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Free Riding in Combinatorial First-Price Sealed-Bid Auctions

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Abstract

We consider an allotted procurement contract awarded by means of a combinatorial first-price sealed-bid auction. Two small firms and a larger firm are competing. Each small firm is interested in a single lot whereas the large firm transmits a global offer. Under a specific informational framework, we derive the asymmetric combinatorial equilibrium bidding strategies and show that they exhibit a free-riding effect. We show that this effect is increasing with the level of uncertainty and decreasing with risk aversion. When all the firms are risk neutral or equally risk averse, the magnitude of the free-riding effect is unaffected by the division of the contract chosen by the public buyer. Nevertheless, when each firm exhibits its own risk aversion parameter, we find that the free-riding effect is reduced (resp. increased) as the more risk averse small firm competes for a larger (resp. smaller) part of the contract.

KEYWORDS: free-riding, combinatorial auctions

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1 Introduction

Since the seminal work of Vickrey (1961) there has been a substantial and rapidly growing literature on auction theory. This literature has *e.g.* focused on the award of multiple homogeneous[1] objects and then on the award of heterogeneous objects when bidders may place bids on combinations of these objects rather than individual items. Such auctions are called combinatorial auctions. They have been used for many decades in, for example, estate auctions, for truckload transportation, bus routes, industrial procurement, and have been proposed for airport arrival and departure slots, as well as for allocating radio spectrum for wireless communications services in the US (see Cramton *et al.*, 2006). A key feature of such a mechanism is that it may yield an endogenous optimal allocation of heterogeneous goods.

One of the most prominent questions addressed during the FCC spectrum auction was that of whether bidders should be permitted to bid on combinations of licenses - that is bids for bundles of licenses together- or should bids be accepted only license-by-license? As noted by Bykovsky *et al.* (2000) when complementary items are auctioned separately, bidders may face a financial exposure problem. Indeed, a bidder who regards two items as complements may transmit an aggressive bid for the first item thinking that he will be able to get the second item. However, he "exposes" himself to the risk of not buying the second item and thus having paid for one item above its standalone value. Thus a noncombinatorial auction can lead to an inefficient allocation. A combinatorial auction has the advantage of solving the financial exposure problem. In such an auction, bidders might use package bidding, *i.e.* might aggregate separated licenses in order to obtain a nationwide aggregation. However, as noted by McMillan (1994):

> "*with nationwide bidding, licenses may not end up with the firms that are willing to pay the most. Nationwide bids can reduce the bidding competition, as the nationwide bidders refrain from driving up the separate-license prices. There is a free-rider problem. Separate-license bidders may not raise their bid enough to beat a nationwide bidder because only part of the gain from raising their bid accrues to them. As a result, a nationwide bidder may win even thought the total value of a license if awarded separately would exceed the nationwide bidder's value. The auction is biased toward the nationwide bidders.*"

This *free-riding* problem has also been called a *threshold* problem since separate-license bidders must come to an agreement on how much each will contribute to

[1] The mechanism design literature has also focused on optimal multi-object auctions (e.g. extensions of the Clarke-Vickrey-Groves mechanism to combinatorial auctions). See, among others, Ausubel and Milgrom (2006).

overcome the threshold created by the nationwide bidder (see *e.g.* Bykovsky *et al.* (2000), Milgrom (2000), Banks *et al.* (2003)). Thus, a prominent drawback of combinatorial auctions was highlighted and the FCC decided not to use any combinatorial auctions until 747-762 and 777-792 band auctions number 31.

Despite this drawback, combinatorial auctions are, as previously noted, commonly used in many other areas, both in ascending and in sealed-bid auctions. For instance, public procurement contracts mainly involve heterogeneous lots being auctioned.[2] If the free-riding problem has been highlighted in *ascending auctions*, a theoretical analysis of the free-riding problem in a context of a *first-price sealed-bid* combinatorial auction has never been undertaken. In order to address this question, we consider the simplest case of a combinatorial auction for a two part contract. We assume that a small firm (hereafter SF) competes with both another SF and a large firm (hereafter LF). Each SF has specific technological skills and so only competes for a specific part of the contract while a LF may try to carry out the whole contract.[3] Then, both SFs can only win if the sum of their cost-bids is lower than the LF's bid. In this context, obviously, both SFs may suffer from a free-riding problem, recognizing that the gain in reducing their own bids does not fully accrue to them.

The threshold problem has been mainly analyzed through the use of examples and experimentations (see *e.g.* Bykovsky *et al.* (2000), Banks *et al.* (2003)). However, to the best of our knowledge, no theoretical model of the threshold problem has been developed. Several authors consider a setting with simultaneous multi-objects auctions where unit bidders compete with bundle bidders who value the different items as complements. For instance, Albano et al. (2006) compare two simultaneous ascending auctions for two objects, namely the Japanese auctions and the simultaneous English auctions. Zheng (2008) analyzes equilibria in simultaneous English auctions that allows jump bidding. Krishna and Rosenthal (1996) have depicted bidding equilibrium in second price simultaneous auctions. However, in Krishna and Rosenthal (1996), Albano et al. (2006) and Zheng (2008) a free-riding effect can not appear since they do not consider a *package auction* where a bidder can submit bids for a package of several items. So, a unit bidder does not need the

[2] Indeed, as ruled by Section 19.202-1 of the US Federal Acquisition Regulation (FAR), public procurement authorities often divide proposed acquisitions of supplies and services into reasonably small lots to permit offers on quantities less than the total requirement. This allotment (or unbundling) favors wide small business participation. But it also leads to competition between small and large firms since SMEs may compete for allotted parts of a contract whereas large firms may compete for aggregated lots.

[3] In our framework, note that each firm places only one bid.

other unit bidder's bid to outbid the global bidder.[4]

One very first question we want to address in this article is whether the SFs, whilst competing for a part of an allotted contract, suffer from a free-riding effect as depicted as well in the context of the FCC spectrum ascending auctions and in subscription games? Moreover, can we quantify the magnitude of inefficiency induced by the free-riding effect? Finally, is the free riding effect affected by the way the contracts are divided, namely by the potential asymmetries regarding the relative sizes of the parts of the contract? Besides, it is often argued that SFs may exhibit a higher risk aversion level than LFs do. Then, what is the influence of firms' risk aversion level on the magnitude of the free-riding effect? Answering theses questions turns out to be crucial in order to determine both efficiency and revenue properties of a first-price sealed-bid combinatorial auction.

Nevertheless, studying first-price sealed-bid combinatorial auctions at a theoretical level is a quite complex problem. Indeed, each bidder has now to determine his strategic bid taking into account not only the expected best offer but also combinations of other bidders' offers. Tackling this issue in the standard informational framework of the Independent Private Value (IPV) model does not allow us to explicitly compute the equilibrium strategies. Thus, following Von Ungern-Sternberg (1991) (hereafter VUS), we choose to adopt a simpler informational paradigm which, in our opinion, does not constitute a great reduction in terms of realism as compared with the standard IPV model. As first depicted by VUS,

> "*the model we shall use combines the simplifying properties of the standard IPV and the common value models. As in the IPV model, we assume that each bidder can predict his own cost of completing the contract with certainty. As in the simplest version of the common value model, we assume that each bidder has no grounds for believing his own cost estimate to be higher or lower on average than his competitor's costs. Formally, we model this by assuming that the different producers' costs are independent drawings from a known distribution with an unknown mean.*"

This prior belief is depicted by Biais and Bossaerts (1998) as the " *average opinion rule*". In a more general setting, these authors also present an informational structure where each agent believes that the private values of the others are

[4] Some connected research analyzes a strategic free-riding effect in the context of the voluntary provision of a public good. Such an effect appears *e.g.* as equilibrium strategy in subscription games (see *e.g.* Alborth *et al.* (2001), Laussel and Palfrey (2003), Menezes *et al.* (2001)). In such games, the sum of individual bids must overcome the fixed threshold value of the public good cost. The auction mechanism considered here departs from these analyses, since in our setting the threshold value corresponds to the LF's bid and so is *endogenously* and *strategically* determined.

i.i.d. drawn from a distribution indexed by a parameter. Then, the agents know the functional form of the distribution but are uncertain about the parameter. They will infer this parameter from their own private valuation. In such a context, the observation of a producer's private cost tells him nothing about his relative position compared with his competitors. So, one of the key features of this assumption is that *there is no reason why a producer should let his strategic mark-up depend on his own cost*. It enables us to explicitly compute the equilibrium outcomes and shed light on strategic issues.

Within this informational paradigm, we are able to consider some new and interesting concerns. Thus, it allows us to simultaneously consider *several asymmetries between bidders, namely concerning the risk aversion level and the cost technology*. Indeed, the allotment of the contract creates *per se* a cost-asymmetry while it is often argued that SFs may face a higher risk aversion level than LFs do.

In the next section, in order to present the informational paradigm and provide some preliminary results useful to our analysis, we derive the asymmetric bidding equilibrium in the context of two risk averse bidders having different relative risk aversion levels in a non-combinatorial first-price sealed-bid auction. We analyze the impact of a bidder's risk aversion on bidding aggressiveness. Secondly, like the well-known result for asymmetric auctions, we find that the less favored bidder bids more aggressively and therefore allocative efficiency is not necessarily attained. Then we turn to the analysis of the combinatorial auction and derive the bidding strategies in such a context. As predicted by intuition, we show that these strategies exhibit a free-riding effect that we specify. This effect has some interesting properties. Namely, we show that the free-riding effect is increasing with the level of uncertainty. If all the firms are equally risk averse, then an increase in relative risk aversion overcomes part of the free-riding problem. Indeed, it tends to increase the aggressiveness of small firms more than that of the LF. However, when all the firms are risk neutral or equally risk averse, the magnitude of the free-riding effect remains unaffected by the heterogeneity of the allotment, *i.e.* unaffected by the division of the contract chosen by the public buyer. This last result only holds in the context of all the firms having the same relative risk aversion parameter. When each firm exhibits its own risk aversion parameter, we find that the free-riding effect is reduced (resp. increased) as the less risk averse SF competes for a larger (resp. smaller) part of the contract. It suggests that the expected cost of the contract for the public buyer is reduced when the larger part of the contract is allocated to the more risk averse SF. Throughout the paper, in order to provide an explicit form of bidding strategies, we will consider the special case of a uniform distribution of bidders' private costs.

2 The asymmetric model

2.1 Assumptions

Let us first depict our specific informational setting. The cost parameter c of each bidder is private information. The true distribution of c is given by $f_\mu(c)$ over $[\mu - a\mu, \mu + a\mu]$ where a is common knowledge whereas the mean μ is unknown. In the sequel, we assume that f_μ is symmetric around μ. Thus, each bidder, only observing his own cost parameter, will determine the best estimator of the unknown mean parameter $\widehat{\mu}$. Let us consider bidders $i \in \{\alpha, \beta, \gamma\}$. From bidder i's point of view, by the maximum likelihood principle, he estimates that $\underset{\mu}{\arg\max} f_\mu(c) = c_i$. Therefore, for bidder, say α, the best estimated distribution of the cost parameter c_β of an opponent β is given by $P(c_\beta \in [x, y]) \approx \int_x^y F_{\widehat{\mu}}(c)dc$ with $\widehat{\mu} = c_\alpha$. In order to keep notation as simple as possible, in the following, in all estimated distributions we will drop $\hat{\mu}$ and implicitly replace it by its estimated value. In our notation, upperscript i will reflect the point of view of bidder i and subscript c_i the considered random variable. Thus, when i learns his own cost c_i, he can infer that $\mu \in [(1-a)c_i, (1+a)c_i]$ according to the cumulative F_μ^i with corresponding density f_μ^i. Since all bidders are ex ante symmetric relative to the informational knowledge, for bidder α, $c_\beta \in [(1-2a)c_\alpha, (1+2a)c_\alpha]$ according to the cumulative distribution $F_{c_\beta}^\alpha$ with corresponding density $f_{c_\beta}^\alpha$.

Let us now consider that bidder $i's$ cost function depends on both his own private cost parameter c_i and a technology parameter τ_i such that

$$C_i(c_i, \tau_i) = \tau_i c_i.$$

In this framework, τ_i is assumed to be *common knowledge* whereas c_i remains *private information*. For instance, think about a contract which requires τ_i hours to be completed at a constant cost per hour c_i. Thus τ_i represents the parameter of *cost-asymmetry*. However, all bidders are ex ante symmetric relative to the informational knowledge. The fact that bidder i privately knows his own cost does not reveal him anything about his relative position and so does not affect his winning probability.[5] Therefore, as in VUS's model, there is no reason why he should let his strategic mark-up depend on the value of his own cost. Consider for instance bidder α observing either $c_\alpha = c_\alpha^1$ or $c_\alpha = c_\alpha^2$. Then α infers that either $\mu \in [(1-a)c_\alpha^1, (1+a)c_\alpha^1]$ or $\mu \in [(1-a)c_\alpha^2, (1+a)c_\alpha^2]$ and that either $c_\beta \in [(1-2a)c_\alpha^1, (1+2a)c_\alpha^1]$ or $c_\beta \in [(1-2a)c_\alpha^2, (1+2a)c_\alpha^2]$. Comparing both observations of c_α, the bidder actually faces the same problem proportionally to c_α. So, the bidding strategy will be

[5] See the appendix for a numerical example.

the same proportionally to c_α. Hence, we can assume that each bidder's equilibrium bid has the following form

$$B_i\left(c_i\right) = \tau_i c_i\left(1 + b_i\right).$$

Since the length of the supports are expressed in relative terms, both the bid and the strategic mark-up b_i are expressed in relative terms too. Thus bidder i will transmit a bid equal to his own cost $\tau_i c_i$ plus a fraction of this cost which represents his mark-up. Some of the seminal papers on auction theory with almost similar informational paradigms have restricted attention to multiplicative bidding strategies (see *e.g.* Rothkopf (1969), Capen *et al.* (1971), Reece (1978) and Case (1979)). The practical relevance of multiplicative bidding strategies was also argued by Rothkopf (1980):

> "*there is reason to believe that in many situations of practical interest the assumption of multiplicative strategies will not introduce significant distortion.*"

Similarly, the simulation models built by Capen *et al.* (1971) of the oil industry and Curtis and Maines (1973) of the construction industry bidding have modeled competitive behavior as multiplicative.

In order to highlight the impact of risk aversion on the bidding strategies, let us assume that each bidder i is characterized by a constant relative risk aversion (CRRA) utility function $u_i(x) = x^{\rho_i}$ (with $0 < \rho_i \leq 1$), where $1 - \rho_i$ is the CRRA parameter of firm i.

2.2 The bidding equilibrium of the two bidder auction

Before moving on to consider the combinatorial framework, and in order to provide some insights and first results about the bidding behavior in an asymmetric framework, we analyze the case with two bidders α and β. Consider a fixed-price procurement contract awarded by means of a first-price sealed-bid auction. Thus, bidder α's expected utility is given by

$$EU_\alpha = \left(\tau_\alpha c_\alpha\left(1 + b_\alpha\right) - \tau_\alpha c_\alpha\right)^{\rho_\alpha} P(b_\alpha, b_\beta),$$

where $P(b_\alpha, b_\beta)$ reflects the probability of winning when bidder α chooses a relative strategic mark-up b_α, while bidder β chooses b_β. Differentiating with respect to b_α

gives the optimal mark-up for bidder α

$$\begin{aligned}\frac{\partial EU_\alpha}{\partial b_\alpha} &= (\tau_\alpha c_\alpha b_\alpha)^{\rho_\alpha} P'(b_\alpha, b_\beta) + (\tau_\alpha c_\alpha)^{\rho_\alpha} \rho_\alpha b_\alpha{}^{(\rho_\alpha - 1)} P(b_\alpha, b_\beta) = 0 \\ &\Leftrightarrow \; b_\alpha P'(b_\alpha, b_\beta) + \rho_\alpha P(b_\alpha, b_\beta) = 0 \\ &\Leftrightarrow \; b_\alpha = -\frac{\rho_\alpha P(b_\alpha, b_\beta)}{P'(b_\alpha, b_\beta)}. \qquad (1)\end{aligned}$$

Let us now derive the winning probability. Bidder α wins if $\tau_\alpha c_\alpha (1 + b_\alpha) < \tau_\beta c_\beta (1 + b_\beta)$ *i.e.* if

$$c_\beta > \underbrace{\frac{\tau_\alpha c_\alpha (1 + b_\alpha)}{\tau_\beta (1 + b_\beta)}}_{\text{A}},$$

which occurs with probability $1 - F_\beta(A)$. Then, ex ante bidder α wins with probability P (defined in expectation over the unknown mean μ)

$$P = \int_\mu (1 - F_\beta(A)) f_\mu(\mu) d\mu.$$

Clearly, if $A > \mu + ac_\alpha$, *i.e.* if $\mu < A - ac_\alpha$, bidder α is certain to lose. Similarly, if $\mu > A + ac_\alpha$, bidder α wins with certainty. Thus P can be rewritten as

$$P = \int_{A - ac_\alpha}^{A + ac_\alpha} (1 - F_\beta(A)) f_\mu(\mu) d\mu + \int_{A + ac_\alpha}^{c_\alpha(1+a)} f_\mu(\mu) d\mu. \qquad (2)$$

Let Q denote the winning probability of bidder β. Obviously from (1), bidder β chooses b_β such that

$$b_\beta = -\frac{\rho_\beta Q(b_\beta, b_\alpha)}{Q'(b_\beta, b_\alpha)}. \qquad (3)$$

In order to provide an explicit form of the winning probabilities, let us now consider the special case of a uniform distribution for F_β and F_α (and therefore for F_μ). Then, we have the following lemma.

Lemma 1 *The optimal strategic mark-up of both bidders are*

$$b_\alpha = \frac{\rho_\alpha \left[(a + 1) \tau_\beta - \tau_\alpha + a \tau_\alpha \rho_\beta (a + 2) \right]}{\tau_\alpha \left[(1 + \rho_\alpha) - \rho_\beta \left[a (a + 2) \rho_\alpha - 1 \right] \right]}$$

and

$$b_\beta = \frac{\rho_\beta \left[(a + 1) \tau_\alpha - \tau_\beta + a \tau_\beta \rho_\alpha (a + 2) \right]}{\tau_\beta \left[(1 + \rho_\alpha) - \rho_\beta \left[a (a + 2) \rho_\alpha - 1 \right] \right]}.$$

Proof of lemma 1: From (2), P and Q become

$$\begin{aligned} P &= \frac{(1+a)(1+b_\beta)\tau_\beta - \tau_\alpha(1+b_\alpha)}{2a(1+b_\beta)\tau_\beta} \\ Q &= \frac{(1+a)(1+b_\alpha)\tau_\alpha - \tau_\beta(1+b_\beta)}{2a(1+b_\alpha)\tau_\alpha}. \end{aligned}$$

A sufficient condition[6] for P and Q to be positive is

$$\begin{cases} a \geq \frac{\tau_\alpha(1+b_\alpha)}{\tau_\beta(1+b_\beta)} - 1 \\ a \geq \frac{\tau_\beta(1+b_\beta)}{\tau_\alpha(1+b_\alpha)} - 1 \end{cases}. \tag{4}$$

We will check ex post that P, Q and the bidding strategies satisfy (4). Given the values of P and Q and the first order conditions (1) and (3), the strategic mark-up of bidders α and β satisfy

$$\begin{cases} b_\alpha = \frac{\rho_\alpha\left[(1+a)\left(1+b_\beta\right)\tau_\beta - \tau_\alpha(1+b_\alpha)\right]}{\tau_\beta} \\ b_\beta = \frac{\rho_\beta\left[(1+a)(1+b_\alpha)\tau_\alpha - \tau_\beta\left(1+b_\beta\right)\right]}{\tau_\alpha} \end{cases}.$$

Solving the latter, we obtain the optimal strategic mark-up of both bidders given by lemma 1. *Q.E.D.*

Remark that the strategic mark-up of a bidder (say α) depends on the level of uncertainty, a, as well as his own risk aversion parameter, ρ_α, and technology parameter τ_α, and those of his opponents, ρ_β and τ_β. In the following, we analyze the impact of these various parameters and show how this informational paradigm fits most of the intuition of the IPV model.

Consider firstly the impact of risk aversion. Let us consider that both bidders are cost-symmetric ($i.e.$ $\tau_\alpha = \tau_\beta$) and assume without loss of generality $\tau_\alpha = \tau_\beta = 1$. We can compute

$$\frac{\partial b_\alpha}{\partial \rho_\alpha} = \frac{a(1+\rho_\beta)\left[1+\rho_\beta(a+2)\right]}{\left[1+\rho_\alpha+\rho_\beta - a(a+2)\rho_\alpha\rho_\beta\right]^2} > 0, \tag{5}$$

and

$$\frac{\partial b_\alpha}{\partial \rho_\beta} = \frac{a(1+a)\rho_\alpha\left[1+\rho_\alpha(a+2)\right]}{\left[1+\rho_\alpha+\rho_\beta - a(a+2)\rho_\alpha\rho_\beta\right]^2} > 0. \tag{6}$$

Intuitively, when ρ_α increases, *i.e.* when bidder α becomes less risk averse, he increases his strategic mark-up. Moreover α incorporates the impact of his opponent's

[6]Implicit in our formulation is that uncertainty is high enough to allow both bidders to win with a positive probability.

risk aversion into his own markup. Indeed, bidder α becomes more aggressive when his opponent is more risk averse, *i.e.* $\frac{\partial b_\alpha}{\partial \rho_\beta} > 0$.

Let us now turn to the analysis of the impact of cost-asymmetry. In order to focus only on cost-asymmetry, assume that both bidders are risk neutral, *i.e.* $\rho_\alpha = \rho_\beta = 1$. We can depict the effect of cost-asymmetry on both relative and total mark-up

$$\frac{\partial b_\alpha}{\partial \tau_\alpha} = \frac{-(1+a)\,\tau_\beta}{(1-a)\,(3+a)\,\tau_\beta^2} < 0,$$

and, for a satisfying (4)

$$\frac{\partial \tau_\alpha b_\alpha}{\partial \tau_\alpha} = \frac{a(a+2)-1}{(1-a)\,(3+a)} > 0.$$

Recall that τ_α reflects the (commonly known) number of hours necessary to accomplish a certain task. Needing more time to undertake the project, the bidder reduces his hourly mark-up, but, as the total amount of time needed increases, the bidder increases his total mark-up.

Assume *e.g.* that $\tau_\beta c_\beta$ is slightly lower than $\tau_\alpha c_\alpha$. Then allocative efficiency would require that bidder β wins. However, if $\tau_\alpha > \tau_\beta$ then $b_\alpha < b_\beta$ and we may have $\tau_\alpha c_\alpha(1 + b_\alpha) < \tau_\beta c_\beta(1 + b_\beta)$ which means that α wins. Thus, since α bids more aggressively than his opponent, allocative efficiency is not necessarily attained. This is a well-known result of auction literature with asymmetric bidders (see *e.g.* Maskin and Riley (2000) and Krishna (2002)).

Note finally that the optimal mark-up b_α and b_β are increasing with respect to the uncertainty parameter a, which is consistent with intuition and conventional results in auction theory.[7]

3 The combinatorial auction

Let us now turn to the combinatorial auction. Suppose the government wishes to undertake a two part contract. This contract can be carried out by three firms: two specialized SFs (SF_α and SF_β) can carry out only a specific single part of the contract (each firm is specialized in a different part), while a LF (say LF_γ) is only interested in undertaking the whole contract. This assumption reflects the specialization of SFs which cannot produce both parts of the contract while the LF (due *e.g.* to outside opportunities, and as the practice confirms) does not want to obtain only a single lot. Let $\tau_\alpha c_\alpha$ reflect the cost of SF_α in implementing the first part of the contract, $\tau_\beta c_\beta$ the cost of SF_β in implementing the second part of the contract and $\tau_\gamma c_\gamma$

[7] See among others Klemperer (2001) or Krishna (2002).

the cost of LF_γ in implementing the whole contract. Each firm i privately knows its own cost c_i whereas τ_i is common knowledge. This cost-asymmetry does not reflect the traditional cost advantage but rather the specialization assumption. Indeed, the technology for carrying out the first lot, the second lot or the whole contract is different. Note that comparative cost advantages can appear in this framework assuming *e.g.* $c_\alpha = c_\beta = c_\gamma = c$ and $\tau_\alpha c + \tau_\beta c > \tau_\gamma c$. In this case, even if all the firms exhibit the same cost parameter, carrying out the whole contract generates some synergies. In comparison with the previous section, the public buyer does not compare bid to bid but balances the sum of the bids of the SFs against the LF's bid. In this context, note that τ_α and τ_β can also be interpreted as the relative sizes of the contract undertaken by SF_α and SF_β.

3.1 The bidding strategies

Let us first consider SFs' strategies. Consider for instance the case of SF_α. It wins if

$$\tau_\alpha c_\alpha (1 + b_\alpha) + \tau_\beta c_\beta (1 + b_\beta) < \tau_\gamma c_\gamma (1 + b_\gamma),$$

where b_γ denotes LF_γ's relative strategic mark-up. The expected utilities of all firms can be written as

$$\begin{aligned} EU_\alpha &= (\tau_\alpha c_\alpha (1 + b_\alpha) - \tau_\alpha c_\alpha)^{\rho_\alpha} P(b_\alpha, b_\beta, b_\gamma) \\ EU_\beta &= (\tau_\beta c_\beta (1 + b_\beta) - \tau_\beta c_\beta)^{\rho_\beta} Q(b_\alpha, b_\beta, b_\gamma) \\ EU_\gamma &= (\tau_\gamma c_\gamma (1 + b_\gamma) - \tau_\gamma c_\gamma)^{\rho_\gamma} R(b_\alpha, b_\beta, b_\gamma), \end{aligned}$$

where Q and R respectively denote the winning probability of SF_β and LF_γ. Following the same developments as the previous section, the strategies are now the solution of the following system

$$\left\{ \begin{array}{l} b_\alpha = -\frac{\rho_\alpha P(b_\alpha, b_\beta, b_\gamma)}{P'(b_\alpha, b_\beta, b_\gamma)} \\ b_\beta = -\frac{\rho_\beta Q(b_\alpha, b_\beta, b_\gamma)}{Q'(b_\alpha, b_\beta, b_\gamma))} \\ b_\gamma = -\frac{\rho_\gamma R(b_\alpha, b_\beta, b_\gamma)}{R'(b_\alpha, b_\beta, b_\gamma))} \end{array} \right. . \tag{7}$$

Then, the following lemma can be stated.[8]

Lemma 2 *Under the combinatorial auction, the optimal strategic mark-up of SF_α, SF_β and LF_γ are*

[8]See the appendix for a proof.

$$b_\alpha = -\frac{\rho_\alpha((1+a)(\tau_\beta+\tau_\gamma(a\rho_\beta-1))+\tau_\alpha(1+a^2\rho_\beta(\rho_\gamma-1)-a(2\rho_\beta+\rho_\gamma)))}{\tau_\alpha(1+\rho_\alpha+\rho_\beta-a(2+a)\rho_\alpha\rho_\beta+\rho_\gamma+a(-\rho_\beta+\rho_\alpha(a\rho_\beta-1))\rho_\gamma)},$$

$$b_\beta = -\frac{\rho_\beta((1+a)\tau_\alpha+(1+a)\tau_\gamma(a\rho_\alpha-1)+\tau_\beta(1+a((a\rho_\alpha-1)\rho_\gamma-(2+a)\rho_\alpha)))}{\tau_\beta(1+\rho_\alpha+\rho_\beta-a(2+a)\rho_\alpha\rho_\beta+\rho_\gamma+a(-\rho_\beta+\rho_\alpha(a\rho_\beta-1))\rho_\gamma)}$$

and

$$b_\gamma = -\frac{\rho_\gamma(\tau_\alpha(a\rho_\beta-1)+(a\rho_\alpha-1)(\tau_\beta+\tau_\gamma(a\rho_\beta-1)))}{\tau_\gamma(1+\rho_\alpha+\rho_\beta-a(2+a)\rho_\alpha\rho_\beta+\rho_\gamma+a(-\rho_\beta+\rho_\alpha(a\rho_\beta-1))\rho_\gamma)}.$$

3.2 Analysis of the free-riding effect

In order to analyze the free riding effect, it is useful to compare the bidding strategies of the LF and the SFs with the two bidder case of subsection 2.2. Indeed, in this case, we can easily show that the bids should equal $a/\left(1-a\right)$ when bidders are cost-symmetric and under risk neutrality. Under the same assumptions but in the context of the combinatorial auction (*i.e.* $\tau_\alpha+\tau_\beta=\tau_\gamma$ and risk neutrality), the LF's bid is equal to $a/4$. Then, the mark-up is lower in the context of the combinatorial auction than in the two bidder case. Considering the uncertainty the LF faces, this result is not striking. Indeed, the LF does not face the same uncertainty in both contexts. In the two bidder case, the cost of its opponent is drawn from a uniform distribution whereas it is drawn from the convolution of two uniform distributions in the combinatorial auction, which is namely a triangular distribution. The law of large numbers implies that the average cost for the sum of SFs' costs with costs drawn from the same distribution is more concentrated near the mean cost. That is, the high and the low costs of SFs tend to average out so that the total cost of both SFs includes proportionately more moderate costs.[9]

Consider now the case of the SFs. SF_α, say, would bid $a/\left(1-a\right)$ in the two bidder case (Under risk neutrality and cost-symmetry). Under the same assumptions but in the context of the combinatorial auction (*i.e.* $\tau_\alpha+\tau_\beta=\tau_\gamma$), the difference between its bid in the combinatorial auction and $a/\left(1-a\right)$ is equal to

$$-\frac{a\left(1+a\right)\tau_\gamma}{4\left(1-a\right)\tau_\alpha}<0.$$

So, like the LF, the SFs choose a lower relative mark-up in the context of the combinatorial auction than in the two bidder case. Nevertheless, as we will see in the following, this does not mean that the SFs do not free-ride.

[9] See e.g. Bakos and Brynjolfsson (1999) for similar arguments in the bundling literature.

Indeed, let us consider that $\tau_\gamma = \tau_\alpha + \tau_\beta$. Under this latter assumption, if each firm observes the same signal $c_\alpha = c_\beta = c_\gamma$, then the LF's cost equals the sum of the individual costs of the SFs. So, if the SFs do not free-ride then the LF's bid should equal the sum of the individual bids of the SFs. Hence, we should have $\tau_\alpha(1+b_\alpha)+\tau_\beta(1+b_\beta) = \tau_\gamma(1+b_\gamma)$. Thus, under the assumptions that $\tau_\gamma = \tau_\alpha+\tau_\beta$ and $c_\alpha = c_\beta = c_\gamma$, we will consider that SFs free-ride when[10]

$$\tau_\alpha(1+b_\alpha)+\tau_\beta(1+b_\beta) > \tau_\gamma(1+b_\gamma). \tag{8}$$

Thus, given the optimal strategies $(b_\alpha, b_\beta, b_\gamma)$ we can derive the following proposition.

Proposition 1 *When all firms are equally risk averse, the bidding strategies of SFs exhibit a free-riding effect.*

Proof of proposition 1: Assume that $\rho_\alpha = \rho_\beta = \rho_\gamma = \rho$. When $\tau_\gamma = \tau_\alpha + \tau_\beta$ and $c_\alpha = c_\beta = c_\gamma$, computing the difference between submitted offers yields

$$\tau_\alpha(1+b_\alpha)+\tau_\beta(1+b_\beta) - \tau_\gamma(1+b_\gamma) = \frac{a\rho\tau_\gamma}{1+\rho\left[3+a(1-\rho)\right]} > 0.$$

Q.E.D.

Hence, combinatorial first-price sealed-bid auctions, like ascending auctions and subscription games, induce SFs to free-ride. This result is rather intuitive. Since both SFs can only win if the sum of their cost-bids is lower than the LF's bid, they do not bid as aggressively as the LF, recognizing that the gain in reducing their own bids does not fully accrue to them. Thus, when switching from the two bidder case to the combinatorial one, two effects can be distinguished:

- The effect of free-riding.
- The competition effect due to the modification of the distribution of the opponent's cost.

Besides, some interesting properties of the free-riding effect can be highlighted in the following. The above discussion sheds light on the relevance of the uncertainty firms face. Let us first focus on the impact of uncertainty on the magnitude of the free riding effect.

[10] At first glance, our definition of the free-riding effect based on the assumption that $c_\alpha = c_\beta = c_\gamma$ may not be too satisfying because this event occurs with probability zero. However, inequality (8) would also hold for a set of costs with positive probability mass (not just if $c_\alpha = c_\beta = c_\gamma$). We thank a referee for pointing out this comment.

Corollary 1 *If bidders are equally risk averse, the free-riding effect is increasing with the level of uncertainty.*

Proof of corollary 1: Assume that $\rho_\alpha = \rho_\beta = \rho_\gamma = \rho$ and $\tau_\gamma = \tau_\alpha + \tau_\beta$. We get

$$\frac{\partial}{\partial a}\left[\frac{a\rho\tau_\gamma}{1+\rho\left[3+a(1-\rho)\right]}\right] = \frac{\rho\tau_\gamma\left(1+3\rho\right)}{\left[1+\rho\left[3+a(1-\rho)\right]\right]^2} > 0.$$

Q.E.D.

This result can easily be explained. Indeed, uncertainty creates an opportunity for firms to obtain informational rents. In our combinatorial framework, the informational rent of the LF does not correspond to the sum of informational rents of the SFs. Technically, the LF internalizes the impact an overstatement of each partial cost has on total cost. For SFs, by contrast, this externality is not internalized and each SF has a stronger incentive to free-ride.[11]

From SF_α's point of view, with $\tau_\alpha > \tau_\beta$,

$$\frac{\partial b_\alpha}{\partial a} = \frac{2(\tau_\alpha - \tau_\beta) + (a-1)^2\tau_\gamma}{4(a-1)^2\tau_\alpha} > 0.$$

So, SF_α's relative mark-up is increasing with uncertainty. We find here the same effect of uncertainty as we found in the two bidder case. Since $\tau_\alpha > \tau_\beta$, SF_α undertakes a larger part of the contract than SF_β and so is mainly concerned with the distribution of c_γ. Roughly speaking, from SF_α's point of view, it is close to a two bidders-asymmetric auction game.

From SF_β's point of view, the effect is less clear-cut

$$\frac{\partial b_\beta}{\partial a} = \frac{-2(\tau_\alpha - \tau_\beta) + (a-1)^2\tau_\gamma}{4(a-1)^2\tau_\beta} \lesseqqgtr 0,$$

which turns out to be negative for τ_α high enough. Since SF_β undertakes a smaller part of the contract than SF_α, it is mainly concerned with the convolution of $c_\gamma - c_\alpha$. So, changing the uncertainty parameter modifies the shape of this distribution.

Let us now analyze the impact of risk aversion on the free-riding effect.

Corollary 2 *If bidders are equally risk averse, then an increase in relative risk aversion overcomes part of the free-riding problem.*

[11] See Morand (2003) for similar results in an optimal allotted procurement mechanism analysis.

Proof of corollary 2: Assume that $\tau_\gamma = \tau_\alpha + \tau_\beta$ and $\rho_\alpha = \rho_\beta = \rho_\gamma = \rho$. We can compute

$$\frac{\partial}{\partial \rho}\left(\frac{(\tau_\gamma) a\rho}{1+\rho\left[3+a(1-\rho)\right]}\right) = \frac{a\left(\tau_\gamma\right)\left(1+a\rho^2\right)}{\left[1+\rho\left(3+a-a\rho\right)\right]^2} > 0.$$

Q.E.D.

When relative risk aversion increases, it tends to increase the aggressiveness of the SFs more than that of the LF. When the LF becomes more risk averse, the simple and traditional trade-off between the profit upon winning and the probability of winning is biased toward the probability. Then, the LF's bid becomes more aggressive. When a SF becomes more risk averse, it realizes the same trade-off than the LF except that it faces another uncertainty relative to the complementary bid of the other SF. Then, each SF's bid becomes even more aggressive.

We now turn to the analysis of the impact of cost-asymmetry on the magnitude of the free riding effect. As an example, assume τ_α close to τ_γ. In this case, SF_β competes only "for a small amount" while SF_α competes with LF_γ almost on an equal footing. Does the need of SF_β handicap SF_α? Does the free-riding effect vanish as τ_α tends to τ_γ, or remain unchanged as long as the auction is a combinatorial one? Tackling this issue, the following proposition can be stated.

Proposition 2 *When all firms are equally risk averse, the magnitude of the free-riding effect remains unaffected by the heterogeneity of the allotment.*

Proof of proposition 2: Assume that $\rho_\alpha = \rho_\beta = \rho_\gamma = \rho$ and $c_\alpha = c_\beta = c_\gamma$. Computing the difference between submitted offers yields

$$\tau_\alpha(1+b_\alpha) + \tau_\beta(1+b_\beta) - \tau_\gamma(1+b_\gamma) = \frac{\tau_\alpha + \tau_\beta + \tau_\gamma\left(a\rho - 1\right)}{1+\rho\left[3+a(1-\rho)\right]} > 0,$$

which only depends on the sum $\tau_\alpha + \tau_\beta$ but not on the relative values of τ_α and τ_β. *Q.E.D.*

This proposition deserves some comments. Assume that $\rho = 1$, $\tau_\alpha = \frac{\tau_\gamma}{2} + \varepsilon$ and $\tau_\beta = \frac{\tau_\gamma}{2} - \varepsilon$. In this case, $\tau_\alpha + \tau_\beta$ remains constant when ε takes values in $[0, \frac{\tau_\gamma}{2}]$. When ε increases, SF_α undertakes a larger part of the contract than SF_β. Then we can compute

$$\frac{\partial b_\alpha}{\partial \varepsilon} = \frac{a(1+a)\tau_\gamma}{(1-a)\left(\tau_\gamma + 2\varepsilon\right)^2} \geq 0.$$

SF_α is more and more concerned with competition against LF_γ. Roughly speaking, the auction game for SF_α tends to a simple asymmetric non-combinatorial one. So, SF_α increases its mark-up as previously argued.

For SF_β, we can compute

$$\frac{\partial b_\beta}{\partial \varepsilon} = \frac{a(1+a)\tau_\gamma}{-(1-a)\left(\tau_\gamma - 2\varepsilon\right)^2} < 0.$$

Obviously, as ε increases, SF_β is more concerned with competition against $LF_\gamma - SF_\alpha$. The auction game for SF_α is “more” combinatorial and SF_β reduces its mark-up.

Note finally that $\frac{\partial b_\gamma}{\partial \varepsilon} = 0$. LF_γ is not concerned with the relative efficiency weights of both SFs.

Figure 1 illustrates the evolution of SFs’ mark-up as the share allocated to SF_α (bold-line) increases (with parameters $a = 1/2$, $\rho = 1/10$ and $\tau_\gamma = 1$).

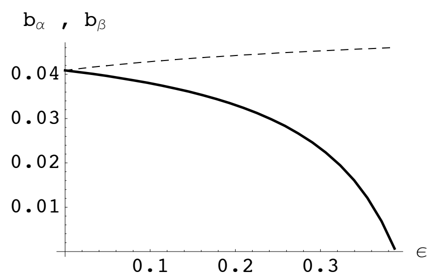

Figure 1: SF’s mark-up

The result of proposition 2 holds in the context of all the firms having the same relative risk aversion parameter (or under risk neutrality). If we relax this assumption, considering that each firm exhibits its own risk aversion parameter $\rho_\alpha \neq \rho_\beta \neq \rho_\gamma$, the following proposition can be stated.

Proposition 3 *The free-riding effect is reduced (resp. increased) as the more risk averse SF competes for a larger (resp. smaller) part of the contract.*

Proof of proposition 3: Once again assume that $\tau_\alpha = \frac{\tau_\gamma}{2} + \varepsilon$ and $\tau_\beta = \frac{\tau_\gamma}{2} - \varepsilon$, we can compute

$$\begin{aligned} & \frac{\partial}{\partial \varepsilon}\left[\tau_\alpha(1+b_\alpha) + \tau_\beta(1+b_\beta) - \tau_\gamma(1+b_\gamma)\right] \\ = \quad & \frac{a(\rho_\alpha - \rho_\beta)}{1 + \rho_\alpha + \rho_\beta + \rho_\gamma - a(2+a)\rho_\alpha\rho_\beta + a\rho_\gamma\left[\rho_\alpha(a\rho_\beta - 1) - \rho_\beta\right]} \end{aligned}$$

The denominator of the latter equation can be rewritten as

$$(1 - a^2\rho_\alpha\rho_\beta)(1-\rho_\gamma) + \rho_\gamma(2 - a\rho_\gamma - a\rho_\beta) + \rho_\alpha(1 - a\rho_\beta) + \rho_\beta(1 - a\rho_\alpha) > 0.$$

Then

$$sign(\frac{\partial}{\partial \varepsilon}[\tau_\alpha(1+b_\alpha)+\tau_\beta(1+b_\beta)-\tau_\gamma(1+b_\gamma)]) = sign(a(\rho_\alpha-\rho_\beta)).$$

Q.E.D.

Recall that we have assumed as an example τ_α close to τ_γ. Therefore, as SF_α is mainly concerned with the asymmetric auction game against LF_γ, in relative terms, it is less aggressive than SF_β. However, as an increase in SF_α's risk aversion level reduces the mark-up, it tends to more than offset this effect. So, this proposition suggests that the expected cost of the contract for the public buyer is reduced when the larger part of the contract is allocated to the more risk averse bidder.

4 Conclusion

The main purpose of this paper was to derive the equilibrium bidding strategies in a combinatorial first-price sealed bid auction when two small business firms compete with a larger firm. In our framework (which departs from the traditional IPV paradigm and borrows from VUS's model), we simultaneously consider two kinds of asymmetries between firms: a cost-asymmetry and a risk-asymmetry. When costs are uniformly distributed, we have depicted two effects on bidding strategies under a combinatorial auction. The first effect deals with the particular distribution of the opponent's cost while the second one is a free-riding effect.

The free-riding effect has some interesting properties. We show that it is increasing with the level of uncertainty. If all the firms are equally risk averse, we find that an increase in relative risk aversion overcomes part of the free-riding problem. Indeed, it tends to increase the aggressiveness of the SFs more than the LF's one. However, when firms are risk neutral or equally risk averse, the magnitude of the free-riding effect remains unaffected by the heterogeneity of the allotment, *i.e.* unaffected by the division of the contract chosen by the public buyer. This result only holds in the context of all the firms having the same relative risk aversion parameter (or being risk neutral). If each firm exhibits its own risk aversion parameter, then we find that the free-riding effect is reduced (resp. increased) as the more risk averse SF competes for a larger (resp. smaller) part of the contract. This suggests that the expected cost of the contract for the public buyer is reduced when the larger part of the contract is allocated to the more risk averse firm. Even if our results have been established under a specific informational paradigm, we believe that they would hold in a more traditional informational setting.

Another way of allotting the contract could be to impose the LF to bid simultaneously on each allotted part of the contract. However, as shown in the context of the FCC spectrum auction, such a rule may create a "financial exposure" since the LF runs the risk of not successfully aggregating all the lots. The impact of the financial exposure in a first-price sealed-bid auction still remains an open question.

Appendix

Example. In order to highlight the relevance of our informational framework, let us present a simple example which shows that VUS's model cannot apply to asymmetric settings whereas our framework can.

Given our notation, consider VUS's model with two bidders $i \in \{\alpha, \beta\}$. Their private costs c_i are drawn from a uniform distribution over $[\mu - a, \mu + a]$, where a is common knowledge whereas the mean μ is unknown. Thus, from his observation c_α, bidder α can infer that $\mu \in [c_\alpha - a, c_\alpha + a]$ and so $c_\beta \in [c_\alpha - 2a, c_\alpha + 2a]$. Since all the bidders are ex ante symmetric, the probability of having the lowest cost for bidder α is the same whatever his cost c_α. Let us now depart from VUS's model by introducing a cost-asymmetry between bidders. Thus, consider that bidders have a cost function $C_i(c_i, \tau_i) = \tau_i c_i$, where τ_i is common knowledge. Assume that $\tau_\alpha = \frac{1}{3}$, $\tau_\beta = \frac{1}{4}$ and $a = 1$.

- If bidder α observes $c_\alpha = 4$, then $\tau_\alpha c_\alpha = \frac{4}{3}$ and α infers that $c_\beta \in [2, 6]$ and $\tau_\beta c_\beta \in [\frac{1}{2}, \frac{3}{2}]$. In this case, α can have the lowest cost with some positive probability.

- If now α observes $c_\alpha = 8$, then $\tau_\alpha c_\alpha = \frac{8}{3}$ and he infers that $c_\beta \in [6, 10]$ and $\tau_\beta c_\beta \in [\frac{3}{2}, \frac{5}{2}]$. Since $\frac{8}{3} > \frac{5}{2}$, α cannot have the lowest cost.

Clearly, VUS's model cannot apply to asymmetric settings because the cost-asymmetry enables each bidder to infer some information about his relative position.

Let us now present the same example but within our framework. Then each bidder i believes that his private information c_i is drawn from a distribution $F^i_{c_i}$ over $[\mu - ac_i, \mu + ac_i]$. When α learns his own cost c_α, he can infer that $\mu \in [(1 - a)c_\alpha, (1+a)c_\alpha]$. Since all bidders are ex ante symmetric relative to the informational knowledge, α infers that $c_\beta \in [(1 - 2a)c_\alpha, (1 + 2a)c_\alpha]$. Assume now that $a = \frac{1}{4}$.

- If bidder α observes $c_\alpha = 4$, then $\tau_\alpha c_\alpha = \frac{4}{3}$ and α infers that $c_\beta \in [2, 6]$ and $\tau_\beta c_\beta \in [\frac{1}{2}, \frac{3}{2}]$.

- But, if α observes $c_\alpha = 8$, then $\tau_\alpha c_\alpha = \frac{8}{3}$ and α infers that $c_\beta \in [4, 12]$ and $\tau_\beta c_\beta \in [1, 3]$.

So, in our framework, the observation of c_α does not affect the winning probability.

Proof of lemma 2. In a combinatorial auction, the winning probability of SF_α is increasing in both its own and SF_β's aggressiveness whereas it is decreasing with LF_γ's aggressiveness. Indeed, SF_α wins *with* SF_β and *against* LF_γ if

$$\tau_\alpha c_\alpha (1 + b_\alpha) < \tau_\gamma c_\gamma (1 + b_\gamma) - \tau_\beta c_\beta (1 + b_\beta),$$

and so, inferring its winning probability, SF_α is concerned with the distribution of $Z = \tau_\gamma c_\gamma (1 + b_\gamma) - \tau_\beta c_\beta (1 + b_\beta)$. As in the previous section, the first step is to compute the winning probability. Recall that SF_α assumes that its own cost parameter is drawn from distribution $F^\alpha_{c_\alpha}(c_\alpha)$ over $[\mu - ac_\alpha, \mu + ac_\alpha]$ with μ unknown and so SF_α can infer that $\tau_\gamma c_\gamma (1 + b_\gamma)$ is drawn from distribution $F^\alpha_{\tau_\gamma c_\gamma (1+b_\gamma)}(.)$ over $[\tau_\gamma(1+b_\gamma)(\mu - ac_\alpha), \tau_\gamma(1+b_\gamma)(\mu + ac_\alpha)]$, while $-\tau_\beta c_\beta (1 + b_\beta)$ is drawn from distribution $F^\alpha_{-\tau_\beta c_\beta \left(1+b_\beta\right)}(.)$ over $[-\tau_\beta(1+b_\beta)(\mu + ac_\alpha), -\tau_\beta(1+b_\beta)(\mu - ac_\alpha)]$. So, the distribution of Z is the convolution of the two previous cumulative distributions with

$$f^\alpha_Z(z) = \int_{-\infty}^{+\infty} f^\alpha_{\tau_\gamma c_\gamma (1+b_\gamma)}(\varepsilon) f^\alpha_{-\tau_\beta c_\beta \left(1+b_\beta\right)}(z - \varepsilon) d\varepsilon.$$

SF_α wins if $\tau_\alpha c_\alpha (1 + b_\alpha) < Z$, and so the winning probability for a given μ is

$$P_\mu = 1 - \int_{-\infty}^{\tau_\alpha c_\alpha (1+b_\alpha)} f^\alpha_Z(s) ds.$$

The winning probability $P(b_\alpha, b_\beta, b_\gamma)$ is determined in expectation over μ. Depending on the value of μ, SF_α can either be certain to win or certain to lose. More precisely, given the parameters τ_β and τ_γ, SF_α wins with certainty if there does not exist any LF_γ's cost parameter low enough and any SF_β's cost parameter high enough to prevent SF_α from winning. Hence

$$\tau_\alpha c_\alpha (1 + b_\alpha) < \tau_\gamma c_\gamma (1 + b_\gamma) - \tau_\beta c_\beta (1 + b_\beta)$$

is always satisfied if

$$\mu > \underbrace{\frac{c_\alpha \left[\tau_\alpha(1 + b_\alpha) + a(\tau_\beta(1 + b_\beta) + \tau_\gamma(1 + b_\gamma))\right]}{\tau_\gamma(1 + b_\gamma) - \tau_\beta(1 + b_\beta)}}_{\mu_1}.$$

Similarly, SF_α can never win if

$$\mu < \underbrace{\frac{c_\alpha \left[a(\tau_\beta(1 + b_\beta) + \tau_\gamma(1 + b_\gamma)) - \tau_\alpha(1 + b_\alpha)\right]}{\tau_\beta(1 + b_\beta) - \tau_\gamma(1 + b_\gamma)}}_{\mu_2}.$$

Therefore, the winning probability is now

$$P(b_\alpha, b_\beta, b_\gamma) = \int_{\mu_1}^{(1+a)c_\alpha} f_\mu^\alpha(\mu) d\mu + \int_{\mu_2}^{\mu_1} f_\mu^\alpha(\mu) \left[1 - \int_{-\infty}^{\tau_\alpha c_\alpha (1+b_\alpha)} f_Z^\alpha(s) ds \right] d\mu. \quad \text{(A.1)}$$

In order to characterize the distribution of the sum and the difference between random variables, let us assume without loss of generality that $\tau_\gamma(1+b_\gamma) > \tau_\alpha(1+b_\alpha) > \tau_\beta(1+b_\beta)$. Since the distribution is bounded above and below, the convolution of Z is defined over different ranges

- $\forall z \in [I_1, I_2] = [\tau_\gamma(1+b_\gamma)(\mu - ac_\alpha) - \tau_\beta(1+b_\beta)(\mu + ac_\alpha),$
  $\tau_\gamma(1+b_\gamma)(\mu - ac_\alpha) - \tau_\beta(1+b_\beta)(\mu - ac_\alpha)],$

  $$f_{Z1}^\alpha(z) = \int_{\tau_\gamma(1+b_\gamma)(\mu - ac_\alpha)}^{z+\tau_\beta(1+b_\beta)(\mu+ac_\alpha)} f^\alpha_{\tau_\gamma c_\gamma(1+b_\gamma)}(\varepsilon) f^\alpha_{-\tau_\beta c_\beta \left(1+b_\beta\right)}(z-\varepsilon) d\varepsilon$$

  and

  $$F_{Z1}^\alpha(x) = \int_{\tau_\gamma(1+b_\gamma)(\mu - ac_\alpha) - \tau_\beta(1+b_\beta)(\mu+ac_\alpha)}^{x} f_{Z1}^\alpha(z) dz.$$

- $\forall z \in [I_2, I_3] = [\tau_\gamma(1+b_\gamma)(\mu - ac_\alpha) - \tau_\beta(1+b_\beta)(\mu - ac_\alpha),$
  $\tau_\gamma(1+b_\gamma)(\mu + ac_\alpha) - \tau_\beta(1+b_\beta)(\mu + ac_\alpha)],$

  $$f_{Z2}^\alpha(z) = \int_{z+\tau_\beta(1+b_\beta)(\mu - ac_\alpha)}^{z+\tau_\beta(1+b_\beta)(\mu+ac_\alpha)} f^\alpha_{\tau_\gamma c_\gamma(1+b_\gamma)}(\varepsilon) f^\alpha_{-\tau_\beta c_\beta \left(1+b_\beta\right)}(z-\varepsilon) d\varepsilon$$

  and

  $$\begin{aligned} F_{Z2}^\alpha(x) = {} & F_{Z1}^\alpha(\tau_\gamma(1+b_\gamma)(\mu - ac_\alpha) - \tau_\beta(1+b_\beta)(\mu - ac_\alpha)) \\ & + \int_{\tau_\gamma(1+b_\gamma)(\mu - ac_\alpha) - \tau_\beta(1+b_\beta)(\mu - ac_\alpha)}^{x} f_{Z2}^\alpha(z) dz. \end{aligned}$$

- $\forall z \in [I_3, I_4] = [\tau_\gamma(1+b_\gamma)(\mu + ac_\alpha) - \tau_\beta(1+b_\beta)(\mu + ac_\alpha),$
  $\tau_\gamma(1+b_\gamma)(\mu + ac_\alpha) - \tau_\beta(1+b_\beta)(\mu - ac_\alpha)],$

  $$f_{Z3}^\alpha(z) = \int_{z+\tau_\beta(1+b_\beta)(\mu - ac_\alpha)}^{\tau_\gamma(1+b_\gamma)(\mu+ac_\alpha)} f^\alpha_{\tau_\gamma c_\gamma(1+b_\gamma)}(\varepsilon) f^\alpha_{-\tau_\beta c_\beta \left(1+b_\beta\right)}(z-\varepsilon) d\varepsilon$$

  and

  $$\begin{aligned} F_{Z3}^\alpha(x) = {} & F_{Z2}^\alpha(\tau_\gamma(1+b_\gamma)(\mu + ac_\alpha) \\ & -\tau_\beta(1+b_\beta)(\mu + ac_\alpha)) + \int_{\tau_\gamma(1+b_\gamma)(\mu + ac_\alpha) - \tau_\beta(1+b_\beta)(\mu+ac_\alpha)}^{x} f_{Z3}^\alpha(z) dz. \end{aligned}$$

Then, the winning probability for a given μ is

$$\begin{aligned} P_\mu = 1 - \int_{-\infty}^{\tau_\alpha c_\alpha(1+b_\alpha)} f_Z^\alpha(s)ds \\ = (1 - F_{Z3}^\alpha(\tau_\alpha c_\alpha(1+b_\alpha)))\left[F_{Z3}^\alpha(I_4) - F_{Z3}^\alpha(I_3)\right] \\ +(1 - F_{Z2}^\alpha(\tau_\alpha c_\alpha(1+b_\alpha)))\left[F_{Z2}^\alpha(I_3) - F_{Z2}^\alpha(I_2)\right] \\ +(1 - F_{Z1}^\alpha(\tau_\alpha c_\alpha(1+b_\alpha)))\left[F_{Z1}^\alpha(I_2) - F_{Z1}^\alpha(I_1)\right]. \end{aligned}$$

So we can compute

$$P_\mu = \frac{1}{2} + \frac{[\tau_\beta(1+b_\beta) - 3\tau_\gamma(1+b_\gamma)]\left[\tau_\alpha(1+b_\alpha)c_\alpha + \mu\left[\tau_\beta(1+b_\beta) - \tau_\gamma(1+b_\gamma)\right]\right]}{4a(1+b_\gamma)^2\tau_\gamma^2 c_\alpha}.$$

Substituting the value of P_μ in (A.1) with $f_\mu^\alpha(\mu) = \frac{1}{2ac_\alpha}$ yields the winning probability

$$P(b_\alpha, b_\beta, b_\gamma) = \frac{\tau_\alpha(1+b_\alpha) + (1+a)[\tau_\beta(1+b_\beta) - \tau_\gamma(1+b_\gamma)]}{2a[\tau_\beta(1+b_\beta) - \tau_\gamma(1+b_\gamma)]}.$$

Similarly for SF_β, the winning probability is

$$Q(b_\alpha, b_\beta, b_\gamma) = \frac{\tau_\beta(1+b_\beta) + (1+a)[\tau_\alpha(1+b_\alpha) - \tau_\gamma(1+b_\gamma)]}{2a[\tau_\alpha(1+b_\alpha) - \tau_\gamma(1+b_\gamma)]}.$$

Consider now the case of LF_γ. Contrary to SF_α and SF_β, LF_γ wins *alone* and *against* the two other SFs. Namely, LF_γ wins if

$$\tau_\gamma c_\gamma(1+b_\gamma) < \tau_\alpha c_\alpha(1+b_\alpha) + \tau_\beta c_\beta(1+b_\beta),$$

and so is concerned with the distribution of $\tau_\alpha c_\alpha(1+b_\alpha) + \tau_\beta c_\beta(1+b_\beta)$. LF_γ assumes that its own cost parameter is drawn from distribution $F_{c_\gamma}^\gamma(c_\gamma)$ over $[\mu - ac_\gamma, \mu + ac_\gamma]$ with μ unknown. So it can infer that $\tau_\alpha c_\alpha(1+b_\alpha) + \tau_\beta c_\beta(1+b_\beta)$ is drawn from the convolution of both random variables $\tau_\alpha c_\alpha(1+b_\alpha)$ and $\tau_\beta c_\beta(1+b_\beta)$ over $[\tau_\alpha(1+b_\alpha)(\mu - ac_\gamma) + \tau_\beta(1+b_\beta)(\mu - ac_\gamma), \tau_\alpha(1+b_\alpha)(\mu + ac_\gamma) + \tau_\beta(1+b_\beta)(\mu - ac_\gamma)]$. Similar developments as those for SF_α yields the winning probability of LF_γ

$$R(b_\alpha, b_\beta, b_\gamma) = \frac{\tau_\alpha(1+b_\alpha) + \tau_\beta(1+b_\beta) - \tau_\gamma(1+b_\gamma)}{2a[\tau_\alpha(1+b_\alpha) + \tau_\beta(1+b_\beta)]}.$$

Given (7) and the values of P, Q and R, the optimal strategic mark-up for SF_α, SF_β and LF_γ solve the following system

$$\left\{ \begin{array}{l} b_\alpha = -\frac{\rho_\alpha[\tau_\alpha(1+b_\alpha)+(1+a)[\tau_\beta(1+b_\beta)-\tau_\gamma(1+b_\gamma)]]}{\tau_\alpha} \\ b_\beta = -\frac{\rho_\beta[\tau_\beta(1+b_\beta)+(1+a)[\tau_\alpha(1+b_\alpha)-\tau_\gamma(1+b_\gamma)]]}{\tau_\beta} \\ b_\gamma = -\frac{\rho_\gamma[\tau_\alpha(1+b_\alpha)+\tau_\beta(1+b_\beta)-\tau_\gamma(1+b_\gamma)]}{\tau_\gamma} \end{array} \right. ,$$

which yields the optimal strategic mark-up of lemma 2. *Q.E.D.*

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