CENTRE DE RECHERCHE SUR LES STRATÉGIES ÉCONOMIQUES

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February 2024

Working paper No. 2024 – 09

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New axiomatizations of the Diversity Owen and Shapley values

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Abstract

The Shapley and Owen values defined respectively for cooperative games with transferable utility (TU-games), and TU-games with coalition structure have recently been extended as allocation rules for TU-games with diversity constraints. This new class of games is introduced by Béal *et al.* (Working paper, 2024). In this new environment, players are divided into disjointed groups called communities. Diversity constraints require a minimum number of members in each community for cooperation to take place. A coalition is diverse if it contains at least the required number of members from each community. The diversity-restricted game is a TU-game which assigns zero to any non-diverse coalition and also assigns the original worth of a coalition if it is diverse. The extensions of the Shapley and Owen values are respectively called the Diversity Shapley value which is defined as the Shapley value of the diversity-restricted game, and the Diversity Owen value which is defined as the Owen value of the diversity-restricted game with coalition structure. Moreover, two axiomatic characterizations of these values are given. In this paper, we also present two new axiomatic characterizations of the Diversity Owen and Shapley values.

Key-words: TU-games, diversity constraints, axiomatic characterization, Diversity Shapley value, Diversity Owen value.

JEL Codes: C71.

1. Introduction

We consider a new class of games called cooperative games with diversity constraints and recently introduced by Béal et al. (2024). A cooperative game with diversity constraints consists of a set of players, a characteristic function that assigns to each coalition its worth, a coalition structure that is simply a partition of the set of players into disjoint

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groups called communities, and a vector of minimum numbers of members of each community who can take part in a cooperation. In this new class of games, diversity is an important and essential criterion for cooperation. A coalition is diverse if it contains a least a required number of members of each community. A game is diverse if all the productive coalitions are diverse. Béal et al. (2024) illustrate this model with three real life situations. The voting system during the constitutional reforms (initiated by the President of the Republic) in the French Parliament of the Fifth Republic can be described as a cooperative game with diversity constraints where the players are the parliamentarians, the characteristic function assigns one to every coalitions containing more than 60% of the parliamentarians and zero otherwise, the coalition structure is composed of two groups (deputies and senators), and a vector of minimum numbers that indicates the quota of members of each group for the decision to be adopted (at least 50% of members from each group). The second example is the participatory budgeting with districts where the players are the projects, the characteristic function assigns to each group of projects their score received from the population during the vote, a partition of these projects according to the different districts that have initiated them, and a minimum number of projects from each district that will be realized. The last example is the well-known voting system in the United Nations Security Council. Moreover, Béal et al. (2024) introduce and axiomatically characterize two allocation rules inspired from the literature of cooperative games with transferable utility (TU-games).

In the literature of TU-games, the Shapley value (Shapley, 1953) is probably the most eminent one-point solution concept. Many authors such as van den Brink (2002), Chun (1991), Myerson (1980), Shapley (1953) and Young (1985) have characterized this value using many different approaches. In addition to this literature, the class of TU-games was extended to the class of TU-games with coalition structure by Aumann and Dreze (1974) where they defined a new solution concept generalizing the Shapley value. Three years later, Owen (1977) introduced and characterized a new allocation rule for TU-games with coalition structure. This allocation rule called the Owen value is a variant of the Shapley value and it is probably the most eminent one-point solution concept for TU-games with coalition structure. Many axiomatic characterizations of the Owen value were proposed. Among these characterizations, we have Calvo et al. (1996), Casajus (2010), Hu (2021), Khmelnitskaya and Yanovskaya (2007), and Owen (1977). Inspired by the approach of Myerson (1977) extending the Shapley value in the class of TU-games with communication structure, Béal et al. (2024) introduced two allocation rules which can be viewed as the extension of the Shapley and Owen values to the class of TU-games with diversity constraints. They consider a particular TU-game called the restricted-diversity game which assigns zero to any non-diverse coalition and the original worth of a coalition if it is diverse. The first value called the Diversity Owen value is defined as the Owen value of the restricted-diversity game with coalition structure. The second allocation rule called the Diversity Shapley value is defined as the Shapley value of the restricted-diversity game without the coalition structure.

Additionally, these allocation rules are characterized by the incorporation of two principles: the spirit of Owen (1977) and Shapley (1953), along with the one of Young (1985) and Khmelnitskaya and Yanovskaya (2007). Following the first approach, three classical axioms are adapted for TU-games with diversity constraints. *Efficiency* states that the worth of the grand coalition is allocated optimally to the players. *Additivity* states that the payoff of any player in the sum of two games should coincide with the sum of its payoffs in the two games. *Intra-coalitional equal treatment of necessary players* states that all necessary players (i.e., those whose absence from a coalition makes it unproductive) within the same community should have the same payoff. The next three axioms are new and indicates the impact of diversity constraints on the allocation process. *Null player out for preserving-diversity games* states that if a player is unproductive in a game and its presence is not necessary to achieve diversity criteria then the payoff of the other players should not change when this player leaves the game. *Equality through diversity* states that the different communities should be treated equally by receiving the same payoffs. *Independence from non-diverse coalitions* states that in a TU-game with diversity constraints, only productive coalitions should matter and then the payoff allocation only depends on the subset of coalitions that meet the diversity requirements. Béal et al. (2024) show that all these six axioms characterize the Diversity Owen value. They also show that when we replace *Intra-coalitional equal treatment of necessary players* by the stronger version *Equal treatment of necessary players* [1] and remove *Equality through diversity*, the rest of the axioms characterizes the Diversity Shapley value. Following the second approach, two classical axioms are stated on the class of diverse games. *Marginality for diverse games* states that if the marginal contributions of any player coincide in two diverse games then this player should have the same payoff in these two games. *Strong monotonicity for diverse games* states that if the marginal contributions of a player in a diverse game are at least equal to its marginal contributions in another diverse game then its payoff in the first game should be at least equal to its payoff in the second game. Béal et al. (2024) show that any one of these last axioms can replace *Null player out for preserving-diversity games* and *Additivity* on the first characterization of the Diversity Owen and Shapley values. They also show that any one of these two last axioms can be replaced by the following axiom introduced by Chun (1989). *Coalitional strategic equivalence for diverse game* states that the payoff of a player in a game should not change in another game if this game is obtained by adding to the original game a diverse game where this player is unproductive. Finally, they show that their two first results remain valid within the class of simple games with diversity constraints by replacing *Additivity* by the axiom of *Transfer* introduced by Dubey (1975).

[1] The only change is that the players are not necessary from the same community.

In this paper, we propose two new characterizations of the Diversity Owen and Shapley values. In our first characterization, we use the fairness approach introduced in the class of TU-games by van den Brink (2002) for the characterization of the Shapley value and latter used by Hu (2021) in the class of TU-games with coalition structure for the characterization of the Owen value. Following the fairness approach, we consider the first characterization of the Diversity Owen value where we show that *Additivity*, *Intra-coalitional equal treatment of necessary players* and *Equality through diversity* can be replaced by the following two new axioms. *Fairness within component* states that the payoffs of two members of the same community in a game change by the same amount if we move to a new game obtained from the original one by adding a game where the two players are symmetric (interchangeable). *Fairness through diversity* states that the amount receives by each community in a game should changes by the the same amount if we move to a new game by adding another game to the original one. When we replace *Fairness within component* by the strongest version simply called *Fairness* (the two players are not necessary the members of the same community) and we also remove *Fairness through diversity*, we obtain a new characterization of the Diversity Shapley value.

In the second approach, we adapt the balanced contributions axiom introduced by Myerson (1980). *Intra-coalitional balanced contributions with out players for preserving diversity* states that if the presence of any one among two members of the same community is not necessary to achieve the diversity criteria then the payoff of any one of this member change by the same amount when the other one leaves the game. Furthermore, if one player among the two proves unproductive in the new game upon the departure of the other player, its payoff in the new game should be zero. In the first characterization of the Diversity Owen value, we show that *Null player out for preserving-diversity games*, *Intra-coalitional equal treatment of necessary players* and *Additivity* can be replaced by *Intra-coalitional balanced contributions with out players for preserving diversity.* Similarly, we derive a new characterization of the Diversity Shapley value by using a stronger version of this new last axiom, where the players are not necessarily members of the same community.

Our paper is organized as follows. In Section 2, we present some basic notions on TU-games with coalition structures, and we also present a literature review on TU-games with diversity constraints. In Section 3, we propose a new characterization of the Diversity Owen and Shapley values by using the fairness approach. In Section 4, we characterize these two allocation rules by using the balanced contributions axiom. Section 5 concludes the paper.

2. Preliminaries

2.1. Cooperative games with coalition structures

A cooperative game with transferable utility (TU-game) is defined as a pair (N, v) where N is a finite set of players and v is a function assigning a real number $v(S)$ to each coalition $S \subseteq N$. This real number can be viewed as the worth of the coalition S. We assume that $v(\emptyset) = 0$. We denote by G the set of TU-games with a finite player set. A player $i \in N$ is null in the game (N, v) if for each $S \subseteq N \backslash \{i\}$, $v(S \cup \{i\}) = v(S)$. Two players $i, j \in N$, are symmetric in the game (N, v) if $v(S \cup \{i\}) = v(S \cup \{j\})$ for every $S \subseteq N \backslash \{i, j\}$. A player $i \in N$ is necessary in the the game (N, v) if for each $S \subseteq N \backslash \{i\}$, $v(S) = 0$. Given two TU-games (N, v) and (N, w), the TU-game $(N, v + w)$ is defined by $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N$. For any $S \subseteq N$, the subgame of (N, v) induced by S is the game $(S, v_{|S})$, where, for each $T \subseteq S$, $v_{|S}(T) = v(T)$. For every coalition $T \subseteq N$, we denote by u_T the unanimity game and it is given by $u_T(S) = 1$ if $T \subseteq S$ and 0 otherwise. Any game can be uniquely represented by a unanimity game. In particular, we have

$$v = \sum_{S \subseteq N, S \neq \emptyset} \Delta_v(S) u_S,$$

where $\Delta_v(S)$ is called the Harsanyi dividend (Harsanyi, 1959) of S and defined recursively as follows $\Delta_v(S) := v(S) - \sum_{T \subsetneq S} \Delta_v(T)$ with $\Delta_v(\emptyset) = 0$.

A value f on G is a function assigning to each TU-game $(N, v) \in G$ a vector of real numbers $(f_i(N, v))_{i \in N}$ such that $f_i(N, v)$ corresponds to the payoff of player $i \in N$ in the game (N, v). The Shapley value Sh (Shapley, 1953) is among the well-known solution concepts for TU-games. It is given by

$$Sh_i(N, v) = \sum_{S \subseteq N, S \ni i} \frac{\Delta_v(S)}{|S|}.$$

The Shapley value equitably distributes the dividend of any coalition among its members. A coalition structure on N denoted by $\mathcal{B} = \{B_1, B_2, \cdots, B_m\}$ is a partition of N into disjoints groups of players B_ℓ. A cooperative game with a coalition structure is a triple $(N, v, \mathcal{B})$ where (N, v) is a TU-game and $\mathcal{B}$ is a coalition structure. We denote by GB the set of TU-games with coalition structure in which the player set is finite. For any $S \subseteq N$, we denote by $\mathcal{B}_{|S} = \{B_1 \cap S, \ldots, B_m \cap S\}$ the coalition structure on S induced by $\mathcal{B}$. For any $S \subseteq N$, the subgame with a coalition structure on S induced by $\mathcal{B}$ is the game with a coalition structure $(S, v_{|S}, \mathcal{B}_{|S})$. The elements of $\mathcal{B}$ are called components. For any $i \in N$, we denote by $\mathcal{B}(i)$ the component containing player i. Many values on GB was defined in the literature and it turns out that the Owen value (Owen, 1977) is probably the best allocation rules for TU games with coalition structures. It is given by

$$Ow_i(N, v, \mathcal{B}) = \sum_{S \subseteq N, S \ni i} \frac{\Delta_v(S)}{|\mathcal{B}(i) \cap S| \cdot |\{B_k \in \mathcal{B} : S \cap B_k \neq \emptyset\}|}.$$

The Owen value is a generalization of the Shapley value to games with coalition structures. It distributes the dividend of any coalition by considering two distinct steps. In the first step, the dividend of a given coalition S is shared equally to all components containing the members of S. In the second step, the dividend received by these components is also shared equally among the members of S belonging to the component.

2.2. Cooperative games with diversity constraints

Béal et al. (2024) enforce some diversity requirements as a prerequisite for cooperation. They assume that the players are partitioned into communities B_k with $k \in \{1, \cdots, m\}$ such that $\mathcal{B}$ is a coalition structure. They predefine a minimal number $d_k \in \{1, \ldots, |B_k|\}$ of each community B_k who can cooperate and then introduce some new notions. A coalition S is a **diverse coalition** if this coalition contains at least the required number of members of each community, i.e., $|S \cap B_k| \geq d_k$ for all $k \in \{1, \cdots, m\}$. We denote by $D(N, \mathcal{B}, d)$ the set of diverse coalitions. A game $(N, v, \mathcal{B}, d)$ is a **diverse game** if $v(S) \neq 0 \implies S \in D(N, \mathcal{B}, d)$. An example of a diverse game is the game $(N, v^d, \mathcal{B}, d)$ where v^d assigns the original worth to every diverse coalition and 0 to every non-diverse coalition. It is given by

$$v^d(S) = \begin{cases} v(S) & \text{if } S \in D(N, \mathcal{B}, d), \\ 0 & \text{otherwise.} \end{cases}$$

The pair (N, v^d) is called the diversity-restricted game. A player $i \in B_k \in \mathcal{B}$ is out in a game $(N, v, \mathcal{B}, d)$ if its presence is not necessary to achieve the diversity requirement, i.e., $|B_k| - d_k \geq 1$. A game $(N, v, \mathcal{B}, d)$ is called ***i*-out diverse**, $i \in B_k$, if $(N, v, \mathcal{B}, d)$ is diverse and $|B_k| - d_k \geq 1$. A **TU-game with diversity constraints** is a four-tuple $(N, v, \mathcal{B}, d)$ where $(N, v, \mathcal{B})$ is a TU-game with a coalition structure and $d = (d_1, d_2, \cdots, d_m)$. We denote by GD the set of all TU-games with diversity constraints and a finite player set. For any $S \subseteq N$, the subgame with diversity constraints of $(N, v, \mathcal{B}, d)$ induced by S is the game with diversity constraints $(S, v_{|S}, \mathcal{B}_{|S}, d)$. As noticed in the introductory section, some real life situations modeled by TU-games with diversity constraints can be found in Béal et al. (2024). Two allocations rules for TU-games with diversity constraints are introduced and axiomatically characterized by Béal et al. (2024). The first allocation rule is called the Diversity Owen value. It is denoted by DOw and defined as the Owen value of the restricted-diverse game with coalition structure, i.e., for any $(N, v, \mathcal{B}, d) \in GD$, $DOw(N, v, \mathcal{B}, d) = Ow(N, v^d, \mathcal{B})$. The second allocation rule is called the Diversity Shapley value. It is denoted by DSh and defined as the Shapley value of the restricted-diverse game without coalition structure, i.e., for any $(N, v, \mathcal{B}, d) \in GD$, $DSh(N, v, \mathcal{B}, d) = Sh(N, v^d)$. Béal et al. (2024) state and interpret the following axioms for TU-games with diversity constraints.

- **Efficiency (E).** For each $(N, v, \mathcal{B}, d) \in GD$, $\sum_{i\in N} f_i(N, v, \mathcal{B}, d) = v(N)$.

- **Additivity (A).** For each $(N, v, \mathcal{B}, d), (N, w, \mathcal{B}, d) \in GD$, $f(N, v + w, \mathcal{B}, d) = f(N, v, \mathcal{B}, d) + f(N, w, \mathcal{B}, d)$.

- **Equal treatment of necessary players (ETNP).** For each $(N, v, \mathcal{B}, d) \in GD$ and each $i, j \in N$, if i and j are necessary in (N, v) then, $f_i(N, v, \mathcal{B}, d) = f_j(N, v, \mathcal{B}, d)$.

- **Intra-coalitional equal treatment of necessary players (ICETNP).** For each $(N, v, \mathcal{B}, d) \in GD$ and each $i, j \in B_h \in \mathcal{B}$, if i and j are necessary in (N, v) then, $f_i(N, v, \mathcal{B}, d) = f_j(N, v, \mathcal{B}, d)$.

- **Independence from non-diverse coalitions (INDC).** For each $(N, v, \mathcal{B}, d)$, $(N, w, \mathcal{B}, d) \in GD$ such that $v(S) = w(S)$ for all $S \in D(N, \mathcal{B}, d)$, $f(N, v, \mathcal{B}, d) = f(N, w, \mathcal{B}, d)$.

- **Intra-coalitional symmetry (ICS).** For any $(N, v, \mathcal{B}, d) \in GD$, if $i, j \in B_k$ and i and j are symmetric in (N, v), then $f_i(N, v, \mathcal{B}, d) = f_j(N, v, \mathcal{B}, d)$.

- **Symmetry (S).** For any $(N, v, \mathcal{B}, d) \in GD$, if i and j are symmetric in (N, v), then $f_i(N, v, \mathcal{B}, d) = f_j(N, v, \mathcal{B}, d)$.

- **Equality through diversity (ED).** For any $(N, v, \mathcal{B}, d) \in GD$ and each $k, q \in M$, $\sum_{i\in B_k} f_i(N, v, \mathcal{B}, d) = \sum_{i\in B_q} f_i(N, v, \mathcal{B}, d)$.

- **Null player out for preserving-diversity games (NPOPD).** If $(N, v, \mathcal{B}, d)$ is i-out diverse and player i is null in (N, v), then $f_j(N, v, \mathcal{B}, d) = f_j(N\backslash\{i\}, v_{|N\backslash\{i\}}, \mathcal{B}_{|N\backslash\{i\}}, d)$ for each $j \in N\backslash\{i\}$.

- **Coalitional strategic equivalence for diverse game (CSEDG).** If $(N, v, \mathcal{B}, d) \in GD$ is diverse and player i is null in (N, v), then for any game $(N, w, \mathcal{B}, d) \in GD$, $f_i(N, v + w, \mathcal{B}, d) = f_i(N, w, \mathcal{B}, d)$.

- **Marginality for diverse games (MDG).** For any pair of diverse games $(N, v, \mathcal{B}, d)$, $(N, w, \mathcal{B}, d) \in GD$, and any $i \in N$ such that $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subseteq N\backslash\{i\}$, we have $f_i(N, v, \mathcal{B}, d) = f_i(N, w, \mathcal{B}, d)$.

- **Strong monotonicity for diverse games (MoDG).** For any pair of diverse games $(N, v, \mathcal{B}, d), (N, w, \mathcal{B}, d) \in GD$, and any $i \in N$ such that $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N\backslash\{i\}$, we have $f_i(N, v, \mathcal{B}, d) \geq f_i(N, w, \mathcal{B}, d)$.

By combining some axioms, they characterize the Diversity Owen and Shapley values as shown below.

Proposition 1. *The Diversity Owen value is the unique value on GD that satisfies Efficiency* **(E)***, Additivity* **(A)***, Intra-coalitional equal treatment of necessary players* **(ICETNP)***, Null player out for preserving-diversity games* **(NPOPD)***, Equality through diversity* **(ED)***, and Independence from non-diverse coalitions* **(INDC)***.*

By using the strongest version of **ICETNP** and by removing **ED**, they characterize the Diversity Shapley value.

Proposition 2. *The Diversity Shapley value is the unique value on GD that satisfies Efficiency* **(E)***, Additivity* **(A)***, Equal treatment of necessary players* **(ETNP)***, Null player out for preserving-diversity games* **(NPOPD)***, and Independence from non-diverse coalitions* **(INDC)***.*

Moreover, they show that **A** and **NPOPD** can be replaced by some new axioms as shown below.

Proposition 3. *The Diversity Owen value is the unique value on GD that satisfies Efficiency* **(E)***, Intra-coalitional equal treatment of necessary players* **(ICETNP)***, Equality through diversity* **(ED)***, Independence from non-diverse coalitions* **(INDC)***, and Coalitional strategic equivalence for diverse game* **(CSEDG)** *or Marginality for diverse games* **(MDG)** *or Strong monotonicity for diverse games* **(MoDG)***.*

Similarly, they propose a second characterization of the Diversity Shapley value.

Proposition 4. *The Diversity Shapley value is the unique value on GD that satisfies Efficiency* **(E)***, Equal treatment of necessary players* **(ETNP)***, Independence from non-diverse coalitions* **(INDC)***, and Coalitional strategic equivalence for diverse game* **(CSEDG)** *or Marginality for diverse games* **(MDG)** *or Strong monotonicity for diverse games* **(MoDG)***.*

3. Axiomatization with fairness approach

We propose a new characterization of the Diversity and Shapley values by using the fairness approach introduced by van den Brink (2002) in the class of TU-games and extended by Hu (2021) in the class of TU-games with coalition structure. We adapt the fairness properties for TU-games with diversity constraints as follows.

Fairness (F). For all $(N, v, \mathcal{B}, d), (N, w, \mathcal{B}, d) \in GD$ and $i, j \in N$,

$$f_i(N, v + w, \mathcal{B}, d) - f_i(N, w, \mathcal{B}, d) = f_j(N, v + w, \mathcal{B}, d) - f_j(N, w, \mathcal{B}, d),$$

whenever i and j are symmetric in (N, v).

This axiom states that the variation of the payoffs of a pair of players should be the same, if we move from one game to another by adding a game in which the two players are symmetric. The next axiom weakens the fairness axiom.

Fairness within component (FwC). For all $(N, v, \mathcal{B}, d), (N, w, \mathcal{B}, d) \in GD$ and $i, j \in B_p \in \mathcal{B}$, if i and j are symmetric in (N, v), then

$$f_i(N, v + w, \mathcal{B}, d) - f_i(N, w, \mathcal{B}, d) = f_j(N, v + w, \mathcal{B}, d) - f_j(N, w, \mathcal{B}, d).$$

It requires that the payoffs of a pair of players belonging to the same community should change by the same amount if we move from one game to another by adding a game in which the two players are symmetric. The next axiom is the last variation of the fairness axiom.

Fairness through diversity (FD). For all $(N, v, \mathcal{B}, d), (N, w, \mathcal{B}, d) \in GD$ and $B_p, B_q \in \mathcal{B}$,

$$\sum_{i \in B_p} f_i(N, v + w, \mathcal{B}, d) - \sum_{i \in B_p} f_i(N, w, \mathcal{B}, d) = \sum_{i \in B_q} f_i(N, v + w, \mathcal{B}, d) - \sum_{i \in B_q} f_i(N, w, \mathcal{B}, d).$$

This axiom requires that the variation of the payoff received by two communities should change by the same amount if we move from one game to another by adding a new game.

The next result states that the Additivity, Intra-coalitional equal treatment of necessary players and Equality through diversity axioms of Proposition 1 can be replaced by Fairness within component and Fairness through diversity axioms.

Proposition 5. *The Diversity Owen value is the unique value on GD that satisfies Efficiency* **(E)***, Fairness within Component* **(FwC)***, Fairness through diversity* **(FD)***, Independence from non-diverse coalitions* **(INDC)***, and Null player out for preserving-diversity games* **(NPOPD)***.*

Proof of Proposition 5. We adapt the approach used by Hu (2021), and Béal et al. (2024).

EXISTENCE. The Diversity Owen value satisfies **E**, **NPOPD**, and **INDC**. Let us also show that it satisfies **FD** and **FwC**.

The Diversity Owen value satisfies **ICS** and **ED**. It also satisfies **FwC** and **FD** since **A** and **ICS** (respectively, **ED**) imply **FwC** and (respectively, **FD**). For any diverse game $(N, v, \mathcal{B}, d) \in GD$, if $i \in N$ is null in (N, v) then i is also null in (N, v^d) since $v^d = v$. Thus, $Ow_i(N, v^d, \mathcal{B}) = 0$.

Uniqueness. Let f be a value on GD that satisfies **E**, **FwC**, **FD**, **INDC**, and **NPOPD**. Let us show that $f = DOw$.

Let us consider a game $(N, v, \mathcal{B}, d) \in GD$. By **INDC**, we have $f(N, v, \mathcal{B}, d) = f(N, v^d, \mathcal{B}, d)$. We set

$$\mathcal{T}(v^d) := \{T \in D(N, \mathcal{B}, d) \mid \Delta_{v^d}(T) \neq 0\}.$$

By induction on the cardinality of $\mathcal{T}(v^d)$, we are going to show that $f = DOw$.

Induction basis (IB): For any game $(N, v, \mathcal{B}, d) \in GD$ such that $\left|\mathcal{T}\left(v^d\right)\right| = 0$, (N, v^d) is a null game.

Let $B_k \in \mathcal{B}$ be a given community and $p \notin N$ an outside player who wants to join the community B_k. Let us define a new game $(N \cup \{p\}, (v^d)_{+p}, \mathcal{B}_{+p}, d) \in GD$ where $(v^d)_{+p}(S) = v^d(S \backslash \{p\})$ for every $S \subseteq N \cup \{p\}$, and $\mathcal{B}_{+p} = \{B_1, B_2, \cdots, B_k \cup \{p\}, B_{k+1}, \cdots, B_m\}$. The TU-game $(N \cup \{p\}, (v^d)_{+p})$ is also a null game. Every player is null in the game $(N \cup \{p\}, (v^d)_{+p})$. For any $j \in B_k \cup \{p\}$, the game $(N \cup \{p\}, (v^d)_{+p}, \mathcal{B}_{+p}, d)$ is j-out diverse. For any $i \in B_k$, by **E** we have

$$\begin{aligned}\sum_{j \in N \cup \{p\}} f_j\big(N \cup \{p\}, (v^d)_{+p}, \mathcal{B}_{+p}, d\big) &= (v^d)_{+p}(N \cup \{p\}) = 0 = (v^d)_{+p}\big((N \backslash \{i\}) \cup \{p\}\big) \\ &= \sum_{j \in (N \backslash \{i\}) \cup \{p\}} f_j\big((N \backslash \{i\}) \cup \{p\}, \big((v^d)_{+p}\big)_{|(N \backslash \{i\}) \cup \{p\}}, \big(\mathcal{B}_{+p}\big)_{|(N \backslash \{i\}) \cup \{p\}}, d\big).\end{aligned}$$

Applying **NPOPD**, we have $f_i(N \cup \{p\}, (v^d)_{+p}, \mathcal{B}_{+p}, d) = 0$. By **NPOPD** again, we have $f_i(N \cup \{p\}, (v^d)_{+p}, \mathcal{B}_{+p}, d) = f_i(N, v^d, \mathcal{B}, d) = 0$.
Similarly with the other communities, we have $f_i(N, v^d, \mathcal{B}, d) = 0 = Ow_i(N, v^d, \mathcal{B}, d)$ for any $i \in N$.
Induction hypothesis (IH): Suppose the claim holds for all $(N, v, \mathcal{B}, d) \in GD$ such that $\left|\mathcal{T}(v^d)\right| \leq \bar{t}$ with $\bar{t} \in \mathbb{N}$.

Induction step: Let $(N, v, \mathcal{B}, d) \in GD$ be a game such that $\left|\mathcal{T}(v^d)\right| = \bar{t} + 1$. We set

$$\mathbb{T}(v^d) := \big\{i \in N \mid i \in T, \text{ for all } T \in \mathcal{T}(v^d)\big\}.$$

This set can be empty or not. So, we consider the following two cases.

1st case: We assume that $\mathbb{T}(v^d) = \emptyset$. If there exists at least one component $B_h \in \mathcal{B}$ such that $|B_h| = d_h$, then $\left|\mathcal{T}\left(v^d\right)\right| = 0$ since the elements of $\mathcal{T}(v^d)$ are supposed to be diverse and also $\mathbb{T}(v^d) = \emptyset$. From (IB), the claim is immediate.

We assume now that there are no $B_h \in \mathcal{B}$ such that $|B_h| = d_h$. For any player $i \in N$, there exists some diverse coalition $T_o \in \mathcal{T}\left(v^d\right)$ such that $i \notin T_o$. Let us fix

$w = \sum_{S \in D(N,\mathcal{B},d) \setminus \{T_o\}} \Delta_{v^d}(S) u_S$. Hence, $v^d = w + \Delta_{v^d}(T_o) u_{T_o}$ and $|\mathcal{T}(w)| = \bar{t}$. By (IH), we deduce that $f(N, w, \mathcal{B}, d) = Ow(N, w^d, \mathcal{B})$.

By applying **FD**, for every $B_p, B_q \in \mathcal{B}$, we have

$$\begin{aligned}\sum_{k \in B_p} f_k(N, v^d, \mathcal{B}, d) - \sum_{k \in B_q} f_k(N, v^d, \mathcal{B}, d) &= \sum_{k \in B_p} f_k(N, w, \mathcal{B}, d) - \sum_{k \in B_q} f_k(N, w, \mathcal{B}, d) \\ &\overset{\text{(IH)}}{=} \sum_{k \in B_p} Ow_k(N, w^d, \mathcal{B}) - \sum_{k \in B_q} Ow_k(N, w^d, \mathcal{B}) \\ &\overset{\text{(ED)}}{=} 0.\end{aligned}$$

Hence, we have

$$\sum_{k \in B_p} f_k(N, v^d, \mathcal{B}, d) = \sum_{k \in B_q} f_k(N, v^d, \mathcal{B}, d).$$

Applying **E** gives

$$\sum_{k \in \mathcal{B}(i)} f_k(N, v^d, \mathcal{B}, d) = \frac{v^d(N)}{m} = \sum_{k \in \mathcal{B}(i)} Ow_k(N, v^d, \mathcal{B}). \tag{1}$$

Recall that $|\mathcal{B}(i)| > 1$ since $d_h < |\mathcal{B}(i)|$ with $B_h = \mathcal{B}(i)$.

Let us assume that $B_h = \mathcal{B}(i) = \{i, i'\}$. So, d_h is necessary equal to 1. There exists a diverse coalition $T_1 \in \mathcal{T}\left(v^d\right)$ such that $i' \notin T_1$. Moreover, $i \in T_1$ and $i' \in T_0$ since T_1 and T_0 are diverse. The players i and i' are symmetric in $\left(N, \Delta_{v^d}(T_o) u_{T_o} + \Delta_{v^d}(T_o) u_{T_1}\right)$. By applying **FwC**, we have

$$\begin{aligned}& f_i(N, v^d, \mathcal{B}, d) - f_{i'}(N, v^d, \mathcal{B}, d) \\ = \;& f_i(N, v^d - \Delta_{v^d}(T_o) u_{T_o} - \Delta_{v^d}(T_o) u_{T_1}, \mathcal{B}, d) - f_{i'}(N, v^d - \Delta_{v^d}(T_o) u_{T_o} - \Delta_{v^d}(T_o) u_{T_1}, \mathcal{B}, d) \\ = \;& f_i(N, w - \Delta_{v^d}(T_o) u_{T_1}, \mathcal{B}, d) - f_{i'}(N, w - \Delta_{v^d}(T_o) u_{T_1}, \mathcal{B}, d) \\ \overset{\text{(IH)}}{=} \;& Ow_i(N, (w - \Delta_{v^d}(T_o) u_{T_1})^d, \mathcal{B}) - Ow_{i'}(N, (w - \Delta_{v^d}(T_o) u_{T_1})^d, \mathcal{B}) \\ = \;& Ow_i(N, v^d, \mathcal{B}) - Ow_{i'}(N, v^d, \mathcal{B}).\end{aligned} \tag{2}$$

Adding Equations (1) and (2) leads to $f_i(N, v^d, \mathcal{B}, d) = Ow_i(N, v^d, \mathcal{B})$.
Now, let us assume that $|\mathcal{B}(i)| > 2$. It is clear that every player $j \in \mathcal{B}(i) \setminus T_o$ and i are symmetric in $\left(N, \Delta_{v^d}(T_o) u_{T_o}\right)$. Similarly to Equation (2), applying **FwC** gives

$$f_i(N, v^d, \mathcal{B}, d) - f_j(N, v^d, \mathcal{B}, d) = Ow_i(N, v^d, \mathcal{B}) - Ow_j(N, v^d, \mathcal{B}). \tag{3}$$

Analogously, every pair of players $j, k \in \mathcal{B}(i) \cap T_o$ are also symmetric in $\left(N, \Delta_{v^d}(T_o) u_{T_o}\right)$, and by **FwC** we have

$$f_k(N, v^d, \mathcal{B}, d) - f_j(N, v^d, \mathcal{B}, d) = Ow_k(N, v^d, \mathcal{B}) - Ow_j(N, v^d, \mathcal{B}). \tag{4}$$

Finally, let $i'' \in \mathcal{B}(i) \cap T_o$ be a player and $T_2 \in \mathcal{T}\left(v^d\right)$ a diverse coalition such that $i'' \notin T_2$, since $v^d = \Delta_{v^d}(T_2)u_{T_2} + \sum_{S \in 2^N \setminus \{T_2, \emptyset\}} \Delta_{v^d}(S)u_S$, by applying (IH) and **FwC**, we have

$$\begin{cases} f_{i''}(N, v^d, \mathcal{B}, d) - f_j(N, v^d, \mathcal{B}, d) \\ \qquad\qquad = Ow_{i''}(N, v^d, \mathcal{B}) - Ow_j(N, v^d, \mathcal{B}) \quad \text{for all } j \in \mathcal{B}(i) \backslash T_2, \\ f_k(N, v^d, \mathcal{B}, d) - f_j(N, v^d, \mathcal{B}, d) \\ \qquad\qquad = Ow_k(N, v^d, \mathcal{B}) - Ow_j(N, v^d, \mathcal{B}) \quad \text{for all } j, k \in \mathcal{B}(i) \cap T_2. \end{cases} \tag{5}$$

Recall that $T_o \neq T_2$. So there are at least $|\mathcal{B}(i)| - 1$ linearly independent equations in Equations (3), (4), and (5). Combining these equations with Equation (1) leads to $f_i(N, v^d, \mathcal{B}, d) = Ow_i(N, v^d, \mathcal{B})$.

2nd case: Now, let us assume that $\mathbb{T}(v^d) \neq \emptyset$. The eventuality where $i \in N \backslash \mathbb{T}(v^d)$ is solved in the *1st case*. So, let us consider that $i \in \mathbb{T}(v^d)$. If $\mathbb{T}(v^d) = \{i\}$, then from the *1st case* and **E**, we have $f_i(N, v^d, \mathcal{B}, d) = Ow_i(N, v^d, \mathcal{B})$ since DOw and f satisfy **E**. We assume that $|\mathbb{T}(v^d)| > 1$. Let us decompose v^d into two functions w_1 and w_2 such that $v^d = w_1 + w_2$ with $w_1 = \sum\limits_{\substack{S \in D(N,\mathcal{B},d) \\ S \not\supseteq \mathbb{T}(v^d)}} \Delta_{v^d}(S)u_S$ and $w_2 = \sum\limits_{\substack{S \in D(N,\mathcal{B},d) \\ S \supseteq \mathbb{T}(v^d)}} \Delta_{v^d}(S)u_S$.

Note that every pair of players in $\mathbb{T}(v^d)$ are symmetric in (N, w_2). For every $j \in \mathbb{T}(v^d) \cap \mathcal{B}(i)$, applying **FwC** gives

$$f_i(N, v^d, \mathcal{B}, d) - f_j(N, v^d, \mathcal{B}, d) = f_i(N, w_1, \mathcal{B}, d) - f_j(N, w_1, \mathcal{B}, d).$$

It is clear that $|\mathcal{T}(w_1)| = 0$. According to (IH) and by applying **FwC**, we have

$$\begin{aligned} f_i(N, v^d, \mathcal{B}, d) - f_j(N, v^d, \mathcal{B}, d) &= f_i(N, w_1, \mathcal{B}, d) - f_j(N, w_1, \mathcal{B}, d) \\ &= Ow_i(N, w_1^d, \mathcal{B}) - Ow_j(N, w_1^d, \mathcal{B}) \\ &= Ow_i(N, v^d, \mathcal{B}) - Ow_j(N, v^d, \mathcal{B}). \end{aligned} \tag{6}$$

We have already shown that $f_j(N, v^d, \mathcal{B}, d) = Ow_j(N, v^d, \mathcal{B})$ for all $j \in \mathcal{B}(i) \backslash \mathbb{T}(v^d)$. Moreover, there are $|\mathbb{T}(v^d)| - 1$ linearly independent equations given by Equation (6). Combining these equations with Equation (1) implies $f_i(N, v^d, \mathcal{B}, d) = Ow_i(N, v^d, \mathcal{B})$. □

Analogously, Additivity and Equal treatment of necessary players axioms of Proposition 2 can be replaced by the Fairness axiom.

Proposition 6. *The Diversity Shapley value is the unique value on GD that satisfies Efficiency* **(E)***, Fairness* **(F)***, Independence from non-diverse coalitions* **(INDC)***, and Null player out for preserving-diversity games* **(NPOPD)***.*

Proof of Proposition 6. EXISTENCE: The Diversity Shapley value satisfies all these axioms.

UNIQUENESS. Let f be a value on GD that satisfies **E**, **F**, **INDC** and **NPOPD**. Let us show that $f = DOw$.

For any game $(N, v, \mathcal{B}, d) \in GD$, by **INDC**, we have $f(N, v, \mathcal{B}, d) = f(N, v^d, \mathcal{B}, d)$. We set

$$\mathcal{T}(v^d) := \{T \in D(N, \mathcal{B}, d) \mid \Delta_{v^d}(T) \neq 0\}.$$

By induction on the cardinality of $\mathcal{T}(v^d)$, let us show that $f = DOw$.

Induction basis (IB): For any game $(N, v, \mathcal{B}, d) \in GD$ such that $\left|\mathcal{T}\left(v^d\right)\right| = 0$, (N, v^d) is a null game. By using **NPOPD**, we have the claim (see the proof of Proposition 5).

For any game $(N, v, \mathcal{B}, d) \in GD$, if $\left|\mathcal{T}\left(v^d\right)\right| = 1$, i.e., $\mathcal{T}\left(v^d\right) = \{S\}$ then $v^d = \Delta_{v^d}(S)u_S$. For any $i \in N \backslash S$, by **NPOPD** and **E**, $f_i(N, v, \mathcal{B}, d) = 0$ (see the proof of Proposition 1 in Béal et al., 2024). Moreover, Béal et al. (2024) in the proof of Proposition 1 show that by **NPOPD**, $f_i(N, v, \mathcal{B}, d) = f_i(S, v_{|S}, \mathcal{B}_{|S}, d)$ for all $i \in S$. Any pair of players in S are symmetric in the game $(S, v_{|S})$. Let consider the null game $(S, \mathbf{0})$ restricted on S, i.e., for any $T \subseteq S$, $\mathbf{0}(T) = 0$. Recall that, $f_i(S, \mathbf{0}, \mathcal{B}_{|S}, d) \overset{(IB)}{=} 0$ for all $i \in S$. For any $i, j \in S$, by **F**, we have $f_i(S, v_{|S}, \mathcal{B}_{|S}, d) = f_i(S, v_{|S} + \mathbf{0}, \mathcal{B}_{|S}, d) - f_i(S, \mathbf{0}, \mathcal{B}_{|S}, d) = f_j(S, v_{|S} + \mathbf{0}, \mathcal{B}_{|S}, d) - f_j(S, \mathbf{0}, \mathcal{B}_{|S}, d) = f_j(S, v_{|S}, \mathcal{B}_{|S}, d)$.
So, by **E**, we have $f_i(S, v_{|S}, \mathcal{B}_{|S}, d) = \frac{\Delta_{v^d}(S)}{|S|}$ for any $i \in S$. Thus, $f = DOw$.

Induction hypothesis (IH): Let the claim holds for all $(N, v, \mathcal{B}, d) \in GD$ such that $\left|\mathcal{T}(v^d)\right| \leq \bar{t}$ with $\bar{t} \in \mathbb{N}$.

Induction step: Let $(N, v, \mathcal{B}, d) \in GD$ be a game such that $2 \leq \left|\mathcal{T}(v^d)\right| = \bar{t} + 1$. By mimicking the proof of Theorem 2.5 given by van den Brink (2002), we have the complete proof. □

4. Axiomatization with balanced contributions property

We propose a new characterization of the Diversity Owen and Shapley values by using the balanced contributions property introduced by Myerson (1980). We adapt this property for TU-games with diversity constraints as follows.

Balanced contributions with out players for preserving diversity (BCOPPD). For any diverse game $(N, v, \mathcal{B}, d) \in GD$ and any out players $i, j \in N$ with $i \neq j$,

$$\begin{aligned} &f_i(N, v, \mathcal{B}, d) - f_i(N \backslash \{j\}, v_{|N \backslash \{j\}}, \mathcal{B}_{|N \backslash \{j\}}, d) \\ &= f_j(N, v, \mathcal{B}, d) - f_j(N \backslash \{i\}, v_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}, d). \end{aligned}$$

Moreover, if i is null in $(N \backslash \{j\}, v_{|N \backslash \{j\}}, \mathcal{B}_{|N \backslash \{j\}}, d)$ then $f_i(N \backslash \{j\}, v_{|N \backslash \{j\}}, \mathcal{B}_{|N \backslash \{j\}}, d) = 0$ (respectively, if j is null in $(N \backslash \{i\}, v_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}, d)$ then $f_j(N \backslash \{i\}, v_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}, d) = 0$).

This axiom requires that for a given pair of players in a diverse game, if the presence of any one among them is not necessary to achieve the diversity requirements then the variation of the payoff of any one among these two players changes by the same amount when the other player leaves the game. Moreover, if one player becomes null when the other leaves the game then its payoff should be zero in the new game.

The next axiom weakens **BCOPPD**.

Intra-coalitional balanced contributions with out players for preserving diversity (IBCOPPD). For any diverse game $(N, v, \mathcal{B}, d) \in GD$ and any out players $i, j \in B_p \in \mathcal{B}$ with $i \neq j$,

$$\begin{aligned} f_i(N, v, \mathcal{B}, d) &- f_i(N \backslash \{j\}, v_{|N \backslash \{j\}}, \mathcal{B}_{|N \backslash \{j\}}, d) \\ &= f_j(N, v, \mathcal{B}, d) - f_j(N \backslash \{i\}, v_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}, d). \end{aligned}$$

Moreover, if i is null in $(N \backslash \{j\}, v_{|N \backslash \{j\}}, \mathcal{B}_{|N \backslash \{j\}}, d)$ then $f_i(N \backslash \{j\}, v_{|N \backslash \{j\}}, \mathcal{B}_{|N \backslash \{j\}}, d) = 0$ (respectively, if j is null in $(N \backslash \{i\}, v_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}, d)$ then $f_j(N \backslash \{i\}, v_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}, d) = 0$).

The interpretation of this axiom remains the same as that of the previous axiom. The only difference is that both players are members of the same community.

The following result shows that **A**, **ICETNP**, and **NPOPD** in Proposition 1 can be replaced by **IBCOPPD**.

Proposition 7. *The Diversity Owen value is the unique value on GD that satisfies Efficiency* **(E)***, Intra-coalitional balanced contributions with out players for preserving diversity* **(IBCOPPD)***, Equality through diversity* **(ED)***, and Independence from non-diverse coalitions* **(INDC)***.*

Proof of Proposition 7. Existence. The Diversity Owen value satisfies **E**, **ED**, and **INDC**. The Diversity Owen value also satisfies **IBCOPPD** since the Owen value satisfies the Intra-coalitionnal balanced contributions axiom (see Calvo et al., 1996). In fact, for any diverse game $(N, v, \mathcal{B}, d) \in GD$, and $i, j \in B_k \in \mathcal{B}$ with $i \neq j$ and $|B_k| > d_k$,

$$\begin{aligned} &Ow_i(N, v^d, \mathcal{B}) - Ow_i(N\backslash\{j\}, v^d_{|N\backslash\{j\}}, \mathcal{B}_{|N\backslash\{j\}}), \\ &\qquad = Ow_j(N, v^d, \mathcal{B}) - Ow_j(N\backslash\{i\}, v^d_{|N\backslash\{i\}}, \mathcal{B}_{|N\backslash\{i\}}) \end{aligned}$$

since v^d is a TU-game on N and $v^d_{|N\backslash\{\ell\}}$ is the TU-game v^d restricted to $N\backslash\{p\}$ with $\ell \in \{i, j\}$. Moreover, $(N\backslash\{\ell\}, v^d_{|N\backslash\{\ell\}}, \mathcal{B}_{|N\backslash\{\ell\}})$ is also a diverse game. If a player $l \in N$ is null in (N, v) then $DOw_l(N, v, \mathcal{B}, d) = 0$.

Uniqueness. Let f be a value on GD that satisfies **E**, **IBCOPPD**, **ED**, and **INDC**. Let us show that f is the Diversity Owen value.
For any $(N, v, \mathcal{B}, d) \in GD$, by **INDC**, we have $f(N, v, \mathcal{B}, d) = f(N, v^d, \mathcal{B}, d)$. For any $B_k \in \mathcal{B}$, by **ED** and **E** we have,

$$\sum_{i \in B_k} f_i(N, v^d, \mathcal{B}, d) = \frac{v(N)}{m} = \sum_{i \in B_k} DOw_i(N, v, \mathcal{B}, d). \tag{7}$$

For any $(N, v, \mathcal{B}, d) \in GD$ and $B_k \in \mathcal{B}$, we are going to show that $f_i(N, v^d, \mathcal{B}, d) = DOw_i(N, v, \mathcal{B}, d)$ for all $i \in B_k$.
If $|B_k| = 1$ then from Eq. (7), we have the claim.
We consider the following two cases.

1st case: We assume that $|B_k| \geq 2$ and $|B_k| > d_k$. By induction on the cardinality of B_k, let us prove the claim.
Induction basis (IB): If $B_k = \{i, j\}$ and $d_k = 1$, applying **IBCOPPD** gives

$$\begin{aligned} &f_i(N, v^d, \mathcal{B}, d) - f_j(N, v^d, \mathcal{B}, d) \\ &\qquad = f_i(N\backslash\{j\}, v^d_{|N\backslash\{j\}}, \mathcal{B}_{|N\backslash\{j\}}, d) - f_j(N\backslash\{i\}, v^d_{|N\backslash\{i\}}, \mathcal{B}_{|N\backslash\{i\}}, d) \\ &\qquad = DOw_i(N\backslash\{j\}, v_{|N\backslash\{j\}}, \mathcal{B}_{|N\backslash\{j\}}, d) - DOw_j(N\backslash\{i\}, v_{|N\backslash\{i\}}, \mathcal{B}_{|N\backslash\{i\}}, d) \\ &\qquad = DOw_i(N, v, \mathcal{B}, d) - DOw_j(N, v, \mathcal{B}, d). \end{aligned} \tag{8}$$

Adding Equations (7) and (8) gives $f_p(N, v^d, \mathcal{B}, d) = DOw_p(N, v, \mathcal{B}, d)$ for all $p \in B_k$.
Induction hypothesis (IH): For any $(N, v, \mathcal{B}, d) \in GD$ and $B_k \in \mathcal{B}$ such that $2 \leq |B_k| \leq \bar{t}$ with $\bar{t} \in \mathbb{N}$, we have $f_i(N, v^d, \mathcal{B}, d) = DOw_i(N, v, \mathcal{B}, d)$, for all $i \in B_k$.
Induction step: Let $(N, v, \mathcal{B}, d) \in GD$ be a game and $B_k \in \mathcal{B}$ a community such that

$2 \le |B_k| = \bar{t}+1$. By applying **IBCOPPD**, we have

$$\begin{aligned} & f_i(N, v^d, \mathcal{B}, d) - f_j(N, v^d, \mathcal{B}, d) \\ & \qquad = f_i(N\backslash\{j\}, v^d_{|N\backslash\{j\}}, \mathcal{B}_{|N\backslash\{j\}}, d) - f_j(N\backslash\{i\}, v^d_{|N\backslash\{i\}}, \mathcal{B}_{|N\backslash\{i\}}, d) \\ & \qquad \overset{(IH)}{=} DOw_i(N\backslash\{j\}, v_{|N\backslash\{j\}}, \mathcal{B}_{|N\backslash\{j\}}, d) - DOw_j(N\backslash\{i\}, v_{|N\backslash\{i\}}, \mathcal{B}_{|N\backslash\{i\}}, d) \\ & \qquad = DOw_i(N, v, \mathcal{B}, d) - DOw_j(N, v, \mathcal{B}, d). \end{aligned}$$

That is, $f_i(N, v^d, \mathcal{B}, d) - DOw_i(N, v, \mathcal{B}, d) = f_j(N, v^d, \mathcal{B}, d) - DOw_j(N, v, \mathcal{B}, d)$.
Moreover,
$\sum_{j\in B_k} \big(f_i(N, v^d, \mathcal{B}, d) - DOw_i(N, v, \mathcal{B}, d)\big) = \sum_{j\in B_k} \big(f_j(N, v^d, \mathcal{B}, d) - DOw_j(N, v, \mathcal{B}, d)\big)$.
This equation implies $|B_k|\big(f_i(N, v^d, \mathcal{B}, d) - DOw_i(N, v, \mathcal{B}, d)\big) \overset{(Eq.7)}{=} 0$.

Finally, we obtain $f_i(N, v^d, \mathcal{B}, d) = DOw_i(N, v, \mathcal{B}, d)$.

2nd case: Let $(N, v, \mathcal{B}, d) \in GD$ be a game and B_k a community. We assume that $|B_k| = d_k$ and $|B_k| \ge 2$. Let us show that we still have the claim.
Let $p \notin N$ be an outside player who wants to join the community B_k. We consider the new diverse game $(N \cup \{p\}, (v^d)_{+p}, \mathcal{B}_{+p}, d)$ defined in the proof of Proposition 5 where $(v^d)_{+p}(S) = v^d(S\backslash\{p\})$ for every $S \subseteq N \cup \{p\}$, and $\mathcal{B}_{+p} = \{B_1, B_2, \cdots, B_k \cup \{p\}, B_{k+1}, \cdots, B_m\}$. In this new game, p is a null and out player.
Note that $|B_k \cup \{p\}| > d_k$. From **case 1**,

$$\begin{aligned} f_i(N \cup \{p\}, (v^d)_{+p}, \mathcal{B}_{+p}, d) \quad & = \quad DOw_i(N \cup \{p\}, (v^d)_{+p}, \mathcal{B}_{+p}, d) \\ & \overset{(NPOPD)}{=} \quad DOw_i(N, v, \mathcal{B}, d) \qquad \text{for all } i \in B_k. \end{aligned}$$

By **E** and **ED**, we have

$$\sum_{j\in B_k\cup\{p\}} f_j(N \cup \{p\}, (v^d)_{+p}, \mathcal{B}_{+p}, d) = \sum_{j\in B_k} DOw_j(N, v, \mathcal{B}, d).$$

From this equation, we can deduce that $f_p(N \cup \{p\}, (v^d)_{+p}, \mathcal{B}_{+p}, d) = 0$.
For any $i \in B_k$, applying **IBCOPPD** gives

$$\begin{aligned} f_i(N \cup \{p\}, (v^d)_{+p}, \mathcal{B}_{+p}, d) - f_i(N, v^d, \mathcal{B}, d) &= f_p(N \cup \{p\}, (v^d)_{+p}, \mathcal{B}_{+p}, d) \\ & \quad - f_p\big((N\backslash\{i\}) \cup \{p\}, \big((v^d)_{+p}\big)_{|(N\backslash\{i\})\cup\{p\}}, \big(\mathcal{B}_{+p}\big)_{|(N\backslash\{i\})\cup\{p\}}, d\big). \end{aligned}$$

Moreover, p is null in $\big((N\backslash\{i\}) \cup \{p\}, \big((v^d)_{+p}\big)_{|(N\backslash\{i\})\cup\{p\}}\big)$, and then

$$f_p\big((N\backslash\{i\}) \cup \{p\}, \big((v^d)_{+p}\big)_{|(N\backslash\{i\})\cup\{p\}}, \big(\mathcal{B}_{+p}\big)_{|(N\backslash\{i\})\cup\{p\}}, d\big) = 0.$$

Hence,

$$DOw_i(N, v, \mathcal{B}, d) - f_i(N, v^d, \mathcal{B}, d) = 0.$$

We conclude that $f_i(N, v^d, \mathcal{B}, d) = DOw_i(N, v, \mathcal{B}, d)$.

□

Similarly, **ETNP**, **A**, and **NPOPD** in Proposition 2 can be replaced by **BCOPPD**.

Proposition 8. *The Diversity Shapley value is the unique value on GD that satisfies Efficiency* **(E)***, Balanced contributions with out players for preserving diversity* **(BCOPPD)***, and Independence from non-diverse coalitions* **(INDC)***.*

Proof of Proposition 8. EXISTENCE. The Diversity Shapley value satisfies **BCOPPD** since the Shapley value satisfies the Balanced contributions axiom (see Myerson, 1980).

UNIQUENESS. Let f be a value on GD that satisfies **E**, **BCOPPD** and **INDC**. Let us show that $f = DSh$. For any $(N, v, \mathcal{B}, d) \in GD$, by **INDC**, $f(N, v, \mathcal{B}, d) = f(N, v^d, \mathcal{B}, d)$.

If $|N| = 1$ then by **E**, we have the result. We consider the following two cases.

Case 1. Let $(N, v, \mathcal{B}, d) \in GD$ be a game. We assume that there are no community $B_k \in \mathcal{B}$ such that $|B_k| = d_k$ (i.e., any player $i \in N$ is out). Let us assume that $N = \{i_1, \cdots, i_n\}$ with $|N| = n$. By induction on n, we are going to show that $f(N, v^d, \mathcal{B}, d) = DSh(N, v, \mathcal{B}, d)$.

Induction basis (IB): If $N = \{i, j\}$ and $m = 1$ such that any player is out then by **E** and **BCOPPD**, we have the result.

Induction hypothesis (IH): We assume that for any game $(N, v, \mathcal{B}, d) \in GD$ such that $n \leq \bar{t}$ with $\bar{t} \in \mathbb{N}$, and $|B_k| > d_k$ for all $B_k \in \mathcal{B}$, we have $f(N, v^d, \mathcal{B}, d) = DSh(N, v, \mathcal{B}, d)$.

Induction step: Let $(N, v, \mathcal{B}, d) \in GD$ be a game such that $2 \leq n = \bar{t} + 1$ and $|B_k| > d_k$ for all $B_k \in \mathcal{B}$.

Applying **BCOPPD** gives the following equations

$$\begin{aligned}
f_{i_1}(N, v^d, \mathcal{B}, d) - f_{i_2}(N, v^d, \mathcal{B}, d) &\overset{\text{(IH)}}{=} DSh_{i_1}(N, v, \mathcal{B}, d) - DSh_{i_2}(N, v, \mathcal{B}, d) \\
f_{i_1}(N, v^d, \mathcal{B}, d) - f_{i_3}(N, v^d, \mathcal{B}, d) &\overset{\text{(IH)}}{=} DSh_{i_1}(N, v, \mathcal{B}, d) - DSh_{i_3}(N, v, \mathcal{B}, d) \\
&\vdots \\
f_{i_1}(N, v^d, \mathcal{B}, d) - f_{i_n}(N, v^d, \mathcal{B}, d) &\overset{\text{(IH)}}{=} DSh_{i_1}(N, v, \mathcal{B}, d) - DSh_{i_n}(N, v, \mathcal{B}, d).
\end{aligned}$$

By adding these equations and by using **E**, we have

$$
\begin{aligned}
&(n-1)f_{i_1}(N,v^d,\mathcal{B},d)-v(N)+f_{i_1}(N,v^d,\mathcal{B},d)\\
&=(n-1)DSh_{i_1}(N,v,\mathcal{B},d)-v(N)+DSh_{i_1}(N,v,\mathcal{B},d).
\end{aligned}
$$

That is,

$$f_{i_1}(N,v^d,\mathcal{B},d)=DSh_{i_1}(N,v,\mathcal{B},d).$$

Similarly, we have $f_i(N,v^d,\mathcal{B},d)=DSh_i(N,v,\mathcal{B},d)$ for all $i\in N$.

Case 2. Let $(N,v,\mathcal{B},d)\in GD$ be a game. We assume that there are at least one community $B_k\in\mathcal{B}$ such that $|B_k|=d_k$. We denote by $\bar{r}=|\{B_k\in\mathcal{B}:|B_k|=d_k\}|$ the number of communities without an out member. By induction on $\bar{r}$, let us show that $f(N,v^d,\mathcal{B},d)=DSh(N,v,\mathcal{B},d)$.

Induction basis (IB): Let $(N,v,\mathcal{B},d)\in GD$ be a game such that $\bar{r}=1$. Let $B_k\in\mathcal{B}$ be the community without an out member. Let $p\notin N$ be an outside player who wants to join the community B_k. We consider the new diverse game $(N\cup\{p\},(v^d)_{+p},\mathcal{B}_{+p},d)$ defined in the proof of the previous proposition. From **Case 1**, we have $f(N\cup\{p\},(v^d)_{+p},\mathcal{B}_{+p},d)=DSh(N\cup\{p\},(v^d)_{+p},\mathcal{B}_{+p},d)$. By **BCOPPD**, we deduce that $f_i(N,v^d,\mathcal{B},d)=DSh_i(N,v,\mathcal{B},d)$ for all $i\in N$.

Induction hypothesis (IH): Let $(N,v,\mathcal{B},d)\in GD$ be a game such that $\bar{r}=z$ with $\bar{r}\geq 1$ and $z\in\mathbb{N}$, and $f(N,v^d,\mathcal{B},d)=DSh(N,v,\mathcal{B},d)$.

Induction step: Let $(N,v,\mathcal{B},d)\in GD$ be a game such that $2\leq\bar{r}=z+1$. Let $B_k\in\mathcal{B}$ be the community without an out member. Let also $p\notin N$ be an outside player who wants to join the community B_k. We consider the new diverse game $(N\cup\{p\},(v^d)_{+p},\mathcal{B}_{+p},d)$. From **(IH)**, we have $f(N\cup\{p\},(v^d)_{+p},\mathcal{B}_{+p},d)=DSh(N\cup\{p\},(v^d)_{+p},\mathcal{B}_{+p},d)$. By **BCOPPD**, it follows that $f_i(N,v^d,\mathcal{B},d)=DSh_i(N,v,\mathcal{B},d)$ for all $i\in N$. □

Through the following remark proposed by Béal et al. (2024), we show that our characterizations are non-redundant.

Remark 1. *i) For Propositions 5 and 7, we consider the following examples:*

- *The value f on GD defined by $f_i(N,v,\mathcal{B},d)=\sum_{S\subseteq N:S\ni i}\frac{\Delta_v(S)}{m|\mathcal{B}(i)\cap S|}$ for each $(N,v,\mathcal{B},d)\in GD$ and $i\in N$ satisfies* **E**, **IBCOPPD**, **ED**, **FwC**, **FD**, *and* **NPOPD**; *but does not satisfy* **INDC**.
- *The value f on GD defined by $f_i(N,v,\mathcal{B},d)=Sh_i(N,v^d)$ for each $(N,v,\mathcal{B},d)\in GD$ and $i\in N$ satisfies* **E**, **FwC**, **IBCOPPD**, **NPOPD**, *and* **INDC**; *but violates* **ED** *and* **FD**.

- *The value f on GD defined by $f_i(N,v,\mathcal{B},d) = \frac{v(N)}{m|\mathcal{B}(i)|}$ for each $(N,v,\mathcal{B},d) \in GD$ and $i \in N$ satisfies* **E**, **FwC**, **FD**, *and* **INDC**; *except* **NPOPD**.
- *The value f on GD defined by*

$$f_i(N,v,\mathcal{B},d) = \sum_{S \subseteq N: S \ni i} \frac{\sum_{j \in \mathcal{B}(i) \cap S} j}{i} \cdot \frac{\Delta_{v^d}(S)}{|\{B_k \in \mathcal{B} : S \cap B_k \neq \emptyset\}|}$$

for each $(N,v,\mathcal{B},d) \in GD$ and $i \in N$ satisfies **E**, **FD**, **NPOPD**, *and* **INDC**; *but does not satisfy* **FwC** *and* **IBCOPPD**.

- *The null value on GD defined by $f_i(N,v,\mathcal{B},d) = 0$ for each $(N,v,\mathcal{B},d) \in GD$ and $i \in N$ satisfies* **ED**, **FwC**, **FD**, **NPOPD**, **IBCOPPD**, *and* **INDC**; *but does not satisfy* **E**.

ii) For Propositions 6 and 8, we consider the following examples:

- *The value f on GD defined by $f_i(N,v,\mathcal{B},d) = Sh_i(N,v)$ for each $(N,v,\mathcal{B},d) \in GD$ and $i \in N$ satisfies* **E**, **BCOPPD**, **F**, *and* **NPOPD**; *but does not satisfy* **INDC**.
- *The value f on GD defined by $f_i(N,v,\mathcal{B},d) = Ow_i(N,v^d,\mathcal{B})$ for each $(N,v,\mathcal{B},d) \in GD$ and $i \in N$ satisfies* **E**, **NPOPD**, *and* **INDC**; *violates* **F**.
- *The equal division value f on GD defined by $f_i(N,v,\mathcal{B},d) = \frac{v(N)}{|N|}$ for each $(N,v,\mathcal{B},d) \in GD$ and $i \in N$ satisfies* **E**, **F**, *and* **INDC**; *except* **NPOPD**.
- *The value f on GD defined by $f_i(N,v,\mathcal{B},d) = Ow_i(N,v^d,\mathcal{B})$ for each $(N,v,\mathcal{B},d) \in GD$ and $i \in N$ satisfies* **E** *and* **INDC**. *It does not satisfy* **BCOPPD**.
- *The null value on GD defined by $f_i(N,v,\mathcal{B},d) = 0$ for each $(N,v,\mathcal{B},d) \in GD$ and $i \in N$ satisfies* **NPOPD**, **BCOPPD**, **F**, *and* **INDC**; *but does not satisfy* **E**.

5. Conclusion

The aim of this paper is to propose two new characterizations of the Diversity Owen and Shapley values. We characterize these two values by using two different approaches: the fairness and balanced contributions approaches. For future research, exploring alternative approaches that consider the impact (gain-loss) of diversity constraints on players' payoffs, especially in scenarios where the minimum required number of members from a specific community undergoes changes, would add valuable insights to the study.

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