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January 2025

Working paper No. 2025 – 01

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Dynamic equilibrium with randomly entering and exiting firms of different types*

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January 10, 2025

Abstract

There exist situations where firms (identical or not) are in a state of renewed interaction and where, at each period, in addition to exits, new firms (identical or not) may arrive. In such cases, no one is able to know ex ante *exactly* how many firms there will be in each period. One of the questions an incumbent firm might therefore ask itself, in this context, is what expected payoff it can expect. Our paper aims to provide an answer to this question, in finite and infinite horizons, using a discrete-time dynamic game with random arrival(s) and exit(s) of different types of firm(s). We first propose a general model, which we then particularize by considering the types as composed of identical players. Within this framework, we address the case of a dynamic Cournot oligopoly with sticky prices, and provide numerical illustrations to underline the interest of this approach and demonstrate its operational character.

Keywords: Oligopoly, Random entries and exits, Types, Dynamic equilibirum, Cournot, sticky prices.

JEL classification: C73, D43, L13

*The authors would like to thank CRESE members Denis Claude, Eric Kamwa, and Aymeric Lardon, as well as SING19 participants for valuable suggestions and comments on a previous version. The usual caveats apply.

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1 Introduction

The great diversity of markets bears witness to the extraordinarily diverse forms of strategic interactions that can exist. One of the most robust conclusions of economic analysis is undoubtedly the importance of studying these interactions in order to better understand and, where necessary, regulate different markets. Our article focuses on contexts where there are renewed interactions on a market, in discrete time, without product differentiation and without information asymmetries, but where nobody, neither the firms, nor the authorities, nor the modeler, knows ex ante *exactly* how many firms there will be in each period. All that each firm knows in our framework are: 1/ its personal characteristics (i.e. its type), 2/ the size of the market, 3/ the possible types of the other firms (i.e. their characteristics), and 4/ the probability laws that determine, in each period, the random exit(s) as well as the random arrival(s) of one (or more) entrant(s).

In addition to the theoretical interest of studying this type of configuration, we think that it can, for example, help us better understand the situation in which a classic restaurant (i.e. with premises, kitchen staff and waiting staff) finds itself in trying to estimate how much it can expect to earn over the course of a year, or over several years. In fact, it seems to us that it is in a position, albeit imperfectly, to know its characteristics (i.e. its cost), the size of the market (i.e. demand), as well as the characteristics of the different types of restaurants that are or may be in competition with it. However, neither it, nor anyone else, is in a position to know with *certainty* how many competitors of each type there will be in each period. This ignorance stems both from the fact that it doesn't know how many restaurants of each type will leave the market, and from the fact that it doesn't know how many restaurants of each type will arrive.

Facing this situation, we consider that the restaurant can use the information provided by statistical institutes, professional chambers or associations, trade unions and courts, to estimate a probability of exit for different types of restaurant. Similarly, when it comes to the potential entry of different types of restaurant, it can find out what commercial premises are available for this type of business in the town or surrounding area (depending on the geographical limits of the relevant market it estimates). However, today, it must also take into account the possible arrival of "dark kitchens".[1] In other words, here again, it has to deal with uncertainty and estimate probabilities.

We then make the simplifying assumption that each restaurant will proceed in the same way and arrive at the same conclusions concerning: market size, types, and entry and exit probabilities for each type of restaurant. We are then able to establish how much a restaurant can expect to obtain, over a finite or infinite horizon, in a dynamic Cournot oligopoly with a homogeneous good, in discrete time, with sticky prices.

[1] This expression refers to independent kitchens that provide a catering service through takeaway sales or deliveries, without generally having a place for on-site consumption for their customers. As they are not subject to specific administrative registration, it is extremely difficult to quantify their number. Most of these restaurants market their products via delivery platforms (e.g. Deliveroo, Grubhub, Just Eat, Uber Eats). These restaurants can be distinguished from set-up restaurants with a stand-alone delivery business, as well as from platforms delivering dishes prepared in their own kitchens. For a more detailed presentation of dark kitchens, we refer readers to da Cunha et al. [2024].

From a theoretical point of view, our dynamic modeling of a market with random arrivals and exits of different types of competitors, where nobody knows ex ante how many competitors will be present in each period, is related to questions concerning entries and exits on a market, those concerning renewed interactions, those concerning the existence of different types of players, and those concerning uncertainty about the number of players. Without claiming to be exhaustive in our review of the literature, we will begin by briefly discussing entry models, taking up the elements provided by Biard and Deschamps [2021]. We will then give an overview of the main classical game-theoretic models with a known number of players, showing that they cannot answer our question. Finally, we present the models that directly address the question of uncertainty in the number of players.

Simplifying Polo's presentation [2020], it is possible to distinguish two main paths within the theoretical literature on entry modeling. Firstly, there are models where entry decisions precede market strategies, in other words situations where firms' strategies cannot be determined with the aim of affecting entry decisions. In the first stage, the (possibly infinite number of) players decide (or not) whether to enter a market where there is no one else, and in the second stage they adopt their strategies, depending in particular on the number of competitors present, which is common knowledge. Alternatively, there are models in which the entry decisions of certain firms are taken after the known number of established firms have chosen their market strategies, and after the first firms have observed these strategies. In this case, the game necessarily comprises several linked stages, and can be conceived at least as a two-period game, in which in the first period the established firms play, and before the start of the second period the potential entrants observe the results and decide (or not) to enter the market. This leaves established firms free to choose their strategies, taking into account not only current competition but also potential competition, in particular by implementing foreclosure strategies. Within these two models, models with free entry occupy a specific place because, on the one hand, they make it possible to endogenize entry when there is an entry cost and, above all, because by using the total surplus criterion it becomes possible to answer the normative question of whether a deregulated market leads to an excessive, insufficient or optimal number of firms on the market (e.g. Belleflamme and Peitz [2015]).

In our model, entries and exits in each period are exogenous, and there may be one (or more) entrant(s) and/or exit(s) with the same (or another) type as the installed firm(s). We limit the number of potential entrants by type, and this number is common knowledge. Furthermore, the installed firm(s) has (have) no opportunity to adopt strategies that deter or slow entries. Nor do we have any normative thoughts on the optimal number of firms to be present in the market. For each type, the strategies played by firms in each period are identical and common knowledge.

The usual game-theoretic models, such as those found for example in the textbooks by Maschler et al. [2020] or Osborne and Rubinstein [1994], are all based on a framework in which the number of players is known. This is naturally the case for full-information games, whether static or dynamic, with perfect or imperfect information, but it is also the case for Bayesian games. Indeed, even though Bayesian games can handle configurations with incomplete information, thanks to the Harsanyi transformation, the uncertainty only concerns the characteristics of the players (i.e. their types) and not their number. Similarly, repeated or stochastic games cannot handle situations where nobody knows ex ante *exactly* the number of players present at each period.

To the best of our knowledge, as already pointed out by Bernhard and Deschamps [2017] whom we follow here, there are currently only two approaches in game theory that directly address the issue of uncertainty in the number of players.

The first, and oldest, approach is that which models this type of situation in games where the number of active players is not common knowledge. According to Levin and Ozdenoren [2004], this approach has been developed in two directions. The first is to consider that there are a number of potential players, and that a stochastic process represented by Nature determines which of these will be active players. The number of potential players and the probability distribution are common knowledge, whereas whether or not a potential player is an active player is private information. The second direction taken in this approach endogenizes the entry process. It considers a market with no one, and a number of potential entrants that is common knowledge. Each potential entrant then privately receives a message from Nature indicating its type (i.e. its cost of entering the market), and each then decides (or not) to enter the market. By way of illustration, the question of how many students are going to take an exam can be answered using this type of modeling. Indeed, the number of students registered is common knowledge, and it can be assumed that each of them knows, privately, whether he or she will take (or not) the exam.

The second approach was developed by Roger Myerson in the late 1990s (e.g. Myerson [1998], and Myerson [2000]), and is known as games with population uncertainty. It models uncertainty in the number of players by considering that the number of players who will actually be on the market is the result of a stochastic process whose probability distribution and mean are common knowledge. Within this type of game, the sub-classes of Poisson games (where the number of players is a random variable following a Poisson distribution of mean n) and extended Poisson games (where the size of the population and the utility functions of the players may depend on an unknown state of the world) have received the most attention. Using this kind of model, Ritzberger [2009] was able to demonstrate that in a Bertrand game with an uncertain population, the presence of two competitors is not sufficient to eliminate profits. And, more recently, De Sinopoli et al. [2023] have proposed a variant of Cournot's oligopoly in which firms are uncertain as to the total number of firms in the industry.

Despite the identical nature of the research question we share with these models, namely uncertainty about the number of players, our modeling differs from them in one respect. Our framework is dynamic, in the sense that the number of players can vary, and no one knows ex ante what the actual number will be at any given time. The only information that is common knowledges in our modelis: the size of the market, the number of incumbent firms at the present time, the types of potential entrants and exits —and the maximum size of these types—, and the probability laws governing entries and exits. Our model is technically in line with the work on piecewise deterministic systems (Davis [1985] and Haurie et al. [1994]) and with the literature on discrete-time dynamic game theory developed by Kordonis and Papavassilopoulos [2015], Bernhard and Hamelin [2016], Bernhard and Deschamps [2017] [2021], and Biard and Deschamps [2021]. Our contribution with regard to the latter is firstly to complement some of their results, to take into account the arrival(s) and exit(s) of one (or more) competitor(s) with a different type(s) to that of the installed firm(s), but above all to propose a dynamic equilibrium and not a sequence of static equilibria.

The rest of our paper is organized in four sections. Section 2 presents the general framework and proves a theorem ensuring the existence of a closed-loop Nash equilibrium with sufficient conditions. As we are unable to prove other results within this general framework, Section 3 makes it more specific by assuming that the different types of possible competitors are made up of players who are identical to each other. We then propose the complete resolution of the case of a dynamic Cournot oligopoly with sticky prices, in discrete time, and both finite and infinite horizons. Section 4 provides numerical illustrations of the model from the previous section, based on Scilab programs, demonstrating both its interest and its operational character. Our final section, Section 5, presents our conclusions, highlights the limitations of our model, and outlines some possible future improvements.

2 The framework

Our model is in discrete time. The time steps are numbered $t \in \mathbb{N}$. Current time is always common knowledge, as are the number and types of players present on the market and those that may enter or leave according to the following description.

2.1 Players

2.1.1 Arrivals and departures

There are $\nu \in \mathbb{N}$ different types of players numbered from one to ν, usually denoted with indices i or j. We denote $\mathsf{T} = \{1, 2 \ldots, \nu\}$ the set of types. This information is common knowledge.

At each instant of time starting with time 1, players may enter the game. Let $a(t) \in \mathcal{A}(t) \subset \mathbb{N}^\nu$ be the vector of arriving players at time t, where $a_i(t)$ is the number of players of type i. Note that there are therefore $\sum_i a_i(t)$ players arriving at time t. Although there may be a large number of players involved, in the general theory the game is not anonymous. (A simplification will occur later on.) Players are numbered by an index or "rank" n, assigned to each arriving player, starting with the first available rank, e.g. in lexicographic order of type numbers. Its rank is used as the name of each player. The maximum rank reached at time t is $N(t) = N(t-1) + \sum_i a_i(t)$. In some cases, we need to bound the maximum number of players potentially involved. Let $\mathcal{N} \subset \mathbb{N}$ be the set of possible ranks, bounded by some $N_{\max}$ or unbounded depending of the precise description on the game.

We need to keep track of the sequence of past arrivals and departures. We denote by t_n the arrival time of player n. The sequence $\{t_n\}_n$ is non-decreasing. Moreover, each arriving player is characterized by its type. We denote $\tau_n \in \mathsf{T}$ the (fixed) type of player n. And we denote $\tau^N(t) = (\tau_1, \tau_2, \dots, \tau_{N(t)})$ the ordered list of length $N(t)$ of the players' types. At time t, $\tau^N(t-1)$ is concatenated with the list of types of the arriving players.

At each instant t, some of the players in the game may leave. The list of the ranks of leaving players is $\ell(t) \subset \{1, 2, \dots, N(t-1)\}$. The list τ^N is then modified by setting the types of these players at zero.

As a consequence of this setup, it is assumed that once a player has left the game, it does not re-enter, or rather if it does, it is with a new "name": its rank, n. The updating of the list τ^N after arrivals and departures as outlined above is denoted $\tau^{N(t)}(t) = \mathcal{T}(\tau^{N(t-1)}(t-1), a(t), \ell(t))$.

2.1.2 The Markov process

Depending on the specific rules of the game, the sets $\mathcal{A}(t)$ of possible arrivals and $\mathcal{L}(t)$ of possible departures may depend on $N(t)$ and $\tau^{N(t)}$ (as $\mathcal{L}(t)$ always do).

A probability law rules the probability $\mathbb{P}(a, \ell; t)$ of each pair $(a, \ell) \in \mathcal{A}(t) \times \mathcal{L}(t)$. These probabilities are to be derived from a more refined description of the possible events, including a stochastic description of how arrivals and departures happen. These rules may only depend on the current list $\tau^{N(t)}$ of players present. They are common knowledge.

The sequence of the τ^N is a Markov process. However, this process has an infinite (if $\mathcal{N}$ is infinite) or exceedingly large set of states ($\nu^{N_{\max}}$ if $\mathcal{N}$ is bounded). We will investigate later a special case where this may be dramatically reduced.

2.2 The dynamic game

2.2.1 Dynamics and payoff

There is a state set $\mathcal{X}$. To distinguish it from the Markov state, we will call it the set of *action states*. The action state of the game at time t is $x(t) \in \mathcal{X}$. There is a disturbance set $\mathcal{W}$, the disturbances being a sequence of i.i.d. random variables $w(t)$ with a known probability law. There are ν strategy sets S_i. Player's n strategy choice at time t is $s_n(t) \in \mathsf{S}_{\tau_n}$. We will use the notation $s^N = (s_1, \dots, s_N) \in \mathsf{S}^N$, $s^{N \setminus k}$ the list s^N deprived of s_k, and $s^N = (s^{N \setminus k}, s_k)$. Note that $\mathsf{S}^N = \mathsf{S}_{\tau_1} \times \cdots \times \mathsf{S}_{\tau_N}$ is a function of τ^N. All this information is common knowledge.

The Markov state, the disturbances and the actions of the players govern the evolution of the action state in the following way: there are functions $f^N : \mathcal{X} \times \mathsf{T}^N \times \mathsf{S}^N \times \mathcal{W} \to \mathcal{X}$, independent of s_k for each k such that $\tau_k = 0$. The action state evolves according to

$$x(t+1) = f^N(x(t), \tau^N(t), s^N(t), w(t)) \,, \quad x(0) = x_0 \,, \qquad \text{where } N = N(t)\,.$$

The payoff is given by a set of bounded functions $L_n^N : \mathcal{X} \times \mathsf{T}^N \times \mathsf{S}^N \times \mathcal{W} \to \mathbb{R}$, independent of $s_k \in s^N$, and most likely equal to zero, if $\tau_k = 0$. Each player gets a payoff discounted according to a discount factor $\rho \in (0,1)$:

$$\Pi_n = \mathbb{E} \sum_{t=t_n}^{\infty} \rho^{t-1} L_n^N(x(t), \tau^N(t), s^N(t), w(t)) \,, \qquad \text{where } N = N(t)\,.$$

(This formalism allows one to define a different payoff function for each player. In an application, the L_n might only depend on the type τ_n of each player. See our example below.)

Mixed strategies We will allow mixed strategies σ_n belonging to the set $\mathcal{S}_n$ of probability distributions over S_n. In that case, we will write with a transparent abuse of notation, and for each $w \in \mathcal{W}$

$$L_n^N(x, \tau^N, \sigma^N, w) = \int_{\mathsf{S}^N} L_n^N(x, \tau^N, s^N, w)\, \mathrm{d}\sigma^N(s^N)\,.$$

2.2.2 Markov Perfect Equilibrium strategies

We allow closed-loop, state feedback —or Markov— strategies, and as the equilibrium strategies are to be determined by dynamic programming, which is Kuhn's backward induction, the equilibrium thus computed are perfect.

Closed-loop strategies (CL-strategies) are thus defined as N-tuples of the form

$$s_n(t) = \varphi_n^N(x(t), \tau^N(t)) \in \mathsf{S}_n \,, \qquad \text{or} \qquad \sigma_n(t) = \varphi_n^N(x(t), \tau^N(t)) \in \mathcal{S}_n\,.$$

Depending on the detailed problem considered, there may be restrictions on the set Σ_n of admissible feedbacks. A complete set $\Phi_n = \{\varphi_n^M(\cdot,\cdot)\}_{M\in\mathcal{N}}$ is called a CL-strategy of player n. A complete set $\Phi^{\mathcal{N}} = \{\Phi_n\}_{n\in\mathcal{N}}$ of CL-strategies defines a strategy set leading to a set of individual profits $\{\Pi_n(\Phi^{\mathcal{N}})\}_{n\in\mathcal{N}}$. We denote $\Phi^{\mathcal{N}\setminus n} = \{\Phi_k\}_{k\in\mathcal{N}\setminus n}$ and $\Phi^{\mathcal{N}} = (\Phi^{\mathcal{N}\setminus n}, \Phi_n)$. The set of admissible strategies of player n is denoted Ψ_n.

Definition 1 *A closed-loop Nash equilibrium is a complete family $\widehat{\Phi}^{\mathcal{N}}$ of CL-strategies such that*

$$\forall n \in \mathcal{N}\,, \quad \forall \Phi_n \in \Psi_n\,, \quad \Pi_n(\widehat{\Phi}^{\mathcal{N}}) \ge \Pi_n(\widehat{\Phi}^{\mathcal{N}\setminus n}, \Phi_n)\,.$$

Theorem 1 *If there exists a family of bounded functions $V_n : \mathcal{X} \times \mathsf{T}^{\mathcal{N}} \to \mathbb{R}$ and an admissible strategy $\widehat{\Phi}^{\mathcal{N}} = \{\hat{\varphi}_n^N\}_{n,N}$ such that*

$$\begin{aligned}
&\forall n \in \mathcal{N}\,,\ \forall x \in \mathcal{X}\,, \forall N \in \mathcal{N}, \forall \tau^N \in \mathsf{T}^N, \forall s_n \in \mathsf{S}_{\tau_n}\,,\\
&V_n(x, \tau^N) =\\
&= \mathbb{E} L_n^N(x, \tau^N, \hat{\varphi}_n^N(x, \tau^N), w) + \rho \mathbb{E} V_n(f_n^N(x, \tau^N, \hat{\varphi}_n^N(x, \tau^N), w), \mathcal{T}(\tau^N, a, \ell))\\
&\ge \mathbb{E} L_n^N(x, \tau^N, (\hat{\varphi}_n^{N\setminus n}, s_n), w) + \rho \mathbb{E} V_n(f_n^N(x, \tau^N, (\hat{\varphi}_n^{N\setminus n}, s_n), w), \mathcal{T}(\tau^N, a, \ell))\,,
\end{aligned}$$

where the expectations are over the disturbances w and the entries and exits (a, ℓ) ruled by the probabilities $\mathbb{P}(a, \ell; t)$, then the strategy $\widehat{\Phi}^{\mathcal{N}}$ is a CL-Nash equilibrium, where the payoff of each player n is $V_n(x_0, \emptyset)$.

Proof By standard dynamic programming.

Remark To save on notation, we have stated the game as an infinite horizon one. It is a simple matter to derive the analogous theorem for a finite horizon game, where the functions f_n^N, L_n^N, the strategies $\hat{\varphi}_n^N$, and the Value functions V_n are time-dependent. The fixed-point stationary Isaacs equation above becomes an explicit backward recursion to be initiated at $V_n(T, X, \tau^N) = 0$. Boundedness is no longer required. Notice also that this same algorithm may be viewed as a Picard algorithm to try and solve the theorem's fixed-point equation.

By its excessive level of generality, the above model borders on the useless. The relevent question is: how to particularize it to create a useful model?

2.3 Types as clones

We particularize the general set-up above to a situation leading to a problem with a much smaller Markov state space.

We assume that the players of a given type are all identical, having the same payment function L, and share the same entry and departure probabilities. Moreover, if m out of a maximum of N players of a given type are present on the market, each one of them has a probability m/N of being among the ones that are present. As a consequence of being perfect clones, they share the same feedback strategy.

To make this precise, we introduce the vector $m(t) \in \mathbb{N}^\nu$ of players present at time t, where m_i is the number of players of type i. We write

$$m(t) = m(t-1) + a(t) - b(t)$$

where the vector of arrivals a has been defined above, and $b(t) = \beta(\ell(t))$ is the vector of the numbers of leaving players by type, both random variables. The detailed rules of the game define the probability laws ruling the random vectors $a(t)$ and $b(t)$, depending only on $m(t)$.

We assume that $f^N(x, s^N, \tau^N)$ depends in fact on m:

$$x(t+1) = f(x(t), m(t), s(t), w(t)) \,,$$

and likewise

$$\Pi_i = \mathbb{E} \sum_{t=1}^{\infty} \rho^{t-1} \frac{m_i(t)}{N_i} L_i(x(t), m(t), s(t), w(t))$$

is the payoff of any player of type number i.

It may be noticed that another interpretation of the same criterion is in terms of teams, each type behaving as a team that *in fine* shares its earnings equally among its members.

In that case, $m(t)$ characterizes the Nerode equivalence class of the arrival sequence $\tau^{N(t)}$ in the automaton transforming this sequence into the sequence of outputs f and $L(t)$. Thus, the pair (x, m) is now the complete state of the game, instead of the much larger (x, τ^N).

Assuming, for instance, that each m_i is bounded by a number N_i, the state space of the Markov process m is finite with $N = \prod_{i=1}^{\nu}(N_i + 1)$ elements. This allows one to number these states by a number $k(m)$. Then, the detailed probabilities of entry and exit are sufficiently characterized by the $N \times N$ *transition matrix* M where M_ℓ^k stands for the probability of reaching state number ℓ at the next step if current state is number k.

Theorem 1 may easily be re-written in terms of this new state. We dispense with this easy re-writing and directly turn to an example.

3 Example: Dynamic Cournot oligopoly with sticky prices

3.1 The problem

There are ν types of producers of an identical good sold on a market. In each time period t, producers of type i all produce a quantity $q_i(t)$. These producers interact in the fashion descibed above as clones within each type.

It must be underlined that upper indices, typically k, ℓ, in the sequel are *not* powers, but indices specifying the Markov state. Accoringly, m_i^k denotes the number m_i of players of type i present on the market in state number k. Upper indices 2 denote squares, often written e.g. as $(\Delta^k)^2$.

We need to introduce an index ε_i^k defined as

$$\varepsilon_i^k = \begin{cases} 0 & \text{if } m_i^k = 0\,, \\ 1 & \text{if } m_i^k > 0\,. \end{cases}$$

The market considered is ruled by an affine inverse demand function

$$P(q) = a_0 - \sum_{i=1}^{\nu}(b_i m_i q_i)\,.$$

But prices are "sticky", so that there exists a positive number θ measuring the stickyness and the production $q(t)$ is sold at an average price $p(t) = \theta P(q(t-1)) + (1-\theta)P(q(t))$. Technical as well as economic considerations lead us to assume that $\theta \leq 1/2$, and consequently the alternative mesure of stickyness $\delta = \theta/(1-\theta) \in [0, 1]$.

Each type has a linear production cost $c_i q_i$ per producer. It is convenient to introduce the *price state* x:

$$x(t) = P(q(t-1)) = a_0 - \sum_{i=1}^{\nu} m_i b_i q_i(t-1)\,.$$

A discount factor $\rho < 1$ is given. We consider a time horizon T which may be finite or infinite. The expected profit of players of type i is:

$$\Pi_i = \mathbb{E}\sum_{t=1}^{T} \rho^{t-1}\frac{m_i(t)}{N_i}\left[\theta x(t) + (1-\theta)\left(a_0 - \sum_{i=1}^{\nu} m_i b_i q_i\right) - c_i\right] q_i(t)\,.$$

Finally, we simplify the notation by posing

$$\frac{\theta}{1-\theta} = \delta\,, \quad a_i = a_0 - \frac{c_i}{1-\theta}\,, \quad b_i m_i q_i = r_i\,, \quad \sum_{i=1}^{\nu} r_i = R\,.$$

And we will consider the equivalent modified profit

$$\frac{N_i b_i}{1-\theta}\Pi_i = \widetilde{\Pi}_i\,.$$

As a result, we get

$$x(t+1) = a_0 - R(t)\,,$$
$$\widetilde{\Pi}_i = \mathbb{E}\sum_{i=1}^{T}\rho^{t-1}[\delta x(t) + a_i - R(t)]r_i(t)\,.$$

3.2 Finite horizon

We seek a dynamic Cournot-Nash equilibrium. We let $\rho^t V_i^k(t,x)$ be the equilibrium expected profit-to-go of player i in terms of the modified profit $\widetilde{\Pi}_i$ from time t onwards if current states are k and x. We ignore the natural constraints $q_i \geq 0$ and $P(q) \geq 0$. We leave them to be checked on any numerical application.

We will show that we can find Value functions as follows (here, x^2 is a square):

$$V_i^k(t,x) = \frac{1}{2}F_i^k(t)x^2 + G_i^k(t)x + H_i^k(t)\,. \tag{1}$$

Value functions V_i^k are given by Isaacs' equation:

$$V_i^k(t,x) = \max_{r_i}\Bigg\{(\delta x + a_i - R)r_i$$
$$+\rho\mathbb{E}\left[\frac{1}{2}F_i^{k(t+1)}(t+1)(a_0-R)^2 + G_i^{k(t+1)}(t+1)(a_0-R) + H_i^{k(t+1)}(t+1)\right]\Bigg\}\,,$$

Using the theorem of embedded conditional expectations, we may condition the expectation above on the current value of k. Let

$$\bar{F}_i^k(t) := \mathbb{E}[F_i^{k(t+1)} \mid k(t) = k] = \sum_{\ell=1}^{N} M_\ell^k F_i^\ell(t) = MF_i^k(t)\,,$$

and likewise

$$\bar{G}_i^k(t) = \mathbb{E}[G_i^\ell(t) \mid k] = MG_i^k(t)\,, \quad \bar{H}_i^k(t) = \mathbb{E}[H_i^\ell(t) \mid k] = MH_i^k(t)\,.$$

Isaacs'equation becomes

$$V_i^k(t,x) = \max_{r_i}\Bigg\{(\delta x + a_i - R)r_i \\ + \rho\left[\frac{1}{2}\bar{F}_i^\ell(t{+}1)(a_0 - R)^2 + \bar{G}_i^\ell(t{+}1)(a_0 - R) + \bar{H}_i^\ell(t{+}1)\right]\Bigg\}\,.$$

Let $r_i^k(t)$ stand for the maximizing r_i above and $R^k(t)$ accordingly. The equilibrium productions are undefined if $m_i^k = 0$, and if $m_i^k > 0$, given by $q_i^k(t) = r_i^k(t)/(m_i^k(t)b_i)$.

Let

$$\gamma_i^k(t) = a_i - \rho\bar{G}_i^k(t)\,, \quad \text{and} \quad \Gamma^k(t) = \sum_{i=1}^{\nu} \varepsilon_i^k \gamma_i^k(t)\,.$$

Performing the maximization, we easily find that

$$r_i^k = \varepsilon_i^k\Big(\delta x - R^k\left[1 - \rho\bar{F}_i^k(t{+}1)\right] + \gamma_i^k(t{+}1) - \rho\bar{F}_i^k(t{+}1)a_0\Big)\,. \tag{2}$$

Summing over the i, and defining

$$\Phi^k(t) = \sum_{i=1}^{\nu} \varepsilon_i^k \bar{F}_i^k(t) \quad \text{and} \quad \Delta^k(t) = 1 + \nu^k - \rho\Phi^k(t)\,,$$

we obtain

$$R^k = \frac{1}{\Delta^k(t{+}1)}\left[\nu^k\delta x + \Gamma^k(t{+}1) - \rho\Phi^k(t{+}1)a_0\right]\,. \tag{3}$$

Placing these in Isaacs' equation and identifying coefficients of like powers of x, we obtain

$$F_i^k(t) = \frac{\delta^2}{(\Delta^k(t{+}1))^2}\Big[2\varepsilon_i^k\left[\rho\nu^k\bar{F}_i^k(t{+}1) + 1 - \rho\Phi^k(t{+}1)\right]\left[1 - \rho\Phi^k(t{+}1)\right] \\ + \rho(\nu^k)^2\bar{F}_i^k(t{+}1)\Big]\,,$$

$$
\begin{aligned}
G_i^k(t) = &- \frac{\delta}{\Delta^k(t+1)}\Big\{\rho\bar G_i^k(t+1)[\varepsilon_i^k(1-\rho\Phi^k(t+1))+\nu^k] \\
&\qquad + \varepsilon_i^k\left[2(1-\rho\Phi^k(t+1))+\nu^k\rho\bar F_i^k(t+1)\right]a_i\Big\} \\
&+ \frac{\delta}{(\Delta^k(t+1))^2}\Big\{\Big[\varepsilon_i^k\rho\Phi^k(t+1)\left[2(1-\rho\Phi^k(t+1))+(1+2\nu^k)\rho\bar F_i^k(t+1)\right] \\
&\qquad - \rho\bar F_i^k(t+1)(\varepsilon_i^k+\nu^k)(1+\nu^k)\Big]a_0 \\
&+ \left[\varepsilon_i^k(1-\rho\Phi^k(t+1))(\rho\bar F_i^k(t+1)-2)+\nu^k\rho\bar F_i^k(t+1)(1-\varepsilon_i^k)\right]\Gamma^k(t+1)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
H_i^k(t) = &\ \varepsilon_i^k\Big[\Delta^k(t+1)a_i-\Gamma^k(t+1)+\rho\Phi^k(t+1)a_0\Big]\times \\
&\Big[\Delta^k(t+1)\big[\gamma_i^k(t+1)-\rho\bar F_i^k(t+1)a_0\big]-\big[\Gamma^k(t+1)-\rho\Phi^k(t+1)a_0\big]\big[1-\rho\bar F_i^k(t+1)\big]\Big] \\
&+\frac{\rho}{2}\bar F_i^k(t+1)\Big[\Delta^k(t+1)a_0-\Gamma^k(t+1)+\rho\Phi^k(t+1)a_0\Big]^2 \\
&+\Delta^k(t+1)\rho\bar G_i^k(t+1)\Big[\Delta^k(t+1)a_0-\Gamma^k(t+1)+\rho\Phi^k(t+1)a_0\Big] \\
&+\rho\bar H_i^k(t+1).
\end{aligned}
$$

and finally

$$
\bar F(t) = MF(t), \quad \bar G(t) = MG(t), \quad \bar H(t) = MH(t).
$$

These equations are to be initialized at $F(T)=0$, $G(T)=0$, $H(T)=0$. They are explicit.

3.3 Infinite horizon

We turn now to the case $T=\infty$, for which we look for a stationary solution.

3.3.1 Equilibrium productions

We seek a Value function of the form

$$
V_i^k(x) = \frac{1}{2}F_i^k x^2 + G_i^k x + H_i^k. \tag{4}
$$

The stationary Isaacs-Bellman equation reads, if $\varepsilon_i^k = 1$:

$$
V_i^k(x) = \max_{r_i}\left\{(\delta x + a_i - R)r_i + \rho\mathbb{E}\left[\frac{1}{2}F_i^\ell(a_0-R)^2 + G_i^\ell(a_0-R)+H_i^\ell\right]\right\},
$$

where ℓ stands for $k(m(t+1))$. The notation $\bar F_i^k$, $\bar G_i^k$ and $\bar H_i^k$ carry over from the finite horizon case, but are now constant. The same applies to the notation γ_i^k, Γ^k, Φ^k, and Δ^k:

$$
\Phi^k = \sum_{i=1}^{\nu}\varepsilon_i^k\bar F_i^k, \quad \gamma_i^k = a_i - \rho\bar G_i^k, \quad \Gamma^k = \sum_{i=1}^{\nu}\varepsilon_i^k\gamma_i^k, \quad \Delta^k = 1+\nu^k-\rho\Phi^k.
$$

Isaacs' equation becomes, as previously,

$$V_i^k(x) = \max_{r_i} \left\{ (\delta x + a_i - R) r_i + \rho \left[\frac{1}{2} \bar{F}_i^k (a_0 - R)^2 + \bar{G}_i^k (a_0 - R) + \bar{H}_i^k \right] \right\} , \tag{5}$$

and if it has a solution, the function (4) being bounded, since $x \in [0, a_0]$, it is indeed a valid Value function and the corresponding strategies r_i^k are equilibrium strategies.

We perform the maximization in r_i. We recover the previous formula (2) for r_i^k, and summing over i, formula (3) for $R^k = \sum_{i=1}^{\nu} r_i^k$, but now both time-independent. As previously, if $m_i^k = 0$, then $r_i^k = 0$ also, while q_i^k is not defined. If $m_i^k > 0$, then $q_i^k = r_i^k / (b_i m_i^k)$.

3.3.2 Determination of F_i and $\bar{F}_i$

Placing back formula (2) in (5) and identifying terms in x^2, we get, before any expansion:

$$\begin{aligned} V_i^k(x) = & \tfrac{1}{(\Delta)^2} \Big\{ \varepsilon_i^k \left[(1 - \rho\Phi^k)\delta x + \Delta^k a_i - \Gamma^k + \rho\Phi^k a_0 \right] \\ & \times \left[(1 - \rho\Phi^k + \nu^k \rho \bar{F}_i^k)\delta x + \Delta^k(\gamma_i^k - \rho F_i^k a_0) - (\Gamma^k - \rho\Phi^k a_0)(1 - \rho\bar{F}_i^k) \right] \\ & + \tfrac{1}{2}\rho \bar{F}_i^k (-\nu^k \delta x + \Delta^k a_0 - \Gamma^k + \rho\Phi^k a_0)^2 \\ & + \Delta^k \rho \bar{G}_i^k (-\nu^k \delta x + \Delta^k a_0 - \Gamma^k + \rho\Phi^k a_0) \Big\} + \rho \bar{H}_i^k . \end{aligned} \tag{6}$$

Identifying the coefficients in x^2, we find

$$\frac{1}{2} F_i^k = \frac{\delta^2}{(\Delta^k)^2} \left[\varepsilon_i^k (\rho \nu^k \bar{F}_i^k + 1 - \rho\Phi^k)(1 - \rho\Phi^k) + \frac{1}{2}\rho(\nu^k)^2 \bar{F}_i^k \right] .$$

This allows one to write the $N \times \nu$ matrix F in terms of $\bar{F} = MF$.

We obtain a complicated fixed-point equation $F = \mathcal{F}(F)$ concerning the matrix F. Its form shows that for δ small enough, the right hand side is a contraction, therefore in that case there is a unique solution to this equation, which can be approached via a Picard iteration. Only numerical experiments can tell what happens for δ close to one and how quickly the algorithm converges when it does. Our experiments exhibit rapid convergence even for $\delta = 1$.

3.3.3 Determination of G_i and $\bar{G}_i$

We remark that the Picard iterations for F are exactly the non-stationary equation carried over until convergence. One might do likewise for G and H. However, the convergence for G is much slower than for F, and that for H again much slower than for G. Indeed, if H is to be apprached to, say 10^{-4}, then we must certainly go up to T such that $\rho^T/(1-\rho) = 10^{-5}$. If $\rho = .95$, this means 283 iterations, and if $\rho = .99$, 1604 iterations, if the algorithm converges at all. (We observed a slower convergence.) We therefore have rather to find explicit formulas.

Identifying terms in x in (1) and (6), we obtain

$$G_i^k = -\frac{\delta}{\Delta^k}\Big\{\rho\bar{G}_i^k[\varepsilon_i^k(1-\rho\Phi^k)+\nu^k] + \varepsilon_i^k\left[2(1-\rho\Phi^k)+\nu^k\rho\bar{F}_i^k\right]a_i\Big\}$$
$$+\frac{\delta}{(\Delta^k)^2}\Big\{\Big[\varepsilon_i^k\rho\Phi^k\left[2(1-\rho\Phi^k)+(1+2\nu^k)\rho\bar{F}_i^k\right] - \rho\bar{F}_i^k(\varepsilon_i^k+\nu^k)(1+\nu^k)\Big]a_0$$
$$+\left[\varepsilon_i^k(1-\rho\Phi^k)(-2+\rho\bar{F}_i^k)+\nu^k\rho\bar{F}_i^k(1-\varepsilon_i^k)\right]\Gamma^k\Big\}$$

For ease of manipulation, we write this as

$$G_i^k = -\delta\rho\zeta_i^k\bar{G}_i^k + \delta\varphi_i^k a_i + \delta\chi_i^k a_0 + \delta\psi_i^k\Gamma^k = -\delta\rho\zeta_i^k\bar{G}_i^k + \delta\omega_i^k + \delta\psi_i^k\Gamma^k\,.$$

where the coefficients ζ_i^k, φ_i^k, χ_i^k, and ψ_i^k, can be read directly from lines 1, 1, 2, and 3 respectively of the three-line equation, and $\omega_i^k = \varphi_i^k a_i + \chi_i^k a_0$ from the above equation.

Use the $N\times N$ diagonal matrices

$$\mathcal{E}_i = \mathrm{diag}_k\{\varepsilon_i^k\}\,,\quad Z_i = \mathrm{diag}_k\{\zeta_i^k\}\quad\text{and}\quad \Psi_i = \mathrm{diag}_k\{\psi_i^k\}$$

and the N-vectors $\bar{G}_i$ and ω_i to write the N equations above as

$$G_i = -\delta\rho Z_i\bar{G}_i + \delta\omega_i + \delta\Psi_i\Gamma = -\delta\rho Z_i M G_i + \delta(\omega_i + \Psi_i\Gamma)\,.$$

Let also the $N\times N$ matrices K_i and L_i be defined as

$$K_i = (I+\delta\rho Z_i M)^{-1}\,,\quad\text{and}\quad L_i = MK_i\,,$$

to get

$$G_i = \delta K_i(\omega_i+\Psi_i\Gamma)\,,\quad \bar{G}_i = \delta L_i(\omega_i+\Psi_i\Gamma)\,,\quad \mathcal{E}_i\bar{G}_i = \delta\mathcal{E}_i L_i(\omega_i+\Psi_i\Gamma)\,.$$

We may now use this equation to find an explicit expression of Γ. The vector $\mathcal{E}_i\bar{G}_i$ has its entry k as $\varepsilon_i^k\bar{G}_i^k$. Remembering that

$$\Gamma^k = \sum_{i=1}^{\nu}\varepsilon_i^k(a_i - \rho\bar{G}_i^k)\,,$$

and using the $N\times\nu$ matrix ε and the ν-vector a of the a_i, we obtain

$$\Gamma = \varepsilon a - \delta\rho\sum_{i=1}^{\nu}\left(\mathcal{E}_i L_i\omega_i + \mathcal{E}_i L_i\Psi_i\Gamma\right)\,.$$

Define finally

$$\Lambda = \sum_{i=1}^{\nu}\mathcal{E}_i L_i\Psi_i$$

to get

$$\Gamma = \varepsilon a - \delta\rho\sum_{i=1}^{\nu}\mathcal{E}_i L_i\omega_i - \delta\rho\Lambda\Gamma\,,$$

hence

$$\Gamma = (I+\delta\rho\Lambda)^{-1}\left[\varepsilon a - \delta\rho\sum_{i=1}^{\nu}\mathcal{E}_i L_i\omega_i\right]\,.$$

3.3.4 Determination of H_i^k and $\bar{H}_i^k$

It remains to identify terms without x in (6). It yields

$$\begin{aligned} H_i^k &= \varepsilon_i^k[\Delta^k a_i - \Gamma^k + \rho\Phi^k a_0][\Delta^k(\gamma_i^k - \rho\bar{F}_i^k a_0) - (\Gamma^k - \rho\Phi^k a_0)(1 - \rho\bar{F}_i^k)] \\ &\quad + \frac{\rho}{2}\bar{F}_i^k(\Delta^k a_0 - \Gamma^k + \rho\Phi^k a_0)^2 + \Delta^k\rho\bar{G}_i^k(\Delta^k a_0 - \Gamma^k + \rho\Phi^k a_0) + \rho\bar{H}_i^k. \end{aligned}$$

There does not seem to be much to gain in expanding and regrouping terms. At this stage all terms in the right hand side are known. Let us write it as

$$H_i^k = h_i^k + \rho\bar{H}_i^k \quad \text{or in matrix form} \quad H = h + \rho\bar{H} = h + \rho MH\,,$$

and thus

$$H = (I - \rho M)^{-1}h, \quad \text{and} \quad \bar{H} = MH\,.$$

The discount coefficient ρ being strictly smaller than one, the matrix $I - \rho M$ is indeed invertible.

3.3.5 A significant simplification

If the entry-and-exit process is such that there is always at least one producer of each type in the market, then all the complexity linked to the case $m_i = 0$ disappears. Several simplifications occur. First of all, the coefficients F_i^k and $\bar{F}_i^k$ are independent of i. Therefore, not only is the fixed-point equation simpler, but more importantly, the unknown is now the unique N-vector $\bar{F}$ instead of the matrix of the $\bar{F}_i^k$. Thus N unknowns instead of νN. Further, it also turns out that in that case, the coefficients ζ_i^k are all equal to one, simplifying the calculation of $\bar{G}_i$, and consequently of Γ.

4 Some numerical illustrations

In order to check the feasibility, and to gain some intuition about the qualitative consequences of the theory, we performed some numerical computations.[2]

The number of free parameters needed to define an experiment is daunting. For example, for the case below with three types of four producers each, the Markov transition matrix has 15,500 free parameters... We made a series of hypotheses to simplify this. But there are also such parameters as θ, ρ, a_0 and the b_i and c_i.

In the following cases, we assumed the following:

- Prices:
  - $\theta = 1/2$. This is the least favorable case for the fixed-point algorithm and other matrix inversions.
  - $a_0 = 10$.

[2]The Scilab programs are provided as an online complement.

- $b_i = 1$ for all types.

- Discounting: $\rho = .95$.
- Time horizon: the same experiments were performed with a time horizon of 12 and for the infinite horizon case.
- Types
  - Number: 3 (a fourth one for the "institution", see below).
  - Number N_i of producers per type: 3 or 4.
  - Characteristics: A "medium" type serves as a reference. One type is less efficient (called "weak") and one more efficient (called "strong"). This translates into the coefficients c_i which are chosen as 3, 2, and 1 respectively.
  - In some cases, one medium producer, called the "institution", is privileged in that it never leaves the market. (We make it a fourth type, with $m_4^k = 1$ for all k.)
- Entering and exiting: at each time step, the probabilities of a given number of producers entering or leaving are independent of the current state m, and only depend on the type. Specifically, to let the efficiency be the only differentiating factor, we have chosen for the three types probabilities of one, two, three or four producers entering equal to .4, .16, .064, 0 respectively, and the corresponding probabilities of leaving once there as .2, .04, .008, 0 respectively.
- Initial conditions:
  - Markov state $(0, 1, 0)$. If there is an institution, it is the initial producer present. (i.e. $m = (0, 0, 0, 1)$.)
  - Price state: the monopoly price of a median player = 6.

The initial condition with one "medium" producer alone present on the market results in the medium producers faring better than the strong ones in some of the experiments.

The numerical results are reported in the following table.

T	N_i	weak	median	strong	Institution
12	3	9.863	23.27	16.25	None
12	3	5.282	10.31	10.45	57.37
12	4	7.219	17.36	12.07	None
12	4	3.751	7.206	7.630	55.81
∞	3	21.88	39.46	37.27	None
∞	3	11.79	23.66	24.24	87.82
∞	4	15.79	29.13	25.56	None
∞	4	8.274	16.25	17.66	84.35

As a comparison, the monopoly infinite horizon payoff of the median player is 320 without viscosity, and 327.4 with $\theta = .5$.

5 Conclusion

We think that our paper provides some answers to the question of how to evaluate the expected profit for a player in an oligopoly model with random arrival(s) and exit(s) of players with the same type (or not). In our view, it offers a flexible model that allows the use of various probability laws and numerical simulations to estimate the sensitivity of results to parameters.

However, our current modelling also has a number of limitations. We are aware of at least four of them. The first stems from the fact that, in our framework, market size (i.e. demand) is constant and exogenous, which means in particular that we do not take into account changes in the latter that might be linked to firms' strategies, as with advertising, for example. The second limitation relates to the fact that we do not model the entry decision. This is exogenous, so it is not possible for incumbent firms to implement strategies to deter or delay entry. The third limitation stems from the implicit assumption that the type of each agent is constant throughout the game. The fourth limit relates to our assumption of common knowledge of stochastic processes concerning market entry and exit. In theory, there's nothing to prevent these stochastic processes from depending on the number of competitors present on the market, but in practice the size of the transition matrix (e.g. 15,500 probabilities in the case of three types of four players) means that we have to make simplifications to construct this matrix. We have therefore chosen, in our numerical applications, to make these probabilities independent of the number of producers present.

Given our current state of knowledge, these limitations seem very difficult to overcome. However, as far as the constant nature of demand is concerned, we can construt a model that takes account of random exogenous demand.

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A Notations of Section 3

Data

- a_0: constant coefficient in inverse demand function.
- b_i: weight of the productoin of type i in the inverse demand function.
- c_i: unit production cost of team number i.
- $k(m)$: numbering of the Markov states. k a particular state number.
- m: a state of the Markov chain. A ν-vector.
- m_i: number of actors of team i present on the market.
- m_i^k: m_i when in Markov state number k.
- M: transition matrix of the Markov chain.
- N: number of distinct Markov states.
- $P(q) = a_0 - \sum_{i=1}^{\nu} b_i m_i q_i$: Inverse demand function.
- q_i: production of any producer of type i.
- $q_i^k(x)$: equlibrium production of producers of type i when in Markov state k, and price state x. One of the quantities sought.
- ν: number of types.
- Π_i: expected intertemporal profit of type i.
- θ: viscosity coefficient.
- ρ: time-discount coefficient.

Other notation

- $a_i = a_0 - c_i/(1-\theta)$.
- $a^k = \sum_{i=1}^{\nu} \varepsilon_i^k a_i$, a the N-vector of the a^k.
- F_i^k, G_i^k, H_i^k: $V_i^k(x) = F_i^k x^2 + G_i^k x + H_i^k$.
- F_i, G_i, $H_i \in \mathbb{R}^N$ (column) N-vectors of coordinates upper-indexed by k.
- $\bar{F}_i^k$, $\bar{G}_i^k$, $\bar{H}_i^k \in \mathbb{R}$ coordinates of $\bar{F}_i = MF_i$, $\bar{G}_i = MG_i$, $\bar{H}_i = MH_i$ respectively.
- h_i^k the r.h.s. in the calculation of H_i^k, and h_i the N-vector of the h_i^k.
- $K_i = (I + \delta\rho Z_i M)^{-1}$.

- $L_i = MK_i$.
- ℓ_i^k Line vector, line number k of L_i.
- $r_i = b_i m_i q_i$, used as the control of producers of type i.
- r_i^k: equilibrium control of producers of type i in Markov state k. (Function of x.)
- $R = \sum_{i=1}^{\nu} r_i. \qquad R^k = \sum_{i=1}^{\nu} r_i^k.$
- $V_i^k(x)$ Isaacs Value function of producers of type i for the performance index $\widetilde{\Pi}_i$.
- $Z_i = \mathrm{diag}_k\{\zeta_i^k\}$ a $N \times N$ diagonal matrix,
- $\alpha^k = \sum_{i=1}^{\nu} \varepsilon_i^k \ell_i^k \varphi_i a_i$ and a the N-vector of the a^k.
- $\beta^k = \sum_{i=1}^{\nu} \varepsilon_i^k \ell_i^k \chi_i$ and β the N-vector of the β^k.
- $\gamma_i^k = a_i - \rho \bar{G}_i^k$.
- $\Gamma^k = \sum_{i=1}^{\nu} \varepsilon_i^k \gamma_i^k$.
- $\delta = \theta/(1-\theta)$.
- $\Delta^k = 1 + \nu^k - \rho \Phi^k$.
- $\varepsilon_i^k = 0$ if $m_i^k = 0$, and $\varepsilon_i^k = 1$ if $m_i^k > 0$.
- $\mathcal{E}_i = \mathrm{diag}_k\{\varepsilon_i^k\}$ a $N \times N$ diagonal matrix.
- ζ_i^k, scalar coefficient of $\bar{G}_i^k$ in r.h.s. of equation for G_i^k. (See Z_i above.)
- $\lambda^k = \sum_{i=1}^{\nu} \varepsilon_i^k \ell_i^k \Psi_i$ a N-line vector.
- $\nu^k = \sum_{i=1}^{\nu} \varepsilon_i^k$.
- $\Phi^k = \sum_{i-1}^{\nu} \varepsilon_i^k \bar{F}_i^k$.
- φ_i^k, χ_i^k, ψ_i^k scalar coefficients in G_i^k.
- φ_i and χ_i, N-vectors of the φ_i^k and χ_i^k
- $\Psi_i = \mathrm{diag}_k\{\psi_i^k\}$, a $N \times N$ diagonal matrix.
- $\omega_i^k = \varphi_i^k a_i + \chi_i^k a_0, \quad \omega_i$ the N-vector of the ω_i^k.