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Veto players, the kernel of the Shapley value and its characterization

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Abstract

In this article, we provide a new basis for the kernel of the Shapley value (Shapley, 1953), which is used to construct a new axiom of invariance, and to provide a new axiomatic characterization of the Shapley value. This characterization only invokes marginalistic principles, and does not rely on classical axioms such as symmetry, efficiency or linearity. Moreover, our approach reveals a new instructive role played by veto players.

Keywords: Veto players, Addition invariance, Basis, Kernel, Shapley value

1. Introduction

The Shapley value (Shapley, 1953) is probably the most eminent allocation rule for cooperative games with transferable utility (simply games henceforth), which is calculated as a player's average marginal contribution to coalitions. The Shapley value has been successfully applied to various situations, in particular to the measure of voting power in committees (Shapley and Shubik, 1954). In a voting situation, coalitions are either winning or losing, and a player belonging to all winning coalitions is called a veto player. Unanimity games are instances of such voting situations in which the unique minimally winning coalition is the coalition of veto players. The collection of unanimity games plays an important role in the classical characterization of the Shapley value by efficiency, additivity, symmetry and the null the player axiom. Each game is decomposed into a weighted sum of unanimity games in which the Shapley value assigns an equal share of a unit to each veto player and zero to each other player. These coefficients are called the Harsanyi dividends (Harsanyi, 1959). The dividend of a singleton is equal to its worth while, recursively, the dividends of all other coalitions are defined as their worth minus the dividends of all proper subcoalitions. In this sense the dividend of a coalition measures the earnings of cooperation of the coalition that was not yet realized by its proper subcoalitions. In terms of Harsanyi dividends, the Shapley value allocates to each player an equal share of the dividend of each coalition he or she belongs to.

Young (1985) provides another elegant characterization of the Shapley value by efficiency, symmetry and strong monotonicity. The latter axiom requires that whenever a player's marginal contributions weakly increase, his payoff weakly increases, and thus highlights the marginalistic nature of the Shapley value. Marginalistic allocation rules in a broad sense are called probabilistic allocation rules in Weber (1988). The differences between marginalistic and egalitarian allocation rules are studied by van den Brink (2007) and Casajus and Huettner (2014). In the latter article, a variant of strong monotonicity is invoked to characterize the egalitarian-Shapley values, i.e. the convex combination of the Shapley value and the equal division value.

In this article, we point out a new instructive function of unanimity games, which reflects another fundamental marginalistic aspect of the Shapley value. Consider a unanimity game with at least two veto

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players and imagine that a given veto player is deprived of his veto power, i.e. a new unanimity game is obtained on the initial coalition without the player. What consequences can we expect in terms of allocation variation? The marginal contributions of the deprived player decrease, the marginal contributions of each other veto player increase, and there is no impact on the marginal contributions of any other (non-veto) player. According to Young's (1985) strong monotonicity principle, the allocation should not increase for the deprived veto player, not decrease for each other veto player and remain the same for each other player. Now suppose that a veto player is chosen randomly in the initial unanimity game. The expected allocation variation should be null for the non-veto players. Since the worth of the grand coalition is unaffected and each veto player is treated symmetrically, the expected allocation variation should be null for these players too. In a sense, this means that a veto player has just as much to lose from the cancellation of his veto power as to gain from removing alternately the veto power of each other veto player. The situation we just considered can be described by a game constructed from a coalition, which measures the impact of randomly removing, at the margin, the veto power of one of its members. In this article, we invoke the previous invariance principle in the following axiom: an allocation rule should be invariant to the addition of such a game. In a sense, we require that an allocation rule should be invariant to the random marginal removal of veto power.

The collection of games involved in our axiom is very useful on many aspects. Firstly, this collection forms a basis of the kernel of the Shapley value, i.e. the set of games for which the Shapley value prescribes a null payoff vectors (see Kleinberg and Weiss, 1985). Put differently, since the Shapley value is a linear function, its kernel describes classes of equivalent games in the sense that two games are equivalent with respect to the Shapley value if their difference belongs the kernel. Therefore, our approach enables to solve the following so-called inverse problem: for a given payoff vector, what are the games in which the Shapley value prescribes this payoff vector?

Secondly, it is well-known that the space of all games with a fixed player set is the direct-sum of the kernel of the Shapley value and the space of all inessential (or additive) games. As a consequence, we obtain a new basis for the space of all games with a fixed player set. Furthermore, this basis is used to provide a new characterization of the Shapley value as the unique allocation rule satisfying the above-mentioned axiom of invariance and the inessential game axiom. The latter axiom requires that each player gets his own stand-alone worth in any inessential game. Contrary to Young (1985), our axiomatic characterization only relies on marginalistic principles. This is true not only for our axiom of invariance as explained above, but also for the inessential game axiom since in any inessential game, every player's marginal contribution is equal to his stand-alone worth. In other words, among all probabilistic allocation rules (which satisfy by definition the inessential game axiom), the Shapley value is singled out by the the marginalistic principle incorporated to our axiom of invariance. As in Hart and Mas-colell (1989) and Myerson (1980), the essence of the Shapley value is revealed by a unique axiom, the other one being rather standard. It is also interesting to note that our characterization does not rely on the classical axioms of efficiency, symmetry and linearity.

Thirdly, we provide a formulation of the coordinates of the games in our new basis in terms of Harsanyi dividends. The resulting expression can be seen as an extension of the Shapley value in the sense each coalition of a given size receives an equal share of the Harsanyi dividend of the coalition it is included in. Therefore, the vector of coordinates of the games constructed from the singleton coalitions coincides with the Shapley value, expressed in terms of the Harsanyi dividends. Faigle and Grabisch (2014) study such extensions of values.

The rest of the article is organized as follows. The notations and definitions are provided in section 2. The results are presented in section 3. Section 4 concludes.

2. Preliminaries

2.1. Notations

Throughout this article, the cardinality of a finite set S will be denoted by the lower case s, the collection of all subsets of S will be denoted by 2^S, and weak set inclusion will be denoted by $\subseteq$. Proper set inclusion will be denoted by $\subset$. For notational convenience, we will write singleton $\{i\}$ as i. Given a real linear space

V, its additive identity element will be denoted by $\mathbf{0}_V$, and its dimension by $\dim(V)$. If a linear space V is the direct sum of the subspaces V^1 and V^2, i.e. $V = V^1 + V^2$ and $V^1 \cap V^2 = \{\mathbf{0}_V\}$, we write $V = V^1 \oplus V^2$. If $f : V \longrightarrow U$ is a linear mapping, then denote by $\mathrm{Ker}(f)$ its kernel, i.e. the set of vectors $v \in V$ such that $f(v) = \mathbf{0}_U$.

2.2. Definitions

Let $N = \{1, 2, \ldots, n\}$ be a fixed and finite set of n players. Each subset S of N is called a *coalition*. A cooperative game with transferable utility or simply a *TU-game* on a fixed player set N is a function $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. The set of TU-games v on N, denoted by V_N, forms a linear space where $\dim(V_N) = 2^n - 1$. For each coalition $S \subseteq N$, $v(S)$ describes the *worth* of the coalition S when its members cooperate. For any two TU-games v and w in V_N and any $\alpha \in \mathbb{R}$, the TU-game $\alpha v + w \in V_N$ is defined as follows: for each $S \subseteq N$, $(\alpha v + w)(S) = \alpha v(S) + w(S)$. For any non-empty coalition $S \subseteq N$, the *S-Dirac TU-game* δ_S is defined as: $\delta_S(T) = 0$ if $T = S$ and $\delta_S(T) = 0$ otherwise. The collection of all S-Dirac TU-games $\{\delta_S\}_{S \subseteq N \setminus \{\emptyset\}}$ is the standard basis for V_N:

$$\forall v \in V_N, \quad v = \sum_{S \in 2^N \setminus \{\emptyset\}} v(S)\delta_S.$$

A *simple TU-game* is a TU-game where each coalition has either worth 0 or worth 1 and the grand coalition N has worth 1. Coalitions with worth 1 are called *winning*, the other coalitions are called *losing*. A player is called a *veto player* in a simple TU-game if he or she belongs to all winning coalitions. For any non-empty coalition $S \subseteq N$, the *S-unanimity TU-game* $u_S \in V_N$ is a simple TU-game defined as: for each $T \supseteq S$, $u_S(T) = 1$, and $u_S(T) = 0$ for each other T. Players in S are the veto players. The collection of S-unanimity TU-game $\{u_S\}_{S \subseteq N \setminus \{\emptyset\}}$ is a well-studied basis for V_N (see Shapley, 1953). Precisely, for each $v \in V_N$, there exists a unique collection of real numbers $\{a_S(v)\}_{S \in 2^N \setminus \{\emptyset\}}$, called the Harsanyi dividends (Harsanyi, 1959), such that:

$$v = \sum_{S \in 2^N \setminus \{\emptyset\}} a_S(v) u_S, \tag{1}$$

where

$$\forall S \in 2^N \setminus \{\emptyset\}, \quad a_S(v) = \sum_{T \in 2^S} (-1)^{s-t} v(T). \tag{2}$$

A TU-game $v \in V_N$ is *inessential* if, for each $S \subseteq N$ such that $S \neq \emptyset$, $v(S) = \sum_{i \in S} v(i)$. The subset of inessential TU-games, denoted by I, forms a subspace of V_N such that $\dim(I) = n$. A basis for I is the collection of i-unanimity TU-games $\{u_i\}_{i \in N}$.

A *payoff vector* $x \in \mathbb{R}^n$ is an n-dimensional vector giving a payoff $x_i \in \mathbb{R}$ to each agent $i \in N$. An *allocation rule* Φ on V_N is a mapping $\Phi : V_N \longrightarrow \mathbb{R}^n$ which uniquely determines, for each $v \in V_N$ and each $i \in N$, a payoff $\Phi_i(v) \in \mathbb{R}$ for participating to $v \in V_N$.

The *Shapley value* (Shapley, 1953), Sh, is the allocation rule defined on V_N as follows:

$$\forall i \in N, \quad \mathrm{Sh}_i(v) = \sum_{T \in 2^N : T \ni i} \frac{(n-t)!(t-1)!}{n!} \big(v(T) - v(T \setminus i)\big). \tag{3}$$

Because of (1) and the fact that Sh is a linear function on V_N, it holds that:

$$\mathrm{Sh}(v) = \sum_{S \in 2^N \setminus \{\emptyset\}} a_S(v) \mathrm{Sh}(u_S)$$

From (3), we obtain:

$$\mathrm{Sh}_i(u_S) = \begin{cases} 1/s & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Thus, an equivalent expression of (3) is:

$$\forall i \in N, \quad \mathrm{Sh}_i(v) = \sum_{S \in 2^N : S \ni i} \frac{a_S(v)}{s}, \tag{5}$$

i.e., the Shapley value of a player in a TU-game is the sum of the Harsanyi dividends accruing to this player from the various coalitions he could participate.

We will refer to the following axiom later on.

Inessential game axiom. An allocation rule Φ satisfies the inessential game axiom if for each $w \in I$, it holds that: $\Phi(w) = (w(1), \ldots, w(n))$.

The Shapley value is a linear function that satisfies the Inessential game axiom.

3. Results

Let us introduce the following collection of TU-games. For each $S \subseteq N$, $s \geq 2$, define $w_S : 2^N \longrightarrow \mathbb{R}$ as:

$$w_S(T) = \begin{cases} -1/s & \text{if } |S \cap T| = s - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Noting that the condition $u_S(T) = 1$ for $T \supseteq S$ is equivalent to $u_S(T) = 1$ for $|S \cap T| = s$ in the definition of the S-unanimity TU-game u_S, it straightforward to verify that:

$$w_S = \sum_{i \in S} \frac{1}{s}\big(u_S - u_{S \setminus \{i\}}\big). \tag{7}$$

Expression (7) points out the role of each player in S. The element in the brackets associated with player $i \in S$ measures the impact, for any coalition $T \in 2^N$, of taking away i's veto power while keeping the veto power of all other members of S. Two cases arise. This cancellation has either no influence on T because T already contains $S \setminus i$ or does not contain S, or a positive impact on T when T contains $S \setminus i$ but not i (the losing coalition T becomes a winning coalition). When i is randomly chosen among the players in S, the worth $w_S(T)$ means that the expected impact on T is null except when T contains all but one player in S.

Proposition 1 *The collection of TU-games $\{w_S\}_{S \in 2^N, s \geq 2}$ forms a basis for Ker(Sh).*

Proof. Because $\mathrm{Sh} : V_N \longrightarrow \mathbb{R}^n$ satisfies the Inessential game property, it is onto. It follows that:

$$\dim(\mathrm{Ker}(\mathrm{Sh})) = 2^n - 1 - n.$$

The collection of all TU-games $\{w_S\}_{S \in 2^N, s \geq 2}$, contains exactly $2^n - 1 - n$. Therefore, to prove the statement of Proposition 1, it remains to show:
(a) For each $S \in 2^N, s \geq 2$, $\mathrm{Sh}(w_S) = \mathbf{0}_{\mathbb{R}^N}$;
(b) The collection of all TU-games $\{w_S\}_{S \in 2^N, s \geq 2}$ are linearly independent.

Proof of (a). Pick any $S \in 2^N, s \geq 2$. By (7) and linearity of the Shapley value:

$$\mathrm{Sh}(w_S) = \mathrm{Sh}(u_S) - \sum_{i \in S} \frac{1}{s} \mathrm{Sh}(u_{S \setminus i}).$$

By (4), we obtain:

$$\forall i \in S, \quad \forall j \in S \setminus i, \quad \mathrm{Sh}_j(u_{S \setminus i}) = \frac{1}{s-1}, \text{ and } \forall j \in N \setminus (S \setminus i), \quad \mathrm{Sh}_j(u_{S \setminus i}) = 0,$$

and

$$\forall j \in S, \quad \mathrm{Sh}_j(u_S) = \frac{1}{s}, \text{ and } \forall j \in N \setminus S, \quad \mathrm{Sh}_j(u_S) = 0.$$

From this, we deduce:

$$\forall j \in S, \quad \mathrm{Sh}_j(w_S) = \frac{1}{s} - \frac{1}{s}(s-1)\frac{1}{s-1} = 0, \text{ and } \forall j \in N \setminus S, \quad \mathrm{Sh}_j(w_S) = 0,$$

which completes the proof of point (a).

Proof of (b). Pick any TU-game w_S. By (1), w_S admits a unique linear decomposition along the T-unanimity TU-games:

$$w_S = \sum_{T \in 2^N \setminus \{\emptyset\}} a_T(w_S) u_T.$$

In view of (7):

$$\forall i \in S, \quad a_{S \setminus i}(w_S) = -\frac{1}{s}, \; a_S(w_S) = 1, \text{ and, for each other } T, \quad a_T(w_S) = 0. \tag{8}$$

Consider any linear order $(2^N, \preceq)$ extending the partial order $(2^N, \subseteq)$ and such that $S \preceq T$ whenever $s \leq t$, and the $(2^n - 1 - n) \times (2^n - 1)$ coalitional-matrix

$$A = [a_{S,T}]_{S \in 2^N, s \geq 2, T \in 2^N \setminus \{\emptyset\}}, \text{ where each } (S,T)\text{-entry is defined as: } \; a_{S,T} = a_T(w_S).$$

Thus, each of the $2^n - 1 - n$ rows is identified with the coordinates of w_S,

$$A = \begin{bmatrix} a_1(w_{\{1,2\}}) & a_2(w_{\{1,2\}}) & a_3(w_{\{1,2\}}) & \dots & a_T(w_{\{1,2\}}) & \dots & a_N(w_{\{1,2\}}) \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_1(w_S) & a_2(w_S) & a_3(w_S) & \dots & a_T(w_S) & \dots & a_N(w_S) \\ \vdots & \vdots & \vdots & & \vdots & & \vdots \\ a_1(w_N) & a_2(w_N) & a_3(w_N) & \dots & a_T(w_N) & \dots & a_N(w_N) \end{bmatrix}$$

The rank of A is at most $2^n - 1 - n$. Consider the $(2^n - 1 - n) \times (2^n - 1 - n)$ submatrix $[a_{S,T}]_{S \in 2^N, s \geq 2, T \in 2^N, s \geq 2}$. By (8), it is lower triangular with non-null diagonal elements, so that its rank is $2^n - 1 - n$. Conclude that A has full rank, which amounts to saying that the collection of all TU-games $\{w_S\}_{S \in 2^N, s \geq 2}$ are linearly independent. This completes the proof of point (b). ■

Consider any TU-game $v \in V_N$, any coalition $S \subseteq N$ such that $s \geq 2$, and any real number $a \in \mathbb{R}$. We say that the TU-game $(v + aw_S) \in V_N$ is obtained from $v \in V_N$ through the (S,a)*-addition*. This operation of addition from a TU-game v keeps the worth $v(N)$ unchanged since $w_S(N) = 0$ for each $S \in 2^N \setminus \{\emptyset\}$. Note also that we only consider coalitions of size at least two when we apply a (S,a)-addition to a TU-game. Based on (S,a)-additions, we consider the following axiom of invariance.

w-addition invariance. An allocation rule Φ satisfies w-addition invariance if, for each $v \in V_N$, and each (S,a)-addition, it holds that: $\Phi(v) = \Phi(v + aw_S)$.

The interpretation that adding a game which measures the random marginal removal of veto power has no influence on the allocated payoffs has already been detailed in the introduction. We immediately deduce from Proposition 1 that the Shapley value satisfies w-addition invariance.

Corollary 1 *The Shapley value satisfies w-addition invariance.*

Combining Proposition 1 and the well-known fact that $\mathrm{Ker}(\mathrm{Sh}) \oplus I = V_N$, one obtains a new basis for V_N. Precisely, the collection of TU-games

$$\left\{w_S : S \in 2^N, s \geq 2\right\} \bigcup \left\{u_i : i \in N\right\} \tag{9}$$

forms a basis for V_N. Using this fact and Proposition 1, we are able to derive a new characterization of the Shapley value.

Proposition 2 *The Shapley value Sh is the unique allocation rule on V_N which satisfies w-addition invariance and the Inessential game axiom.*

Proof. By corollary 1, Sh satisfies w-addition invariance. It has been already underlined that it also satisfies the Inessential game axiom. So, there is at least one allocation rule satisfying these two axioms on V_N. Next, let Φ be any allocation rule satisfying w-addition invariance and the Inessential game axiom. Pick any $v \in V_N$. The TU-game v admits a unique linear decomposition along the elements of the set given by (9):

$$v = \sum_{i \in N} b_i(v) u_i + \sum_{S \in 2^N, s \geq 2} b_S(v) w_S.$$

By successive applications of w-addition invariance, we obtain:

$$\Phi(v) = \Phi\left(\sum_{i \in N} b_i(v) u_i + \sum_{S \in 2^N, s \geq 2} b_S(v) w_S\right) = \Phi\left(\sum_{i \in N} b_i(v) u_i\right).$$

Because Φ also satisfies the Inessential game axiom, we get:

$$\Phi\left(\sum_{i \in N} b_i(v) u_i\right) = (b_1(v), \ldots, b_n(v)),$$

which shows that Φ is uniquely determined. This completes the proof of Proposition 2. ■

Equation (2) expresses the Harsanyi dividends in terms of the worths of the coalitions, i.e. the coordinates of v in the basis of Dirac TU-games. This transformation is known as the Möbius transform of a TU-game (see, e.g., Grabisch, 2002). The next result provides such a transformation from the Harsanyi dividends to the coordinates of v in the basis given by (9). This transformation extends the Shapley value in the sense that each individual coordinate $b_i(v)$ of a TU-game v coincides with $\mathrm{Sh}_i(v)$ expressed as in (5). More precisely, whereas the Shapley value allocates an equal share of the Harsanyi dividend $a_T(v)$ to each $i \in T$, its extension to each coalition, given by (11) below, allocates to each coalition $S \subseteq T$ of size s an equal share of the dividend $a_T(v)$.

For the sake of the presentation, set $w_i := u_i$ for the rest of the article. So, we will write $v \in V_N$ as:

$$v = \sum_{i \in N} b_i(v) u_i + \sum_{S \in 2^N : s \geq 2} b_S(v) w_S = \sum_{S \in 2^N \setminus \{\emptyset\}} b_S(v) w_S. \tag{10}$$

Proposition 3 *Let $\{a_S(v)\}_{S \in 2^N \setminus \{\emptyset\}}$ be the Harsanyi dividends of $v \in V_N$, and let $\{b_S(v)\}_{S \in 2^N \setminus \{\emptyset\}}\}$ be its coordinates with respect to the basis (9). It holds that:*

$$\forall S \in 2^N \setminus \{\emptyset\}, \quad b_S(v) = \sum_{T \in 2^N : T \supseteq S} \frac{a_T(v)}{\binom{t}{s}} \tag{11}$$

or by (2),

$$\forall S \in 2^N \setminus \{\emptyset\}, \quad b_S(v) = \sum_{T \in 2^N : T \supseteq S} \frac{1}{\binom{t}{s}} \sum_{K \in 2^T} (-1)^{t-k} v(K). \tag{12}$$

Before proving Proposition 3, we formulate we following two informative remarks.

Remarks

- The transformation $v = \{v(S)\}_{S\in 2^N\setminus\{\emptyset\}} \longmapsto \{b_S(v)\}_{S\in 2^N\setminus\emptyset}$ (or the transformation $\{a_S(v)\}_{S\in 2^N\setminus\{\emptyset\}} \longmapsto \{b_S(v)\}_{S\in 2^N\setminus\emptyset}$) given by (12) (given by (11) respectively) is a linear and invertible function which extends the Shapley value in the sense that:

$$\begin{aligned}
\forall i \in N, \quad b_i(v) &= \sum_{T\in 2^N:T\supseteq i} \frac{1}{\binom{t}{1}} \sum_{K\in 2^T} (-1)^{t-k} v(K) \\
&= \sum_{T\in 2^N:T\supseteq i} \frac{1}{t} \sum_{K\in 2^T} (-1)^{t-k} v(K) \\
&= \sum_{T\in 2^N:T\ni i} \frac{a_T(v)}{t} \\
&= \mathrm{Sh}_i(v).
\end{aligned}$$

  Faigle and Grabisch (2014) study various linear and invertible functions extending an allocation rule. For instance, they present another linear and invertible function extending the Shapley value, called the Shapley interaction transform, and defined as:

$$\forall S \in 2^N \setminus \{\emptyset\}, \quad I_S(v) = \sum_{T\in 2^N:T\subseteq N\setminus S} \frac{(n-t-s)!t!}{(n-s-1)!} \sum_{K\in 2^S} (-1)^{s-k} v(T\cup K). \tag{13}$$

- From the above remark and Proposition 1, we can solve the following inverse problem: given a vector of real numbers $(r_1, \ldots, r_n)$, find all $v \in V_N$ such that $\mathrm{Sh}(v) = (r_1, \ldots, r_n)$. The solution set is the subset of TU-games $v \in V_N$ such that:

$$v = \sum_{i\in N} r_i w_i + \sum_{S\in 2^N:s\geq 2} b_S(v) w_S.$$

Proof. (of Proposition 3) We first prove the following claim:

$$\forall S \in 2^N \setminus \{\emptyset\}, \quad u_S = \sum_{T\in 2^S\setminus\{\emptyset\}} \frac{1}{\binom{s}{t}} w_T. \tag{14}$$

We prove this claim by induction on the size $s \geq 1$ of S.

Initial step Pick any $i \in N$. By (14) and the convention $w_i := u_i$,

$$u_i = \frac{1}{\binom{1}{1}} w_i.$$

Induction hypothesis Assume that (14) holds for each coalition of size at most $k < n$.

Induction step Pick any coalition S of size $s = k+1$. By (7):

$$u_S = w_S + \frac{1}{s} \sum_{i\in S} u_{S\setminus i}$$

Using the induction hypothesis, we obtain:

$$\begin{aligned}
u_S &= w_S + \frac{1}{s}\sum_{i\in S}\sum_{T\in 2^{S\setminus i}\setminus\{\emptyset\}} \frac{1}{\binom{s-1}{t}} w_T \\
&= w_S + \frac{1}{s}\sum_{T\subset S, T\neq\emptyset}\sum_{i\in S\setminus T} \frac{1}{\binom{s-1}{t}} w_T \\
&= w_S + \sum_{T\subset S, T\neq\emptyset} \frac{(s-t)}{s\binom{s-1}{t}} w_T \\
&= \frac{1}{\binom{s}{s}} w_S + \sum_{T\subset S, T\neq\emptyset} \frac{1}{\binom{s}{t}} w_T \\
&= \sum_{T\in 2^S\setminus\{\emptyset\}} \frac{1}{\binom{s}{t}} w_T,
\end{aligned}$$

as desired. Thus, (14) holds for any nonempty coalition.

Next, pick any $v \in V_N$. We have:

$$v = \sum_{S\in 2^N\setminus\{\emptyset\}} a_S(v)u_S = \sum_{S\in 2^N\setminus\{\emptyset\}} b_S(v)w_S \tag{15}$$

By (14):

$$\begin{aligned}
\sum_{S\in 2^N\setminus\{\emptyset\}} a_S(v)u_S &= \sum_{S\in 2^N\setminus\{\emptyset\}} a_S(v) \sum_{T\in 2^S\setminus\{\emptyset\}} \frac{1}{\binom{s}{t}} w_T \\
&= \sum_{T\in 2^N\setminus\{\emptyset\}} \sum_{S\in 2^N : S\supseteq T} a_S(v) \frac{1}{\binom{s}{t}} w_T \\
&= \sum_{S\in 2^N\setminus\{\emptyset\}} \sum_{T\in 2^N : T\supseteq S} a_T(v) \frac{1}{\binom{t}{s}} w_S
\end{aligned}$$

Using (15), we get the desired result:

$$\forall S \in 2^N \setminus \{\emptyset\}, \quad b_S(v) = \sum_{T\in 2^N : T\supseteq S} \frac{a_T(v)}{\binom{t}{s}}.$$

■

4. Conclusion

We shall conclude this article by mentioning the close work by Yokote (2014) who provides an alternative basis of the kernel of the Shapley value, which has perhaps a less transparent interpretation. Yokote (2014) considers another axiom of invariance, called Strong addition Invariance, which exploits his basis of the kernel of the Shapley value. He characterizes the Shapley values by the combination of Strong addition invariance and the Dummy player axiom.

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