

crese

CENTRE DE RECHERCHE
SUR LES STRATÉGIES ÉCONOMIQUES

Decomposition of the space of TU-games, Strong Transfer Invariance and the Banzhaf value

SYLVAIN BÉAL, ERIC RÉMILA, PHILIPPE SOLAL

November 2014

Working paper No. 2014–05

CRESE 30, avenue de l'Observatoire
25009 Besançon
France
<http://crese.univ-fcomte.fr/>

The views expressed are those of the authors
and do not necessarily reflect those of CRESE.

 **UFC**
UNIVERSITÉ
DE FRANCHE-COMTÉ

Decomposition of the space of TU-games, Strong Transfer Invariance and the Banzhaf value

Sylvain Béal^{a,*}, Eric Rémila^b, Philippe Solal^b

^a*Université de Franche-Comté, CRESE, 30 Avenue de l'Observatoire, 25009 Besançon, France*

^b*Université de Saint-Etienne, CNRS UMR 5824 GATE Lyon Saint-Etienne, France*

Abstract

We provide a new and concise characterization of the Banzhaf value on the (linear) space of all TU-games on a fixed player set by means of two transparent axioms. The first one is the well-known Dummy player axiom. The second axiom, called Strong transfer invariance, indicates that a player's payoff is invariant to a transfer of worth between two coalitions he or she belongs to. To prove this result we derive direct-sum decompositions of the space of all TU-games. We show that, for each player, the space of all TU-games is the direct sum of the subspace of TU-games where this player is dummy and the subspace spanned by the TU-games used to construct the transfers of worth. This decomposition method has several advantages listed as concluding remarks.

Keywords: Banzhaf value, Dummy player axiom, Direct-sum decomposition, Strong Transfer invariance

1. Introduction

The Banzhaf value, initially introduced in the context of voting games by Banzhaf (1965), and later on extended to arbitrary games by, e.g., Owen (1975) and Dubey and Shapley (1979), is one of the most popular values in cooperative games with transferable utilities (henceforth, TU-games). The Banzhaf value assigns to each player the average of the marginal contribution to any coalition he or she belongs to, and considers that each player is equally likely to enter to any coalition. It has received numerous characterizations both on restricted domains and the full domain as well as on a fixed player set and on variable player sets (e.g. Dubey, Shapley, 1979, Lehrer 1988, Haller 1994, Malawski 2002, Casajus 2012, 2014). It has also been extended to TU-games equipped with a coalition structure (Owen, 1981) or a coalition configuration (Albizuri, Aurrekoetxea, 2006), to TU-games on antimatroids (Algaba et al., 2004), to TU-games on convex geometries (Bilbao et al., 1998), etc. Beyond the application to the measurement of voting power, the Banzhaf value has also been successfully employed to model control relationships in corporate structures (see Crama and Leruth, 2013, for a recent survey on this question).

In this note we provide a new and concise characterization of the Banzhaf value on the (linear) space of all TU-games on a fixed player set by means of two transparent axioms. The first one

*Corresponding author

Email addresses: sylvain.beal@univ-fcomte.fr (Sylvain Béal), eric.remila@univ-st-etienne.fr (Eric Rémila), philippe.solal@univ-st-etienne.fr (Philippe Solal)

URL: <https://sites.google.com/site/bealpage/> (Sylvain Béal)

is the well-known Dummy player axiom. The second axiom, called Strong transfer invariance, indicates that a player's payoff is invariant to a transfer of worth between two coalitions of which he or she is a member. This means that if the worth of coalition varies of a certain amount and, at the same time, the worth of another coalition varies by the opposite amount, then for members of both coalitions the payoff allocation does not change.

Characterizations of the Banzhaf value which in addition to the Dummy player axiom invoke only one another axiom can be found in Casajus (2012, 2014). The added axiom relies on a different principle than the one included in the axiom of Strong transfer invariance. To be precise, these characterizations use either an amalgamation principle between two players when the player set is variable or a similar principle of collusion when the player set is fixed. Theorem 7 in Casajus (2012) establishes that the Banzhaf value on variable player sets is characterized by the Dummy player axiom and 2-efficiency. The axiom of 2-efficiency is an axiom of amalgamation due to Lehrer (1988) which requires that the payoff of a player after amalgamation with another player is the sum of the individual payoffs of the two players obtained before the operation of amalgamation. Theorem 4 in Casajus (2014) establishes that the Banzhaf value on a fixed player set is the unique value that satisfies the Dummy player axiom and Proxy neutrality or Association neutrality or Distrust neutrality, provided that the player set does not contain two players. Proxy neutrality and Association neutrality are borrowed from Haller (1994) and Distrust neutrality is invoked by Malawski (2002). These axioms of collusion between two players are in the same spirit of 2-efficiency, but works on a fixed player set.

To prove our characterization, and contrary to the above-mentioned results by Casajus, we derive direct-sum decompositions of the space of all TU-games. We show that, for each player, the space of all TU-games is the direct sum of the subspace of TU-games where this player is dummy and the subspace spanned by the TU-games used to construct the transfers of worth. This decomposition method has several advantages that we list as a conclusion. This approach has been used in Béal et al. (2013) by means of additions and transfers of worth, and in Yokote (2014) by means of additions of worth. Axioms of invariance constructed from the considered additions and transfer are invoked to characterize the Shapley value (Shapley, 1953) and the equal division solution, but are not satisfied by the Banzhaf value.

The rest of this note is organized as follows. Section 2 contains the basic definitions and introduce the axiom of Strong transfer invariance. Section 3 contains the main results. Section 4 makes some concluding remarks on the method of decomposition of the space of TU-games.

2. Preliminaries

Throughout this note, the cardinality of a finite set S will be denoted by $|S|$, the collection of all subsets of S will be denoted by 2^S , weak set inclusion will be denoted by \subseteq , and proper set inclusion will be denoted by \subset . Also for notational convenience, we will write singleton $\{i\}$ as i . Let V be a real linear space equipped with an inner product “ \cdot ”. Its additive identity element is denoted by $\mathbf{0}$ and its dimension by $\dim(V)$. Given a linear subspace U of V , we denote by U^\perp its orthogonal complement. If V is the direct sum of the subspaces V^1 and V^2 , i.e. $V = V^1 + V^2$ and $V^1 \cap V^2 = \{\mathbf{0}\}$, we write $V = V^1 \oplus V^2$. If X is a non-empty subset of V , then $\text{Sp}(X)$ denotes the smallest subspace containing X .

Let $N = \{1, 2, \dots, n\}$ be a fixed and finite set of n players. Subsets of N are called *coalitions*, while N is called the *grand coalition*. A cooperative game with transferable utility or simply a *TU-game* on N is a function $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. The set of TU-games v on N , denoted

by V_N , forms a linear space where $\dim(V_N) = 2^n - 1$. For each coalition $S \subseteq N$, $v(S)$ describes the *worth* of the coalition S when its members cooperate. The set $H(v)$, called the *support* of v , is the set of coalitions $S \subseteq N$ such that $v(S) \neq 0$. For any two TU-games v and w in V_N and any $\alpha \in \mathbb{R}$, the TU-game $\alpha v + w \in V_N$ is defined as follows: for each $S \subseteq N$, $(\alpha v + w)(S) = \alpha v(S) + w(S)$. The inner product $v \cdot w$ is defined as $\sum_{S \subseteq N} v(S)w(S)$.

For any non-empty coalition $T \subseteq N$, the *Dirac TU-game* $\delta_T \in V_N$ is defined as: $\delta_T(T) = 1$, and $\delta_T(S) = 0$ for each other $S \subseteq N$. Clearly, the collection of all Dirac TU-games is a basis for V_N . For any $i \in N$, the *i -dictator TU-game* $u_i \in V_N$ is defined as $u_i(S) = 1$ if $S \ni i$, and $u_i(S) = 0$ for each $S \not\ni i$.

Player $i \in N$ is a *dummy player* in v if:

$$\forall S \ni i, \quad v(S) - v(S \setminus i) - v(i) = 0.$$

Denote by D_i the subspace of V_N where player $i \in N$ is dummy.

Player is *null* in v if:

$$\forall S \ni i, \quad v(S) - v(S \setminus i) = 0.$$

Denote by D_i^0 the subspace of V_N where player $i \in N$ is null.

A *value* Φ on V_N is a mapping $\Phi : V_N \rightarrow \mathbb{R}^n$ which uniquely determines, for each $v \in V_N$ and each $i \in N$, a payoff $\Phi_i(v) \in \mathbb{R}$ for participating to $v \in V_N$.

Dummy player axiom. A value Φ satisfies the Dummy player axiom if, for each $i \in N$ and each $v \in D_i$, it holds that: $\Phi_i(v) = v(i)$.

Null player axiom. A value Φ satisfies the Null player axiom if, for each $i \in N$ and each $v \in D_i^0$, it holds that: $\Phi_i(v) = 0$.

The Dummy player axiom implies the Null player axiom. A well-known value satisfying the Dummy player axiom is the *Banzhaf value* (Banzhaf, 1965), Bz , defined on V_N as:

$$\forall i \in N, \quad Bz_i(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N: S \ni i} (v(S) - v(S \setminus i)).$$

Because $Bz_i(v)$ only depends on the sum of the contributions to all i 's coalitions, Bz_i is also invariant to a transfer of worth between any two coalitions containing i . Let us elaborate on this property. Consider any TU-game $v \in V_N$, any pair of distinct coalitions $S^+ \subseteq N$ and $S^- \subseteq N$ and any real number $\alpha \in \mathbb{R}$. The TU-game $v + \alpha(\delta_{S^+} - \delta_{S^-}) \in V_N$ is obtained from $v \in V_N$ through the (S^+, S^-, α) -*transfer*. We also write that a (S^+, S^-, α) -transfer *involves* player $i \in N$ if $i \in S^+ \cap S^-$. By definition, Bz_i is not altered by a (S^+, S^-, α) -transfer involving i , which implies that Bz satisfies the following axiom.

Strong transfer invariance. A value Φ on V_N satisfies Strong transfer invariance if, for each $v \in V_N$ and each (S^+, S^-, α) -transfer, it holds that: for each $i \in S^+ \cap S^-$, $\Phi_i(v) = \Phi_i(v + \alpha(\delta_{S^+} - \delta_{S^-}))$.

Strong transfer invariance can be interpreted as follows. Suppose that the size of the coalitions does not matter in the coalition formation process. Imagine further that each player owns a part

of the worth of each coalitions he belongs to, which he can freely transfer from one of his coalitions to another. Then, the axiom simply requires that such a transfer of worth is neither beneficial nor detrimental for the player. In a sense, the axiom is weak since it is silent of the influence of the transfer on the other players.

3. Results

It turns out that the combination of Strong transfer invariance and the Dummy player axiom characterizes Bz on V_N .

Proposition 1 *A value Φ on V_N satisfies Strong transfer invariance and the Dummy player axiom if and only if $\Phi = Bz$.*

To prove the statement of Proposition 1, we will use the following decomposition result of the space V_N .

Proposition 2 *For each $i \in N$, it holds that $V_N = \Delta_i \oplus D_i$, where Δ_i is the subspace of V_N defined as $\Delta_i = Sp(Y_i)$, where*

$$Y_i = \left\{ \delta_{S^+} - \delta_{S^-} : S^+ \neq S^-, i \in S^+ \cap S^- \right\}.$$

Furthermore, $\dim(\Delta_i) = 2^{n-1} - 1$.

Proof. We first prove that $\Delta_i = U_i^\perp$, where

$$U_i = Sp\left(\left\{ \delta_S : S \not\ni i \right\} \cup \{u_i\}\right),$$

and $\dim(\Delta_i) = 2^{n-1} - 1$.

(i) We show that $\Delta_i \subseteq U_i^\perp$. It suffices to verify that: $Y_i \subseteq U_i^\perp$. Pick any element $(\delta_{S^+} - \delta_{S^-}) \in Y_i$. First, consider any $S \not\ni i$. We obviously have:

$$(\delta_{S^+} - \delta_{S^-}) \cdot \delta_S = 0.$$

Second, it is also clear that:

$$(\delta_{S^+} - \delta_{S^-}) \cdot u_i = 0.$$

Therefore, $\Delta_i \subseteq U_i^\perp$.

(ii) We show that $\Delta_i \supseteq U_i^\perp$. We proceed by induction on the size of the support $H(v)$ of $v \in U_i^\perp$. If $|H(v)| = 0$, then $v = \mathbf{0}$ and so $v \in \Delta_i$. Assume $v \in \Delta_i$ for all $v \in U_i^\perp$ such that $|H(v)| \leq h$, and pick any $v \in U_i^\perp$ such that $|H(v)| = h + 1$. Consider any $S^+ \in H(v)$. On the one hand, because $v \in U_i^\perp$, we have $v \cdot \delta_S = v(S) = 0$ for each $S \not\ni i$, and so $S^+ \ni i$. On the other hand, $v \in U_i^\perp$ implies $u_i \cdot v = 0$, i.e.,

$$\sum_{S \subseteq N: S \ni i} v(S) = 0.$$

Therefore, there exists a coalition $S^- \in H(v)$, distinct from S^+ such that $S^- \ni i$. Consider the TU-game $w = v - v(S^+)(\delta_{S^+} - \delta_{S^-})$. We have $w \in U_i^\perp$ and $|H(w)| < |H(v)|$. By the induction

hypothesis $w \in \Delta_i$. It is also clear that $-v(S^+)(\delta_{S^+} - \delta_{S^-}) \in \Delta_i$ so that $v = w - v(S^+)(\delta_{S^+} - \delta_{S^-}) \in \Delta_i$. Thus, $\Delta_i \supseteq U_i^\perp$.

From (i)-(ii), we conclude that $U_i^\perp = \Delta_i$. To determine the dimension of Δ_i , we use the fact that $\dim(U_i \oplus U_i^\perp) = \dim(V_N) = 2^n - 1$, and that the elements of $\{\delta_S : S \not\ni i\} \cup \{u_i\}$ are obviously linearly independent. Therefore,

$$\dim(\Delta_i) = \dim(U_i^\perp) = \dim(V_N) - \dim(U_i) = 2^n - 1 - 2^{n-1} = 2^{n-1} - 1,$$

as claimed.

It remains to prove show that $\Delta_i \cap D_i = \{\mathbf{0}\}$. Pick any $v \in \Delta_i \cap D_i$. On the one hand, $v \in \Delta_i$ and $\Delta_i = U_i^\perp$ mean that:

$$\forall S \not\ni i, \quad v(S) = 0, \quad \text{and} \quad \sum_{S \subseteq N: S \ni i} v(S) = 0. \quad (1)$$

On the other hand, $v \in D_i$ means that:

$$\forall S \ni i, \quad v(S) - v(S \setminus i) - v(i) = 0. \quad (2)$$

Combining (1) and (2), we get:

$$\forall S \ni i, \quad v(S) = v(i) \quad \text{and} \quad \sum_{S \subseteq N: S \ni i} v(S) = 0.$$

Therefore, the unique possibility is $v = \mathbf{0}$, as desired. ■

Proof. (of Proposition 1) Consider any value Φ satisfying Strong addition invariance and the Dummy player axiom on V_N . Pick any $v \in V_N$ and any $i \in N$. By Proposition 2, there is exactly one $r^i \in \Delta_i$ and exactly one $w^i \in D_i$ such that:

$$v = r^i + w^i.$$

Applying Strong addition invariance and the Dummy player axiom on Φ_i , we obtain:

$$\Phi_i(v) = \Phi_i(r^i + w^i) = \Phi_i(w^i) = w^i(i),$$

which proves that Φ_i is uniquely determines on V_N . As player i has been chosen arbitrarily in N , conclude that Φ is uniquely determined on V_N . Because Bz satisfies also Strong addition invariance and the Dummy player axiom, we obtain $\text{Bz} = \Phi$ on V_N , as desired. ■

4. Concluding remarks

The logic behind the result contained in Proposition 1 is very intuitive. Pick any $v \in V_N$ and any $i \in N$. For each coalition $S^+ \ni i$, if $v(S^+) - v(S^+ \setminus i) \neq \text{Bz}_i(v)$, then there necessarily exists another coalition $S^- \ni i$ such that $v(S^-) - v(S^- \setminus i) \neq \text{Bz}_i(v)$. Apply the $(S^+, S^-, \text{Bz}_i(v) - v(S^+) + v(S^+ \setminus i))$ -transfer to v . It follows that the number of coalitions $S \ni i$ such that $v(S) - v(S \setminus i) \neq \text{Bz}_i(v)$ decreases of at least one unit. This operation can be repeated as many times as necessary to reach a TU-game v^i in which i is dummy and such that $v^i(i) = \text{Bz}_i(v)$. By the Dummy player axiom $\Phi_i(v^i) = \text{Bz}_i(v)$. Because at each step of this procedure we apply admissible transfers with respect to the axiom of Strong addition invariance, we obtain $\Phi_i(v) = \Phi_i(v^i) = \text{Bz}_i(v)$. However, the direct-sum decompositions of the space of TU-games contained in Proposition 2 allows to obtain a stronger result and to understand the effects of each axiom on V_N and Bz, which would have been hidden otherwise. Here is a list of consequences of Proposition 2.

1. Proposition 2 ensures that the Dummy player axiom cannot be replaced by the Null player axiom in Proposition 1. Indeed, because $D_i^0 \subset D_i$, we obtain $V_N \neq \Delta_i + D_i^0$ so that the proof of Proposition 1 would not longer hold. In this sense, the characterization obtained in Proposition 1 is sharp.
2. If we want to characterize Bz by using the Null player axiom instead of the Dummy player axiom, the previous remark indicates that we need (at least one) another axiom. Consider the following version of the axiom of covariance:

$$\forall v \in V_N, \forall \alpha \in \mathbb{R}, \forall i \in N, \quad \Phi_i(v + \alpha u_i) = \Phi_i(v) + \alpha. \quad (3)$$

Obviously, Bz satisfies this axiom since Bz is a linear function on V_N and $Bz_i(u_i) = 1$ for i , and $Bz_j(u_i) = 0$ for each $j \in N \setminus i$. Next, noting that $\dim(D_i^0) = 2^{n-1} - 1$, it is not difficult to see that $D_i = D_i^0 \oplus \text{Sp}(u_i)$ so that:

$$\forall i \in N, \quad V_N = \Delta_i \oplus D_i^0 \oplus \text{Sp}(u_i).$$

Using the above direct-sum decompositions of V_N and proceeding as in the proof of Proposition 2, we conclude that Bz is the unique value on V_N satisfying Strong transfer invariance, the Null player axiom and the version of covariance as expressed in (3).

3. It is known that $Bz_i : V_N \rightarrow \mathbb{R}$ is linear and onto so that the dimension of its kernel is $\dim(V_N) - \dim(\mathbb{R}) = 2^n - 2$. Consider any TU-game $v \in \Delta_i \oplus D_i^0$. Using the unique decomposition $v = r^i + w^{0,i}$, where $r^i \in \Delta_i$ and $w^{0,i} \in D_i^0$, we obtain:

$$Bz_i(v) = Bz_i(r^i + w^{0,i}) = Bz_i(w^{0,i}) = 0,$$

where the second inequality is obtained by Strong transfer invariance and the third equality by the Null player axiom. Therefore, $\Delta_i \oplus D_i^0$ is contained in the kernel of Bz_i . But as

$$\dim(\Delta_i \oplus D_i^0) = \dim(\Delta_i) + \dim(D_i^0) = 2^{n-1} - 1 + 2^{n-1} - 1 = 2^n - 2,$$

we conclude that $\Delta_i \oplus D_i^0$ spans the kernel of Bz_i . From this result and the fact that $V_N = \Delta_i \oplus D_i^0 \oplus \text{Sp}(u_i)$, we are able to solve the following inverse problem: for each $i \in N$, for each $\alpha_i \in \mathbb{R}$, find all $v \in V_N$ that $Bz_i(v) = \alpha_i$. For each $i \in N$, the solution set is the subset of TU-games $v \in V_N$ such that $v = y^i + \alpha_i u_i$ for some $y^i \in \Delta_i \oplus D_i^0$.

4. The direct-sum decompositions $\Delta_i \oplus D_i$, $i \in N$, also ensures that the axiom of Strong invariance and the Dummy player axiom are logically independent. Indeed, define the value Φ on V_N , where $V_N = \Delta_i \oplus D_i$ for each $i \in N$, as follows:

$$\forall v \in V_N, \forall i \in N, \quad \Phi_i(v) = Bz_i(r^i) + w^i(N),$$

where $r^i + w^i$ is the unique decomposition of v along Δ_i and D_i . This value satisfies Strong addition invariance but violates the Dummy player axiom. Now assume that Φ is defined as:

$$\forall v \in V_N, \forall i \in N, \quad \Phi_i(v) = r^i(i) + Bz_i(w^i).$$

This value satisfies the Dummy player axiom but violates Strong addition invariance.

Acknowledgement

Financial support by the National Agency for Research (ANR) – research program “DynaMITE: Dynamic Matching and Interactions: Theory and Experiments”, contract ANR-13-BSHS1-0010 – and the “Mathématiques de la décision pour l’ingénierie physique et sociale” (MODMAD) project is gratefully acknowledged.

References

- [1] Albizuri MJ, and J Aurrekoetxea (2006) Coalition configurations and the Banzhaf index. *Soc Choice Welfare* 26: 571-596.
- [2] Algaba E, JM Bilbao, R van den Brink, and A Jiménez-Losada (2004), An axiomatization of the Banzhaf value for cooperative games on antimatroids, *Math Meth of Oper Res* 59: 147-166.
- [3] Banzhaf JF (1965) Weighted voting does not work: a mathematical analysis. *Rutgers Law Rev* 19:317-343.
- [4] Béal S, E Rémila, P Solal (2013) A decomposition of the space of TU-games using addition and transfer invariance. CRESE working paper no 2013-08.
- [5] Bilbao JM, A Jiménez, and JJ López (1998) The Banzhaf power index on convex geometries. *Math Soc Sc* 36:157-173
- [6] Casajus A (2012) Amalgamating players, symmetry, and the Banzhaf value. *Int J Game Theory* 41:497-515.
- [7] Casajus A (2014) Collusion, quarrel, and the Banzhaf value. *Int J Game Theory* 43:1-11.
- [8] Crama Y, L. Leruth (2013) Power indices and the measurement of control in corporate structures. *Int Game Theory Rev* 15:1340017.
- [9] Dubey P, LS Shapley (1979) Mathematical properties of the Banzhaf power index. *Math Oper Res* 4:99-131
- [10] Haller H (1994) Collusion properties of values. *Int. J Game Theory* 23:261-281.
- [11] Lehrer E (1988) An axiomatization of the Banzhaf value. *Int J Game Theory* 17:89-99.
- [12] Malawski M (2002) Equal treatment, symmetry and Banzhaf value axiomatizations. *Int J Game Theory* 31:47-67.
- [13] Owen G (1975) Multilinear extensions and the Banzhaf value. *Naval Research Logistics Quart* 22:741-750.
- [14] Owen G (1981) Modification of the Banzhaf-Coleman index for games with a priori unions. In: Holler MJ (ed) *Power, voting and voting power*. Physica-Verlag, Wurzburg, pp 232-238.
- [15] Shapley LS (1953) A value for n -person games. In: *Contributions to the Theory of Games II*, Ann. Math. Stud., Kuhn HW, Tucker AW (eds.), Princeton University Press, Princeton, 307-317.
- [16] Yokote K (2014) Weak addition invariance and axiomatization of the weighted Shapley value, *Int J Game Theory* (forthcoming, Published online: 16 May 2014, DOI 10.1007/s00182-014-0429-7).