighted voting games with n voters. Data come from Kurz (2012, 2018) who provides all minimum sum representations of weighted voting games for up to 9 voters, where the weights and the quota are restricted to integers. In the case of two voters, all of the k different weighted voting games G_k in $\mathcal{WG}_2$ are listed in Table 9 together with the set $\mathcal{SYM}(G_k)$ of distinct classes of symmetric voters, $W^*(G_k)$ of shift-minimal winning coalitions and $L^*(G_k)$ of shift-maximal losing coalitions. Recall that, in this table, each weighted voting game is given by a unique weighted representation with a minimal sum of positive integer weights.

Table 9: List of all two-voter weighted voting games G

k	G_k	$L^*(G_k)$	$W^*(G_k)$	$\mathcal{SYM}(G_k)$
1	$[2;1,1]$	$[\{1\}]$	$[\{1,2\}]$	$\{1,2\}$
2	$[1;1,0]$	$[\{2\}]$	$[\{1\}]$	$\{1\},\{2\}$
3	$[1;1,1]$	$[]$	$[\{2\}]$	$\{1,2\}$

In the case of three voters, the data are listed in Table 10 and all four-voter possible weighted voting games are listed in Table 14 in the appendix.

Table 10: List of all three-voter weighted voting games G

k	G_k	$L^*(G_k)$	$W^*(G_k)$	$\mathcal{SYM}(G_k)$
1	$[3;1,1,1]$	$[\{1,2\}]$	$[\{1,2,3\}]$	$\{1,2,3\}$
2	$[2;1,1,0]$	$[\{1,3\}]$	$[\{1,2\}]$	$\{1,2\},\{3\}$
3	$[3;2,1,1]$	$[\{1\},\{2,3\}]$	$[\{1,3\}]$	$\{1\},\{2,3\}$
4	$[1;1,0,0]$	$[\{2,3\}]$	$[\{1\}]$	$\{1\},\{2,3\}$
5	$[2;2,1,1]$	$[\{2\}]$	$[\{1\},\{2,3\}]$	$\{1\},\{2,3\}$
6	$[2;1,1,1]$	$[\{1\}]$	$[\{2,3\}]$	$\{1,2,3\}$
7	$[1;1,1,0]$	$[\{3\}]$	$[\{2\}]$	$\{1,2\},\{3\}$
8	$[1;1,1,1]$	$[]$	$[\{3\}]$	$\{1,2,3\}$

The system (18) will consist of k different systems $(\mathcal{C}_{n,w,q}(G_k))$, one for every possible representation G_k of the game G. The notation $\mathcal{C}_{n,w,q}(G_k)$ corresponds to the set of all consistent weighted voting games having G_k as a representation with minimum sum, and let us denote by $|\mathcal{C}_{n,w,q}(G_k)|$ the cardinality of this set. Since each of the k weighted representations describes a unique weighted voting game, it follows that

$$|\mathcal{C}_{n,w,q}| = \sum_{G_k \in \mathcal{WG}_n} |\mathcal{C}_{n,w,q}(G_k)| \qquad \text{and} \qquad I(n,w,q) = 1 - \sum_{G_k \in \mathcal{WG}_n} \frac{|\mathcal{C}_{n,w,q}(G_k)|}{|\mathcal{G}_{n,w,q}|} \tag{19}$$

To illustrate our approach, let us take the case of $n = 2$ (see Table 9). The systems that we have to deal with are as follows:

$$(\mathcal{G}_{2,w,q}) : \begin{cases} w - q \geq 0 \\ q \geq 1 \\ w_1 - w_2 \geq 0 \\ w - w_1 - w_2 = 0 \end{cases} \qquad (\mathcal{C}_{2,w,q}(G_1)) : \begin{cases} w_1 + w_2 - q \geq 0 \\ q - w_1 \geq 1 \\ w_1 - w_2 = 0 \\ w - q \geq 0 \\ q \geq 1 \\ w - w_1 - w_2 = 0 \end{cases}$$

$$(\mathcal{C}_{2,w,q}(G_2)) : \begin{cases} w_1 - q \geq 0 \\ q - w_2 \geq 1 \\ w_1 - w_2 \geq 1 \\ w - q \geq 0 \\ q \geq 1 \\ w - w_1 - w_2 = 0 \end{cases} \qquad (\mathcal{C}_{2,w,q}(G_3)) : \begin{cases} w_2 - q \geq 0 \\ w_1 - w_2 = 0 \\ w - q \geq 0 \\ q \geq 1 \\ w - w_1 - w_2 = 0 \end{cases}$$

In order to use (19), the numbers $|\mathcal{G}_{n,w,q}|$ and $|\mathcal{C}_{n,w,q}(G_k)|$, for any given form G_k, of integer solutions of the systems $(\mathcal{G}_{n,w,q})$ and $(\mathcal{C}_{n,w,q}(G_k))$ are calculated by means of Ehrhart polynomials, a method introduced in the social choice literature by Lepelley et al. (2008) and Wilson and Pritchard (2007) in order to estimate the probabilities of some voting events in social choice theory. This theory teaches us that the two numbers $|\mathcal{G}_{n,w,q}|$ and $|\mathcal{C}_{n,w,q}(G_k)|$ are pseudo-polynomials with two parameters w and q. We use the parametrized Barvinok's algorithm (Barvinok, 1994, Barvinok and Pommersheim, 1999, Verdoolaege et al., 2004) in order to solve those systems[9] and we refer the reader to Gehrlein and Lepelley (2011, 2017) for more details on the use of these tools in social choice theory.

Note that Table 11 (see Kurz, 2012, 2018) enumerates the number of weighted voting games with minimum integer representations up to 9 voters.[10] This means that the number of distinct weighted voting games that we need to consider is equal to 117 if the number of voters is 5 and makes our computations very tedious. For this reason, we only consider a number of voters up to 4 in the case of finite total weight w. For an infinite total weight, a simplification will allow us to consider all cases up to 5 voters. Despite this fact, we believe, however, that the considered values of n, w, and q give us enough information regarding the probability of inconsistent weighting scenarios in the class of weighted voting games.

Table 11: Number of weighted voting games with minimum integer representations

n	2	3	4	5	6	7	8	9
Number of games	3	8	25	117	1 111	29 373	20 730 164	993 061 482

Finally, let us recall that a weighted form of the dual game $\widehat{G}$ of G is $[\widehat{q}; w_1, w_2, \ldots, w_n]$ where $w = w_N$ is the total weight and $\widehat{q} = w - q + 1$. The next remark tells us that the two frequencies $I(n, w, q)$ and $I(n, w, \widehat{q})$ are the same for all $w \geq 2$ and for all $q \leq w$.

Remark 10 *For the dual game $\widehat{G} = (N, \widehat{W})$ of the weighted game $G = (N, W)$, it can be easily checked that the set $\widehat{W}^*$ of all shift-minimal winning coalitions and the set $\widehat{L}^*$ of all shift-maximal losing coalitions in $\widehat{W}^*$ are such that*

$$\widehat{W}^* = \{N \backslash S \mid S \in L^*\} \text{ and } \widehat{L}^* = \{N \backslash S \mid S \in W^*\}. \tag{20}$$

Moreover,

$$(w_S - q \geq 0, \text{ for all } S \in W^*) \Longleftrightarrow (\widehat{q} - w_T \geq 1, \text{ for all } T \in \widehat{L}^*) \tag{21}$$

and

$$(q - w_S \geq 1, \text{ for all } S \in L^*) \Longleftrightarrow (w_T - \widehat{q} \geq 0, \text{ for all } T \in \widehat{W}^*). \tag{22}$$

If follows that $[q; w_1, w_2, \ldots, w_n] \in \mathcal{G}_{n,w,q} \Longleftrightarrow [\widehat{q}; w_1, w_2, \ldots, w_n] \in \mathcal{G}_{n,w,\widehat{q}}$ and similarly, $[q; w_1, w_2, \ldots, w_n] \in \mathcal{C}_{n,w,q} \Longleftrightarrow [\widehat{q}; w_1, w_2, \ldots, w_n] \in \mathcal{C}_{n,w,\widehat{q}}$. This proves that $|\mathcal{G}_{n,w,q}| = |\mathcal{G}_{n,w,\widehat{q}}|$ and for all games $[q; w_1, w_2, \ldots, w_n]$, $|\mathcal{C}_{n,w,q}| = |\mathcal{C}_{n,w,\widehat{q}}|$. Therefore,

$$I(n, w, q) = I(n, w, \widehat{q}) \text{ for all } w \geq 2 \text{ and for all } q \leq w. \tag{23}$$

[9] The free software to calculate the integer points under the Parameterized Barvinok's algorithm can be found at http://freecode.com/projects/barvinok. The algorithm allows one to quantify the number of integer solutions for systems of (in)equalities with parameters.

[10] We would like to gratefully acknowledge Sascha Kurz for having provided us with the lists of weighted voting games up to 7 voters. Those lists are available upon request.

5.2 Finite total weights

The equation (19) makes it possible to derive the representations of $I(n,w,q)$ for the considered values of n. Let us start with the case of $n=2$.

Proposition 11 *For $n=2$, the probability of inconsistent weighting is given as:*

- *For w even*

$$I(2,w,q)=\begin{cases}\frac{w-2q}{w+2} & \text{if } q\in[1,\frac{w}{2}]\\ \frac{2q-w-2}{w+2} & \text{if } q\in[\frac{w}{2}+1,w]\end{cases}$$

- *For w odd*

$$I(2,w,q)=\begin{cases}\frac{w-2q+1}{w+1} & \text{if } q\in[1,\frac{w+1}{2}]\\ \frac{2q-w-1}{w+1} & \text{if } q\in[\frac{w+1}{2}+1,w]\end{cases}$$

Proof. To obtain these representations, we compute first the total number $|\mathcal{G}_{2,w,q}|$ of solutions of the system $(\mathcal{G}_{2,w,q})$. As noted before, we use the parametrized Barvinok's algorithm in order to solve this system. The program indicates that the corresponding quasi-polynomial is as follows: $|\mathcal{G}_{2,w,q}|=\frac{w}{2}+[1,\frac{1}{2}]_w$ for $1\leq q\leq w$. The number $[1,\frac{1}{2}]_w$ is a 2-periodic coefficient, meaning that such a coefficient depends on the parity of the parameter w: the coefficient is equal to 1 for even w and to $\frac{1}{2}$ for odd w. Consider now the number $|\mathcal{C}_{2,w,q}|$. This is equal to the number of solutions for three systems $(\mathcal{C}_{2,w,q}(G_k))$ as indicated before. The polynomial giving the number of solutions for each case differs depending on whether w is odd or even, but also on the value of q. For $k=1$, the program indicates that the number of solutions $|\mathcal{C}_{2,w,q}(G_1)|$ is equal to 1 if w is even and $\frac{w+2}{2}\leq q\leq w$ and it is equal to 0 otherwise. For $k=2$, the number of solutions $|\mathcal{C}_{2,w,q}(G_2)|$ is equal to $w-q+1$ if $\frac{w+2}{2}\leq q\leq w$ and it is equal to q if $1\leq q\leq\frac{w+1}{2}$. For $k=3$, the number of solutions $|\mathcal{C}_{2,w,q}(G_3)|$ is equal to 1 if w is even and $1\leq q\leq\frac{w}{2}$ and it is equal to 0 otherwise. The number $|\mathcal{C}_{2,w,q}|$ is then obtained by summing the number of solutions of the three cases above. The desired representation is obtained by dividing $|\mathcal{C}_{2,w,q}|$ by the total number $|\mathcal{G}_{2,w,q}|$ and using (19). ∎

The next proposition deals with the probability representations for the 3-voter case.

Proposition 12 *For $n=3$, the probability of inconsistent weighting $I(3,w,q)$ is given* *as:*

$$\begin{bmatrix} [a,b] & 0<q\leq\frac{w}{3} & \frac{w}{3}<q\leq\frac{w}{2} & \frac{w}{2}<q\leq\frac{2w}{3} & \frac{2w}{3}<q\leq w] \\
[0,0] & \frac{w^2+6\,w-18\,q}{w^2+6\,w+12} & \frac{w^2-6\,w-24+18\,q}{w^2+6\,w+12} & \frac{w^2+12\,w-18\,q}{w^2+6\,w+12} & \frac{w^2-12\,w-24+18\,q}{w^2+6\,w+12} \\
[1,0] & \frac{w^2+6\,w-6-18\,q}{w^2+6\,w+12} & \frac{w^2-6\,w-18+18\,q}{w^2+6\,w+12} & \frac{w^2+12\,w-6-18\,q}{w^2+6\,w+12} & \frac{w^2-12\,w-18+18\,q}{w^2+6\,w+12} \\
[0,1] & \frac{w^2+6\,w+5-18\,q}{w^2+6\,w+5} & \frac{w^2-6\,w-7+18\,q}{w^2+6\,w+5} & \frac{w^2+12\,w+11-18\,q}{w^2+6\,w+5} & \frac{w^2-12\,w-13+18\,q}{w^2+6\,w+5} \\
[1,1] & \frac{w^2+6\,w+11-18\,q}{w^2+6\,w+5} & \frac{w^2-6\,w-13+18\,q}{w^2+6\,w+5} & \frac{w^2+12\,w+5-18\,q}{w^2+6\,w+5} & \frac{w^2-12\,w-7+18\,q}{w^2+6\,w+5} \\
[0,2] & \frac{w^2+6\,w+8-18\,q}{w^2+6\,w+8} & \frac{w^2-6\,w-16+18\,q}{w^2+6\,w+8} & \frac{w^2+12\,w+8-18\,q}{w^2+6\,w+8} & \frac{w^2-12\,w-16+18\,q}{w^2+6\,w+8} \\
[1,2] & \frac{w^2+6\,w+2-18\,q}{w^2+6\,w+8} & \frac{w^2-6\,w-10+18\,q}{w^2+6\,w+8} & \frac{w^2+12\,w+2-18\,q}{w^2+6\,w+8} & \frac{w^2-12\,w-10+18\,q}{w^2+6\,w+8} \\
[0,3] & \frac{w^2+6\,w-3-18\,q}{w^2+6\,w+9} & \frac{w^2-6\,w-15+18\,q}{w^2+6\,w+9} & \frac{w^2+12\,w+3-18\,q}{w^2+6\,w+9} & \frac{w^2-12\,w-21+18\,q}{w^2+6\,w+9} \\
[1,3] & \frac{w^2+6\,w+3-18\,q}{w^2+6\,w+9} & \frac{w^2-6\,w-21+18\,q}{w^2+6\,w+9} & \frac{w^2+12\,w-3-18\,q}{w^2+6\,w+9} & \frac{w^2-12\,w-15+18\,q}{w^2+6\,w+9} \\
[0,4] & \frac{w^2+6\,w+8-18\,q}{w^2+6\,w+8} & \frac{w^2-6\,w-16+18\,q}{w^2+6\,w+8} & \frac{w^2+12\,w+8-18\,q}{w^2+6\,w+8} & \frac{w^2-12\,w-16+18\,q}{w^2+6\,w+8} \\
[1,4] & \frac{w^2+6\,w+2-18\,q}{w^2+6\,w+8} & \frac{w^2-6\,w-10+18\,q}{w^2+6\,w+8} & \frac{w^2+12\,w+2-18\,q}{w^2+6\,w+8} & \frac{w^2-12\,w-10+18\,q}{w^2+6\,w+8} \\
[0,5] & \frac{w^2+6\,w+5-18\,q}{w^2+6\,w+5} & \frac{w^2-6\,w-7+18\,q}{w^2+6\,w+5} & \frac{w^2+12\,w+11-18\,q}{w^2+6\,w+5} & \frac{w^2-12\,w-13+18\,q}{w^2+6\,w+5} \\
[1,5] & \frac{w^2+6\,w+11-18\,q}{w^2+6\,w+5} & \frac{w^2-6\,w-13+18\,q}{w^2+6\,w+5} & \frac{w^2+12\,w+5-18\,q}{w^2+6\,w+5} & \frac{w^2-12\,w-7+18\,q}{w^2+6\,w+5}
\end{bmatrix}$$

In the first column, a list $[a,b]$ of two integers is reported. The results should be read as follows: if q mod 2 is a while w mod 6 is b, then the required frequency of inconsistent weighting is provided in the row $[a,b]$ and the appropriate column.

Proof. We first compute the total number $|\mathcal{G}_{3,w,q}|$. We again use the parametrized Barvinok's algorithm in order to solve the system $(\mathcal{G}_{3,w,q})$. The program indicates that the corresponding quasi-polynomial is as follows: $|\mathcal{G}_{3,w,q}| = \frac{w^2}{12} + \frac{w}{2} + [1, \frac{5}{12}, \frac{2}{3}, \frac{3}{4}, \frac{2}{3}, \frac{5}{12}]_w$ for $1 \leq q \leq w$. Regarding the number $|\mathcal{C}_{2,w,q}|$, Table 10 indicates that eight cases have to be considered. For $k = 1$, the program indicates that the number of solutions $|\mathcal{C}_{3,w,q}(G_1)|$ is equal to 1 if w is a multiple of 3 and $\frac{2w+3}{3} \leq q \leq w$ and it is equal to 0 otherwise. For $k = 2$, the number $|\mathcal{C}_{3,w,q}(G_2)|$ is equal to $q - \frac{w}{2} + [0, -\frac{1}{2}]_w$ if $\frac{w+2}{2} \leq q \leq \frac{2w+2}{3}$, it is equal to $-\frac{q}{2} + \frac{w}{2} + [[1, \frac{1}{2}]_w, [\frac{1}{2}, 0]_w]_q$ if $\frac{2w+3}{3} \leq q \leq w$, and it is equal to 0 otherwise. For $k = 3$, the number $|\mathcal{C}_{3,w,q}(G_3)|$ is equal to $q - \frac{w}{2} + [[-1, -\frac{1}{2}]_w, [0, -\frac{1}{2}]_w]_q$ if $\frac{w+2}{2} \leq q \leq \frac{2w+1}{3}$, it is equal to $-\frac{q}{2} + \frac{w}{2} + [[0, \frac{1}{2}]_w, [\frac{1}{2}, 0]_w]_q$ if $\frac{2w+2}{3} \leq q \leq w - 1$, and it is equal to 0 otherwise. For $k = 4$, the number $|\mathcal{C}_{3,w,q}(G_4)|$ is equal to $\frac{q}{2} + [0, \frac{1}{2}]_q$ if $1 \leq q \leq \frac{w+1}{2}$ and it is equal to $-\frac{q}{2} + \frac{w}{2} + [[1, \frac{1}{2}]_w, [\frac{1}{2}, 1]_w]_q$ if $\frac{w+2}{2} \leq q \leq w$. For $k = 5$, the number $|\mathcal{C}_{3,w,q}(G_5)|$ is equal to $\frac{q}{2} + [0, -\frac{1}{2}]_q$ if $2 \leq q \leq \frac{w+1}{3}$, it is equal to $-q + \frac{w}{2} + [[1, \frac{1}{2}]_w, [0, \frac{1}{2}]_w]_q$ if $\frac{w+2}{3} \leq q \leq \frac{w}{2}$, and it is equal to 0 otherwise. For $k = 6$, the number $|\mathcal{C}_{3,w,q}(G_6)|$ is equal to 1 if w is a multiple of 3 and $\frac{w+3}{3} \leq q \leq \frac{2w}{3}$ and it is equal to 0 otherwise. For $k = 7$, the number $|\mathcal{C}_{3,w,q}(G_7)|$ is equal to $\frac{q}{2} + [[0, 0]_w, [\frac{1}{2}, -\frac{1}{2}]_w]_q$ if $1 \leq q \leq \frac{w}{3}$, it is equal to $-q + \frac{w}{2} + [1, \frac{1}{2}]_w$ if $\frac{w+1}{3} \leq q \leq \frac{w}{2}$, and it is equal to 0 otherwise. Finally, for $k = 8$, the number $|\mathcal{C}_{3,w,q}(G_8)|$ is equal to 1 if w is a multiple of 3 and $1 \leq q \leq \frac{w}{3}$ and it is equal to 0 otherwise. The number $|\mathcal{C}_{3,w,q}|$ is obtained by summing the number of solutions of the eight cases above. The desired representation is obtained by dividing $|\mathcal{C}_{3,w,q}|$ by the total number $|\mathcal{G}_{3,w,q}|$ and using (19). ■

We deduce from the above two theorems the computed values of $I(2, w, q)$ and $I(3, w, q)$ that are displayed in Tables 16 and 17 (see Appendix) for various values of w and some values of q varying from 1 to w. The four-voter case is more complex and we cannot find exact formulas for $I(4, w, q)$ since the periods of the 25 quasi-polynomials that we have to deal with are very large. However, we report the computed values of $I(4, w, q)$ in Table 18 (see Appendix) for the same values of w and q.[11] Figures 1-3 illustrate our results for some values of w. We also report in Tables 16-18 the optimal probabilities denoted $I_{min}(q)$ and $I_{max}(q)$ corresponding respectively to the minimal and the maximal probabilities as well as the corresponding quotas q running from 1 to w for every value of w. For $n = 2$, for instance, the minimum of $I(n, w, q)$ is 0 and corresponds to $q = \frac{w}{2}$ and $q = \frac{w}{2} + 1$ for even w and $q = \frac{w+1}{2}$ for odd w. The maximum of $I(n, w, q)$ corresponds to the probability obtained for $q = 1$ and $q = w$.

A couple of points should be stressed when studying those results closely. First, we note that, for a given value of w, the probability of inconsistent weighting tends to decrease when q is small while it increases when q goes to w. Recall that our probabilities are symmetric around a quota corresponding to simple majority (see Remark 10). Second, it is worth noticing that for the three-voter and four-voter cases, our probabilities tend to slightly increase when q is around simple majority. Third, from Tables 16 to 18, we also find that, for given quota q and total weight w, the probability of inconsistent weighting increases with the number of voters. Fourth, recall that Proposition 6 tells us that each weighted voting game that admits a pair of symmetric voters has some consistent weighted forms as well as some inconsistent weighted forms. However, it should be noted that our probabilistic results show that an increase in w will clearly lead to a very high probability of inconsistent weighting even for a small number of voters.

[11] The representations of the 25 cases are available upon request from the authors.

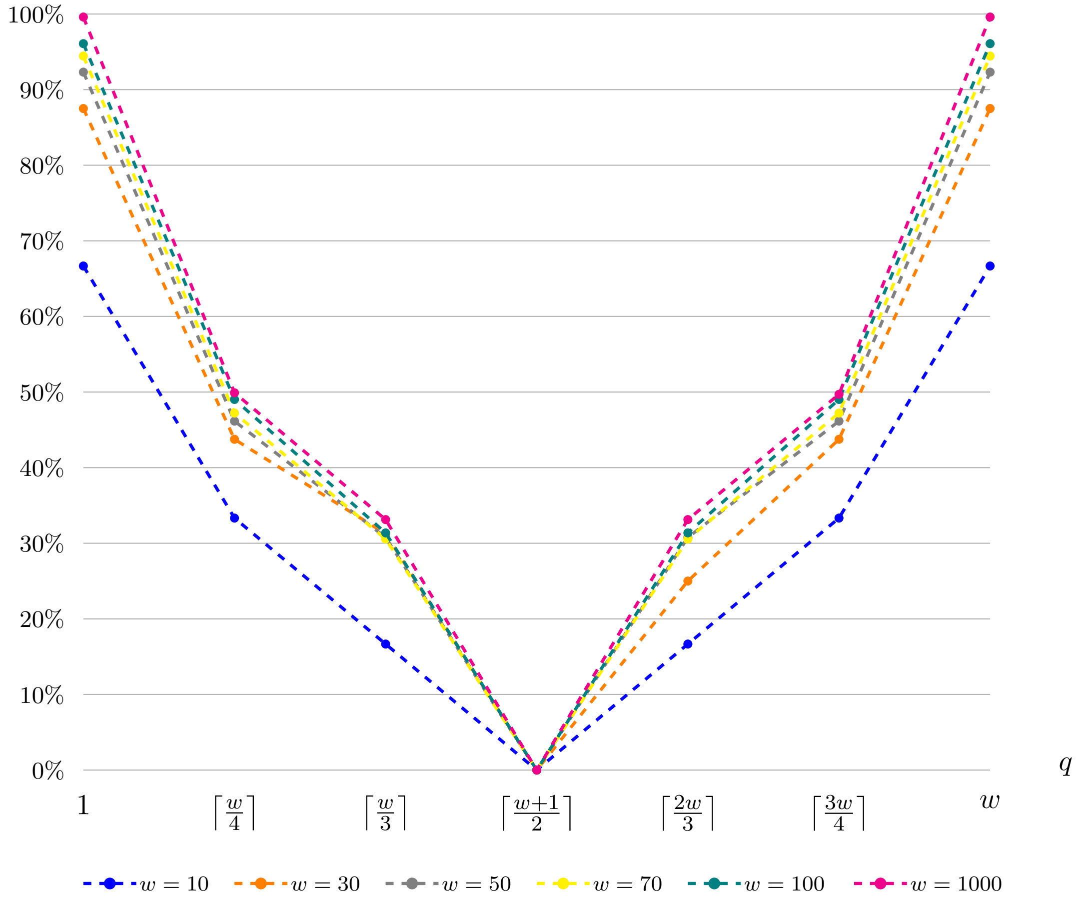

Figure 1: Probability of having inconsistent weighted forms with 2 voters for some fixed values of w and q.

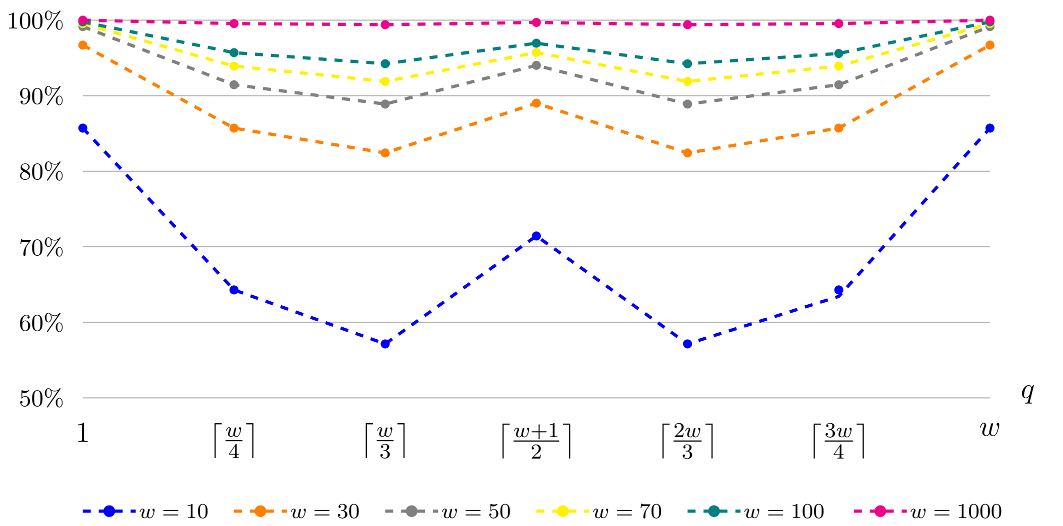

Figure 2: Probability of having inconsistent weighted forms with 3 voters for some fixed values of w and q.

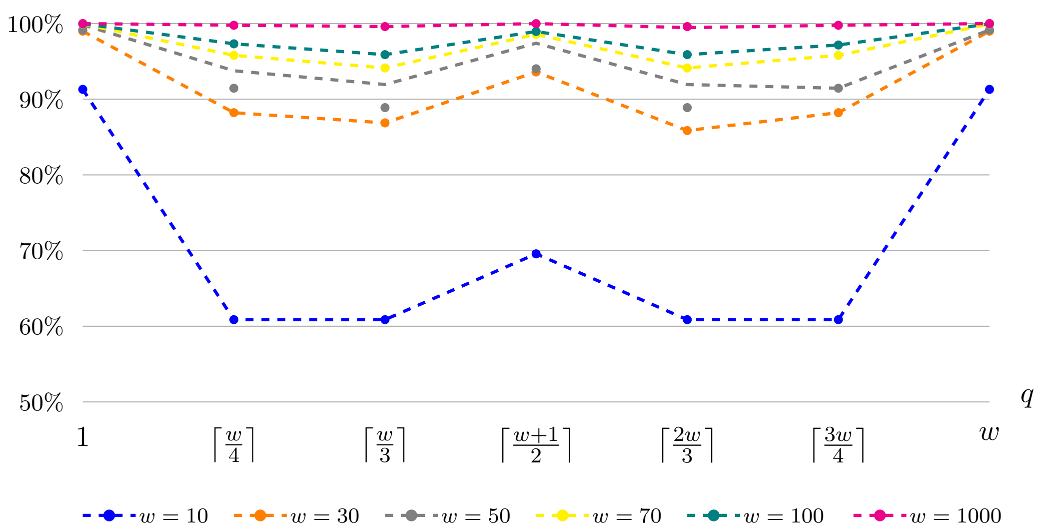

Figure 3: Probability of having inconsistent weighted forms with 4 voters for some fixed values of w and q.

5.3 Infinite total weights

This section deals with the case of a total weight w tending to infinity. Let us consider the ratio $\frac{q}{w}$ which tends to r such that $0 < r \leq 1$ as w tends to infinity. We focus on the limit $I(n, \infty, r)$ of $I(n, w, q)$ as w tends to infinity and $\frac{w}{q}$ tends to r. As mentioned before, given $G \in \mathcal{WG}_n$, the total numbers $|\mathcal{G}_{n,w,q}|$ and $|\mathcal{C}_{n,w,q}(G)|$ of solution of the sets $(\mathcal{G}_{n,w,q})$ and $(\mathcal{C}_{n,w,q}(G))$ of constraints are pseudo-polynomials. It is well known that the leading terms of $|\mathcal{G}_{n,w,q}|$ and $|\mathcal{C}_{n,w,q}(G)|$ are the n-dimensional volumes $vol\,(\mathcal{G}_n)$ and $vol\,(\mathcal{C}_n(G))$ of the supporting polytope $(\mathcal{G}_n)$ and $(\mathcal{C}_n(G))$ associated with $(\mathcal{G}_{n,w,q})$ and $(\mathcal{C}_{n,w,q}(G))$ such that:

$$(\mathcal{C}_n(G)) : \begin{cases} x_S - y \geq 0 & \text{for all } S \in W^* \\ y - x_S \geq 0 & \text{for all } S \in L^* \\ x_i - x_j = 0 & \text{for all } i, j \in N_t,\ t = 1, 2, \ldots, s \\ x_i - x_j \geq 0 & \text{for all } i \in N_t,\ j \in N_{t+1},\ t = 1, 2, \ldots, s-1 \\ 1 - y \geq 0 & \\ y \geq 0 & \\ 1 - \sum_{i \in N} x_i = 0 & \end{cases}$$

and

$$(\mathcal{G}_n) : \begin{cases} 1 - y \geq 0 & \\ y \geq 0 & \\ x_i \geq x_{i+1} & i = 1, 2, \ldots, n-1 \\ 1 - \sum_{i \in N} x_i = 0 & \end{cases}$$

where $y = \frac{q}{w}$, $x_S = \frac{w_S}{w}$ for every $S \in N$, and $x_i = \frac{w_i}{w}$ for $i = 1, 2, \ldots, n$. Note that from $(\mathcal{G}_{n,w,q})$ and $(\mathcal{C}_{n,w,q}(G))$ to $(\mathcal{G}_n)$ and $(\mathcal{C}_n(G))$, each $\frac{c}{w}$ is set to 0 for each second term c as w tends to infinity. It follows from equation (19) that

$$I(n, \infty, r) = 1 - \sum_{G_k \in \mathcal{WG}_n} \frac{vol\,(\mathcal{C}_n\,(G_k))}{vol\,(\mathcal{G}_n)} \tag{24}$$

Several methods and algorithms can be found in the literature for the computation of polytopes' volumes. As noted before, the volume of every rational polytope can be calculated by considering the leading coefficient of the Ehrhart quasi-polynomial associated with it. We will consider this method throughout the article. However, it is worth noting that the volume of a rational polytope can also be obtained by a direct use of various volume computation algorithms that we can find in the literature. For more details on these algorithms and their use in social choice theory, the reader may refer to Cervone et al. (2005) and Moyouwou and Tchantcho (2017). These techniques have recently been used under different forms by Bubboloni et al. (2020), Diss and Doghmi (2016), El Ouafdi et al. (2020), Kamwa (2019), Kamwa and Moyouwou (2020), Lepelley et al. (2018), and Lepelley and Smaoui (2019), among others.

Recall now that $\mathcal{WG}_n^*$ is the set of all weighted voting games G with n voters that admit no pair of symmetric voters; that is, all representation forms with $\mathcal{SYM}(G) = (\{1\}, \{2\}, \ldots, \{n\})$. To compute $I(n, \infty, r)$, we need again the complete list $\mathcal{WG}_n$ of all weighted voting games with n voters that was obtained from Kurz (2012). However, the next proposition teaches us that a simplification can be made in the infinite case, and this plays an important role in obtaining our probabilities of inconsistent weighting.

Proposition 13 *Given any number $n \geq 2$ of voters,*

$$I(n, \infty, r) = 1 - \sum_{G_k \in \mathcal{WG}_n^*} \frac{vol(\mathcal{C}_n(G_k))}{vol(\mathcal{G}_n)}. \tag{25}$$

Proof. Note first that the dimension of the supporting polytope $(\mathcal{G}_n)$ is n. In addition, for each game $G_k \in \mathcal{WG}_n \backslash \mathcal{WG}_n^*$, $\mathcal{SYM}(G_k) \neq (\{1\}, \{2\}, \ldots, \{n\})$ and for each game $G_k \in \mathcal{WG}_n^*$, $\mathcal{SYM}(G_k) = (\{1\}, \{2\}, \ldots, \{n\})$. Therefore, for all games $G_k \in \mathcal{WG}_n \backslash \mathcal{WG}_n^*$, the constraints in $(\mathcal{C}_{n,w,q}(G_k))$ include at least one equality between the weights of symmetric voters since at least two symmetric voters exist. Each possible such equality is reported in the constraints of the supporting polytope $\mathcal{C}_n(G_k)$ of $(\mathcal{C}_{n,w,q}(G_k))$. As a result, $\mathcal{C}_n(G_k)$ will have dimension less than n. Hence its n-dimensional volume is 0; that is $vol(\mathcal{C}_n(G_k)) = 0$. This completes the proof. ∎

Intuitively, Proposition 13 tells us that the probability of an inconsistent weighting in G as w tends to infinity is equal to one minus the probability of the cases where G_k leads to $\mathcal{SYM}(G_k) = (\{1\}, \{2\}, \ldots, \{n\})$. This result allows us to find the probability representations as a function of the ratio r for the considered values of n.

Proposition 14 *For $n = 2$, $w \to +\infty$, and $\frac{q}{w} \to r$ with $0 < r \leq 1$, the probability of inconsistent weighting is given as:*

$$I(2, \infty, r) = \begin{cases} 1 - 2r & \text{if } r \in]0, \frac{1}{2}] \\ 2r - 1 & \text{if } r \in [\frac{1}{2}, 1] \end{cases}$$

Proof. Recall that, in Table 9, each weighted voting game is given by a unique weighted representation with a minimal sum of positive integer weights. It follows that $\mathcal{WG}_2^*$ is a singleton and consists in the game G_2 in which voter 1 is the dictator. Recall now from the proof of Proposition 11 that $|\mathcal{G}_{2,w,q}| = \frac{w}{2} + [1, \frac{1}{2}]_w$ and, for $k = 2$, the number of solutions of the system $(\mathcal{C}_{2,w,q}(G_2))$ is equal to $w - q + 1$ if $\frac{w+2}{2} \leq q \leq w$ and it is equal to q if $1 \leq q \leq \frac{w+1}{2}$. If we assume large weights, replace q by rw in the results, and only consider the term of higher degree in w, we have $vol(\mathcal{G}_n) = \frac{1}{2}$, $vol(\mathcal{C}_n(G_2)) = 1 - r$ if

$\frac{1}{2} \leq r \leq 1$, and $vol\left(\mathcal{C}_n\left(G_2\right)\right) = r$ if $0 < r \leq \frac{1}{2}$. Then $\frac{vol(\mathcal{C}_n(G_2))}{vol(\mathcal{G}_n)}$ is equal to $(2-2r)$ if $\frac{1}{2} \leq r \leq 1$ and it is equal to $2r$ if $0 < r \leq \frac{1}{2}$. We complete the proof using Proposition 13. ■

The next proposition deals with the cases of $n = 3$ and $n = 4$.

Proposition 15 *For $w \to +\infty$, and $\frac{q}{w} \to r$ with $(0 < r \leq 1)$, the probability of inconsistent weighting is given as:*

$$I\left(n, \infty, r\right) = 1, \quad for \quad n = 3, 4.$$

Proof. Let us start with the case of $n = 3$. In Table 10, each weighted voting game admits at least a pair of symmetric voters. It follows that $\mathcal{WG}_3^*$ is empty. In other words, the set of constraints associated with consistent representations of any weighted voting game with three voters admits an equality between the weights of two symmetric voters. Therefore, $\mathcal{C}_n\left(G_k\right)$ for every k of Table 10 will have dimension less than 3. Hence, its 3-dimensional volume is 0. Therefore, using Proposition 13, we get $I\left(3, \infty, r\right) = 1$. The same proof is used for $n = 4$ since $\mathcal{WG}_4^*$ is also empty (see Table 14). ■

Remark that $\mathcal{WG}_3^* = \mathcal{WG}_4^* = \emptyset$ also comes from Proposition 9. The next proposition is related to the case of $n = 5$.

Proposition 16 *For $n = 5$, $w \to +\infty$, and $\frac{q}{w} \to r$ with $0 < r \leq 1$, the probability of inconsistent weighting is given as:*

$$I\left(5, \infty, r\right) = \begin{cases} 1 & if\ r \in]0, \frac{1}{3}] \\ -810\,r^4 + 1080\,r^3 - 540\,r^2 + 120\,r - 9 & if\ r \in [\frac{1}{3}, \frac{2}{5}] \\ 17940\,r^4 - 28920\,r^3 + 17460\,r^2 - 4680\,r + 471 & if\ r \in [\frac{2}{5}, \frac{3}{7}] \\ -54090\,r^4 + 94560\,r^3 - 61920\,r^2 + 18000\,r - 1959 & if\ r \in [\frac{3}{7}, \frac{4}{9}] \\ 11520\,r^4 - 22080\,r^3 + 15840\,r^2 - 5040\,r + 601 & if\ r \in [\frac{4}{9}, \frac{1}{2}] \\ 11520\,r^4 - 24000\,r^3 + 18720\,r^2 - 6480\,r + 841 & if\ r \in [\frac{1}{2}, \frac{5}{9}] \\ -54090\,r^4 + 121800\,r^3 - 102780\,r^2 + 38520\,r - 5409 & if\ r \in [\frac{5}{9}, \frac{4}{7}] \\ 17940\,r^4 - 42840\,r^3 + 38340\,r^2 - 15240\,r + 2271 & if\ r \in [\frac{4}{7}, \frac{3}{5}] \\ -810\,r^4 + 2160\,r^3 - 2160\,r^2 + 960\,r - 159 & if\ r \in [\frac{3}{5}, \frac{2}{3}] \\ 1 & if\ r \in]\frac{2}{3}, 1] \end{cases}$$

Proof. For $n = 5$, there are only two possible weighted voting games such that $\mathcal{SYM}\left(G\right) = \left(\{1\}, \{2\}, \ldots, \{n\}\right)$: $[9; 5, 4, 3, 2, 1]$ and $[7; 5, 4, 3, 2, 1]$. The two games as well as their shift-minimal winning coalitions and shift-maximal losing coalitions are described in Table 15 (see the appendix). As we assume large weights, we can again replace q by rw and then consider the term of higher degree in w. Using the parametrized Barvinok's algorithm, the term of higher degree in w of $|\mathcal{G}_{5,w,q}|$ is $\frac{1}{2880}$ for $1 \leq q \leq w$. Then, this value defines our first volume $vol\left(\mathcal{G}_n\right)$. In addition, for $k = 1$, the term of higher degree in w of $|\mathcal{C}_{n,w,q}(G_1)|$ defines the second volume

$$vol\left(\mathcal{C}_n\left(G_1\right)\right)=\begin{cases}-4r^4+\frac{25}{3}r^3-\frac{13}{2}r^2+\frac{9}{4}r-\frac{7}{24} & \text{for } \frac{1}{2}\leq r\leq\frac{5}{9}\\ \frac{601}{32}r^4-\frac{1015}{24}r^3+\frac{571}{16}r^2-\frac{107}{8}r+\frac{541}{288} & \text{for } \frac{5}{9}\leq r\leq\frac{4}{7}\\ -\frac{299}{48}r^4+\frac{119}{8}r^3-\frac{213}{16}r^2+\frac{127}{24}r-\frac{227}{288} & \text{for } \frac{4}{7}\leq r\leq\frac{3}{5}\\ \frac{9}{32}r^4-\frac{3}{4}r^3+\frac{3}{4}r^2-\frac{1}{3}r+\frac{1}{18} & \text{for } \frac{3}{5}\leq r\leq\frac{2}{3}\\ 0 & \text{otherwise}\end{cases}$$

Finally, for $k=2$, the term of higher degree in w of $|\mathcal{C}_{n,w,q}(G_2)|$ defines the last volume

$$vol\left(\mathcal{C}_n\left(G_2\right)\right)=\begin{cases}\frac{9}{32}r^4-\frac{3}{8}r^3+\frac{3}{16}r^2-\frac{1}{24}r+\frac{1}{288} & \text{for } \frac{1}{3}\leq r\leq\frac{2}{5}\\ -\frac{299}{48}r^4+\frac{241}{24}r^3-\frac{97}{16}r^2+\frac{13}{8}r-\frac{47}{288} & \text{for } \frac{2}{5}\leq r\leq\frac{3}{7}\\ \frac{601}{32}r^4-\frac{197}{6}r^3+\frac{43}{2}r^2-\frac{25}{4}r+\frac{49}{72} & \text{for } \frac{3}{7}\leq r\leq\frac{4}{9}\\ -4r^4+\frac{23}{3}r^3-\frac{11}{2}r^2+\frac{7}{4}r-\frac{5}{24} & \text{for } \frac{4}{9}\leq r\leq\frac{1}{2}\\ 0 & \text{otherwise}\end{cases}$$

Using Proposition 13, the desired probabilities are obtained as a function of r by summing the outputs of $k=1$ and $k=2$ and dividing by $vol\left(\mathcal{G}_n\right)=\frac{1}{2880}$. ■

The computed values of $I(n,\infty,r)$ are displayed in Table 12 for different values of the ratio r. For the five-voter case, Table 13 provides the limiting probability $I(5,\infty,r)$ for other interesting values of r. A couple of points should be stressed when looking closely at those results. First, it turns out that when a weighted representation is chosen from $\mathcal{G}_{n,w,q}$ with $n=2$, the limiting probability of inconsistent weighting tends to 0 as soon as r tends to $\frac{1}{2}$. Put another way, inconsistent weightings with two voters are rare as soon as the quota represents half of the total sum of weights. This result is consistent with our previous results obtained for a finite total weight w. Second, it also appears that, when a weighted representation is chosen from $\mathcal{G}_{n,w,q}$ with $n=3$ and $n=4$, the limiting probability of observing an inconsistent weighting is equal to 1. In other words, one is quite sure to obtain an inconsistent weighting by choosing a random weighted representation with three and four voters. Third, it seems that similarly strange behavior is observed for the five-voter case with infinite total weight w when it is compared to the one that we already observed for the three-voter and four-voter cases with finite total weight w. More exactly, our results show that $I(5,\infty,r)$ tends to decrease when the value of r exceeds $\frac{1}{3}$ while it increases when r goes to $\frac{2}{3}$. However, $I(5,\infty,r)$ also increases when r goes to $\frac{1}{3}$ and it slightly decreases when r exceeds $\frac{1}{2}$. Note finally that the number of distinct weighted voting games with $\mathcal{SYM}\left(G_k\right)=\left(\{1\},\{2\},\ldots,\{n\}\right)$ that we have to deal with is equal to 76 and 5601 for $n=6$ and $n=7$, respectively. The cases that we deal with in our article illustrate how very tedious it will be to perform all the calculations. This again makes the computations very complicated. However, we believe that they lead to identical results or small differences in the probability for having inconsistent weighting for both finite and infinite total weight w.

Table 12: The limiting probability $I(n, \infty, r)$ of having inconsistent weighted forms with 2, 3, 4, and 5 voters (in %)

	Number of voters n			
r	2	3	4	5
0^+	100	100	100	100
0.1	80	100	100	100
0.2	60	100	100	100
0.3	40	100	100	100
0.4	20	100	100	98.40
0.5	0	100	100	100
0.6	20	100	100	98.40
0.7	40	100	100	100
0.8	60	100	100	100
0.9	80	100	100	100
1	100	100	100	100

Table 13: The limiting probability $I(5, \infty, r)$ of having inconsistent weighted forms with $r \in [\frac{1}{3}, \frac{2}{3}]$ (in %)

r	$\frac{1}{3}$	0.35	0.40	0.45	0.46	0.47	0.48	0.49	0.50
$I(n, \infty, r)$	100	99.99	98.40	95.20	96.81	98.34	99.42	99.92	100
r	0.51	0.52	0.53	0.54	0.55	0.60	0.65	$\frac{2}{3}$	
$I(n, \infty, r)$	99.92	99.42	98.34	96.81	95.20	98.40	99.99	100	

6 Concluding remarks

Our study raises the following additional questions.

Firstly, regarding the probabilistic approach, our probabilities have been obtained by assuming that all representations of games in $\mathcal{G}_{n,w,q}$ are equally likely to be observed. If the weights are chosen to reflect the size differences between voters, then it is not very realistic that all representations are equally likely. It is obviously natural to ask whether our probabilities behave similarly when considering other distributions of the weighted voting games. The extension of our results to other distributions remains open.

Secondly, it is also worth mentioning that our probabilities are computed for a given quota and for a known total sum of weights. An inconsistent weighting arises from an inappropriate distribution of weights. An alternative problem we did not consider here would have consisted in assuming that only the total sum of weights is known. In this latter case, an inconsistent weighting results both from the choice of a quota and the distribution of weights. The corresponding probabilities of inconsistent weighting in this second setting can be obtained by summing the ones reported here, provided that each of our probabilities for a given quota is multiplied by the proportion of weighted voting games with that quota among all weighted voting games having the same sum of weights.

Thirdly, we did not consider power indices in our framework although they can be used to test or characterize inconsistent weighted forms. Two prominent such power indices are the Shapley-Shubik power index (Shapley and Shubik, 1954) and the Banzhaf power index (Banzhaf, 1965). It is well known that these power indices assign the same power to two voters in a weighted voting game if and only if they are symmetric. As a consequence, a weighted form is inconsistent for a weighted voting game if and only if there are two voters with distinct weights but the same Shapley-Shubik or Banzhaf index. In future works, it would make sense to develop the connections between inconsistent weighting and power indices.

Appendix

Table 14: List of all four-voter weighted voting games G

k	G_k	$L^*(G_k)$	$W^*(G_k)$	$\mathcal{SYM}(G_k)$
1	$[4;1,1,1,1]$	$[\{1,2,3\}]$	$[\{1,2,3,4\}]$	$\{1,2,3,4\}$
2	$[3;1,1,1,0]$	$[\{1,2,4\}]$	$[\{1,2,3\}]$	$\{1,2,3\},\{4\}$
3	$[5;2,2,1,1]$	$[\{1,2\},\{1,3,4\}]$	$[\{1,2,4\}]$	$\{1,2\},\{3,4\}$
4	$[2;1,1,0,0]$	$[\{1,3,4\}]$	$[\{1,2\}]$	$\{1,2\},\{3,4\}$
5	$[5;3,2,1,1]$	$[\{1,3\},\{2,3,4\}]$	$[\{1,2\},\{1,3,4\}]$	$\{1\},\{2\},\{3,4\}$
6	$[4;2,2,1,1]$	$[\{1,3\}]$	$[\{1,2\},\{2,3,4\}]$	$\{1,2\},\{3,4\}$
7	$[4;2,1,1,1]$	$[\{1,2\},\{2,3,4\}]$	$[\{1,3,4\}]$	$\{1\},\{2,3,4\}$
8	$[3;2,1,1,0]$	$[\{1,4\},\{2,3,4\}]$	$[\{1,3\}]$	$\{1\},\{2,3\},\{4\}$
9	$[5;3,2,2,1]$	$[\{1,4\},\{2,3\}]$	$[\{1,3\},\{2,3,4\}]$	$\{1\},\{2,3\},\{4\}$
10	$[4;3,1,1,1]$	$[\{1\},\{2,3,4\}]$	$[\{1,4\}]$	$\{1\},\{2,3,4\}$
11	$[3;2,1,1,1]$	$[\{1\},\{2,3\}]$	$[\{1,4\},\{2,3,4\}]$	$\{1\},\{2,3,4\}$
12	$[4;3,2,2,1]$	$[\{1\},\{2,4\}]$	$[\{1,4\},\{2,3\}]$	$\{1\},\{2,3\},\{4\}$
13	$[1;1,0,0,0]$	$[\{2,3,4\}]$	$[\{1\}]$	$\{1\},\{2,3,4\}$
14	$[3;3,1,1,1]$	$[\{2,3\}]$	$[\{1\},\{2,3,4\}]$	$\{1\},\{2,3,4\}$
15	$[2;2,1,1,0]$	$[\{2,4\}]$	$[\{1\},\{2,3\}]$	$\{1\},\{2,3\},\{4\}$
16	$[3;3,2,1,1]$	$[\{2\},\{3,4\}]$	$[\{1\},\{2,4\}]$	$\{1\},\{2\},\{3,4\}$
17	$[2;2,1,1,1]$	$[\{2\}]$	$[\{1\},\{3,4\}]$	$\{1\},\{2,3,4\}$
18	$[3;1,1,1,1]$	$[\{1,2\}]$	$[\{2,3,4\}]$	$\{1,2,3,4\}$
19	$[2;1,1,1,0]$	$[\{1,4\}]$	$[\{2,3\}]$	$\{1,2,3\},\{4\}$
20	$[3;2,2,1,1]$	$[\{1\},\{3,4\}]$	$[\{2,4\}]$	$\{1,2\},\{3,4\}$
21	$[1;1,1,0,0]$	$[\{3,4\}]$	$[\{2\}]$	$\{1,2\},\{3,4\}$
22	$[2;2,2,1,1]$	$[\{3\}]$	$[\{2\},\{3,4\}]$	$\{1,2\},\{3,4\}$
23	$[2;1,1,1,1]$	$[\{1\}]$	$[\{3,4\}]$	$\{1,2,3,4\}$
24	$[1;1,1,1,0]$	$[\{4\}]$	$[\{3\}]$	$\{1,2,3\},\{4\}$
25	$[1;1,1,1,1]$	$[]$	$[\{4\}]$	$\{1,2,3,4\}$

Table 15: List of all five-voter weighted voting games G with no pair of symmetric voters

k	G_k	$L^*(G_k)$	$W^*(G_k)$	$\mathcal{SYM}(G_k)$
1	$[9;5,4,3,2,1]$	$[\{1,3\},\{1,4,5\},\{2,3,5\}]$	$[\{1,2\},\{1,3,5\},\{2,3,4\}]$	$\{1\},\ldots,\{5\}$
2	$[7;5,4,3,2,1]$	$[\{1,5\},\{2,4\},\{3,4,5\}]$	$[\{1,4\},\{2,3\},\{2,4,5\}]$	$\{1\},\ldots,\{5\}$

Table 16: Probability of having inconsistent weighted forms with 2 voters (in %)

$w \downarrow q \rightarrow$	1	$\lceil \frac{w}{4} \rceil$	$\lceil \frac{w}{3} \rceil$	$\lceil \frac{w+1}{2} \rceil$	$\lceil \frac{2w}{3} \rceil$	$\lceil \frac{3w}{4} \rceil$	w	I_{min} (q)	I_{max} (q)
5	66.67	33.33	33.33	0	33.33	33.33	66.67	0 (3)	66.67 (1 & 5)
10	66.67	33.33	16.67	0	16.67	33.33	66.67	0 (5 & 6)	66.67 (1 & 10)
15	87.50	50.00	37.50	0	25.00	50.00	87.50	0 (8)	87.50 (1 & 15)
20	81.82	45.45	27.27	0	27.27	36.36	81.82	0 (10 & 11)	81.82 (1 & 20)
25	92.31	46.15	30.77	0	30.77	46.15	92.31	0 (13)	92.31 (1 & 25)
30	87.50	43.75	31.25	0	25.00	43.75	87.50	0 (15 & 16)	87.50 (1 & 30)
35	94.44	50.00	33.33	0	33.33	50.00	94.44	0 (18)	94.44 (1 & 35)
40	90.48	47.62	28.57	0	28.57	42.86	90.48	0 (20 & 21)	90.48 (1 & 40)
45	95.65	47.83	34.78	0	30.43	47.83	95.65	0 (23)	95.65 (1 & 45)
50	92.31	46.15	30.77	0	30.77	46.15	92.31	0 (25 &26)	92.31 (1 & 50)
55	96.43	50.00	32.14	0	32.14	50.00	96.43	0 (28)	96.43 (1 & 55)
60	93.55	48.39	32.26	0	29.03	45.16	93.55	0 (30 & 31)	93.55 (1 & 60)
65	96.97	48.48	33.33	0	33.33	48.48	96.97	0 (33)	96.97 (1 & 65)
70	94.44	47.22	30.56	0	30.56	47.22	94.44	0 (35 & 36)	94.44 (1 & 70)
75	97.37	50.00	34.21	0	31.58	50.00	97.37	0 (38)	97.37 (1 & 75)
80	95.12	48.78	31.71	0	31.71	46.34	95.12	0 (40 & 41)	95.12 (1 & 80)
85	97.67	48.84	32.56	0	32.56	48.84	97.67	0 (43)	97.67 (1 & 85)
90	95.65	47.83	32.61	0	30.43	47.83	95.65	0 (45 & 46)	95.65 (1 & 90)
95	97.92	50.00	33.33	0	33.33	50.00	97.92	0 (48)	97.92 (1 & 95)
100	96.08	49.02	31.37	0	31.37	47.06	96.08	0 (50 & 51)	96.08 (1 & 100)
105	98.11	49.06	33.96	0	32.08	49.06	98.11	0 (53)	98.11 (1 & 105)
1000	99.60	49.90	33.13	0	33.13	49.70	99.60	0 (500 & 501)	99.60 (1 & 1000)
1005	99.80	49.90	33.40	0	33.20	49.90	99.80	0 (503)	99.80 (1 & 1005)

Table 17: Probability of having inconsistent weighted forms with 3 voters (in %)

$w \downarrow q \rightarrow$	1	$\lceil \frac{w}{4} \rceil$	$\lceil \frac{w}{3} \rceil$	$\lceil \frac{w+1}{2} \rceil$	$\lceil \frac{2w}{3} \rceil$	$\lceil \frac{3w}{4} \rceil$	w	I_{min} (q)	I_{max} (q)
5	80.00	40.00	40.00	60.00	40.00	40.00	80.00	40.00 (2 & 3)	80.00 (1 & 5)
10	85.71	64.29	57.14	71.43	57.14	64.29	85.71	57.14 (4 & 7)	85,71 (1 & 10)
15	92.59	74.07	70.37	81.48	70.37	74.07	92.59	70.37 (5, 6, 10 & 11)	92,59 (1 & 15)
20	95.45	81.82	75.00	84.09	75.00	79.55	95.45	75.00 (7 & 14)	95.45 (1 & 20)
25	98.46	84.62	80.00	89.23	80.00	84.62	98.46	80.00 (9 & 17)	98.46 (1 & 25)
30	96.70	85.71	82.42	89.01	82.42	85.71	96.70	82.42 (10, 11, 20 & 21)	96.70 (1 & 30)
35	99.17	89.17	85.00	92.50	85.00	89.17	99.17	85.00 (12 & 24)	99.17 (1 & 35)
40	98.70	90.26	86.36	92.21	86.36	88.96	98.70	86.36 (14 & 27)	98.70 (1 & 40)
45	98.96	90.10	88.02	93.23	88.02	90.10	98.96	88.02 (15, 16, 30 & 31)	98.96 (1 & 45)
50	99.15	91.45	88.89	94.02	88.89	91.45	99.15	88.89 (17 & 34)	99.15 (1 & 50)
55	99.64	92.50	90.00	95.00	90.00	92.50	99.64	90.00 (19 & 37)	99.64 (1 & 55)
60	99.09	92.75	90.63	94.56	90.63	92.45	99.09	90,63 (20, 21, 40 & 41)	99.09 (1 & 60)
65	99.74	93.51	91.43	95.58	91.43	93.51	99.74	91.43 (22 & 44)	99.74 (1 & 65)
70	99.55	93.92	91.89	95.72	91.89	93.92	99.55	91.89 (24 & 47)	99.55 (1 & 70)
75	99.61	94.28	92.50	96.06	92.50	94.28	99.61	92.50 (25, 26, 50 & 51)	99.61 (1 & 75)
80	99.65	94.77	92.86	96.17	92.86	94.43	99.65	92.86 (27 & 54)	99.65 (1 & 80)
85	99.84	94.88	93.33	96.59	93.33	94.88	99.84	93.33 (29 & 57)	99.84 (1 & 85)
90	99.58	95.01	93.62	96.53	93.62	95.01	99.58	93.62 (30, 31, 60 & 61)	99.58 (1 & 90)
95	99.88	95.50	94.00	97.00	94.00	95.50	99.88	94.00 (32 & 64)	99.88 (1 & 95)
100	99.77	95.70	94.23	96.95	94.23	95.59	99.77	94.23 (34 & 67)	99.77 (1 & 100)
105	99.79	95.78	94.55	97.12	94.55	95.78	99.79	94.55 (35, 36, 70 & 71)	99.79 (1 & 105)
1000	100.00	99.55	99.40	99.70	99.40	99.55	100.00	99.40 (334 & 667)	100.00 (1 & 1000)
1005	100.00	99.55	99.41	99.70	99.41	99.55	100.00	99.41 (335, 336, 670 & 671)	100.00 (1 & 1005)

Table 18: Probability of having inconsistent weighted forms with 4 voters (in %)

$w \downarrow q \rightarrow$	1	$\lceil \frac{w}{4} \rceil$	$\lceil \frac{w}{3} \rceil$	$\lceil \frac{w+1}{2} \rceil$	$\lceil \frac{2w}{3} \rceil$	$\lceil \frac{3w}{4} \rceil$	w	I_{min} (q)	I_{max} (q)
5	83.33	50.00	50.00	66.67	50.00	50.00	83.33	50.00 (2 & 4)	83.33 (1 & 5)
10	91.30	60.87	60.87	69.57	60.87	60.87	91.30	60.87 (3, 4, 7 & 8)	91.30 (1 & 10)
15	96.30	81.48	79.63	88.89	75.93	81.48	96.30	75.93 (6 & 10)	96.30 (1 & 15)
20	97.22	83.33	79.63	84.26	79.63	80.56	97.22	77.78 (8 & 13)	97.22 (1 & 20)
25	99.46	88.65	85.95	95.14	85.95	88.65	99.46	85.41 (10 & 16)	99.46 (1 & 25)
30	98.99	88.22	86.87	93.60	85.86	88.22	98.99	85.52 (12 & 19)	98.99 (1 & 30)
35	99.77	92.74	90.02	97.28	90.02	92.74	99.77	89.34 (13 & 23)	99.77 (1 & 35)
40	99.53	92.25	89.56	94.62	89.56	91.46	99.53	89.24 (15, 16, 25 & 26)	99.53 (1 & 40)
45	99.77	93.98	92.13	98.15	91.44	93.98	99.77	91.44 (16, 17, 29 & 30)	99.77 (1 & 45)
50	99.83	93.76	91.94	97.40	91.94	93.76	99.83	91.42 (18 & 33)	99.83 (1 & 50)
55	99.93	95.45	93.11	98.73	93.11	95.45	99.93	92.78 (20 & 36)	99.93 (1 & 55)
60	99.79	95.17	93.34	97.32	92.92	94.75	99.79	92.65 (22 & 39)	99.79 (1 & 60)
65	99.96	96.04	94.28	99.07	94.28	96.04	99.96	93.81 (24 & 42)	99.96 (1 & 65)
70	99.93	95.80	94.13	98.57	94.13	95.80	99.93	93.75 (26 & 45)	99.93 (1 & 70)
75	99.94	96.73	95.10	99.27	94.76	96.73	99.94	94.59 (28 & 48)	99.94 (1 & 75)
80	99.93	96.55	94.96	98.43	94.96	96.32	99.93	94.53 (30 & 51)	99.93 (1 & 80)
85	99.98	97.03	95.47	99.43	95.47	97.03	99.98	95.21 (31 & 55)	99.98 (1 & 85)
90	99.95	96.88	95.58	99.09	95.38	96.88	99.95	95,19 (32 & 59)	99.95 (1 & 90)
95	99.99	97.46	95.99	99.54	95.99	97.46	99.99	95.69 (34 & 62)	99.99 (1 & 95)
100	99.96	97.32	95.89	98.95	95.89	97.16	99.96	95.63 (36 & 65)	99.96 (1 & 100)
105	99.98	97.64	96.42	99.61	96.23	97.64	99.98	96.06 (38 & 68)	99.98 (1 & 105)
1000	100.00	99.77	99.60	99.99	99.60	99.77	100.00	99.57 (356 & 645)	100.00 (1 & 1000)
1005	100.00	99.77	99.60	100.00	99.60	99.77	100.00	99,57 (360 & 646)	100.00 (1 & 1005)

References

Algaba, E., Bilbao, J.M. and Fernandez, J. (2007) The distribution of power in the European Constitution. European Journal of Operational Research 176(3), 1752-1755.

Alonso-Meijide, J. (2005) Generating functions for coalition power indices: An application to the IMF. Annals of Operations Research 137, 21-44.

Arcaini, G. and Gambarelli, G. (1986) Algorithm for automatic computation of the power variations in share tradings. Calcolo 23(1), 13-19.

Banzhaf, J.F. (1965) Weighted voting doesn't work: a mathematical analysis. Rutgers Law Review 19, 317-343.

Barthélemy, F., Lepelley, D., Martin, M. and Smaoui, H. (2020) Dummy voters and the quota in weighted voting games. Group Decision and Negotiation, forthcoming.

Barthélemy, F., Lepelley, D. and Martin, M. (2013) On the likelihood of dummy voters in weighted majority games. Social Choice and Welfare 41, 263-279.

Barthélemy, F. and Martin, M. (2020) Dummy voters and the quota in weighted voting games: Some further results. In Evaluating Voting Systems with Probability Models, Essays by and in honor of William Gehrlein and Dominique Lepelley (Chapter 13). Diss, M. and Merlin, V. (editors). Springer, Berlin.

Barthélemy, F. and Martin, M. (2007) Critères pour une meilleure répartition des sièges du Val d'Oise au sein des structures intercommunales: une application au cas du Val d'Oise. Revue Economique 58, 399-426.

Barvinok, A. (1994) Polynomial time algorithm for counting integral points in polyhedra when the dimension is fixed. Mathematics of Operations Research 19, 769-779.

Barvinok, A. and Pommersheim, J. (1999) An algorithmic theory of lattice points in polyhedra, in : New perspectives in algebraic combinatorics, Berkeley, Ca, 1996-1997. Mathematical Sciences Research Institute Publications 38, 91-147.

Bilbao, J., Fernandez, J., Jimenez, N. and Lopez, J. (2002). Voting power in the European Union Enlargement. European Journal of Operational Research 143(1), 181-196.

Bison, F., Bonnet, J. and Lepelley, D. (2004) The determination of the number of delegates inside the inter-communities structures: an application of the Banzhaf power index. Revue d'Economie Régionale et Urbaine 2, 259-282.

Blancard, S., Lepelley, D. and Smaoui, H. (2020) L'évolution du pouvoir de vote des communes au sein des communautés d'agglomération de La Réunion. Working paper CEMOI.

Bonnet, J. and Lepelley, D. (2001) Pouvoir de vote et Intercommunalité : le cas des Etablissements Publics de Coopération Intercommunale de la région de Basse-Normandie. In : R. Le Duff, J.-P. Rigal et G. Schmidt (eds.), Démocratie et management local, Dalloz, pp. 519-532.

Bubboloni, D., Diss, M. and Gori, M. (2020) Extensions of the Simpson voting rule to the committee selection setting. Public Choice 183, 151-185.

Cervone, D.P., Gehrlein, W.V. and Zwicker, W.S. (2005) Which scoring rule maximizes Condorcet efficiency under IAC? Theory and Decision 58(2), 145-185.

de Keijzer, B., Klos, T. and Zhang, Y. (2010) Enumeration and exact design of weighted voting games. In Proceedings of the 9th international conference on autonomous agents and multiagent systems (Vol. 1, pp. 391-398).

Dia, I. and Kamwa, E. (2019) Le Pouvoir de Vote dans les Etablissements Publics de Coopération Intercommunales de la Martinique et de la Guadeloupe. Economie et Institutions, Forthcoming.

Diss, M. and Doghmi, A. (2016) Multi-winner scoring election methods: Condorcet consistency and paradoxes. Public Choice 169, 97-116.

Diss, M. and Steffen, F. (2018) The Distribution of Power in the Lebanese Parliament Revisited. Working Paper, GATE.

Diss, M. and Zouache, A. (2015) Une étude de la répartition du pouvoir confessionnel au Liban. Revue d'économie politique 125, 527-546.

El Ouafdi, A., Lepelley, D. and Smaoui, H. (2020) Probabilities of electoral outcomes: from three-candidate to four-candidate elections. Theory and Decision 88, 205-229.

Felsenthal, D.S. and Machover, M. (2004) Analysis of QM rules in the draft constitution for Europe proposed by the European Convention, 2003. Social Choice and Welfare 23(1), 1-20.

Freixas, J. and Molinero, X. (2010) Weighted games without a unique minimal representation in integers. Optimization Methods & Software 25(2), 203-215.

Freixas, J. and Molinero, X. (2009) On the existence of a minimum integer representation for weighted voting games. Annals of Operations Research 166, 243-260.

Freixas, J. and Zwicker, W.S. (2003) Weighted voting, abstention, and multiple levels of approval. Social Choice and Welfare 21, 399-431.

Gambarelli, G. (1994) Power indices for political and financial decision making: A review. Annals of Operations Research 51, 1572-9338.

Gehrlein, W.V. and Lepelley, D. (2011) Voting paradoxes and group coherence. Studies in Choice and Welfare - Springer.

Gehrlein, W.V. and Lepelley, D. (2017) Elections, Voting Rules and Paradoxical Outcomes. Studies in Choice and Welfare - Springer.

Kamwa, E. (2019) On the likelihood of the Borda effect: the overall probabilities for general weighted scoring rules and scoring runoff rules. Group Decision and Negotiation 28(3), 519-541.

Kamwa, E. and Moyouwou, I. (2020) Susceptibility to manipulation by sincere truncation: the case of scoring rules and scoring runoff systems. In Evaluating Voting Systems with Probability Models, Essays by and in honor of William Gehrlein and Dominique Lepelley. Diss, M. and Merlin, V. (editors). Springer, Berlin

Koki, C. and Leonardos, S. (2019) Coalitions and Voting Power in the Greek Parliament of 2012: A Case-Study. Homo Oeconomicus 35, 295-313.

Kurz, S. (2012) On minimum sum representations for weighted voting games. Annals of Operations Research 196, 361-369.

Kurz, S. (2018) Correction to: On minimum sum representations for weighted voting games. Annals of Operations Research 271, 1087-1089.

Lane, J.-E. and Maeland, R. (2000) Constitutional analysis: The power index approach. European Journal of Political Research 37, 31-56.

Laruelle, A. and Widgren, M. (1998) Is the allocation of voting power among EU states fair? Public Choice 94(3-4), 317-339.

Leech, D. (2002a) Designing the voting system for the council of the European Union volume. Public Choice 113(3), 437-464.

Leech, D. (2002b) Voting power in the governance of the international monetary fund. Annals of Operations Research 109(1), 375-397.

Leech, D. (1992) Empirical analysis of the distribution of a priori voting power: some results of the British Labour Party conference and Electoral College, European Journal of Political Research 21(3), 245-65.

Lepelley, D., Moyouwou, I. and Smaoui, H. (2018) Monotonicity paradoxes in three-candidate elections using scoring elimination rules. Social Choice and Welfare 50(1), 1-33.

Lepelley, D., Louichi, A. and Smaoui, H. (2008) On Ehrhart polynomials and probability calculations in voting theory. Social Choice and Welfare 30, 363-383.

Lepelley, D. and Smaoui, H. (2019) Comparing two ways for eliminating candidates in three-alternative elections using sequential scoring rules. Group Decision and Negotiation 28(4), 787-804.

Mann, I. and Shapley L.S. (1960) Values of large games IV: Evaluating the electoral college by Monte-Carlo techniques. Technical report, The RAND Corporation.

Moyouwou, I. and Tchantcho, H. (2017) Asymptotic vulnerability of positional voting rules to coalitional manipulation. Mathematical Social Sciences 89, 70-82.

O'Neill, B. (1996) Power and satisfaction in the United Nations Security Council. Journal of Conflict Resolution 40, 219-237.

Shapley, L.S. and Shubik, M. (1954) A method for evaluating the distribution of power in a committee system. American Political Science Review 48, 787-792.

Strand, J.R. and Rapkin, D.P. (2010) Weighted Voting in the United Nations Security Council: A Simulation. Simulation and Gaming 42(6), 1-30.

Taylor, A.D. and Zwicker, W.S. (1992) A characterization of weighted voting. Proceedings of the American Mathematical Society 115, 1089-1094.

Van Deemen, A. and Rusinowska, A. (2003) Paradoxes of Voting Power in Dutch Politics. Public Choice 115, 109-137.

Verdoolaege, S., Seghir, R. Beyls, K., Loechner, V. and M. Bruynooghe (2004) Analytical computation of Ehrhart polynomials: Enabling more compiler analysis and optimizations. Proceedings of International Conference on Compilers, Architecture, and Synthesis for Embedded Systems, 248-258. Washington D.C.

Wilson, M.C. and Pritchard, G. (2007) Probability calculations under the IAC hypothesis. Mathematical Social Sciences 54, 244-256.