CENTRE DE RECHERCHE SUR LES STRATÉGIES ÉCONOMIQUES

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February 2015

Working paper No. 2015 – 01

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Efficient extensions of the Myerson value☆

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Abstract

We study values for transferable utility games enriched by a communication graph (CO-games) where the graph does not necessarily affect the productivity but can influence the way the players distribute the worth generated by the grand coalition. Thus, we can envisage values that are efficient instead of values that are component efficient. For CO-games with connected graphs, efficiency and component efficiency coincide. In particular, the Myerson value (Myerson, 1977) is efficient for such games. Moreover, fairness is characteristic of the Myerson value. We identify the value that is efficient for all CO-games, coincides with the Myerson value for CO-games with connected graphs, and satisfies fairness.

Keywords: communication graph, fairness, efficiency, efficient extension, Shapley value, Myerson value
2010 MSC: 91A12, JEL: C71, D60

1. Introduction

The players involved in a cooperative game with transferable utilities, or simply TU-game, only differ with respect to the worth that the coalitions they belong to can obtain from cooperation. Nonetheless, an essential characteristic of many natural situations is that the players organize themselves into some hierarchical, technical, or communicational structure. It is therefore crucial to understand how the distribution of payoffs among the players can be affected by their social organization. Myerson (1977) proposes to model the

☆We are grateful to René van den Brink for valuable comments on this article. Financial support for Frank Huettner from the Deutsche Forschungsgemeinschaft (DFG) grant HU 2205/1-1 is gratefully acknowledged. Moreover, financial support from research programs "DynaMITE: Dynamic Matching and Interactions: Theory and Experiments", contract ANR-13-BSHS1-0010, and MODMAD is gratefully acknowledged.

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affinities between the players by an undirected graph. The combination of a TU-game and a graph is called a communication game, or simply CO-game, and a communication value, henceforth CO-value, evaluates the payoff that each player can claim for his participation in a CO-game.

Myerson (1977) assumes that a coalition is feasible if and only if its members are connected directly or indirectly through their links in the graph. Thus, the communication between the players is necessary to enable their cooperation. This interpretation leads to CO-values that are *component efficient*, i.e., the worth of each component of the graph is distributed among its members. The most prominent component efficient CO-values probably is the Myerson value (Myerson, 1977). The Myerson value can be characterized by component efficiency and fairness. Fairness states that adding a link to the graph changes the payoffs of the players forming this link by the same amount.

An alternative natural interpretation of the communication graph is that the players use their social links in order to improve their bargaining position in the negotiation process underlying a CO-game, as highlighted by Owen (1977) and Hart and Kurz (1983) in the framework of TU-games with a coalition structure. Communication among players is therefore not regarded to be necessary for establishing cooperation. As a consequence, this interpretation supports CO-values that are *efficient*, i.e., the worth of the grand coalition is distributed among its members. Efficient CO-values have been introduced by Casajus (2007) and more recently by Hamiache (2012), Béal et al. (2012), and van den Brink et al. (2012).

In this article, further developments on the design of efficient CO-values are investigated. Our study is motivated by two facts. First, we feel that the fairness property also is reasonable if a communication graph is mainly understood as a means of bargaining. This is particularly desirable because this property conveys a natural principle of distributive justice. Two players who establish a new channel for negotiations should equally benefit. Second, for CO-games with connected graphs, component efficiency and efficiency coincide. Thus, even though the two interpretations of a communication graph are incompatible for CO-games with unconnected graphs, they generally agree on the payoffs for CO-games with connected graphs.

A first step towards the analysis of efficient CO-values for CO-games is to consider CO-values on CO-games with connected graphs and to look for their *efficient extensions* to the class of CO-games with arbitrary graphs. By an efficient extension, we mean an *(a)* efficient CO-value that *(b)* coincides with the underlying CO-value on connected graphs, and *(c)* satisfies the fairness property characterizing the underlying CO-value. Hence, we are interested in the CO-values that are efficient, coincide with the Myerson value for CO-games with connected graphs, and satisfy fairness.

One may argue that the Myerson value is component efficient by nature. However, we support the argument that the Myerson value is an efficient and fair CO-value for CO-games with connected graphs by providing a new characterization of the Myerson value on this class of CO-games that uses efficiency instead of component efficiency. Then, we turn to the only efficient and fair extension of the Myerson value that has been provided in the literature (van den Brink et al., 2012). We prove its uniqueness as a consequence of the following more general result: There exists at most one extension of any fair and efficient CO-value for CO-games with connected graphs to the class of all CO-games. Based on these results, we then present a new characterization of the CO-value introduced by van den Brink et al. (2012).

It is worth to mention that our study exhibits some similarities with the literature on TU-games with a coalition structure, in which two analogous conflicting interpretations of the coalition structure coexist. The Owen value (Owen, 1977) is efficient while the value of Aumann and Drèze (1974) is component efficient, i.e., the worth of each component of the coalition structure is distributed among its members. For an efficient CO-value in the spirit of the Owen value, the reader is referred to Casajus (2007).

The article is organized as follows. Section 2 gives basic definitions and notations. The (component-)fair and efficient extensions of the Myerson value is studied in Section 3. Some remarks conclude the paper.

2. Cooperative games and graphs

Fix an infinite set $\mathfrak{U}$, the universe of players, and let $\mathcal{N}$ denote the set of non-empty and finite subsets of $\mathfrak{U}$.

A **TU-game** is a pair (N, v) consisting of a set of players $N \in \mathcal{N}$ and a **coalition function** $v \in \{f : 2^N \longrightarrow \mathbb{R} \mid f(\emptyset) = 0\}$, where 2^N denotes the power set of N. Subsets of N are called **coalitions**, and $v(S)$ is called the worth of coalition S. For any TU-game (N, v) and any $S \subseteq N$, the sub-game of (N, v) induced by S is denoted by $(S, v|_S)$, where $v|_S$ is the restriction of v to 2^S.

A **value** on $\mathcal{N}$ is an operator φ that assigns a payoff vector $\varphi(N, v) \in \mathbb{R}^N$ to any TU-game (N, v). The **Shapley value** (Shapley, 1953) is the value given by

$$\mathrm{SH}_i\left(N, v\right) = \sum_{S \subseteq N \setminus \{i\}} \frac{1}{|N|} \cdot \binom{|N|-1}{|S|}^{-1} \cdot \left(v\left(S \cup \{i\}\right) - v\left(S\right)\right)$$

for all TU-games (N, v), and $i \in N$.

A **communication graph** for $N \in \mathcal{N}$ is an undirected graph (N, L), $L \subseteq \mathcal{L}^N := \{\{i, j\} \,|\, i, j \in N, i \neq j\}$; a typical element (**link**) of L is written as $ij := \{i, j\}$. Player's $i, j \in N$ are called connected in (N, L) if there is a sequence of players $(i_1, i_2, \ldots, i_k)$, $k \in \mathbb{N}$, $k > 1$ from N such that $i_1 = i$, $i_k = j$, and $i_\ell i_{\ell+1} \in L$ for all $\ell \in \{1, \ldots, k-1\}$. It is clear that connectedness is an equivalence relation. Hence, it induces a partition $\mathcal{C}\left(N, L\right)$ of N, the set of **components** of (N, L), such that $C \in \mathcal{C}\left(N, L\right)$, $i, j \in C$, $k \in N \setminus C$, $i \neq j$ implies that i and j are connected and that i and k are not connected in (N, L). The component of (N, L) containing $i \in N$ is denoted by $C_i\left(N, L\right)$. The graph (N, L) is called **connected** if $\mathcal{C}\left(N, L\right) = \{N\}$.

A **CO-game** is a triple (N, v, L), where (N, v) is a TU-game and $L \subseteq \mathcal{L}^N$. We denote by $\mathcal{G}$ the set of all such CO-games. A CO-game is called connected if the associated graph is connected, and cycle-free is the associated graph is cycle-free. We denote by $\mathcal{G}_C \subseteq \mathcal{G}$ the class of all **connected CO-games**. A **CO-value** on some class of CO-games $\mathcal{G}^* \subseteq \mathcal{G}$ is an operator φ that assigns a payoff vector $\varphi(N,v,L) \in \mathbb{R}^N$ to every CO-game $(N,v,L) \in \mathcal{G}^*$.

The **Myerson value** (Myerson, 1977) is the CO-value on $\mathcal{G}$ given by

$$\mathrm{MY}\,(N,v,L) := \mathrm{SH}\left(N,v^L\right), \qquad v^L\,(S) := \sum_{T \in \mathcal{C}(S,L|_S)} v\,(T)\,, \quad S \subseteq N$$

It is characterized by component efficiency and fairness. Throughout this article, we sometimes invoke axioms on different subclasses of CO-games indicated by "$|_{\mathcal{G}^*}$", $\mathcal{G}^* \subseteq \mathcal{G}$ in their definition. For any such subclass, all the CO-games used in the axiom belong to the subclass.

Component efficiency, $\mathbf{CE}|_{\mathcal{G}^*}$. A value φ satisfies $\mathbf{CE}|_{\mathcal{G}^*}$ if for all $(N,v,L) \in \mathcal{G}^*$ and $C \in \mathcal{C}(N,L)$,

$$\sum_{i \in C} \varphi_i(N,v,L) = v(C).$$

Fairness, $\mathbf{F}|_{\mathcal{G}^*}$. A value φ satisfies $\mathbf{F}|_{\mathcal{G}^*}$ if for all $(N,v,L) \in \mathcal{G}^*$, and $ij \in L$ such that $(N,v,L \setminus \{ij\}) \in \mathcal{G}^*$,

$$\varphi_i\,(N,v,L) - \varphi_i\,(N,v,L \setminus \{ij\}) = \varphi_j\,(N,v,L) - \varphi_j\,(N,v,L \setminus \{ij\})\,.$$

Component efficiency states that the worth of each component of the graph is distributed among its members. Fairness requires that removing a link from the graph changes the payoffs of the players forming this link by the same amount.

Theorem 1. (Myerson, 1977) *The Myerson value is the unique CO-value on $\mathcal{G}$ that satisfies component efficiency ($\mathbf{CE}|_{\mathcal{G}}$) and fairness ($\mathbf{F}|_{\mathcal{G}}$).*

3. Efficient and fair extension of the Myerson value

For connected CO-games, the Myerson value is efficient, i.e., it satisfies the following property on $\mathcal{G}_C$.

Efficiency, $\mathbf{E}|_{\mathcal{G}^*}$. A value φ satisfies $\mathbf{E}|_{\mathcal{G}^*}$ if for all $(N,v,L) \in \mathcal{G}^*$, we have $\sum_{i \in N} \varphi_i\,(N,v,L) = v\,(N)\,.$

In this section, we explore the possibility of an efficient extension of the Myerson value for connected CO-games to the class of all CO-games. This extension shall be in the same spirit as the Myerson value. Since the Myerson value is efficient on the class of connected CO-games, one possibly would like this CO-value to coincide with the Myerson value for such games. Moreover, Myerson (1977) characterizes his CO-value via component efficiency and fairness, i.e., fairness can be considered to be the soul of his value. Thus, fairness seems to be a desirable property for the efficient version of the Myerson value.

In order to clarify that the Myerson value is an efficient and fair CO-value for connected CO-games, we invoke the following relative of fairness that has been suggested by Casajus (2009).[1]

[1] Note that the original statement of this property as "Weak fairness 2" contains a typo.

Connected fairness, $\mathbf{CNF}|_{\mathcal{G}^*}$. A value φ satisfies $\mathbf{CNF}|_{\mathcal{G}^*}$ if for all $(N, v, L) \in \mathcal{G}_C \cap \mathcal{G}^*$ and $ij \in L$, we have

$$\begin{aligned}&\varphi_i(N, v, L) - \varphi_i\left(C_i(N, L \setminus \{ij\}), v|_{C_i(N,L\setminus\{ij\})}, L \setminus \{ij\}|_{C_i(N,L\setminus\{ij\})}\right)\\&\quad= \varphi_j(N, v, L) - \varphi_j\left(C_j(N, L \setminus \{ij\}), v|_{C_j(N,L\setminus\{ij\})}, L \setminus \{ij\}|_{C_j(N,L\setminus\{ij\})}\right).\end{aligned}$$

Similarly to fairness, connected fairness considers the change of the payoffs of two players i and j if the link ij is removed. Either the players remain in the same component and connected fairness imposes the same condition as fairness. Or the players end up in different components. In this case, connected fairness compares the original payoffs with the payoffs obtained if the CO-game is restricted to each player's component, respectively, and imposes an equal change of the payoffs. Note that all graphs involved in this axiom are connected. We provide the following characterization of the Myerson value on the class of connected CO-games.[2]

Theorem 2. *A CO-value φ on $\mathcal{G}_C$ satisfies efficiency ($\boldsymbol{E}|_{\mathcal{G}_C}$) and connected fairness ($\boldsymbol{CNF}|_{\mathcal{G}_C}$) if and only if $\varphi = \mathrm{MY}$ on $\mathcal{G}_C$.*

Proof. By Theorem 1, the Myerson value on $\mathcal{G}_C$ satisfies $\mathbf{E}|_{\mathcal{G}_C}$ and $\mathbf{F}|_{\mathcal{G}_C}$. Concerning $\mathbf{CNF}|_{\mathcal{G}_C}$, let $(N, v, L) \in \mathcal{G}_C$ and $ij \in L$ be as in the axiom. If removing ij does not split the component, $\mathbf{CNF}|_{\mathcal{G}_C}$ is implied by $\mathbf{F}|_{\mathcal{G}_C}$.

If removing ij splits the component, then

$$\begin{aligned}&\mathrm{MY}_i(N, v, L) - \mathrm{MY}_j(N, v, L)\\=&\mathrm{MY}_i(N, v, L \setminus \{ij\}) - \mathrm{MY}_j(N, v, L \setminus \{ij\})\\=&\mathrm{MY}_i\left(C_i(N, L \setminus \{ij\}), v|_{C_i(N,L\setminus\{ij\})}, L \setminus \{ij\}|_{C_i(N,L\setminus\{ij\})}\right)\\&- \mathrm{MY}_j\left(C_j(N, L \setminus \{ij\}), v|_{C_j(N,L\setminus\{ij\})}, L \setminus \{ij\}|_{C_j(N,L\setminus\{ij\})}\right).\end{aligned}$$

where the first equation follows from $\mathbf{F}|_{\mathcal{G}_C}$ and the second equation drops from the fact that MY satisfies component decomposability[3] (van den Nouweland, 1993, pp. 28-29).

Hence, we only have to prove that at most one value satisfies these two axioms. By contradiction, assume that two different CO-values φ and ψ on $\mathcal{G}_C$ satisfy $\mathbf{E}|_{\mathcal{G}_C}$ and $\mathbf{CNF}|_{\mathcal{G}_C}$. Let $N \in \mathcal{N}$ be such that $\varphi(N', v, L) = \psi(N', v, L)$ for all $(N', v, L) \in \mathcal{G}_C$ with $|N'| < |N|$. Let $L \subseteq \mathcal{L}^N$ be such that $\varphi(N, v, L') = \psi(N, v, L')$ for all $(N, v, L') \in \mathcal{G}_C$ with $L' \subsetneq L$. By $\mathbf{E}|_{\mathcal{G}_C}$, it holds that $|N| \geq 2$. Furthermore, $L \neq \emptyset$ since $(N, v, L) \in \mathcal{G}_C$. For each $ij \in L$,

[2] According to Theorem 2, there exists a redundancy in the similar but weaker result due to Casajus (2009, Lemma 4.2).

[3] Component decomposability: For all $(N, v, L) \in \mathcal{G}^*$, $C \in \mathcal{C}(N, L)$, and $i \in C$ such that $(C, v|_C, L|_C) \in \mathcal{G}^*$, we have $\varphi_i(N, v, L) = \varphi_i(C, v|_C, L|_C)$.

$\mathbf{CNF}|_{\mathcal{G}_C}$ and the assumptions on N and L yield that

$$\begin{aligned}\varphi_i(N,v,L)-\varphi_j(N,v,L) &= \varphi_i\left(C_i\left(N,L\setminus\{ij\}\right),v|_{C_i(N,L\setminus\{ij\})},L\setminus\{ij\}\,|_{C_i(N,L\setminus\{ij\})}\right)\\ &\quad -\varphi_j\left(C_j\left(N,L\setminus\{ij\}\right),v|_{C_j(N,L\setminus\{ij\})},L\setminus\{ij\}\,|_{C_j(N,L\setminus\{ij\})}\right)\\ &= \psi_i\left(C_i\left(N,L\setminus\{ij\}\right),v|_{C_i(N,L\setminus\{ij\})},L\setminus\{ij\}\,|_{C_i(N,L\setminus\{ij\})}\right)\\ &\quad -\psi_j\left(C_j\left(N,L\setminus\{ij\}\right),v|_{C_j(N,L\setminus\{ij\})},L\setminus\{ij\}\,|_{C_j(N,L\setminus\{ij\})}\right)\\ &= \psi_i(N,v,L)-\psi_j(N,v,L),\end{aligned}$$

i.e., $\varphi_i(N,v,L)-\psi_i(N,v,L)=\varphi_j(N,v,L)-\psi_j(N,v,L)$. Since (N,L) is connected, there is a chain of links connecting any $k,\ell\in N$. Hence, we have $\varphi_k(N,v,L)-\psi_k(N,v,L)=\varphi_\ell(N,v,L)-\psi_\ell(N,v,L)=\Delta$ for all $k,\ell\in N$ and some $\Delta\in\mathbb{R}$. Now, $\mathbf{E}|_{\mathcal{G}_C}$ implies $\Delta=0$, a contradiction. ☐

Among the efficient CO-values that coincide with the Myerson value on connected CO-games and that have been discussed in the literature, only one CO-value satisfies fairness. In the following, we call this value the **Efficient Egalitarian Myerson value**, given by

$$\mathrm{EEMY}_i\left(N,v,L\right):=\mathrm{MY}_i\left(N,v,L\right)+\frac{v\left(N\right)-v^L(N)}{|N|}.$$

van den Brink et al. (2012) introduce and characterize the Efficient Egalitarian Myerson value using the following axiom.

Fair distribution of surplus, $\mathbf{FDS}|_{\mathcal{G}^*}$. A value φ satisfies $\mathbf{FDS}|_{\mathcal{G}^*}$ if for all $(N,v,L)\in\mathcal{G}^*$, and $C,C'\in\mathcal{C}\left(N,L\right)$, we have

$$\sum_{i\in C}\frac{\varphi_i\left(N,v,L\right)-\varphi_i\left(C,v|_C,L|_C\right)}{|C|}=\sum_{i\in C'}\frac{\varphi_i\left(N,v,L\right)-\varphi_i\left(C',v|_{C'},L|_{C'}\right)}{|C'|}.$$

Fair distribution of surplus requires that the average change of the payoffs of the players in a component $C\in\mathcal{C}\left(N,L\right)$ equals the average change of the players in any other component $C'\in\mathcal{C}\left(N,L\right)$ if one compares the payoffs in the restriction of the CO-game to the respective component with the payoffs of the original CO-game.

Theorem 3. (van den Brink et al., 2012) *A CO-value φ on $\mathcal{G}$ satisfies efficiency ($\boldsymbol{E}|_{\mathcal{G}}$), fairness ($\boldsymbol{F}|_{\mathcal{G}}$), and fair distribution of surplus ($\boldsymbol{FDS}|_{\mathcal{G}}$) if and only if $\varphi=\mathrm{EEMY}$.*

Since the Efficient Egalitarian Myerson value coincides with the Myerson value for connected CO-games, Theorem 3 implies that the restriction of the Myerson value to connected CO-games has an efficient and fair extension to the whole domain of CO-games. This triggers the question on whether there exist other (possibly less egalitarian) efficient and fair extensions of the Myerson value from connected CO-games to the class of all CO-games. Such extensions do not exist. This is a direct consequence of the next result, which shows that for every fair and efficient CO-value on the class of connected CO-games, there exists at most one fair and efficient extension to the class of all CO-games.[4]

[4]The proof of Theorem 4 indicates that it can be sharpened by just requiring $\varphi\left(N,v,L\right)=\psi\left(N,v,L\right)$ for all $L\subseteq\mathcal{L}^N$ such that $\left|\mathcal{C}\left(N,\mathcal{L}^N\setminus L\right)\right|>1$. Note that $\left|\mathcal{C}\left(N,\mathcal{L}^N\setminus L\right)\right|>1$ entails $\left|\mathcal{C}\left(N,L\right)\right|=1$.

Theorem 4. *Let φ and ψ be two CO-values on $\mathcal{G}$ that satisfy efficiency ($\boldsymbol{E}|_{\mathcal{G}}$) and fairness ($\boldsymbol{F}|_{\mathcal{G}}$). If $\varphi(N,v,L) = \psi(N,v,L)$ for all $(N,v,L) \in \mathcal{G}_C$, then $\varphi = \psi$ for all $(N,v,L) \in \mathcal{G}$.*

Proof. Let the CO-values φ and ψ be as in the theorem. Suppose $\varphi \neq \psi$ and consider any $N \in \mathcal{N}$ such that $\varphi(N,v,L) \neq \psi(N,v,L)$ for some $(N,v,L) \in \mathcal{G}$. There is some maximal $L \subseteq \mathcal{L}^N$ and some $i \in N$ such that $\varphi_i(N,v,L) \neq \psi_i(N,v,L)$. By assumption, $(N,v,L) \in \mathcal{G} \setminus \mathcal{G}_C$ so that $|\mathcal{C}(N,L)| > 1$. Let $C := C_i(N,L)$ and choose any $j \in N \setminus C$. By $\mathbf{F}|_{\mathcal{G}}$ and the maximality of L, we have

$$\begin{aligned}\varphi_i(N,v,L) - \varphi_k(N,v,L) &= \varphi_i(N,v,L\cup\{ik\}) - \varphi_k(N,v,L\cup\{ik\}) \\ &= \psi_i(N,v,L\cup\{ik\}) - \psi_k(N,v,L\cup\{ik\}) \\ &= \psi_i(N,v,L) - \psi_k(N,v,L)\end{aligned}$$

for all $k \in N \setminus C$. Analogously, one shows

$$\varphi_j(N,v,L) - \varphi_\ell(N,v,L) = \psi_j(N,v,L) - \psi_\ell(N,v,L)$$

for all $\ell \in C \setminus \{i\}$. Hence, we have

$$\varphi_i(N,v,L) - \psi_i(N,v,L) = \varphi_j(N,v,L) - \psi_j(N,v,L)$$

for all $j \in N$. Summing up over $j \in N$ gives

$$|N| \cdot (\varphi_i(N,v,L) - \psi_i(N,v,L)) = \sum_{j\in N} \varphi_j(N,v,L) - \sum_{j\in N} \psi_j(N,v,L) \overset{\mathbf{E}|_{\mathcal{G}}}{=} 0,$$

i.e., $\varphi_i(N,v,L) = \psi_i(N,v,L)$. Contradiction. □

Applying the previous Theorem 4 to the Myerson value immediately yields the following theorem.

Theorem 5. *The Efficient Egalitarian Myerson value* EEMY *is the unique efficient and fair extension of the Myerson value, i.e., φ satisfies $\varphi =$ MY on $\mathcal{G}_C$ and meets efficiency ($\boldsymbol{E}|_{\mathcal{G}}$) and fairness ($\boldsymbol{F}|_{\mathcal{G}}$), if and only if $\varphi =$ EEMY on $\mathcal{G}$.*

Finally, we obtain a new characterization of the Efficient Egalitarian Myerson value.

Theorem 6. *The Efficient Egalitarian Myerson value* EEMY *is the unique CO-value on $\mathcal{G}$ that satisfies efficiency ($\boldsymbol{E}|_{\mathcal{G}}$), fairness ($\boldsymbol{F}|_{\mathcal{G}}$), and connected fairness ($\boldsymbol{CNF}|_{\mathcal{G}}$).*

Proof. *Existence*: By Theorem 3, EEMY satisfies $\mathbf{E}|_{\mathcal{G}}$ and $\mathbf{F}|_{\mathcal{G}}$. From Theorem 2, it is immediate that EEMY satisfies $\mathbf{CNF}|_{\mathcal{G}}$. *Uniqueness*: Let φ satisfy $\mathbf{E}|_{\mathcal{G}}$, $\mathbf{F}|_{\mathcal{G}}$, and $\mathbf{CNF}|_{\mathcal{G}}$. By Theorem 2, $\varphi =$ MY $=$ EEMY on $\mathcal{G}_C$. Using Theorem 4, one obtains $\varphi =$ EEMY on $\mathcal{G}$. □

Remark 1. *This characterization of the Efficient Egalitarian Myerson value is non-redundant. The null value* Null *given by* $\text{Null}_i(N, v, L) = 0$ *for all* $(N, v, L) \in \mathcal{G}$ *and* $i \in N$ *satisfies fairness and connected fairness but not efficiency. The equal division solution* ED *given by* $\text{ED}_i(N, v, L) = v(N) / |N|$ *for all* $(N, v, L) \in \mathcal{G}$ *and* $i \in N$ *satisfies efficiency and fairness but not connected fairness. The CO-value* $\varphi^{\heartsuit}$ *given by*

$$\varphi_i^{\heartsuit}(N, v, L) = \text{MY}_i(v) + \frac{v(N) - v^L(N)}{|\mathcal{C}(N, L)| \cdot |C_i(N, L)|}$$

for all $(N, v, L) \in \mathcal{G}$ *and* $i \in N$ *satisfies efficiency and connected component fairness but not connected fairness. Note that the examples also show that fairness neither implies nor is implied by connected fairness.*

Remark 2. *Among all the efficient CO-values, the Efficient Egalitarian Myerson value is the unique value that minimizes the euclidean distance to the Myerson value. More specifically, it is straightforward to show that*

$$\text{EEMY}(N, v, L) = \underset{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N)}{\mathit{argmin}} \; d(x, \text{MY}(N, v, L))$$

for all $(N, v, L) \in \mathcal{G}$, *where* $d(x, y) := \sqrt{\sum_{i \in N} (x_i - y_i)^2}$ *denotes the Euclidean distance between* $x \in \mathbb{R}^N$ *and* $y \in \mathbb{R}^N$.[5]

4. Concluding remarks

We emphasize that the Myerson value is an efficient CO-value for connected CO-games by providing a characterization that employs efficiency instead of component efficiency, and that works within the class of connected CO-games. Further, we consider fair and efficient extensions to the class of all CO-games. It turns out that only one such extension exists. It assigns to every player an equal share of the surplus created between the components, $\left(v(N) - v^L(N)\right) / |N|$, plus his payoff according to the Myerson value. We provide a characterization of this CO-value that rests on fairness properties and on efficiency.

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[5] The Lagrangian $L(x, \lambda) = d(x, \text{MY}(N, v, L)) + \lambda \cdot \left(v(N) - \sum_{i \in N} x_i\right)$ yields the first order conditions $\sum_{i \in N} x_i = v(N)$ and $\lambda = 2 \cdot (x_i - \text{MY}_i(N, v, L))$ for all $i \in N$. Summing up the latter equations over $i \in N$ and inserting $\sum_{i \in N} x_i = v(N)$ and $\sum_{i \in N} \text{MY}_i(N, v, L) = v^L(N)$ gives $|N| \cdot \lambda = 2 \cdot \left(v(N) - v^L(N)\right)$. This implies $x_i = \text{EEMY}_i(N, v, L)$ for all $i \in N$.

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