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for TU-games

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# A Class of Solidarity Allocation Rules for TU-games

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## Abstract

A new class of allocation rules combining marginalistic and egalitarian principles is introduced for cooperative TU-games. It includes some modes of solidarity among the players by taking the collective contribution of some coalitions to the grand coalition into account. Relationships with other class of allocation rules such as the Egalitarian Shapley values and the Procedural values are discussed. Two axiomatic characterizations are provided: one of the whole class of allocation rules, and one of each of its extreme points.

*Keywords:* TU-games, Solidarity, null player, Egalitarian Shapley value, Procedural values.

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## 1. Introduction

One of the main issues in economic allocation problems is the trade-off between marginalism and egalitarianism, which can be tackled by cooperative games with transferable utility. A cooperative game with transferable utility (TU-game) on a given player set specifies, for each coalition of players, a worth measuring the best possible result for the coalition should its members cooperate without the help of any other player. An allocation rule assigns a payoff vector to each such game, which can be interpreted as the payoffs given by a regulator to the players for participating in the game.

The Shapley value (Shapley, 1953) and the egalitarian division rule are two well-known allocation rules, but each only incorporates one of the two above-mentioned principles. Assuming that the grand coalition has formed by a succession of one-by-one arrivals, the Shapley value of an player is equal to his expected contribution to the coalition of players he joins upon arriving. Therefore, the Shapley value is exclusively based on a marginalistic principle. As a consequence, unproductive players get zero payoff, which means that the Shapley value rules out every kind of solidarity between the players. By contrast, the Egalitarian Division rule, which divides the worth achieved by the grand coalition equally among all players, does not depend on the players' contributions at all, and as such can be seen as too solidaristic. Although these two allocation rules seem to be rather opposite, they both satisfy basic axioms such as Efficiency, Anonymity and Linearity. The class of allocation rules satisfying these three axioms has been studied and characterized by Ruiz et al. (1998) and Radzik and Driessen (2013), among others. In the latter article, it is proved that

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any efficient, anonymous and linear allocation rule can be formulated as the Shapley value of an appropriately modified game.

A growing literature in which less extreme visions of the solidarity principle are invoked to design allocation rules has emerged within the class of efficient, anonymous and linear allocation rules. Such allocation rules incorporate some modes of solidarity: the most productive players should obtain a better treatment, but a solidarity principle should ensure a reduction of the payoffs inequalities with the less productive players. The Equal Surplus Division rule (Driessen and Funaki, 1991) first assigns to each player his stand-alone worth and then splits equally what remains of the worth of the grand coalition. The Solidarity value (Sprumont, 1990; Nowak and Radzik, 1994) is similar to the Shapley value, except that the contribution of a player to a coalition is replaced by the average contribution over the coalition's members. The Least Square Prenucleolus (Ruiz et al., 1996) first assigns to each player his Banzhaf value (Banzhaf, 1965) and then splits equally what remains of the worth of the grand coalition. The Consensus values (Ju et al., 2007) is the class of all convex combinations between the Shapley value and the Equal Surplus Division rule. The Egalitarian Shapley values (Joosten, 1996; van den Brink et al., 2013) is the class of all convex combinations between the Shapley value and the Egalitarian Division rule. The allocation rules belonging to the class introduced in Casajus and Huettner (2014a) are distinguished by the type of player whose removal from a game does not affect the remaining players' payoffs. This class contains the Shapley value and the Egalitarian Division rule as extreme points, and the Solidarity value is its center. The Procedural values (Malawski, 2013) is a class of allocation rules similar in spirit to the Shapley value except that the contribution of the arriving player can be arbitrarily shared among him and the players arrived before him. The Solidarity value, the Egalitarian Division rule, the Shapley value and the Egalitarian Shapley values are instances of the Procedural values. These relationships have allowed for comparable axiomatic characterizations of these allocation rules (see van den Brink, 2007; Kamijo and Kongo, 2012; Chameni Nembua, 2012; Casajus and Huettner, 2013, 2014a,b, in addition to the aforementioned articles). The Equal Surplus Division rule, the Consensus values and the Least Square Prenucleolus are not Procedural values; and the class studied in Casajus and Huettner (2014a) is not related to the class of Procedural values by set inclusion.

In this article, we introduce a new class of allocation rules which combines marginalistic and egalitarian principles and which is included in the class of efficient, anonymous and linear allocation rules. As for the Solidarity value and the Procedural values, we keep Shapley's idea that the one-by-one formation of the grand coalition is modeled by permutations of the players. Nonetheless, instead of rewarding every player with his (individual) contribution to the coalition he joins upon entering, we also rely on the notion of collective contribution to the grand coalition so as to reflect some aspects of solidarity. More specifically, for a coalition  $S$ , the collective contribution of  $S$  to the grand coalition  $N$  is measured by the difference between the worth of  $N$  and the worth of the coalition of players in  $N$  but not in  $S$ . In a sense, if  $S$  was considered as a single entity, then the collective contribution of  $S$  to  $N$  would reduce to the individual contribution of player  $S$  to  $N$ . The computation of the Shapley value only involves individual contributions. To the contrary, the Egalitarian Division rule only rests on the collective contribution of the grand coalition to itself, which then is split evenly.

In the building blocks of our class of allocation rules, the collective contribution to the grand coalition replaces the individual contributions as soon as the currently formed coalition has reached some size  $p$ . Thus, there are two distinct steps. Before attaining the critical size  $p$ , each entering

player gets his individual contribution to the coalition he joins. When the critical size  $p$  is reached, the remaining players keep on entering one by one, but instead of rewarding each of them when entering, they cumulate their contributions until the grand coalition is formed. Then, the collective contribution of the coalition of remaining players to the grand coalition is evenly distributed among them. This procedure can be interpreted as the creation of a mutual fund by these remaining players, which is used for promoting equality among them, creating de facto some solidarity. Our construction procedure can be justified by two phenomena. On the one hand, the fact that the mutual fund is established when some size  $p$  is attained seems consistent with both empirical and theoretical findings as emphasized by García and Vanden (2009, pp. 1980). On the other hand, the appeal to a regulator to ensure some solidarity among the players is sometimes necessary. For instance, in the context of health insurances, Stone (1993) points out that a mutual insurance can hardly be implemented without the coercive authority of a state. The author underlines that the competitive insurance industry in the U.S. often leads to fragmentation of the society into ever-smaller, more homogeneous groups, which in turn implies the destruction of mutual aids.

We call  $\text{Sol}^p$  the allocation rule defined by averaging the payoff vector described in the previous paragraph over all permutations of the players. The class of solidarity allocation rules that we study, denoted by  $\mathbf{Sol}_N$ , is the convex combination of all  $\text{Sol}^p$  allocation rules. The class  $\mathbf{Sol}_N$  and its elements are investigated through the following two types of results.

Firstly, we relate our class to the previously mentioned allocation rules and class of allocation rules. Proposition 6 shows that  $\mathbf{Sol}_N$  is (strictly) included in the class of Procedural values. Nevertheless, although the construction of  $\mathbf{Sol}_N$  and the Procedural values are different, Proposition 6 also provides an alternative formulation of each extreme point  $\text{Sol}^p$  of  $\mathbf{Sol}_N$  in terms of Procedural values. More specifically,  $\text{Sol}^p$  coincides with the Procedural value in which the contribution of the entering player is assigned to himself if the size of the current coalition is not larger than  $p + 1$ , and to the player entered in position  $p + 1$  otherwise. Since  $\text{Sol}^0$  and  $\text{Sol}^{n-1}$  coincide with the Egalitarian Division rule and the Shapley value, respectively,  $\mathbf{Sol}_N$  also contains the class of all Egalitarian Shapley values. Furthermore,  $\mathbf{Sol}_N$  includes the Solidarity value. This result follows from Proposition 8, which characterizes  $\mathbf{Sol}_N$  by means of the modified game studied in Radzik and Driessen (2013). Based on this result, we derive closed form expression of each  $\text{Sol}^p$ .

Secondly, we provide an axiomatic characterization of  $\mathbf{Sol}_N$  and of each of its extreme points  $\text{Sol}^p$ . Proposition 9 characterizes each  $\text{Sol}^p$  allocation rule by the standard axioms of Efficiency, Equal treatment of equals and Additivity together with the new  $p$ -null player axiom. The latter axiom falls in line with other parametrized alterations of the null player axiom (see Ju et al., 2007; Kamijo and Kongo, 2012; Chameni Nembua, 2012; Casajus and Huettner, 2014a; Béal et al., 2015).<sup>1</sup> It assigns a null payoff to a player who has null contribution to coalitions of size less than  $p$  and such that every coalition of size  $p$  without this player has the same worth as the grand coalition. In order to characterize the class  $\mathbf{Sol}_N$ , we invoke in Proposition 10 the four axioms Efficiency, Additivity, Desirability and Monotonicity used by Malawski (2013) to characterize the larger class of Procedural values, and add the new axiom of Null player in a null environment for positive games. The latter axiom imposes that a null player does not obtain a positive payoff if both the worth of all coalitions are non-negative and if the grand coalition achieves a zero worth. This axiom aims at emphasizing the limits of the solidarity when the resources available to the society are not sufficient to redistribute monetary payoffs to unproductive players. Our characterization is also comparable to Theorem 2 in Casajus and Huettner (2013), which characterizes the Egalitarian

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<sup>1</sup>The modified version of the null player axiom invoked in Nowak and Radzik (1994) does not rely on a parameter.

Shapley values by Efficiency, Additivity, Desirability and Null player in a productive environment. The latter axiom points out situations in which solidarity is possible by requiring that null players obtain non-negative payoff if the grand coalition has a non-negative worth. One can move from our Proposition 10 to Theorem 2 in Casajus and Huettner (2013) by dropping Monotonicity and by replacing Null player in a null environment for positive games by Null player in a productive environment. Finally, Proposition 10 implies that  $\mathbf{Sol}_N$  neither includes nor is included in the class of allocation rules studied in Casajus and Huettner (2014a).

The rest of the article is organized as follows. Section 2 gives the basic definitions and contextualizes our study by stating the closest results. The class  $\mathbf{Sol}_N$  and its extreme points are constructed in section 3. Their properties are studied in section 4. Section 5 contains the axiomatic characterizations. Section 6 concludes. All proofs are relegated to the Appendix.

## 2. Preliminaries

### 2.1. TU-games

Throughout this article, the cardinality of a finite set  $S$  will be denoted by the lower case  $s$ , the collection of all subsets of  $S$  will be denoted by  $2^S$ , and, for notational convenience, we will write singleton  $\{i\}$  as  $i$ .

Let  $N = \{1, 2, \dots, n\}$  be a fixed and finite set of  $n$  players. Each subset  $S$  of  $N$  is called a *coalition* while  $N$  is called the *grand coalition*. A cooperative game with transferable utility or simply a *TU-game* on a fixed player set  $N$  is a function  $v : 2^N \rightarrow \mathbb{R}$  such that  $v(\emptyset) = 0$ . For each coalition  $S \subseteq N$ ,  $v(S)$  describes the *worth* of the coalition  $S$  when its members cooperate. For any two TU-games  $v$  and  $w$  in  $V_N$  and any  $\alpha \in \mathbb{R}$ , the TU-game  $\alpha v + w \in V_N$  is defined as follows: for each  $S \subseteq N$ ,  $(\alpha v + w)(S) = \alpha v(S) + w(S)$ . A TU-game  $v \in V_N$  is *positive* if, for each  $S \subseteq N$ ,  $v(S) \geq 0$ . A TU-game  $v \in V_N$  is *monotone* if, for each  $S \subseteq T \subseteq N$ ,  $v(S) \leq v(T)$ .

A permutation  $\sigma$  on  $N$  assigns a position  $\sigma(i)$  to each player  $i \in N$ . Let  $\Sigma_N$  be the set of  $n!$  permutations on  $N$ . Given  $v \in V_N$  and  $\sigma \in \Sigma_N$ ,  $\sigma v \in V_N$  is defined as: for each nonempty  $S \subseteq N$ ,  $\sigma v(\cup_{i \in S} \sigma(i)) = v(S)$ .

Two distinct players  $i \in N$  and  $j \in N$  are *equals* in  $v \in V_N$  if for each  $S \subseteq N \setminus \{i, j\}$ , it holds that  $v(S \cup i) = v(S \cup j)$ . Player  $i \in N$  is a *nullifying player* in  $v$  if for each coalition  $S \ni i$ , it holds that  $v(S) = 0$ . Player is *null* in  $v$  if, for each coalition  $S \ni i$ ,  $v(S) = v(S \setminus i)$ .

### 2.2. Allocation rules

An *allocation rule*  $\Phi$  on  $V_N$  is a mapping  $\Phi : V_N \rightarrow \mathbb{R}^n$  which uniquely determines, for each  $v \in V_N$  and each  $i \in N$ , a payoff  $\Phi_i(v) \in \mathbb{R}$  for participating to  $v \in V_N$ . In this article we consider the following allocation rules.

The *Egalitarian Division rule*, ED, is defined on  $V_N$  as:

$$\forall i \in N, \quad \text{ED}_i(v) = \frac{v(N)}{n}.$$

For any permutation  $\sigma$  and any player  $i \in N$ , define the coalition containing player  $i$  and the set of its predecessors in  $\sigma$  as  $P_i^\sigma = \{j \in N : \sigma(j) \leq \sigma(i)\}$ . The *Shapley value* (Shapley, 1953), Sh, is defined on  $V_N$  as follows:

$$\forall i \in N, \quad \text{Sh}_i(v) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} (v(P_i^\sigma) - v(P_i^\sigma \setminus i)) = \sum_{S \in 2^N : S \ni i} \frac{(n-s)!(s-1)!}{n!} (v(S) - v(S \setminus i)).$$

The set of Egalitarian Shapley values has been suggested by Joosten (1996). Each allocation rule belonging to this class, denoted by  $\text{EDSh}^\alpha$ , is a convex combination of Sh and ED, i.e. there is  $\alpha \in [0, 1]$  such that:

$$\text{EDSh}^\alpha(v) = \alpha \text{Sh}(v) + (1 - \alpha) \text{ED}(v).$$

Malawski (2013) introduces a set of allocation rules, called the Procedural values. A Procedural value is the average of contribution vectors associated with all permutations of the player set, where, for each permutation and each player, a procedure specifies how the contribution of this player is shared among him and all his predecessors in the permutation. Formally, a procedure  $l$  on  $V_N$  is a collection of nonnegative coefficients  $((l_{p,q})_{q=1}^p)_{p=1}^n$  such that for each  $p \in \{1, \dots, n\}$ ,  $\sum_{q=1}^p l_{q,p} = 1$ . The coefficient  $l_{p,q}$  specifies the share of player at position  $q \leq p$  in the contribution of player at position  $p$  in the permutation. Obviously,  $l_{11} = 1$ . For each permutation  $\sigma \in \Sigma_N$ , and each  $v \in V_N$ , the procedure  $l$  generates a contribution vector  $r^{\sigma,l}(v)$  defined as follows:

$$\forall i \in N, \quad r_i^{\sigma,l}(v) = \sum_{j \in (N \setminus P_i^\sigma) \cup i} l_{\sigma(j), \sigma(i)} (v(P_j^\sigma) - v(P_j^\sigma \setminus j)). \quad (1)$$

The Procedural value associated with procedure  $l$  is the allocation rule  $\text{PV}^l$  on  $V_N$  defined as follows:

$$\forall i \in N, \quad \text{PV}_i^l(v) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} r_i^{\sigma,l}(v).$$

### 2.3. Axioms for allocation rules

An allocation rule  $\Phi$  on  $V_N$  satisfies:

**Efficiency** if for each  $v \in V_N$ , it holds that:  $\sum_{i \in N} \Phi_i(v) = v(N)$ ;

**Anonymity** if for each  $v \in V_N$  and each  $\sigma \in \Sigma_N$ , it holds that:  $\Phi_i(v) = \Phi_{\sigma(i)}(\sigma v)$ ;

**Equal treatment of equals** if for each  $v \in V_N$  and each pair  $\{i, j\} \subseteq N$  of equal players in  $v$ , it holds that:  $\Phi_i(v) = \Phi_j(v)$ ;

**Desirability**<sup>2</sup> if for each  $v \in V_N$  and each pair  $\{i, j\} \subseteq N$  such that, for each  $S \subseteq N \setminus \{i, j\}$ ,  $v(S \cup i) \geq v(S \cup j)$ , it holds that:  $\Phi_i(v) \geq \Phi_j(v)$ ;

**Null player axiom** if for each  $v \in V_N$  and each null player  $i \in N$  in  $v$ , it holds that:  $\Phi_i(v) = 0$ ;

**Null player axiom in a productive environment** if for each  $v \in V_N$  such that  $v(N) \geq 0$ , and each null player  $i \in N$  in  $v$ , it holds that:  $\Phi_i(v) \geq 0$ ;

**Nullifying player axiom** if for each  $v \in V_N$  and each nullifying player  $i \in N$  in  $v$ , it holds that:  $\Phi_i(v) = 0$ ;

**Linearity** if for each  $v$  and  $w$  in  $V_N$  and each  $\alpha \in \mathbb{R}$ , it holds that:  $\Phi(\alpha v + w) = \alpha \Phi(v) + \Phi(w)$ ;

**Additivity** if for each  $v$  and  $w$  in  $V_N$ , it holds that:  $\Phi(v + w) = \Phi(v) + \Phi(w)$ ;

**Monotonicity**<sup>3</sup> if in each monotone game  $v \in V_N$  and each  $i \in N$ , it holds that:  $\Phi_i(v) \geq 0$ .

<sup>2</sup>Desirability appears in the literature under different names, such as local monotonicity (e.g. Malawski, 2013; van den Brink et al., 2013) or Fair treatment (e.g. Radzik and Driessen, 2013).

<sup>3</sup>Monotonicity is also known as Positivity (e.g. Kalai and Samet, 1987) and Weak monotonicity (e.g. Malawski, 2013). We refrain from using the latter name because Weak monotonicity is used in van den Brink et al. (2013) for a weak version of Strong monotonicity as introduced by Young (1985).

Note that Anonymity implies Equal treatment of equals, Desirability implies Equal treatment of equals, Linearity implies Additivity, the Null player axiom implies the Null player axiom in a productive environment. In the rest of the article, we use the acronyms  $\mathbf{EAL}_N$  for the set of all allocation rules satisfying Efficiency, Anonymity and Linearity on  $V_N$ ,  $\mathbf{Pv}_N$  for the set of all Procedural values on  $V_N$ , and  $\mathbf{EDSh}_N$  for the set of all Egalitarian-Shapley values on  $V_N$ .

#### 2.4. Some existing results

Various characterizations of the Shapley value have been given in the literature. One of the most famous characterizations uses Additivity, Efficiency, Equal Treatment of Equals, and the Null player axiom. It can be easily derived from the seminal article by Shapley (1953).

**Proposition 1** (Shapley, 1953)

*An allocation rule  $\Phi$  on  $V_N$  is equal to Sh if and only if it satisfies Efficiency, Equal treatment of equals, Additivity, and the Null player axiom.*

Deleting the Null player axiom and adding the Nullifying player axiom in the statement of Proposition 1 yields the Egalitarian Division rule. This result is due to van den Brink (2007).

**Proposition 2** (van den Brink, 2007)

*An allocation rule  $\Phi$  on  $V_N$  is equal to ED if and only if it satisfies Efficiency, Equal treatment of equals, Additivity, and the Nullifying player axiom.*

Casajus and Huettner (2013) show that substituting Equal treatment of equals and the Null player axiom in the statement of Proposition 1 by Desirability and the Null player axiom in productive environment selects the set of Egalitarian-Shapley values.

**Proposition 3** (Casajus and Huettner, 2013, Theorem 2)

*An allocation rule  $\Phi$  on  $V_N$  belongs to  $\mathbf{EDSh}_N$  if and only if it satisfies Efficiency, Desirability, Additivity, and the Null player axiom in a productive environment.*

In order to select the Procedural values, Malawski (2013) replaces in Proposition 3 the Null player axiom in a productive environment by Monotonicity.

**Proposition 4** (Malawski, 2013, Theorem 3)

*An allocation rule  $\Phi$  on  $V_N$  belongs to  $\mathbf{Pv}_N$  if and only if it satisfies Efficiency, Desirability, Additivity, and Monotonicity.*

Propositions 1-4 offer comparable characterizations of different types of allocation rules mixing marginalist and egalitarian principles. In fact, Theorem 3 in Malawski (2013) invokes Linearity instead of Additivity. In general, Additivity does not imply Linearity. However, (Casajus and Huettner, 2013, Lemma 5) show that this implication holds in presence of Efficiency and Desirability. Therefore, Linearity can be replaced by Additivity in the original result by Malawski (2013). It should be also clear that each Egalitarian Shapley value satisfies Monotonicity, which implies that that the set of Procedural values contains the set of Egalitarian Shapley values. In general, Equal treatment of equals does not implies Anonymity. It turns out that for allocation rules satisfying Efficiency and Linearity, Anonymity is equivalent to Equal treatment of equals (see Malawski, 2007, Theorem 2). From this and the fact that Desirability implies Equal treatment of equals, we deduce from Propositions 1-4 that the Shapley value, the Egalitarian Division rule, the Egalitarian Shapley

values, and the Procedural values belong to the larger set of allocation rules satisfying Efficiency, Anonymity and Linearity. Note also that  $\mathbf{Pv}_N$  is a convex set of allocation rules (see Malawski, 2013, Remark 1). From these comments, we obtain the following corollary.

**Corollary 1** *It holds that:  $\mathbf{EAL}_N \supseteq \mathbf{Pv}_N \supseteq \mathbf{EDSh}_N \supseteq \{Sh, ED\}$ . Furthermore,  $\mathbf{EAL}_N$ ,  $\mathbf{Pv}_N$ , and  $\mathbf{EDSh}_N$  are convex sets.*

The set of allocation rules satisfying Efficiency, Anonymity (or Equal treatment of equals) and Linearity has been studied by Ruiz et al. (1998), Driessen and Radzik (2003) and Radzik and Driessen (2013) among others.

**Proposition 5** (Driessen and Radzik, 2003; Radzik and Driessen, 2013)

*An allocation rule  $\Phi$  on  $V_N$  belongs to  $\mathbf{EAL}_N$  if and only if there exists a unique vector of constants  $B^\Phi = (b_s^\Phi : s \in \{0, 1, \dots, n\})$  such that  $b_0^\Phi = 0$ ,  $b_n^\Phi = 1$ , and*

$$\Phi(v) = Sh(B^\Phi v), \quad (2)$$

where  $(B^\Phi v)(S) = b_s^\Phi v(S)$  for each coalition  $S$  of size  $s$ ,  $s \in \{0, 1, \dots, n\}$ .

Since the Shapley value, the Egalitarian Division rule, the Egalitarian Shapley values and the Procedural values satisfy all Efficiency, Linearity and Anonymity, each of them has its representation in the form of (2) with some constants  $b_s$ .

**Remark 1** The previous results allow for several specifications.

1. For the Shapley value we obviously have, for each  $s \in \{1, \dots, n-1\}$ ,  $b_s^{Sh} = 1$ .
2. For the Egalitarian Division rule, we have, for each  $s \in \{1, \dots, n-1\}$ ,  $b_s^{ED} = 0$ .
3. For each Egalitarian-Shapley value  $\mathbf{EDSh}^\alpha$ ,  $\alpha \in [0, 1]$ , we have, for each  $s \in \{1, \dots, n-1\}$ ,  $b_s^\alpha = \alpha$ .
4. For each Procedural value  $\mathbf{Pv}^l$ , Lemma 2 in Malawski (2013) shows that, for each  $s \in \{1, \dots, n-1\}$ ,  $b_s^l = l_{s+1, s+1}$ , where, by definition of procedure  $l$ ,  $l_{s+1, s+1} \in [0, 1]$ . In particular, if  $l$  is such that  $l_{ss} = 1$  for each size  $s$ , we have  $\mathbf{Pv}^l = Sh$ ; if the procedure  $l$  is such that  $l_{ss} = 0$  for each  $s > 1$ , we obtain  $\mathbf{Pv}^l = ED$ .

□

**Remark 2** Point 4 in Remark 1 suggests that the allocation determined by a procedure  $l$  only depends on the real numbers  $(l_{ss} : s \in \{1, \dots, n\})$  where  $l_{11} = 1$ . This property of Procedural values has been proved by (Malawski, 2013, Theorem 1). □

**Remark 3** Note that the function

$$\Psi_v : \mathbb{R}^{n+1} \longrightarrow V_N \text{ defined as } \Psi_v(B) = Bv,$$

where  $Bv$  is defined as in Proposition 5, is linear. So, for any  $\alpha \in \mathbb{R}$ , and any two vectors  $B^{(1)}$  and  $B^{(2)}$ , we have:

$$(\alpha B^{(1)} + B^{(2)})(v) = \Psi_v(\alpha B^{(1)} + B^{(2)}) = \alpha \Psi_v(B^{(1)}) + \Psi_v(B^{(2)}) = \alpha(B^{(1)}v) + B^{(2)}v.$$

□



### 3. Construction of the Solidarity allocation rules

We introduce a new set of solidarity allocation rules which rely on both marginalist and egalitarian principles. The scenario envisaged to define and compute these allocation rules consists of the following steps.

1. Consider any integer  $p$  between 0 and  $n - 1$ .
2. Choose any  $v \in V_N$  and any permutation  $\sigma \in \Sigma_N$  in order to gradually form the grand coalition  $N$ .
3. Each player  $i \in N$  arriving at position  $\sigma(i) \leq p$  obtains his contribution  $v(P_i^\sigma) - v(P_i^\sigma \setminus i)$  upon entering.
4. Each player  $i \in N$  arriving at position  $\sigma(i) > p$  obtains an equal share of the remaining worth  $v(N) - v(P_{\sigma^{-1}(p)}^\sigma)$ .
5. Steps 1-4 determine a payoff vector denoted by  $c^{\sigma,p}(v) \in \mathbb{R}^n$ .
6. Define the payoff vector  $\text{Sol}^p(v)$  as the average of the payoff vectors  $c^{\sigma,p}(v)$  over the  $n!$  permutations  $\sigma \in \Sigma_N$ .
7. Then, assume that the integer  $p$  is drawn from  $\{0, \dots, n - 1\}$  according to the probability distribution  $\alpha = (\alpha_p : p \in \{0, \dots, n - 1\})$ . The Solidarity allocation rule induced by the probability distribution  $\alpha$  is defined for each  $v \in V_N$  as the expected payoff vector  $\text{Sol}^\alpha(v)$  given by:

$$\text{Sol}^\alpha(v) = \sum_{p=0}^{n-1} \alpha_p \text{Sol}^p(v). \quad (3)$$

Step 3 is based on the classical contribution on the entering (single) player. It indicates that each player entering at one of the first  $p$  positions receives his contribution to the coalition formed by his entry according to permutation  $\sigma$ . Similarly, step 4 can be seen as relying on the collective contribution of the remaining players, *i.e.* players arriving after position  $p$  according to  $\sigma$ . Each player still brings his contribution to the current coalition, but these contributions are gathered into a mutual fund until the grand coalition is formed. The coalition consisting of the players arrived after position  $p$  behaves as a single entity: the accumulated worth is its collective contribution to  $N$ , and it makes sense to share it equally among the coalition's members. Latter, we will propose another interpretation which does not rely on the idea of a collective contribution. Step 5 collects the payoffs received by each player with respect to  $\sigma$  and  $p$ . In step 6, these payoffs are averaged over all possible permutations of the players. Step 7 envisages the situation where the threshold  $p$  from which the remaining players enter collectively in a coalition of size  $p$ , is chosen according to a probability distribution. Taking into account this random event, the solidarity allocation rule computes the expected payoff of each player under this probability distribution.

Formally,  $\text{Sol}^p(v)$  is defined as:

$$\forall v \in V_N, \quad \text{Sol}^p(v) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} c^{\sigma,p}(v), \quad (4)$$

where

$$\forall \sigma \in \Sigma_N, \forall i \in N, \quad c_i^{\sigma,p}(v) = \begin{cases} v(P_i^\sigma) - v(P_i^\sigma \setminus i) & \text{if } \sigma(i) \leq p, \\ \frac{v(N) - v(P_{\sigma^{-1}(p)}^\sigma)}{n - p} & \text{if } \sigma(i) > p. \end{cases} \quad (5)$$

and, by convention and abusing notation,  $P_{\sigma^{-1}(0)}^\sigma = \emptyset$ .

From (4) and (5), we have for each  $i \in N$ :

$$\text{Sol}_i^p(v) = \frac{1}{n!} \left[ \sum_{\sigma \in \Sigma_N: \sigma(i) \leq p} (v(P_i^\sigma) - v(P_i^\sigma \setminus i)) + \sum_{\sigma \in \Sigma_N: \sigma(i) > p} \frac{v(N) - v(P_{\sigma^{-1}(p)}^\sigma)}{n-p} \right],$$

where

$$\begin{aligned} \sum_{\sigma \in \Sigma_N: \sigma(i) \leq p} (v(P_i^\sigma) - v(P_i^\sigma \setminus i)) &= \sum_{\substack{S \in 2^N: S \ni i, \\ s \leq p}} \sum_{\sigma \in \Sigma_N: P_i^\sigma = S} (v(S) - v(S \setminus i)) \\ &= \sum_{\substack{S \in 2^N: S \ni i, \\ s \leq p}} (n-s)!(s-1)! (v(S) - v(S \setminus i)) \end{aligned}$$

and

$$\begin{aligned} \sum_{\sigma \in \Sigma_N: \sigma(i) > p} (v(N) - v(P_{\sigma^{-1}(p)}^\sigma)) &= \sum_{\substack{S \in 2^N: S \not\ni i, \\ s=p}} \sum_{\sigma \in \Sigma_N: P_{\sigma^{-1}(p)}^\sigma = S} (v(N) - v(S)) \\ &= \sum_{\substack{S \in 2^N: S \not\ni i, \\ s=p}} (n-s)!s! (v(N) - v(S)). \end{aligned}$$

Therefore, an equivalent representation of  $\text{Sol}^p(v)$  is given by:

$$\forall i \in N, \text{Sol}_i^p(v) = \sum_{\substack{S \in 2^N: S \ni i, \\ s \leq p}} \frac{(n-s)!(s-1)!}{n!} (v(S) - v(S \setminus i)) + \sum_{\substack{S \in 2^N: S \not\ni i, \\ s=p}} \frac{(n-s-1)!s!}{n!} (v(N) - v(S)), \quad (6)$$

or

$$\forall i \in N, \text{Sol}_i^p(v) = \sum_{\substack{S \in 2^N: S \ni i, \\ s \leq p}} \frac{(n-s)!(s-1)!}{n!} (v(S) - v(S \setminus i)) + \sum_{\substack{S \in 2^N: S \ni i, \\ s=p+1}} \frac{(n-s)!(s-1)!}{n!} (v(N) - v(S \setminus i)). \quad (7)$$

#### 4. Properties

Let  $\mathbf{Sol}_N$  be the set of all possible Solidarity allocation rules  $\text{Sol}^\alpha$  on  $V_N$  defined as in (3). By construction, each  $\text{Sol}^p$ ,  $p \in \{0, \dots, n-1\}$ , is an extreme point of the convex set of allocation rules  $\mathbf{Sol}_N$ . We begin by describing some useful properties of these extreme points.

**Proposition 6** Fix any  $p \in \{0, \dots, n-1\}$ .

1. If  $p = 0$ , then  $\text{Sol}^0 = \text{ED}$ ; if  $p = n-1$ , then  $\text{Sol}^{n-1} = \text{Sh}$ .
2. For each  $v \in V_N$ ,  $\text{Sol}^p(v)$  coincides with  $\text{Sh}(B^p v)$ , where  $B^p = (b_s^p : s \in \{0, 1, \dots, n\})$  is such that:

$$b_0^p = 0, \quad b_n^p = 1, \quad b_s^p = \begin{cases} 1 & \text{if } s \in \{1, \dots, p\}, \\ 0 & \text{if } s \in \{p+1, \dots, n-1\}. \end{cases}$$

3.  $\text{Sol}^p$  coincides with the Procedural value  $\text{Pv}^{l^p}$  generated by the procedure  $l^p$  defined as:

$$l_{k,q}^p = \begin{cases} 1 & \text{if } k > p + 1 \text{ and } q = p + 1 \\ 1 & \text{if } k \leq p + 1 \text{ and } q = k \\ 0 & \text{otherwise.} \end{cases}$$

By Remark 2, the procedure  $l^p$  described in Proposition 6 is only one of the possible procedures leading to  $\text{Sol}^p$ . In  $l^p$ , each player entering at position  $k > p + 1$  transfers his or her own contribution to the player entered at position  $p + 1$  while each other player entering at position  $k \leq p + 1$  keeps his or her own contribution. Notice that Point 3 of Proposition 6 follows from Point 2 of Proposition 6 by using the fourth item of Remark 1. Nevertheless, we prefer to provide a detailed and more instructive proof in Appendix.

Combining Proposition 6 with Corollary 1 and keeping in mind that  $\mathbf{Sol}_N$  is a convex set of allocation rules, we obtain the following result.

**Proposition 7** *It holds that:  $\mathbf{EAL}_N \supseteq \mathbf{PV}_N \supseteq \mathbf{Sol}_N \supseteq \mathbf{EDSh}_N \supseteq \{Sh, ED\}$ .*

**Remark 4** The class  $\mathbf{Sol}_N$  and the class of allocation rules studied in Casajus and Huettner (2014a) have a nonempty intersection since they both contain the Shapley value, the Egalitarian Division rule and the Solidarity value (see also Remark 6 below). However, the two classes are not related to each other by set inclusion. On the one hand,  $\mathbf{Sol}_N$  is not included in the class of allocation rules considered in Casajus and Huettner (2014a). This follows from expression (7) in Theorem 3 in Casajus and Huettner (2014a), which implies that the constants  $b_s$  associated with any allocation in the class are linear fractional transformations of  $s$ . As a consequence, any allocation rule  $\text{Sol}^p$  with  $p \notin \{0, n - 1\}$  cannot be formulated as (7) in Theorem 3 in Casajus and Huettner (2014a). On the other hand, the class of allocation rules introduced in Casajus and Huettner (2014a) is not included in  $\mathbf{Sol}_N$ . This property comes from the fact that any element of  $\mathbf{Sol}_N$  satisfies Desirability and Monotonicity as a Procedural value, while this is not the case for all allocation rules in the class of allocation rules examined in Casajus and Huettner (2014a) as shown by their Theorem 6.  $\square$

The second result in this section characterizes the class  $\mathbf{Sol}_N$  in terms of the constants  $b_s$  in the representation (2).

**Proposition 8** *An allocation rule  $\Phi$  on  $V_N$  belongs to  $\mathbf{Sol}_N$  if and only if it can be represented by (2) with constants  $B^\Phi = (b_s^\Phi : s \in \{0, \dots, n\})$  such that:*

$$b_0^\Phi = 0, \quad b_n^\Phi = 1, \quad \text{and } \forall s \in \{1, \dots, n - 1\}, \quad 1 \geq b_1^\Phi \geq b_2^\Phi \geq \dots \geq b_{n-1}^\Phi \geq 0.$$

Furthermore,  $\Phi = \text{Sol}^\alpha$  where  $\alpha = (\alpha_s : s \in \{0, \dots, n - 1\})$  is obtained from the transformation  $B^\Phi \mapsto \alpha$  such that:

$$\alpha_0 = 1 - b_1^\Phi, \quad \alpha_{n-1} = b_{n-1}^\Phi, \quad \text{and } \forall s \in \{1, \dots, n - 2\}, \quad \alpha_s = b_s^\Phi - b_{s+1}^\Phi.$$

**Remark 5** From Proposition 8, the inclusion  $\mathbf{PV}_N \supseteq \mathbf{Sol}_N$ , and the fourth item of Remark 1, it is possible to interpret each  $\text{Sol}^\alpha$  as the Procedural value  $\text{Pv}^{l^\alpha}$  where:

$$l_{k,q}^\alpha = \begin{cases} \alpha_{q-1} & \text{if } q < k, \\ \sum_{j=q-1}^{n-1} \alpha_j & \text{if } q = k. \end{cases}$$

In other words, the contribution of the player entering at position  $k$  is shared as follows: for  $q < k$ , the player entering at position  $q$  receives a share  $\alpha_{q-1}$  of the contribution, and the player  $k$  keeps the remaining part. As such, the coefficients  $\alpha_p$ ,  $p \in \{0, \dots, n-1\}$ , can be interpreted as transfer rates of the contribution of the entering player to his or her predecessors in the permutation. For a given  $p \in \{0, \dots, n-1\}$ , by setting  $\alpha_p = 1$  and  $\alpha_j = 0$  for each  $j \in \{0, \dots, p-1, p+1, \dots, n-1\}$ , we easily recover that  $l_{k,q}^{\alpha} = l_{k,q}^p$ , and thus  $\text{Sol}^{\alpha} = \text{Sol}^p$ .  $\square$

**Remark 6** The Solidarity value introduced by Nowak and Radzik (1994) belongs to  $\mathbf{EAL}_N$ . The associated constants  $b_s^{Sv}$  are given by (see Radzik and Driessen, 2013, Corollary 1):

$$b_0^{Sv} = 0, \quad b_n^{Sv} = 1, \quad \text{and} \quad \forall s \in \{1, \dots, n-1\}, \quad b_s^{Sv} = \frac{1}{s+1}.$$

By Proposition 8, conclude that the Solidarity value belongs to  $\mathbf{Sol}_N$ .  $\square$

**Remark 7** One might want to determine the barycenter of  $\mathbf{Sol}_N$ . By Proposition 8, it is given by the  $\mathbf{EAL}_N$  allocation rule  $\Phi$  with constants

$$b_0^{\Phi} = 0, \quad b_n^{\Phi} = 1, \quad \text{and} \quad \forall s \in \{1, \dots, n-1\}, \quad b_s^{\Phi} = \frac{n-s}{n},$$

or equivalently, it is equal to  $\text{Sol}^{\alpha}$ , where  $\alpha_s = 1/n$  for all  $s \in \{0, \dots, n-1\}$ . As a consequence, the barycenter of  $\mathbf{Sol}_N$  possesses a natural interpretation in terms of Procedural values. More specifically, the entering player gives up a share  $1/n$  of his contribution to each of his predecessors, and keeps what remains. As such, the fraction that the entering players keeps for himself gradually decreases with the number of his predecessors. This means that the degree of solidarity increases with the size of the coalition to which the entering player contributes.  $\square$

## 5. Axiomatic characterizations

This section offers two characterizations: one of each  $\text{Sol}^p$ ,  $p \neq 0$ , and one of the class  $\mathbf{Sol}_N$ . As a start, recall that Point 2 of Proposition 6 indicates that each  $\text{Sol}^p$  has a representation in terms of the Shapley value as expressed in Proposition 5. The corresponding vector of constants  $B^p$  is binary. More precisely, the constants in  $B^p$  coincide with the unitary constants  $B^{Sh}$  of the Shapley value  $\text{Sh}$  up to  $p$ , and then shrink to zero. Thus,  $B^{n-1} = B^{Sh}$  (point 1 of Proposition 6). From this point of view,  $\text{Sol}^p$  is a generalization of the Shapley value  $\text{Sh}$ . As a consequence, it is possible to generalize the classical characterization of the Shapley value contained in Proposition 1. To that end, we introduce the following variant of the null player axiom.

Given  $p \in \{1, \dots, n-1\}$ , and  $v \in V_N$ , we say that  $i \in N$  is a  $p$ -null player in  $v$  if:

$$\forall S \ni i, \quad s \leq p, \quad v(S) = v(S \setminus i) \quad \text{and} \quad \forall S \not\ni i, \quad s = p, \quad v(N) = v(S).$$

**$p$ -null player axiom** An allocation rule  $\Phi$  on  $V_N$  satisfies the  $p$ -null player axiom if, for each  $v \in V_N$  and each  $p$ -null player  $i \in N$  in  $v$ , it holds that:  $\Phi_i(v) = 0$ .

In case  $p = n-1$ , the  $p$ -null player axiom is identical to the null player axiom. Modifications of the null player axiom in the same spirit are invoked in Nowak and Radzik (1994), Ju et al. (2007), Kamijo and Kongo (2012), Chameni Nembua (2012), Casajus and Huettner (2014a) and Béal et al. (2015).

**Proposition 9** *An allocation rule  $\Phi$  on  $V_N$  is equal to  $Sol^p$ ,  $p \in \{1, \dots, n-1\}$ , if and only if it satisfies Efficiency, Equal treatment of equals, Additivity, and the  $p$ -null player axiom.*

The logical independence of the axioms can be demonstrated as follows:

- The Shapley value  $Sh$  satisfies Efficiency, Equal treatment of equals, and Additivity, but violates the  $p$ -null player axiom.
- The null allocation rule on  $V_N$ , which assigns to each  $v \in V_N$  and each  $i \in N$  the payoff vector  $\Phi_i(v) = 0$  satisfies Equal treatment of equals, Additivity, and the  $p$ -null player axiom, but violates Efficiency.
- Fix  $\sigma \in \Sigma_N$  and  $p \in \{1, \dots, n-1\}$ . The allocation rule  $c^{\sigma,p}$  on  $V_N$  satisfies Efficiency, Additivity, and the  $p$ -null player axiom, but violates Equal treatment of equals.
- The Equal Surplus Division rule ESD (Driessen and Funaki, 1991) is defined on  $V_N$  by

$$\forall i \in N, \quad ESD_i(v) = v(i) + \frac{1}{n} \left( v(N) - \sum_{j \in N} v(j) \right).$$

The allocation rule  $\Phi$  on  $V_N$  such that  $\Phi(v) = ESD(v)$  if  $v(i) \neq 0$  for all  $i \in N$  and  $\Phi(v) = Sol^p(v)$  otherwise satisfies Efficiency, Equal treatment of equals, and the  $p$ -null player axiom, but violates Additivity.

It is interesting to note that Proposition 2 and Proposition 9 are comparable characterizations of allocation rules based on egalitarian and marginalist principles.

We are now in a position to provide a characterization of  $Sol_N$  comparable to the characterizations of  $Pv_N$  and  $EDSh_N$  given in Proposition 4 and Proposition 3, respectively. In order to characterize  $EDSh_N$ , Casajus and Huettner (2013) use the axiom of Null player in a productive environment in Proposition 3. This axiom indicates that a null player obtains a non-negative payoff whenever the worth generated by the grand coalition  $N$  is non-negative. Here, deviations from the Shapley payoffs are perceived as an expression of a certain degree of solidarity among players. Since Null players do not exert negative effects when they join a coalition, it is not necessary that they receive negative payoffs. As underlined by the authors, relaxing this axiom of Null player in productive environment by imposing that Null players receive a non-negative payoff when the worth of the grand coalition is null (i.e. in a null environment) has a strong implication on the nature of solidarity among the players within the set of allocation rules  $EAL_N$ : the combination of Efficiency, Linearity, Anonymity (or Equal treatment of equals) and the axiom of Null player in a null environment characterizes the set of all affine combinations of  $Sh$  and  $ED$  (see Casajus and Huettner, 2013, footnote 4). There exist allocation rules in  $Sol_N$  that do not satisfy the axiom of Null player in a null environment. Whenever the environment is null, one can also estimate that the resources of the society are not sufficient to redistribute monetary payoffs to unproductive players even if the other coalitions are productive. Consequently, one can impose that null players should not receive positive payoffs. This point of view expresses the limits of the solidarity principle among the players in a situation where the cooperation of all the members of the society is unproductive.

**Null player in a null environment for positive games.** An allocation rule  $\Phi$  on  $V_N$  satisfies the Null player in a null environment for positive games axiom if for each positive  $v \in V_N$  such that

$v(N) = 0$  and each null player  $i \in N$  in  $v$ , it holds that:  $\Phi_i(v) \leq 0$ .

It turns out that this axiom has a strong implication on the nature of solidarity among the players within the set of allocation rules  $\mathbf{Pv}_N$ .

**Proposition 10** *An allocation rule  $\Phi$  on  $V_N$  belongs to  $\mathbf{Sol}_N$  if and only if it satisfies Efficiency, Additivity, Desirability, Monotonicity, and Null player in a null environment for positive games.*

Thanks to Proposition 4, an equivalent statement of Proposition 10 is that an allocation rule  $\Phi$  on  $V_N$  belongs to  $\mathbf{Sol}$  if and only if  $\Phi$  is a Procedural value satisfying Null player in a null environment for positive TU-games.

The logical independence of the axioms can be demonstrated as follows:

- The Equal Surplus Division rule ESD satisfies Efficiency, Additivity, Desirability, and Null player in a null environment for positive games, but violates Monotonicity.
- The null allocation rule on  $V_N$ , which assigns to each  $v \in V_N$  and each  $i \in N$  the payoff vector  $\Phi_i(v) = 0$  satisfies Additivity, Desirability, Monotonicity, and Null player in a null environment for positive games, but violates Efficiency.
- Fix  $\sigma \in \Sigma_N$  and  $p \in \{1, \dots, n-1\}$ . The allocation rule  $c^{\sigma,p}$  on  $V_N$  satisfies Efficiency, Additivity, Monotonicity, and Null player in a null environment for positive games, but violates Desirability.
- Consider any allocation rule  $\Phi$  on  $V_N$ , which assigns to all games  $v \in V_N$  the payoff vector  $\Phi(v) = \text{Sh}(B^\Phi v)$ , where  $B^\Phi = (b_s^\Phi : s \in \{0, 1, \dots, n\})$  is such that: for each  $s \in \{0, \dots, n\}$ ,  $b_s^\Phi \in [0, 1]$ ,  $b_0^\Phi = 0$ ,  $b_n^\Phi = 1$ , and  $b_1^\Phi < b_2^\Phi < \dots < b_{n-1}^\Phi$ . Then,  $\Phi$  satisfies Efficiency, Additivity, Desirability, Monotonicity, but violates Null player in a null environment for positive games.
- The allocation rule  $\Phi$  on  $V_N$  defined in Example 6 in Malawski (2013) satisfies Efficiency, Desirability, Monotonicity, and Null player in a null environment for positive games, but violates Additivity.

## 6. Conclusion

Different conceptions of the solidarity principle are envisaged in the literature, and a contribution of our article is to propose a new one, based on the idea of collective contributions of a coalition. In the proof of the logical independence of the axioms invoked in our characterizations, we called on the Equal Surplus Division rule. This allocation rule reveals another mode of solidarity: the amount of the mutual fund, which is evenly distributed among all the players, is what remains of the worth of the grand coalition after each player has received his stand-alone worth. Nonetheless, the Equal Surplus Division rule is excluded from all the classes of allocation rules discussed in this article, except the largest class of allocation rule satisfying Efficiency, Anonymity and Linearity. A challenging issue for future research is therefore to design a class of allocation rules relying on solidarity principles, and that would include the Equal Surplus Division rule.

## Appendix

In this Appendix, we employ the following extra definition. For any non-empty coalition  $T \subseteq N$ , the Dirac TU-game  $\delta_T \in V_N$  is defined as:  $\delta_T(T) = 1$ , and  $\delta_T(S) = 0$  for each other  $S$ .

*Proof. (Proposition 6)*

**Point 1** follows from (6).

**Point 2.** Consider the vector of constants  $B^p$  as defined in point 2 of Proposition 6. From the definition of  $\text{Sh}(B^p v)$ , for each  $i \in N$ , we have:

$$\begin{aligned}
\text{Sh}_i(B^p v) &= \sum_{S \in 2^N: S \ni i} \frac{(n-s)!(s-1)!}{n!} (b_s v(S) - b_{s-1} v(S \setminus i)) \\
&= \sum_{\substack{S \in 2^N: S \ni i, \\ s \leq p}} \frac{(n-s)!(s-1)!}{n!} (b_s v(S) - b_{s-1} v(S \setminus i)) \\
&\quad + \sum_{\substack{S \in 2^N: S \ni i, \\ s > p}} \frac{(n-s)!(s-1)!}{n!} (b_s v(S) - b_{s-1} v(S \setminus i)) \\
&= \sum_{\substack{S \in 2^N: S \ni i, \\ s \leq p}} \frac{(n-s)!(s-1)!}{n!} (v(S) - v(S \setminus i)) + \sum_{\substack{S \in 2^N: S \ni i, \\ s = p+1}} \frac{(n-s)!(s-1)!}{n!} (-v(S \setminus i)) + \frac{1}{n} v(N) \\
&= \sum_{\substack{S \in 2^N: S \ni i, \\ s \leq p}} \frac{(n-s)!(s-1)!}{n!} (v(S) - v(S \setminus i)) + \sum_{\substack{S \in 2^N: S \ni i, \\ s = p+1}} \frac{(n-s)!(s-1)!}{n!} (-v(S \setminus i)) \\
&\quad + \sum_{\substack{S \in 2^N: S \ni i, \\ s = p+1}} \frac{(n-s)!(s-1)!}{n!} v(N) \\
&= \sum_{\substack{S \in 2^N: S \ni i, \\ s \leq p}} \frac{(n-s)!(s-1)!}{n!} (v(S) - v(S \setminus i)) + \sum_{\substack{S \in 2^N: S \ni i, \\ s = p+1}} \frac{(n-s)!(s-1)!}{n!} (v(N) - v(S \setminus i)) \\
&= \text{Sol}_i^p(v),
\end{aligned}$$

where the last equality follows from (7).

**Point 3.** Consider the procedure  $l^p$  as defined in point 3 of Proposition 6. First, for any permutation  $\sigma \in \Sigma_N$ , the contribution vector  $r^{\sigma, l^p}$  given by (1) writes:

$$\forall i \in N, \quad r_i^{\sigma, l^p}(v) = \begin{cases} v(P_i^\sigma) - v(P_i^\sigma \setminus i) & \text{if } \sigma(j) \leq p \text{ and } \sigma(i) = \sigma(j), \\ \sum_{j \in (N \setminus P_i^\sigma) \cup i} v(P_j^\sigma) - v(P_j^\sigma \setminus j) & \text{if } \sigma(j) \geq p+1 \text{ and } \sigma(i) = p+1, \\ 0 & \text{otherwise.} \end{cases}$$

or equivalently,

$$\forall i \in N, \quad r_i^{\sigma, l^p}(v) = \begin{cases} v(P_i^\sigma) - v(P_i^\sigma \setminus i) & \text{if } \sigma(i) = \sigma(j) \leq p, \\ v(N) - v(P_{\sigma^{-1}(p)}^\sigma) & \text{if } \sigma(i) = p+1 \leq \sigma(j), \\ 0 & \text{otherwise.} \end{cases}$$

Because the expression does not depend on  $\sigma(j)$ , we have:

$$\forall i \in N, \quad r_i^{\sigma, l^p}(v) = \begin{cases} v(P_i^\sigma) - v(P_i^\sigma \setminus i) & \text{if } \sigma(i) \leq p, \\ v(N) - v(P_{\sigma^{-1}(p)}^\sigma) & \text{if } \sigma(i) = p + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Second, the previous observation implies that the Procedural value induced by  $l^p$  assigns to a player  $i \in N$  in a game  $v \in V_N$ , the payoff

$$\begin{aligned} \text{Pv}_i^{l^p}(v) &= \frac{1}{n!} \left( \sum_{\sigma \in \Sigma_N: \sigma(i) \leq p} (v(P_i^\sigma) - v(P_i^\sigma \setminus i)) + \sum_{\sigma \in \Sigma_N: \sigma(i) = p+1} (v(N) - v(P_{\sigma^{-1}(p)}^\sigma)) \right) \\ &= \frac{1}{n!} \left( \sum_{\sigma \in \Sigma_N: \sigma(i) \leq p} (v(P_i^\sigma) - v(P_i^\sigma \setminus i)) + \sum_{\sigma \in \Sigma_N: \sigma(i) > p} \frac{v(N) - v(P_{\sigma^{-1}(p)}^\sigma)}{n-p} \right) \\ &= \frac{1}{n!} \sum_{\sigma \in \Sigma_N} c_i^{\sigma, p}(v) \\ &= \text{Sol}_i^p(v), \end{aligned}$$

where the second equality follows from the fact that the number of permutations in which  $\sigma(i)$  is greater than  $p$  is  $(n-p)$  times larger than the number of permutations in which  $\sigma(i)$  is equal to  $p+1$ .  $\blacksquare$

*Proof. (Proposition 8)* Assume that  $\Phi \in \mathbf{Sol}_N$ . Then there is a probability distribution  $(\alpha_0, \dots, \alpha_{n-1})$  such that:

$$\Phi = \sum_{p=0}^{n-1} \alpha_p \text{Sol}^p.$$

By point 2 of Proposition 6, we have:

$$\forall v \in V_N, \quad \Phi(v) = \sum_{p=0}^{n-1} \alpha_p \text{Sh}(B^p v).$$

Define the vector of constants  $B^\Phi$  as:

$$B^\Phi = \sum_{p=0}^{n-1} \alpha_p B^p.$$

By Linearity of the Shapley value and Remark 3, we get:

$$\Phi(v) = \sum_{p=0}^{n-1} \alpha_p \text{Sh}(B^p v) = \text{Sh} \left( \sum_{p=0}^{n-1} \alpha_p (B^p v) \right) = \text{Sh} \left( \left( \sum_{p=0}^{n-1} \alpha_p B^p \right) v \right) = \text{Sh}(B^\Phi v)$$

Using the definition of  $B^p$ , we obtain:

$$b_0^\Phi = 0, \quad b_n^\Phi = 1, \quad \text{and, for each } s \in \{1, \dots, n-1\}, \quad b_s^\Phi = \sum_{p=s}^{n-1} \alpha_p.$$



Because, for each  $p \in \{0, \dots, n-1\}$ ,  $\alpha_p \in [0, 1]$ , we conclude that

$$1 = b_n^\Phi \geq b_1^\Phi \geq b_2^\Phi \geq \dots \geq b_{n-1}^\Phi \geq b_0^\Phi = 0,$$

as desired.

Reciprocally, consider an allocation rule  $\Phi$  on  $V_N$  such that  $\Phi(v) = \text{Sh}(B^\Phi v)$ , where the vector of constants  $B^\Phi$  is such that:

$$1 = b_n^\Phi \geq b_1^\Phi \geq b_2^\Phi \geq \dots \geq b_{n-1}^\Phi \geq b_0^\Phi = 0. \quad (8)$$

Define the collection of real numbers  $(\alpha_p : p \in \{0, \dots, n-1\})$  as follows:

$$\forall p \in \{1, \dots, n-2\}, \alpha_p = b_p^\Phi - b_{p+1}^\Phi, \alpha_{n-1} = b_{n-1}^\Phi \text{ and } \alpha_0 = 1 - b_1^\Phi.$$

By (8):

$$\forall p \in \{0, 1, \dots, n-1\}, \alpha_p \in [0, 1], \text{ and } \sum_{p=0}^{n-1} \alpha_p = 1,$$

which means that  $(\alpha_p : p \in \{0, \dots, n-1\})$  can be viewed as a probability distribution over the sizes  $p \in \{0, \dots, n-1\}$ . Furthermore, it holds that:

$$\forall s \in \{1, \dots, n-1\}, \sum_{p=s}^{n-1} \alpha_p = b_s^\Phi.$$

From this, it follows that:

$$\sum_{p=0}^{n-1} \alpha_p B^p = \left( 0, \sum_{p=1}^{n-1} \alpha_p, \sum_{p=2}^{n-1} \alpha_p, \dots, \sum_{p=n-2}^{n-1} \alpha_p, \alpha_{n-1}, \sum_{p=0}^{n-1} \alpha_p \right) = B^\Phi.$$

Therefore, using Remark 3, we obtain:

$$\Phi(v) = \text{Sh}(B^\Phi v) = \text{Sh}\left(\left(\sum_{p=0}^{n-1} \alpha_p B^p\right)v\right) = \text{Sh}\left(\sum_{p=0}^{n-1} \alpha_p (B^p v)\right) = \sum_{p=0}^{n-1} \alpha_p \text{Sh}(B^p v) = \text{Sol}^\alpha(v),$$

as desired. Finally, the second statement of Proposition 8 immediately follows from the previous steps.  $\blacksquare$

*Proof. (Proposition 9)* By Propositions 5 and 6,  $\text{Sol}^p$  satisfies Efficiency, Linearity, Anonymity, and so it also satisfies Additivity and Equal treatment of equals. From the definition of  $c^{p,\sigma}$  given in (5), we easily conclude that  $\text{Sol}^p$  satisfies the  $p$ -null player axiom.

To prove that there exists a unique allocation rule that satisfies Efficiency, Equal treatment of equals, Additivity, and the  $p$ -null player axiom for some  $p \in \{1, \dots, n-1\}$ , consider any such allocation rule  $\Phi$  on  $V_N$ . We have already underlined that the combination of these axioms implies that  $\Phi$  also satisfies Linearity and Anonymity. By Proposition 5, there exists a unique vector of constants  $B^\Phi = (b_s^\Phi : s \in \{0, 1, \dots, n\})$  such that  $b_0^\Phi = 0$ ,  $b_n^\Phi = 1$ , and  $\Phi(v) = \text{Sh}(B^\Phi v)$ . By Proposition 6, it remains to show that  $B^\Phi = B^p$ .

Fix any player  $i \in N$ . For any size  $s \in \{1, \dots, n-1\}$ , any coalition  $S \subseteq N$  of size  $s$  such that  $S \ni i$ , consider the TU-game  $\delta_S + \delta_{S \setminus i}$ . For each  $s \neq p+1$ , player  $i$  is  $p$ -null. By the  $p$ -null player axiom, we have:

$$0 = \Phi_i(\delta_S + \delta_{S \setminus i}) = \text{Sh}_i(B^\Phi(\delta_S + \delta_{S \setminus i})) = \frac{(n-s)!(s-1)!}{n!}(b_s^\Phi - b_{s-1}^\Phi), \quad \text{and so } b_s^\Phi = b_{s-1}^\Phi.$$

We conclude that:

$$b_1^\Phi = \dots = b_p^\Phi \quad \text{and} \quad b_{p+1}^\Phi = \dots = b_{n-1}^\Phi. \quad (9)$$

Next, define the TU-game  $v_{p,i}$  as follows:

$$\forall S \in 2^N, \quad v_{p,i}(S) = \begin{cases} 1 & \text{if } s \geq p+1, \\ 1 & \text{if } s = p \text{ and } S \not\ni i, \\ 0 & \text{if } s = p \text{ and } S \ni i, \\ 0 & \text{if } s < p. \end{cases}$$

Player  $i$  is  $p$ -null in  $v_{p,i}$ . By the  $p$ -null player axiom, we have:

$$\begin{aligned} 0 &= \Phi_i(v_{p,i}) \\ &= \text{Sh}_i(B^\Phi v_{p,i}) \\ &= \sum_{S \in 2^N: S \ni i} \frac{(n-s)!(s-1)!}{n!} (b_s^\Phi - b_{s-1}^\Phi) \\ &= \sum_{S \in 2^N: S \ni i, s \geq p+1} \frac{(n-s)!(s-1)!}{n!} (b_s^\Phi - b_{s-1}^\Phi) \\ &= \sum_{s=p+1}^n \binom{n-1}{s-1} \frac{(n-s)!(s-1)!}{n!} (b_s^\Phi - b_{s-1}^\Phi) \\ &= \frac{1}{n} \sum_{s=p+1}^n (b_s^\Phi - b_{s-1}^\Phi) \\ &= \frac{1}{n} (b_n^\Phi - b_p^\Phi). \end{aligned}$$

Since  $b_n^\Phi = 1$ , it follows that  $b_p^\Phi = 1$ . By (9), we obtain:

$$b_1^\Phi = \dots = b_p^\Phi = 1. \quad (10)$$

This gives the result for  $p = n-1$ . To complete the proof for each other  $p \leq n-2$ , note that player  $i \in N$  is  $p$ -null for each  $p \leq n-2$  in the TU-game  $\delta_{N \setminus i}$ . By the  $p$ -null player axiom, we have:

$$0 = \Phi_i(\delta_{N \setminus i}) = \text{Sh}_i(B^\Phi \delta_{N \setminus i}) = -\frac{b_{n-1}^\Phi}{n}, \quad \text{and so } b_{n-1}^\Phi = 0.$$

By (9), we obtain:

$$b_{p+1}^\Phi = \dots = b_{n-1}^\Phi = 0. \quad (11)$$

By (10) and (11), we conclude that  $B^\Phi = B^p$ , as desired. ■

*Proof. (Proposition 10)* We first prove that each  $\Phi \in \mathbf{Sol}_N$  satisfies all the axioms of the statement of the Proposition 10. By Corollary 1, we have  $\mathbf{Pv}_N \supseteq \mathbf{Sol}_N$ . Therefore, by Proposition 4, each  $\Phi \in \mathbf{Sol}_N$  satisfies Efficiency, Additivity, Desirability, Monotonicity. It remains to show that any  $\Phi \in \mathbf{Sol}_N$  satisfies Null player in a null environment for positive games. Pick any positive  $v \in V_N$  such that  $v(N) = 0$ , and any null player  $i \in N$  in  $v$ . For any  $\sigma \in \Sigma_N$  and any  $p \in \{0, \dots, n-1\}$ , the fact that  $i$  is null in  $v$  and  $v(N) = 0$  implies that (5) can be rewritten as follows:

$$c_i^{\sigma,p}(v) = \begin{cases} 0 & \text{if } \sigma(i) \leq p, \\ -\frac{v(P_{\sigma^{-1}(p)}^\sigma)}{n-p} & \text{if } \sigma(i) > p. \end{cases}$$

By positivity of  $v$ ,  $c_i^{\sigma,p}(v) \leq 0$ , which in turn implies that  $\text{Sol}_i^p(v) \leq 0$  for each  $p \in \{0, \dots, n-1\}$ , and consequently  $\Phi_i(v) \leq 0$ , as desired.

Reciprocally, pick any  $\Phi$  which satisfies Efficiency, Additivity, Desirability, Monotonicity, and Null player in a null environment for positive games. By Proposition 4,  $\Phi \in \mathbf{Pv}_N$ , and by Corollary,  $\Phi \in \mathbf{EAL}_N$ . By Proposition 5, there is a unique vector of constants  $B^\Phi = (b_s^\Phi : s \in \{0, 1, \dots, n\})$  such that  $b_0^\Phi = 0$ ,  $b_n^\Phi = 1$  and:

$$\forall v \in V_N, \quad \Phi(v) = \text{Sh}(B^\Phi v).$$

By Theorem 2 in Radzik and Driessen (2013), we also know that each real number  $b_s^\Phi \in [0, 1]$ . By Proposition 8, it remains to show that, for each  $s \in \{1, \dots, n-2\}$ ,  $b_s^\Phi \geq b_{s+1}^\Phi$ . To that end, pick any  $i \in N$  and any non-empty coalition  $S \subseteq N \setminus i$ ,  $S \neq N \setminus i$ , and consider the TU-game  $\delta_{S \cup i} + \delta_S$ . This TU-game is positive, the environment is null  $(\delta_{S \cup i} + \delta_S)(N) = 0$  since  $S \neq N \setminus i$ , and  $i$  is a null player in  $\delta_{S \cup i} + \delta_S$ . Therefore, Null player in a null environment for positive games yields:

$$\Phi_i(\delta_{S \cup i} + \delta_S) \leq 0. \quad (12)$$

On the other hand, we have:

$$\Phi_i(\delta_{S \cup i} + \delta_S) = \text{Sh}_i(B^\Phi(\delta_{S \cup i} + \delta_S)) = \frac{s!(n-s-1)!}{n!}(b_{s+1}^\Phi - b_s^\Phi). \quad (13)$$

Combining (12) and (13) yields  $b_s^\Phi \geq b_{s+1}^\Phi$  for each  $s \in \{1, \dots, n-2\}$ , as desired.  $\blacksquare$

## References

- Banzhaf, J. F., 1965. Weighted voting doesn't work: a mathematical analysis. *Rutgers Law Review* 19, 317–343.
- Béal, S., Rémila, E., Solal, P., 2015. Preserving or removing special players: what keeps your payoff unchanged in TU-games? *Mathematical Social Sciences* 73, 23–31.
- Casajus, A., Huettner, F., 2013. Null players, solidarity, and the egalitarian shapley values. *Journal of Mathematical Economics* 49, 58–61.
- Casajus, A., Huettner, F., 2014a. On a class of solidarity values. *European Journal of Operational Research* 236, 583–591.
- Casajus, A., Huettner, F., 2014b. Weakly monotonic solutions for cooperative games. *Journal of Economic Theory* 154, 162–172.
- Chameni Nembua, C., 2012. Linear efficient and symmetric values for TU-games: Sharing the joint gain of cooperation. *Games and Economic Behavior* 74, 431–433.
- Driessen, T., Radzik, T., 2003. Extensions of Hart and Mas-Colell's consistency to efficient, linear, and symmetric values for TU-games. In: L.A., P., D.W.K., Y. (Eds.), *ICM Millennium Lectures on Games*. Springer-Verlag, Heidelberg, Germany, pp. 147–166, volume dedicated to the International Congress of Mathematicians, Game Theory and Applications Satellite Conference, August 14–17, 2002, Quindao, China.

- Driessen, T. S. H., Funaki, Y., 1991. Coincidence of and collinearity between game theoretic solutions. *OR Spektrum* 13, 15–30.
- García, D., Vanden, J. M., 2009. Information acquisition and mutual funds. *Journal of Economic Theory* 144, 1965–1995.
- Joosten, R., 1996. Dynamics, equilibria and values, dissertation no. 96-37, Faculty of Economics and Business, Maastricht University.
- Ju, Y., Borm, P., Ruys, P., 2007. The consensus value: A new solution concept for cooperative games. *Social Choice and Welfare* 28, 685–703.
- Kalai, E., Samet, D., 1987. On weighted Shapley values. *International Journal of Game Theory* 16, 205–222.
- Kamijo, Y., Kongo, T., 2012. Whose deletion does not affect your payoff? the difference between the Shapley value, the egalitarian value, the solidarity value, and the Banzhaf value. *European Journal of Operational Research* 216, 638–646.
- Malawski, M., 2007. A note on equal treatment and symmetry of values, Report, Institute of Computer Science PAS 2007.
- Malawski, M., 2013. “Procedural” values for cooperative games. *International Journal of Game Theory* 42, 305–324.
- Nowak, A. S., Radzik, T., 1994. A solidarity value for  $n$ -person transferable utility games. *International Journal of Game Theory* 23, 43–48.
- Radzik, T., Driessen, T., 2013. On a Family of Values for TU-games generalizing the Shapley Value. *Mathematical Social Sciences* 65, 105–111.
- Ruiz, L. M., Valenciano, F., Zarzuelo, J. M., 1996. The least square prenucleolus and the least square nucleolus. Two values for TU games based on the excess vector. *International Journal of Game Theory* 25, 113–134.
- Ruiz, L. M., Valenciano, F., Zarzuelo, J. M., 1998. The family of least square values for transferable utility games. *Games and Economic Behavior* 24, 109–130.
- Shapley, L. S., 1953. A value for  $n$ -person games. In: *Contribution to the Theory of Games vol. II* (H.W. Kuhn and A.W. Tucker eds). *Annals of Mathematics Studies* 28. Princeton University Press, Princeton.
- Sprumont, Y., 1990. Population monotonic allocation schemes for cooperative games with transferable utility. *Games and Economic Behavior* 2, 378–394.
- Stone, D. A., 1993. The struggle for the soul of health insurance. *Journal of Health Politics, Policy and Law* 18, 287–317.
- van den Brink, R., 2007. Null players or nullifying players: the difference between the Shapley value and equal division solutions. *Journal of Economic Theory* 136, 767–775.
- van den Brink, R., Funaki, Y., Ju, Y., 2013. Reconciling marginalism with egalitarianism: Consistency, monotonicity, and implementation of egalitarian Shapley values. *Social Choice and Welfare* 40, 693–714.
- Young, H. P., 1985. Monotonic solutions of cooperative games. *International Journal of Game Theory* 14, 65–72.