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# Axiomatic and bargaining foundations of an allocation rule for ordered tree TU-games[1]

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## Abstract

We introduce the class of tree TU-games augmented by a total order over the links which reflects the formation process of the tree. We first characterize a new allocation rule for this class of cooperative games by means of three axioms: Standardness, Top-consistency and Link Amalgamation. Then, we provide a bargaining foundation for this allocation rule by designing a mechanism, including a bidding stage followed by a bargaining stage, which supports this allocation rule in subgame Nash equilibrium provided that the underlying game is superadditive.

*Keywords:* Amalgamation – Bargaining – Consistency – Tree TU-games – Total order.

## 1. Introduction

An abundant and growing literature examines the influence of economical, hierarchical or communicational structures on the payoff allocation in cooperative game theory. In a seminal contribution, Myerson (1977) enriches the classical model of a cooperative game with transferable utility by an undirected graph which is interpreted as the bilateral communication possibilities among the players. The so-called Myerson value assigns to each player her Shapley value in a game in which only the worth of the coalitions connected through the graph are taken into account, emphasizing the influence of the links in the graph. Ever since, other allocation rules have been successfully introduced and studied, including the Position value (Meessen, 1988, Borm, Owen and Tijs, 1992) and the Average tree solution (Herings, van der Laan and Talman, 2008).

In each of these works, the decision maker who must determine the payoff allocation has no information about the history of the graph associated with the players. In other words, it is not known whether a link of the graph is recent or was formed much earlier, even if the date of creation of a link can be important in the way the players are paid. For instance, the two players who

initiated the formation of the graph by setting the very first link might be rewarded for that. There are several real situations in which such information is (publicly) available. If the players are cities and the links in the graph represent motorway connections, then it is easy to know the construction dates. If the players are countries and the links in the graph represent bilateral trade agreements, then the dates on which these agreements were signed is also known. If the players are Facebook members and a link exists in the graph if two members are friends on the social network, then Facebook lists friend request dates. In all examples, the oldest links probably reflect stronger affinities between the corresponding players. Finally, the links created fifty years ago by the founding members of the European Union are probably very important to determine today's distribution of power among the current members.

In this article, we further enrich Myerson's model by adding information about the formation of the graph. More specifically, we model that by a total order over the link set in which the greatest element is the most recent link added to construct the graph. The four-tuple given by a player set, a characteristic function, a graph and a total order over the link set is called an ordered tree TU-game. We restrict ourselves to the case where the graph is a tree, i.e. it is minimally connected in the sense that it is connected and contains no cycle. There is no equivalent model in the literature, to the best of our knowledge. The works by Demange (2004) and Khmelnitskaya (2010) may be seen as exceptions if the hierarchical structure (a rooted tree in both cases) associated with the cooperative game is interpreted as the sequence by which the graph has formed. The total order over the link set that we consider allows for much more flexibility in modeling the graph formation.

Our contribution is twofold.

In a first step, we introduce three axioms for an allocation rule on the class of ordered tree TU-games: Standardness, Top-consistency and Link amalgamation. Standardness is a classical axiom introduced in Hart and Mas-Colell (1989), which requires that in the two-player case, each one receives her stand-one worth plus an equal share of the surplus resulting from cooperation. Many allocation rules satisfy this axiom. Top-consistency implements a variant of the popular consistency principle (see Thomson, 2013). It is an invariance axiom with respect to a restricted ordered-tree TU-game defined over the set of players contained in one of the two components existing before the addition of the top link (the link eventually added). The worth of this component is computed by assuming that the players outside the component leave the game with their payoffs. The worth of each sub-coalition of the component is not affected. The tree and the order in this restricted situation are defined as the restriction of the original tree and order. Top-consistency demands that the payoffs of the remaining players are invariant to this restriction. Link Amalgamation is also an invariance axiom that follows the tradition of the axioms of amalgamation initiated by Lehrer (1988). Consider an operation of link contraction which removes a link from the tree while simultaneously amalgamating its two incident players. As an example, imagine that two adjacent municipalities merge as is frequently the case on French territory. The coalition function is altered accordingly: the worth a coalition not containing the amalgamated player is not affected; otherwise it is equal to the worth of the corresponding coalition containing the two amalgamated agents. Link amalgamation imposes that the payoffs of the players incident to a link built after the contracted link are not affected by these operations of amalgamation and contraction.

It turns out that the combination of these three axioms yields a unique efficient allocation rule, for which we provide a natural expression constructed recursively by following the order over the links. This allocation rule relies on the following intuitive principle. Imagine that the links of the graph are severed one by one according to the total order. After the cut of a link, each of the

two newly created components receives its worth plus half of the surplus generated by their former union. This payoff is only temporary if the current graph still contains links, and becomes the final payoff of the component when the graph is empty after successive applications of the above standard principle. Since an empty graph contains as many (singleton) components as the number of players, we obtain a final payoff for each player.

In a second step, we provide a bargaining foundation of the allocation rule characterized in the part of the article by designing a bidding mechanism. First, the order of the links is taken into account in the construction of the bidding mechanism. The latter starts with the top link. Both players incident to this links play in a bidding stage and, in a second stage, bargain over the surplus of cooperation. At this end of the bargaining stage, both players obtain an intermediary payoff. Then, the mechanism continues its route on both components that the top link connects, and so on until there is no link to consider. We show that this bidding mechanism supports the above-mentioned allocation rule in subgame perfect equilibrium (SPE) when the underlying TU-game is superadditive.

This result is in line with the Nash program, which intends to bridge the gap between cooperative and noncooperative game theory. This research agenda has been recently influenced much by the work of Pérez-Castrillo and Wettstein (2001), where the Shapley value (Shapley, 1953) for TU-games is "implemented" through a mechanism consisting of a bidding stage followed by a proposal stage. Follow-up this seminal article, Ju and Wettstein (2009) and Béal et al. (2017c) provide a class of bidding mechanisms for implementing and comparing several allocation rules for TU-games. Each of these bidding mechanisms is modeled through a non-cooperative extensive form game. The outcome of each SPE of this game coincides with the allocation of a payoff among a set of players that a solution concept for TU-games recommends. Our bidding mechanism belongs to this category.

The rest of the article is organized as follows. Section 2 gives the basic definitions. In section 3, we introduce the axioms and proceed to the axiomatic study. Section 4 presents the bidding mechanism and shows that it supports our allocation rule in SPE. Section 5 concludes.

## 2. Preliminaries

### 2.1. TU-games

Throughout this article, the cardinality of a finite set $S$ will be denoted by the lower case $s$, the collection of all subsets of $S$ will be denoted by $2^S$, and, for notational convenience, we will write singleton $\{i\}$ as $i$.

Let $\mathbb{N}$ be the universe of potential players and let $N \subseteq \mathbb{N}$ be a finite set of $n$ players. Each subset $S$ of $N$ is called a *coalition* while $N$ is called the *grand coalition*. A cooperative game with transferable utility or simply a *TU-game* on the player set $N$ is a coalition function $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. For each coalition $S \subseteq N$, $v(S)$ describes the *worth* of the coalition $S$ when its members cooperate.

A TU-game is *superadditive* if, for each pair of coalitions $\{S, T\} \subseteq 2^N$ such that $S \cap T = \emptyset$, it holds that $v(S \cup T) \geq v(S) + v(T)$.

### 2.2. Ordered Tree TU-games

A *(simple) graph* is a pair $(N, L)$, where $N \subseteq \mathbb{N} = \{1, 2 \ldots\}$ is a finite set of $n$ nodes, representing a set of players, and $L$ is a set of *links*. Each link is formed by a set of two distinct nodes associated

with it, which are called its *endpoints*. A link with $i \in N$ and $j \in N \setminus i$ as endpoints is denoted by $ij$. For each $i \in N$, the set $N(i)$ defined as $\{j \in N \setminus i : ij \in L\}$ is the set of *neighbors* of $i$ in $(N, L)$. A *(simple) path* from $i \in N$ to $j \in N \setminus i$ in $(N, L)$ is a sequence of distinct players $(i_1, \ldots, i_r)$ such that $r \geq 2$, $i_1 = i$, $i_k i_{k+1} \in L$ for $k = 1, \ldots, r-1$, and $i_r = j$. A *tree* is a graph $(N, L)$ where either $N$ contains only one element or such that for any two distinct players $i, j \in N$, there exists a unique path $(i_1, i_2, \ldots, i_r)$ from $i_1 = i$ to $i_r = j$. In a tree, the cardinality $\ell$ of the set of links $L$ is equal to $n-1$.

A TU-game on a tree with ordered links or simply an *ordered tree TU-game* is a quadruple $(N, v, L, \preceq)$ such that $(N, v)$ is a TU-game, $(N, L)$ is a tree and $(L, \preceq)$ is a total order. The interpretation is that links enter one-by-one to form $L$ according to $\preceq$, where the greatest or top (lowest or bottom, respectively) element of $(L, \preceq)$ represents the last (first, respectively) link added. The *strict* part of $\preceq$ is denoted by $\prec$. Note that $(L, \preceq)$ is significant only when the tree $(N, L)$ contains at least two links, i.e. $n \geq 3$. Denote by $G$ the class of all ordered tree TU-games of the form $(N, v, L, \preceq)$.

Sometimes, we will use the following decomposition of $G$: for each $n \in \mathbb{N}$, let $G^n$ be the subclass of $G$ formed by the ordered tree TU-games $(N, v, L, \preceq) \in G$ containing exactly $n$ players. We have:

$$G = \bigcup_{n \in \mathbb{N}} G^n.$$

*2.3. Allocation rules*

In an *ordered tree TU-game* $(N, v, L, \preceq) \in G$, each player $i \in N$ may receive a payoff $x_i \in \mathbb{R}$. A payoff vector $x = (x_i)_{i \in N} \in \mathbb{R}^n$ lists a payoff $x_i \in \mathbb{R}$ for each $i \in N$. For any nonempty coalition $S \in 2^N$, the notation $x_S$ stands for $\sum_{i \in S} x_i$. An *allocation rule* $\Phi$ on $G$ is a mapping $\Phi : G \longrightarrow \cup_{n \in \mathbb{N}} \mathbb{R}^n$ which uniquely determines, for each $n \in \mathbb{N}$ and each $(N, v, L, \preceq) \in G^n$, a payoff vector $\Phi(N, v, L, \preceq) \in \mathbb{R}^n$.

## 3. Standardness, Link amalgamation and Top-consistency

In this section, we introduce four axioms for a solution $\Phi$ on $G$ and then proceed to an axiomatic study. The first two axioms are standard in the literature.

**Efficiency** An allocation rule $\Phi$ on $G$ is efficient if, for each $(N, v, L, \preceq) \in G$, it holds that:

$$\Phi_N(N, v, L, \preceq) = v(N).$$

**Standardness** An allocation rule $\Phi$ on $G$ is standard if, for each $(N, v, L, \preceq) \in G$ containing two players, say $N = \{i, j\}$, it holds that:

$$\Phi_i(N, v, L, \preceq) = v(i) + \frac{v(N) - v(i) - v(j)}{2} \text{ and } \Phi_j(N, v, L, \preceq) = v(j) + \frac{v(N) - v(i) - v(j)}{2}.$$

Note that Standardness implies Efficiency for the subclass of ordered tree TU-games containing two players.

The third axiom belongs to the family of axioms that incorporate an amalgamation principle. Such an axiom says something about the changes in payoffs when two players are amalgamated to act as if they were a single player. Precisely, our axiom indicates that if two neighbors are

amalgamated and the link joining them contracted, then the payoff of each player incident to a link entering after the contracted link is unchanged. Formally, pick any $(N, v, L, \preceq) \in G$ containing at least three players, and any link $ij \in L$. Define the ordered tree TU-game $(N^{ij}, v^{ij}, L^{ij}, \preceq^{ij}) \in G$ resulting from the amalgamation of the two neighbors $i$ and $j$ and the contraction of the link $ij$ as follows:

1. The player set is $N^{ij} = (N \setminus \{i, j\}) \cup \{\overline{ij}\}$. Because $i$ and $j$ collude and act as a single entity, they are amalgamated into a new single entity $\overline{ij}$;

2. The coalition function $v^{ij} : 2^{N^{ij}} \longrightarrow \mathbb{R}$ takes into account the fact that $\overline{ij}$ results from the amalgamation of two neighbors who act as a single entity:

$$v^{ij}(S) = \begin{cases} v((S \cup \{i, j\}) \setminus \overline{ij}) & \text{if } \overline{ij} \in S, \\ \\ v(S) & \text{if } \overline{ij} \notin S. \end{cases}$$

3. The set of links is modified accordingly. Because $\overline{ij}$ is a new single entity, the link $ij$ is contracted:

$$L^{ij} = \left( L \setminus \{pq \in L : \{p, q\} \cap \{i, j\} \neq \emptyset\} \right) \cup \{\overline{ij}q : iq \in L \text{ or } jq \in L, q \in N \setminus \{i, j\}\}.$$

4. Because $(N, L)$ is a tree, for each $p \in N \setminus \{i, j\}$, $ij \in L$ and $ip \in L$, we have $jp \notin L$. It follows that there is a bijective function $o : L \setminus ij \longrightarrow L^{ij}$ defined as follows:

$$o(pq) = \begin{cases} pq & \text{if } \{i, j\} \cap \{p, q\} = \emptyset, \\ \\ \overline{ij}q & \text{if } p = i \text{ and } q \in N(i), \\ \\ \overline{ij}q & \text{if } p = j \text{ and } q \in N(j). \end{cases}$$

The total order $(L^{ij}, \preceq^{ij})$, induced by $(L, \preceq)$, $ij \in L$ and the function $o$, is as follows:

$$\forall pq, uz \in L \setminus ij, \quad o(pq) \preceq^{ij} o(uz) :\Longleftrightarrow pq \preceq uz,$$

which means that the order in which links have been created is preserved.

**Link amalgamation** An allocation rule $\Phi$ on $G$ satisfies the axiom of Link amalgamation if, for each $(N, v, L, \preceq) \in G$, each pair of neighbors $\{i, j\} \subseteq N$ and each $p \in N \setminus \{i, j\}$ where there is $i_k i_{k+1} \in L$ such that $ij \prec i_k i_{k+1}$ on the unique path $(i_1, i_2, \ldots, i_r)$ from $i_1 = p$ to $i_r = i$ in $(N, L)$, it holds that:
$$\Phi_p(N^{ij}, v^{ij}, L^{ij}, \preceq^{ij}) = \Phi_p(N, v, L, \preceq).$$

Note that the axiom of Link amalgamation applies in ordered tree TU-games containing at least three players.

The last axiom incorporates a consistency principle. Informally, a consistency principle states the following. Fix an allocation rule for a class of cooperative games, consider an element of this class of cooperative games as well as the payoff vector chosen by the allocation rule for this game. Assume that a coalition of agents are paid according this vector and leave the game. Then, the remaining players examine the possibility to renegotiate the payoff allocation between them. Such a situation may be described by a reduced game on the remaining players in which the worth of each

coalition needs to be re-evaluated. The solution is consistent if, for this reduced game, the payoffs distributed by the allocation rule coincide with the payoffs allocated in the original game. As underlined by Aumann (2008) and Thomson (2013), the consistency principle has been examined in the context of a great variety of concrete problems of resource allocation. In one form or another, it is common to almost all solutions and often plays a key role in axiomatic characterizations of the solutions. In the context of ordered tree TU-games, we design a consistency axiom which takes into account the order in which links are added.

Pick any $(N, v, L, \preceq) \in G$ and the top link of $(N, \preceq)$, say the link $ij \in L$. Define $C_i^N$ as the set of players, including $i$, such that the unique path connecting them to $i$ does not contain $j$ ; define $C_j^N$ in a similar way. Note that $\{C_i^N, C_j^N\}$ forms a partition of $N$. Fix a payoff vector $x = (x_i)_{i \in N} \in \mathbb{R}^n$. Assume that the payoffs have been distributed according to the payoff vector $x$ and that the members of $C_j^N$ leave the game with their component of the vector $x$. Let us re-evaluate the situation of the members of $C_i^N$ at this point. To do this, we define the reduced game $(N_{i,x}, v_{i,x}, L_{i,x}, \preceq_{i,x}) \in G$ they face as follows:

1. The player set $N_{i,x} = C_i^N$ ;
2. The coalition function $v_{i,x} : 2^{N_{i,x}} \longrightarrow \mathbb{R}$ takes into account that the players in $C_j^N$ are already paid according to $x$:
$$v_{i,x}(S) = \begin{cases} v(N) - x_{C_j^N} & \text{if} \quad S = N_{i,x}, \\ v(S) & \text{if} \quad S \subset N_{i,x}. \end{cases}$$

   The TU-game $(N_{i,x}, v_{i,x})$ is sometimes called the projected reduced game (see, for instance, Funaki and Yamato, 2001).
3. The tree $(N_{i,x}, L_{i,x})$ is the subtree induced by $N^{i,x}$ on $(N, L)$, i.e.

$$L_{i,x} = \left\{ pq \in L : p, q \in C_i^N \right\}.$$

4. The total order $(L_{i,x}, \preceq_{i,x})$ is the restriction of $(L, \preceq)$ to $L_{i,x}$.

Therefore, $v_{i,x}(N_{i,x})$ is the total worth left for the remaining players $N_{i,x} = C_{i,x}$ who interact according to $(N_{i,x}, v_{i,x}, L_{i,x}, \preceq_{i,x})$; this worth is the only parameter which depends on values $x_p$, $p \in C_j^N$. Fix an allocation rule $\Phi$ on $G$. For the sake of notation, we will denote the reduced game $(N_{i,\Phi(N,v,L,\preceq)}, v_{i,\Phi(N,v,L,\preceq)}, L_{i,\Phi(N,v,L,\preceq)}, \preceq_{i,\Phi(N,v,L,\preceq)})$ by $(N_{i,\Phi}, v_{i,\Phi}, L_{i,\Phi}, \preceq_{i,\Phi})$.

**Top consistency** An allocation rule $\Phi$ on $G$ is Top consistent if, for any $(N, v, L \preceq) \in G$, where $ij \in L$ denotes the top link of $(N, \preceq)$, it holds that:

$$\forall p \in N_{i,\Phi}, \quad \Phi_p(N_{i,\Phi}, v_{i,\Phi}, L_{i,\Phi}, \preceq_{i,\Phi}) = \Phi_p(N, v, L, \preceq).$$

The first result of this section establishes that the combination of Standardness and Top-consistency implies Efficiency.

**Proposition 1** *Let $\Phi$ be an allocation rule on $G$ which satisfies Standardness and Top consistency. Then, $\Phi$ satisfies Efficiency.*

*Proof.* Consider any $(N, v, L, \preceq) \in G$. We distinguish two cases according to the number of elements in $N$.

**Case 1** Assume that $N$ contains only one elements, say $N = \{i\}$, which implies that $L = \emptyset$, and so $(L, \preceq)$ is void. From $(\{i\}, v, \emptyset, \emptyset)$, construct the ordered tree TU-games $(\{i, j\}, w, \{ij\}, \preceq)$, where $w(i) = v(i)$, $w(j) = 0$ and $w(\{i, j\}) = v(i)$. Because there is only one link $ij$, we have $ij \preceq ij$. Applying Standardness, we have:

$$\Phi_i(\{i, j\}, w, \{ij\}, \preceq) = v(i) \text{ and } \Phi_j(\{i, j\}, w, \{ij\}, \preceq) = 0 \tag{1}$$

By Top consistency, we have:

$$\Phi_i(\{i, j\}_{i,\Phi}, w_{i,\Phi}, \{ij\}_{i,\Phi}, \preceq_{i,\Phi}) = \Phi_i(\{i, j\}, w, \{ij\}, \preceq), \tag{2}$$

where $\{i, j\}_{i,\Phi} = \{i\}$,

$$w_{i,\Phi}(\{i\}) = w(\{i, j\}) - \Phi_j(\{i, j\}, w, \{ij\}, \preceq) = v(i) - 0 = v(i),$$

and $\preceq_{i,\Phi}$ is void. Therefore, $(\{i, j\}_{i,\Phi}, w_{i,\Phi}, \{ij\}_{i,\Phi}, \preceq_{i,\Phi}) = (\{i\}, v, \emptyset, \emptyset)$ and so:

$$\Phi_i(\{i, j\}_{i,\Phi}, w_{i,\Phi}, \{ij\}_{i,\Phi}, \preceq_{i,\Phi}) = \Phi_i(\{i\}, v, \emptyset, \emptyset) \tag{3}$$

By (1), (2) and (3), we conclude that:

$$\Phi_i(\{i\}, v, \emptyset, \emptyset) = v(i),$$

as desired.

**Case 2** Assume that $N$ contains at least two players. We proceed by induction on the number $n \geq 2$ of players.

*Initial step* Consider any $(N, v, L, \preceq) \in G^2$. The result follows by Standardness.

*Induction hypothesis* Assume that $\Phi$ is Efficient for each $(N, v, L, \preceq) \in G^n$, $2 \leq n \leq r$, $r \geq 2$.

*Induction step* Consider any $(N, v, L, \preceq) \in G^{r+1}$. Let $ij \in L$ be the top link of $(L, \preceq)$. By Top consistency, we have:

$$\forall p \in C_i^N, \quad \Phi_p(N, v, L, \preceq) = \Phi_p(N_{i,\Phi}, v_{i,\Phi}, L_{i,\Phi}, \preceq_{i,\Phi}) \tag{4}$$

Because $c_i^N \leq r$ we can apply the induction hypothesis to obtain:

$$
\begin{aligned}
\Phi_{C_i^N}(N_{i,\Phi}, v_{i,\Phi}, L_{i,\Phi}, \preceq_{i,\Phi}) &= v_{i,\Phi}(C_i^N) \\
&= v(N) - \Phi_{C_j^N}(N, v, L, \preceq),
\end{aligned}
$$

where the second equality follows from the definition of $(N_{i,\Phi}, v_{i,\Phi})$. Combining the above equality with equation (4), we obtain:

$$
\begin{aligned}
\Phi_{C_i^N}(N_{i,\Phi}, v_{i,\Phi}, L_{i,\Phi}, \preceq_{i,\Phi}) &= \Phi_{C_i^N}(N, v, L, \preceq) \\
&= v(N) - \Phi_{C_j^N}(N, v, L, \preceq).
\end{aligned}
$$

Because $\{C_i^N, C_j^N\}$ forms a partition of $N$, we conclude that $\Phi_N(N, v, L, \preceq) = v(N)$, as desired. This completes the proof of Proposition 1. ∎

The following proposition establishes that there is at most allocation rule on $G$ which satisfies Standardness, Link amalgamation, and Top consistency.

**Proposition 2** *There exists at most one allocation rule $\Phi$ on $G$ which satisfies Standardness, Link amalgamation, and Top consistency.*

*Proof.* Consider any allocation rule $\Phi$ on $G$ which satisfies Standardness, Link amalgamation and Top consistency. To show: $\Phi$ is uniquely determined on $G$. The proof is done by induction on the number of players in $(N, v, L, \preceq) \in G$.

*Initial step* If $(N, v, L, \preceq) \in G^1$, the result holds by Proposition 1. If $(N, v, L, \preceq) \in G^2$, the result holds by Standardness.

*Induction hypothesis* Assume that $\Phi$ is uniquely determined for each subclass $G^n$, $1 \leq n \leq r$, where $r \geq 2$.

*Induction step* Consider any $(N, v, L \preceq) \in G^{r+1}$. Let $ij \in L$ be the bottom link of $(L, \preceq)$. Pick any $p \in N \setminus \{i, j\}$, which is possible since $r \geq 3$. Assume that the neighbors $i$ and $j$ collude and act as a single entity $\overline{ij}$. The player set $N^{ij}$ contains exactly $r$ players. By the induction hypothesis, $\Phi_p(N^{ij}, v^{ij}, L^{ij}, \preceq^{ij})$ is uniquely determined. Because $ij$ is the first link to be created, we conclude by Link amalgamation that:

$$\forall p \in N \setminus \{i, j\}, \quad \Phi_p(N, v, L, \preceq) = \Phi_p(N^{ij}, v^{ij}, L^{ij}, \preceq^{ij}),$$

which proves that, for each $p \in N \setminus \{i, j\}$, $\Phi_p(N, v, L, \preceq)$ is uniquely determined. It remains to show that $\Phi_i(N, v, L, \preceq)$ and $\Phi_j(N, v, L, \preceq)$ are uniquely determined. To this end, consider the top link $uz$ of $(N, \preceq)$. Because $r \geq 3$, it is the case that $uz \neq ij$. It follows that either $\{i, j\} \subseteq C_u^N$ or $\{i, j\} \subseteq C_z^N$. Assume, without loss of generality, that $\{i, j\} \subseteq C_u^N$. From above, we know that, for each $p \in C_z^N \subseteq N \setminus \{i, j\}$, $\Phi_p(N, v, L, \preceq)$ is uniquely determined. Therefore, the worth $v_{u,\Phi}(N_{u,\Phi})$ of the coalition $N_{u,\Phi}$,

$$v_{u,\Phi}(N_{u,\Phi}) = v(N) - \Phi_{C_z^N}(N, v, L, \preceq),$$

is well-defined, so is the reduced game $(N_{u,\Phi}, v_{u,\Phi}, L_{u,\Phi}, \preceq_{u,\Phi})$. The player set $N_{u,\Phi} = C_u^N$ contains at most $r$ players. Thus, we can apply the induction hypothesis to conclude that:

$$\forall p \in C_u^p, \quad \Phi_p(N_{u,\Phi}, v_{u,\Phi}, L_{u,\Phi}, \preceq_{u,\Phi})$$

is uniquely determined. In particular, $\Phi_i(N_{u,\Phi}, v_{u,\Phi}, L_{u,\Phi}, \preceq_{u,\Phi})$ and $\Phi_j(N_{u,\Phi}, v_{u,\Phi}, L_{u,\Phi}, \preceq_{u,\Phi})$ are uniquely determined. From Top consistency, it holds that:

$$\Phi_i(N, v, L, \preceq) = \Phi_i(N_{u,\Phi}, v_{u,\Phi}, L_{u,\Phi}, \preceq_{u,\Phi}) \text{ and } \Phi_j(N, v, L, \preceq) = \Phi_j(N_{u,\Phi}, v_{u,\Phi}, L_{u,\Phi}, \preceq_{u,\Phi}),$$

which ensures that $\Phi_i(N, v, L, \preceq)$ and $\Phi_j(N, v, L, \preceq)$ are uniquely determined. This completes the induction step.

Conclude that $\Phi$ is uniquely determined on $G$. ∎

Notice that in the proof of Proposition 2, we only amalgamate the two endpoints of the bottom link. Therefore, we can weaken the axiom of Link amalgamation by considering only the bottom link. This results in the following axiom.

**Bottom link amalgamation** An allocation rule $\Phi$ on $G$ satisfies the axiom of Bottom link amalgamation if, for each $(N, v, L, \preceq) \in G$ and $p \in N \setminus \{i, j\}$ where $ij \in L$ denotes the bottom link of $(L, \preceq)$, it holds that:

$$\forall p \in N \setminus \{i, j\}, \quad \Phi_p(N^{ij}, v^{ij}, L^{ij}, \preceq^{ij}) = \Phi_p(N, v, L, \preceq).$$

**Proposition 3** *There exists at most one allocation rule $\Phi$ on $G$ which satisfies Standardness, Bottom link amalgamation, and Top consistency.*

**Remark** Bottom link amalgamation and Link amalgamation are in line with other axioms of amalgamation, collusion or merging initiated by Lehrer (1988) for TU-games. There is however a difference. Our axioms do not compare the payoff of the amalgamated agent with the sum of the payoffs of its constituents in the original situation. For instance, many characterizations of the Banzhaf value (1965) for TU-games, the Banzhaf graph value for graph TU-games (Alonso-Meijide, Fiestras-Janeiro 2006) and the Banzhaf-Owen graph value for graph TU-games with a priori unions (Alonso-Meijide et al. 2009) use a principle of neutrality meaning that the sum of the payoffs of two players/neighbors does not change if these two players/neighbors merge into a single player. In this sense, two players/neighbors never benefit from acting as one entity. In contrast, Link amalgamation and Bottom link amalgamation indicate that if two neighbors are amalgamated into one player, then the payoffs of certain other agents are invariant. This principle of invariance is inspired from the axiom of Amalgamation in Béal et al (2015) for rooted tree TU-games and the axiom of Invariance with respect to cone amalgamation in Béal et al. (2017a) for tree TU-games. The main difference between Link amalgamation and these axioms is that in Link amalgamation all pairs of neighbors, except those involved in the top link, are allowed to collude, whereas in Amalgamation and Invariance with respect to cone amalgamation only particular sets of players can collude (the set of subordinates of a player and the so-called cone a tree, respectively). Nevertheless, combining Bottom link amalgamation with Efficiency allows to restore the above-mentioned principle of neutrality. Precisely, assume that an allocation rule $\Phi$ on $G$ satisfies Bottom link amalgamation and Efficiency. Then, we have:

$$\Phi_i(N, v, L \preceq) + \Phi_j(N, v, L \preceq) = \Phi_{\overline{ij}}(N^{ij}, v^{ij}, L^{ij} \preceq^{ij}).$$

To see this, consider the bottom link $ij \in L$ and the associated game $(N^{ij}, v^{ij}, L^{ij} \preceq^{ij})$. On the one hand, by Bottom link amalgamation, we have:

$$\forall p \in N \setminus \{i, j\}, \quad \Phi_p(N^{ij}, v^{ij}, L^{ij}, \preceq^{ij}) = \Phi_p(N, v, L, \preceq).$$

On the other hand, by Efficiency, we have:

$$\sum_{p \in N \setminus \{i,j\}} \Phi_p(N^{ij}, v^{ij}, L^{ij}, \preceq^{ij}) + \Phi_{\overline{ij}}(N^{ij}, v^{ij}, L^{ij} \preceq^{ij}) = v^{ij}(N^{ij})$$

and

$$\sum_{p \in N \setminus \{i,j\}} \Phi_p(N, v, L \preceq) + \Phi_i(N, v, L \preceq) + \Phi_j(N, v, L \preceq) = v(N).$$

Because $v^{ij}(N^{ij}) = v(N)$, we get:

$$\Phi_i(N, v, L \preceq) + \Phi_j(N, v, L \preceq) = \Phi_{\overline{ij}}(N^{ij}, v^{ij}, L^{ij} \preceq^{ij}).$$

It is also worth mentioning the work by van den Brink (2012) who considers the possibility that two neighbors collude in a tree TU-game. The main difference with our operation of amalgamation is that that the operation of collusion envisaged by van den Brink leaves the player set and the link set unchanged. $\square$

Having proved that Standardness, (Bottom) Link amalgamation, and Top consistency determine at most one allocation rule on $G$, we are now going to prove that such an allocation rule exists and provide a closed form expression for it. To this end, we need a definition. From any $(N, v, L, \preceq) \in G$ containing at least two agents, we create an associated ordered tree TU-games $(N_i, v_i, L_i, \preceq_i) \in G$, where $i \in N$ is one of the two endpoints of the top link of $(N, \preceq)$, as follows:

1. The player set $N_i = C_i^N$ ;
2. The coalition function $v_i : 2^{N_i} \longrightarrow \mathbb{R}$ is such that:

$$
v_i(S) = \begin{cases} v(C_i^N) + \dfrac{v(N) - v(C_i^N) - v(C_j^N)}{2} & \text{if} \quad S = N_i, \\[2mm] v(S) & \text{if} \quad S \subset N_i; \end{cases}
$$

3. The tree $(N_i, L_i)$ is the subtree induced by $N^i$ on $(N, L)$, i.e.

$$
L_i = \{ pq \in L : p, q \in C_i^N \};
$$

4. The total order $(L_i, \preceq_i)$ is the restriction of $(L, \preceq)$ to $L^i$.

On the class of ordered tree TU-games $G$, define recursively the *allocation rule* $\Phi^e$ as follows:

(a) For any $(N, v, L, \preceq) \in G$ such that $N = \{i\}$, $i \in \mathbb{N}$, $\Phi_i^e(N, v, L, \preceq) = v(N)$;

(b) For any $(N, v, L, \preceq) \in G$ such that $n \geq 2$,

$$
\forall p \in C_i^N, \quad \Phi_p^e(N, v, L, \preceq) = \Phi_p^e(N_i, v_i, L_i, \preceq_i),
$$

where $i \in N$ is one of the two endpoints of the top link of $(N, \preceq)$.

The following example illustrates the computation of $\Phi^e$.

**Example 1** *Let the ordered tree TU-game $(N, v, L, \preceq)$ be given by $N = \{1, 2, 3, 4, 5\}$, for each $i \in N$, $v(i) = i$, for each other nonempty coalition $S \in 2^N$, $v(S) = s^2$, $L = \{12, 23, 34, 25\}$ and $12 \preceq 25 \preceq 34 \preceq 23$. Figure 1 describes the formation process of the tree from left to right. Thus,*
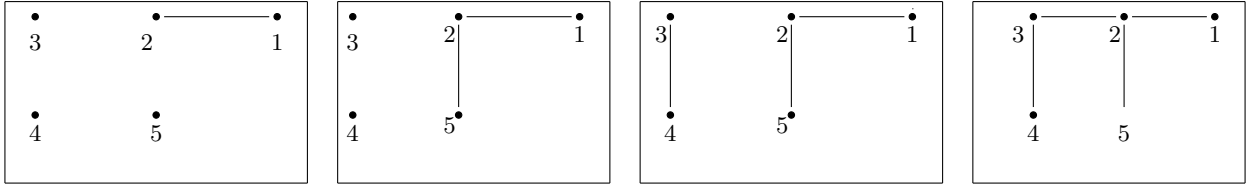


Figure 1: Formation process of $(N, L)$ according to $\preceq$.

*the top link is 23 and so $C_2^N = \{1, 2, 5\}$, $C_3^N = \{3, 4\}$. By point (b) of the definition of $\Phi^e$ we have:*

$$
\forall i \in \{1, 2, 5\} \quad \Phi_i^e(N, v, L, \preceq) = \Phi_i^e(N_2, v_2, L_2, \preceq_2),
$$

and
$$\forall i \in \{3, 4\}, \quad \Phi_i^e(N, v, L, \preceq) = \Phi_i^e(N_3, v_3, L_3, \preceq_3).$$

First, let us compute $(N_2, v_2, L_2, \preceq_2)$. We have $N_2 = C_2^N$, $v_2(S) = v(S)$ for each nonempty coalition $S$ of size one or two, and
$$v_2(N_2) = 9 + \frac{25 - 9 - 4}{2} = 15.$$

In $(N_2, v_2, L_2, \preceq_2)$, the top link is 25 so that $C_2^{N_2} = \{1, 2\}$, $C_5^{N_2} = \{5\}$. By point (b) of the definition of $\Phi^e$ we have,

$$\forall i \in \{1, 2\}, \quad \Phi_i^e(N_2, v_2, L_2, \preceq_2) = \Phi_i^e((N_2)_2, (v_2)_2, (L_2)_2, (\preceq_2)_2)$$

and
$$\Phi_5^e(N_2, v_2, L_2, \preceq_2) = \Phi_5^e((N_2)_5, (v_2)_5, (L_2)_5, (\preceq_2)_5).$$

Note that $((N_2)_5, (v_2)_5, (L_2)_5, (\preceq_2)_5)$ contains only one player, i.e. player 5, so that by point (a) of the definition of $\Phi^e$, we obtain:

$$\begin{aligned}
\Phi_5^e((N_2)_5, (v_2)_5, (L_2)_5, (\preceq_2)_5) &= (v_2)_5(5) \\
&= v_2(5) + \frac{v_2(N_2) - v((N_2)_2) - v((N_2)_5)}{2} \\
&= 5 + \frac{15 - 4 - 5}{2} \\
&= 8.
\end{aligned}$$

Therefore, we obtain:

$$\Phi_5^e(N, v, L, \preceq) = \Phi_5^e(N_2, v_2, L_2, \preceq_2) = \Phi_5^e((N_2)_5, (v_2)_5, (L_2)_5, (\preceq_2)_5) = 8.$$

Regarding the ordered tree TU-game $((N_2)_2, (v_2)_2, (L_2)_2, (\preceq_2)_2)$, we have $(N_2)_2 = C_2^{N_2} = \{1, 2\}$, $(v_2)_2(S) = v(S)$ for each nonempty coalition $S$ of size one, and

$$(v_2)_2((N_2)_2) = 4 + \frac{15 - 4 - 5}{2} = 7.$$

We also have $(L_2)_2 = \{12\}$ and so $12(\preceq_2)_2 12$. By point (b) of definition of $\Phi^e$,

$$\forall i \in \{1, 2\}, \quad \Phi_i^e((N_2)_2, (v_2)_2, (L_2)_2, (\preceq_2)_2) = \Phi_i^e(((N_2)_2)_i, ((v_2)_2)_i, ((L_2)_2)_i, ((\preceq_2)_2)_i).$$

For each $i \in \{1, 2\}$, $(((N_2)_2)_i, ((v_2)_2)_i, ((L_2)_2)_i, ((\preceq_2)_2)_i)$ contains only player $i$. By applying point (a) of the definition of $\Phi^e$, we have:

$$\begin{aligned}
\forall i \in \{1, 2\}, \quad \Phi_i^e((N_2)_2, (v_2)_2, (L_2)_2, (\preceq_2)_2) &= ((v_2)_2)_i(i) \\
&= i + \frac{7 - 1 - 2}{2} \\
&= i + 2.
\end{aligned}$$

We are able to conclude that:

$$\Phi_1^e(N, v, L, \preceq) = 3 \text{ and } \Phi_2^e(N, v, L, \preceq) = 4.$$

*Proceeding in a similar way from the ordered tee TU-game* $(N_3, v_3, L_3, \preceq_3)$*, we obtain the following payoffs:*

$$\Phi_3^e(N, v, L, \preceq) = \frac{9}{2} \text{ and } \Phi_4^e(N, v, L, \preceq) = \frac{11}{2}.$$

*To sum up:*

$$\Phi^e(N, v, L \preceq) = \left(3, 4, \frac{9}{2}, \frac{11}{2}, 8\right).$$

$\square$

**Proposition 4** *The allocation rule* $\Phi^e$ *satisfies Standardness, Link amalgamation and Consistency on* $G$.

Proof. Consider the allocation rule $\Phi^e$ on $G$.

**Standardness** Pick any $(N, v, L \preceq) \in G$ containing two players, say $N = \{i, j\}$. We have: $C_i^N = \{i\}$, $C_j^N = \{j\}$, and

$$v_i(S) = \begin{cases} v(i) + \dfrac{v(N) - v(i) - v(j)}{2} & \text{if } S = \{i\}, \\[2mm] 0 & \text{if } S = \emptyset. \end{cases}$$

On the one hand, because $(N_i, v_i, L_i \preceq_i) \in G$ contains only one player, by point (a) of the definition of $\Phi^e$, we obtain:

$$\Phi_i^e(N_i, v_i, L_i \preceq_i) = v(i) + \frac{v(N) - v(i) - v(j)}{2}.$$

On the other hand, by point (b) of the definition of $\Phi^e$, we get:

$$\Phi_i^e(N, v, L, \preceq) = \Phi_i^e(N_i, v_i, L_i, \preceq_i),$$

and so

$$\Phi_i^e(N, v, L, \preceq) = v(i) + \frac{v(N) - v(i) - v(j)}{2},$$

as desired. Proceeding in the same way for player $j$, we conclude that $\Phi^e$ satisfies Standardness.

**Link amalgamation** In case the player set contains one or two elements, there is nothing to prove. For the other cases, we prove that $\Phi^e$ satisfies Link amalgamation by induction on the number of players in an ordered tree TU-game.

*Initial step* Consider any $(N, v, L \preceq) \in G$ containing three players, say $N = \{1, 2, 3\}$, and assume, without loss of generality, that $L = \{12, 23\}$ where $12 \preceq 23$. By definition, the axiom of Link amalgamation does not apply when the two neighbors involve in the top link collude. So, assume that neighbors 1 and 2 collude to form a single entity $\overline{12}$ so that $N^{12} = \{\overline{12}, 3\}$, $L^{12} = \{\overline{123}\}$, $\overline{123} \preceq^{12} \overline{123}$ and $v^{12}(\overline{123}) = v(N)$, $v^{12}(3) = v(3)$ and $v^{12}(\overline{12}) = v(\{1, 2\})$. Because the link 12 lies on the unique path going from 3 to 1, Link amalgamation holds if and only if:

$$\Phi_3^e(N, v, L \preceq) = \Phi_3^e(N^{12}, v^{12}, L^{12}, \preceq^{12}).$$

On the other hand, since $N_3 = \{3\}$, by point (b) of the definition of $\Phi^e$, we have:

$$\Phi_3^e(N, v, L \preceq) = \Phi_3^e(N_3, v_3, L_3, \preceq_3),$$

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and

$$v_3(S) = \begin{cases} v(3) + \dfrac{v(N) - v(3) - v(\{1,2\})}{2} & \text{if} \quad S = \{3\}, \\ \\ 0 & \text{if} \quad S = \emptyset. \end{cases}$$

Because $(N_3, v_3, L_3, \preceq_3) \in G$ contains only one player, by point (a) of the definition of $\Phi^e$, we have:

$$\Phi_3^e(N_3, v_3, L_3, \preceq_3) = v(3) + \frac{v(N) - v(3) - v(\{1,2\})}{2}.$$

On the other hand, by Standardness of $\Phi^e$ (see the previous point), we have:

$$\begin{aligned} \Phi_3^e(N^{12}, v^{12}, L^{12}, \preceq^{12}) &= v^{12}(3) + \frac{v^{12}(N^{12}) - v^{12}(3) - v^{12}(\overline{12})}{2} \\ &= v(3) + \frac{v(N) - v(3) - v(\{1,2\})}{2} \\ &= \Phi_3^e(N_3, v_3, L_3, \preceq_3). \end{aligned}$$

Therefore, we get:

$$\Phi_3^e(N, v, L \preceq) = \Phi_3^e(N_3, v_3, L_3, \preceq_3) = \Phi_3^e(N^{12}, v^{12}, L^{12}, \preceq^{12}),$$

as desired.

*Induction hypothesis* Assume that $\Phi^e$ satisfies Link amalgamation for each subclass $G^n$, $3 \leq n \leq r$, where $r \geq 3$.

*Induction step* Consider any $(N, v, L \preceq) \in G^{r+1}$ and denote by $uz \in L$ the top link of $(N, \preceq)$. Pick any link $ij \in L \setminus \{uz\}$, and assume that neighbors $i$ and $j$ decide to collude to form a single entity $\overline{ij}$. Pick any $p \in N^{ij}$ such that there is $i_k i_{k+1} \in L$ where $ij \prec i_k i_{k+1}$ and $i_k i_{k+1}$ lies on the unique path from $p$ to $i$ in $(N, L)$. To show: $\Phi_p^e(N, v, L \preceq) = \Phi_p^e(N^{ij}, v^{ij}, L^{ij}, \preceq^{ij})$. Assume, without loss of generality, that $p \in C_u^N$. By definition of $\Phi^e$, we have:

$$\Phi_p^e(N, v, L \preceq) = \Phi_p(N_u, v_u, L_u \preceq_u) \text{ and } \Phi_p^e(N^{ij}, v^{ij}, L^{ij} \preceq^{ij}) = \Phi_p^e((N^{ij})_u, (v^{ij})_u, (L^{ij})_u, (\preceq^{ij})_u) \tag{5}$$

We distinguish two exclusive cases.

**Case 1** The top link $uz \in L$ does not lies on the unique path from $p$ to $i$. In such a case, all the players lying on this path belong to $C_u^N$. In particular, $p, i, j \in C_u^N$. The ordered tree TU-game $(N_u, v_u, L_u \preceq_u)$ contains at least three players and at most $r \geq 4$ players. Because all the players on the unique path from $p$ to $i$ in $(N, L)$ still belong to $C_u^N$, it remains true that there is $i_k i_{k+1} \in L_u$ where $ij \prec_u i_k i_{k+1}$ and $i_k i_{k+1}$ lies on the unique path from $p$ to $i$ in the subtree $(N_u, L_u)$. Thus, we can apply the induction hypothesis to obtain:

$$\Phi_p^e(N_u, v_u, L_u \preceq_u) = \Phi_p^e((N_u)^{ij}, (v_u)^{ij}, (L_u)^{ij}, (\preceq_u)^{ij}) \tag{6}$$

It is not difficult to verify that $((N_u)^{ij}, (v_u)^{ij}, (L_u)^{ij}, (\preceq_u)^{ij}) = ((N^{ij})_u, (v^{ij})_u, (L^{ij})_u, (\preceq^{ij})_u)$ so that equality (6) becomes:

$$\Phi_p^e(N_u, v_u, L_u \preceq_u) = \Phi_p^e((N^{ij})_u, (v^{ij})_u, (L^{ij})_u, (\preceq^{ij})_u). \tag{7}$$

Combining (5) with (7), we obtain the desired equality:

$$\Phi_p^e(N, v, L \preceq) = \Phi_p^e(N^{ij}, v^{ij}, L^{ij} \preceq^{ij}).$$

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**Case 2** The top link $uz \in L$ lies on the unique path from $p$ to $i$. By assumption $p \in C_u^N$, and so $i \in C_z^N$. In such a case, it is obvious that $((N^{ij})_u, (v^{ij})_u, (L^{ij})_u, (\preceq^{ij})_u) = (N_u, v_u, L_u \preceq_u)$, so that, by (5), we directly obtain the desired equality.

This completes the induction step. So, conclude that $\Phi^e$ satisfies Link amalgamation.

**Top consistency** To show: for each $(N, v, L, \preceq) \in G$ where $ij \in L$ denotes the top link of $(N, \preceq)$, it holds that

$$\forall p \in N_{i,\Phi}, \quad \Phi_p^e(N_{i,\Phi^e}, v_{i,\Phi^e}, L_{i,\Phi^e}, \preceq_{i,\Phi^e}) = \Phi_p^e(N, v, L, \preceq).$$

To show these equalities, it is useful to prove first that $\Phi^e$ is Efficient. This statement is true for ordered tree TU-games in $G^1 \cup G^2$ by point (a) of the definition of $\Phi^e$ and by the fact that $\Phi^e$ satisfies Standardness. To complete the proof, assume the statement holds for all ordered tree TU-games in $G^k$ where $1 \le k \le r$ and $r \ge 3$. Pick any $(N, v, L, \preceq) \in G^{r+1}$ where $ij \in L$ stands for the top link. By definition of $\Phi^e$, we have:

$$\sum_{p \in C_i^N} \Phi_p^e(N, v, L \preceq) = \sum_{p \in C_i^N} \Phi_p(N_i, v_i, L_i \preceq_i) \text{ and } \sum_{p \in C_j^N} \Phi_p^e(N, v, L \preceq) = \sum_{p \in C_j^N} \Phi_p(N_j, v_j, L_j \preceq_j).$$

Because the cardinality of the sets $N_i = C_i^N$ and $N_j = C_j^N$ is equal at most to $r$, the induction hypothesis applies:

$$\sum_{p \in C_i^N} \Phi_p(N_i, v_i, L_i \preceq_i) = v_i(N_i) \text{ and } \sum_{p \in C_j^N} \Phi_p(N_j, v_j, L_j \preceq_j) = v_j(N_j),$$

where

$$v_i(N_i) = v(C_i^N) + \frac{v(N) - v(C_i^N) - v(C_j^N)}{2} \text{ and } v_j(N_j) = v(C_j^N) + \frac{v(N) - v(C_i^N) - v(C_j^N)}{2}.$$

Because $\{C_i^N, C_j^N\}$ forms a partition of $N$, we get

$$\sum_{p \in N} \Phi_p^e(N, v, L \preceq) = \sum_{p \in C_i^N} \Phi_p^e(N, v, L \preceq) + \sum_{p \in C_j^N} \Phi_p(N_i, v_i, L_i \preceq_i) = v_i(N_i) + v_j(N_j) = v(N),$$

from which we conclude that $\Phi^e$ is Efficient. Next, from Efficiency of $\Phi^e$, we deduce that:

$$\sum_{p \in C_j^N} \Phi_p(N_i, v_i, L_i \preceq_i) = v(C_j^N) + \frac{v(N) - v(C_i^N) - v(C_j^N)}{2}.$$

From this, we get that $(N_{i,\Phi^e}, v_{i,\Phi^e})$ is such that $N_{i,\Phi^e} = N_i = C_i^N$, and

$$v_{i,\Phi^e}(S) = \begin{cases} v(N) - v(C_j^N) - \dfrac{v(N) - v(C_i^N) - v(C_j^N)}{2} & \text{if } S = C_i^N, \\ v(S) & \text{if } S \subset C_i^N, \end{cases}$$

so that $(N_i, v_i) = (N_{i,\Phi^e}, v_{i,\Phi^e})$. Because $L_{i,\Phi^e}$ and $\preceq_{i,\Phi^e}$ do not depend on $\Phi^e$, it is immediate to verify that $L_{i,\Phi^e} = L_i$ and $\preceq_{i,\Phi^e} = \preceq_i$. Therefore, $(N_i, v_i, L_i, \preceq_i) = (N_{i,\Phi^e}, v_{i,\Phi^e}, L_{i,\Phi^e}, \preceq_{i,\Phi^e})$, and so by definition of $\Phi^e$ we get:

$$\forall p \in N_{i,\Phi^e}, \quad \Phi_p^e(N, v, L, \preceq) = \Phi_p^e(N_i, v_i, L_i, \preceq_i) = \Phi_p^e(N_{i,\Phi^e}, v_{i,\Phi^e}, L_{i,\Phi^e}, \preceq_{i,\Phi^e}),$$

which proves that $\Phi^e$ satisfies Top consistency. ∎

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The next corollary follows immediately from Proposition 2 (or Proposition 3) and Proposition 4.

**Corollary 1** *The allocation rule $\Phi^e$ is the only allocation rule on $G$ that satisfies Standardness, (Bottom) Link amalgamation and Top consistency.*

The logical independence of the axioms is demonstrated as follows:

1. The Equal Surplus Division rule applied on $G$, denoted by ESD and defined as:

$$\forall i \in N, \quad \text{ESD}_i(N, v, L, \preceq) = v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n},$$

   satisfies Standardness, Consistency but violates Link amalgamation.

2. The allocation rule $\Psi^1$ on $G$ defined as $\Phi^e$ except for the point (b) when the associated game $(N_i, v_i, L_i, \preceq_i)$ contains only two players, say $N_i = \{i, k\}$. In such a case, $v_i(N_i)$ is fully allocated to player $i$ if $i < k$; otherwise the worth is allocated to player $k$. The allocation rule $\Psi^1$ satisfies Standardness, Amalgamation but violates Consistency.

3. Consider the null allocation rule on $G$ which assigns a null payoff vector to each ordered tree TU-game in $G$. This allocation rule violates Standardness, but satisfies Consistency and Link Amalgamation.

**Remark** To conclude this section, two remarks are in order.

1. It should be noted that, in very special cases, the payoffs allocated by $\Phi^e$ coincide with the payoffs allocated by the Sequential Surplus Equal Division (SESD) rule introduced by Béal et al. (2015) for rooted tree TU-games. The SESD rule generalizes the individual standardized remainder vectors proposed by Ju et al. (2007) for the class of all TU-games to the class of rooted tree TU-games. Consider a rooted tree TU-game $(N, v, D)$ where the rooted tree $(N, D)$ is a directed line, say $D = \{(i, i+1) : i \in \{1, \ldots, n-1\}\}$. In this case, it is easy to see that the SESD rule applied to $(N, v, D)$ coincides with the payoffs distributed by $\Phi^e$ applied to the situation $(N, v, L, \preceq)$, where $L = \{i(i+1) : i \in \{1, \ldots, n-1\}\}$ and the total order over the links is $12 \succeq 23 \succeq \ldots \succeq (n-1)n$.

2. The rule $\Phi^e$, the SESD rule for rooted tree TU-games and the sequential sharing (bankrupcty) rules introduced by Ansik and Weikkard (2012) for river sharing problems (where the river is represented by a directed line), have in common to be computed recursively from the Top link.

## 4. Bargaining foundation

We address the problem of reaching an allocation of the worth of the grand coalition by suggesting a bidding mechanism and focusing on its subgame perfect Nash equilibria. We design a bidding mechanism where pairs of neighbors are successively involved in a two-stage bargaining game where the first stage is a bidding stage and the second stage is a 'Take-it or leave-it procedure'. The distinctiveness of our mechanism is its recursive structure.

The idea to introduce a bidding stage in a bargaining process is due to Demange (1984). In the context of exchange economies, she devised a bidding mechanism to implement efficient egalitarian allocations in subgame perfect equililbrium. Later, Pérez-Castrillo and Wettstein (2001) revisited

the bidding mechanisms to support the (weighted) Shapley value(s) in subgame perfect Nash equilibrium. Recently, Navarro and Perea (2013), van den Brink et al. (2016) and Béal et al. (2017c) design bidding mechanisms to support allocation rules defined on (directed) graph TU-games in subgame perfect Nash equilibrium.

We first define a simple two-player mechanism, called Mechanism (A). Then, we will embed it into a mechanism for ordered tree TU-games with $n$ players, called Mechanism (B), whose structure is recursive.

**Mechanism (A)** Let $Q$ be a non-negative real value which has to be shared between a set $N$ of two players,

**Stage 1** Each player $i \in N$ makes bids $h^i \in \mathbb{R}$.[2] Denote by $\Omega$ the subset of players with the *highest bids*. Pick at random any player $j \in \Omega$. Each such a $j$ induces a sequential strategic game $G_j$ whose payoffs are denoted by $(g_j^i)_{i \in N}$.

**Stage 2** $G_j$ describes a 'Take-it-or-leave-it' procedure or an ultimatum game, where player $j$ makes a proposal $x_j \in [0, Q]$ to the other player. If the other player $i \neq j$ accepts the proposal, then he or she receives $g_j^i = x_j$, and player $j$ receives $g_j^j = Q - x_j$. If $i$ refuses the proposal, both player obtain $g_j^i = g_j^j = 0$.

**Stage 3** Rewards $(z_j^i)_{i \in N}$ resulting from **Stage 1** and **Stage 2** associated with $G_j$, $j \in \Omega$, are defined as:

$$z_j^j = g_j^j - h^j + \frac{h^j - h^i}{2} \text{ and } z_j^i = g_j^i - (-h^i) + \frac{h^j - h^i}{2} \text{ for } i \neq j,$$

or equivalently

$$z_j^j = g_j^j - \frac{h^i + h^j}{2} \text{ and } z_j^i = g_j^i + \frac{h^i + h^j}{2} \text{ for } i \neq j.$$

The explanation is that each agent pays her bid, receives an equal share of the aggregate bid plus the payoff resulting from the interaction in $G_j$, $j \in \Omega$.

Finally, since $j$ is chosen randomly in $\Omega$, the expected payoff of each player playing **Mechanism (A)** is given by:

$$\forall i \in N, \quad m^i = \frac{\sum_{j \in \Omega} z_j^i}{|\Omega|}.$$

Mechanism (A) is a particular case of a class of general bidding mechanisms studied in Béal et al. (2017b). In particular, it is well-known that $G_j$, $j \in \Omega$, admits a unique subgame perfect Nash equilibrium with payoffs $(\hat{g}_j^i)_{i \in N}$ and where the offer made by the proposer $j$ is equal to zero. From Proposition 9 in Béal al (2017b), we obtain that Mechanism (A) admits a unique subgame perfect equilibrium with the following properties:

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[2] To be precise, in the family of bidding mechanisms we consider — see e.g., Béal et al. (2017b) —, it is customary to assume that each $i \in N$ bids over $i$ and $j \in N \setminus i$ under the 'budget' constraint $h_i^i/2 + \sum_{j \in N \setminus i} h_j^i/2 = 0$. Because here $N$ contains only two players, $h_i^i$ determines uniquely $h_{-i}^i$ under the 'budget' constraint, where $-i$ denotes $i$'s opponent. That is the reason why we simplify a little bit the notation. The weight $1/2$ means that the designer values each agent equally. .

1. The equilibrium bids $(\hat{h}^i)_{i \in N}$ coincide. Precisely, for each $i \in N$, $\hat{h}^i = Q/2$, so that $\Omega = N$;
2. The equilibrium proposals $(\hat{x}_j)_{j \in N}$ coincide. Precisely, for each $j \in N$, $\hat{x}_j = 0$; and each player accepts any proposal from the proposer.
3. The equilibrium rewards $(\hat{z}^i_j)_{i \in N}$ in $G_j$ do not depend on $j \in \Omega$, and, therefore,

$$\forall j \in N, \forall i \in N, \quad \hat{z}^i_j = \frac{Q}{2}.$$

4. From the previous item, we conclude that the expected payoff $\hat{m}^i$ of each player $i \in N$ in Mechanism (A) is given by:

$$\forall i \in N, \quad \hat{m}^i = \frac{Q}{2}.$$

We have now the material to design Mechanism (B) which supports $\Phi^e$ in subgame perfect Nash equilibrium. As underlined above, Mechanism (B) has a recursive structure, as it is the case for $\Phi^e$. More specifically, Mechanism (B) applied to $(N, v, L, \preceq)$ first calls Mechanism (A) and then calls Mechanism (B) in a self-similar way but applied to an ordered tree TU-game constructed from the outcome of Mechanism (A) and $(N, v, L, \preceq)$. In the following, given $(N, v, L, \preceq) \in G$ and the top link $ij \in L$, the notation $(N_i, v^c_i)$ stands for the TU-game on $N_i = C^N_i$ such that $c \in \mathbb{R}$, $v^c_i(N_i) = v(N_i) + c$, and, for each other coalition $S \subset N_i$, $v^c_i(S) = v(S)$.

**Mechanism (B)** Let $(N, v, L, \preceq) \in G$ such that $(N, v)$ is superadditive.

1. If $N = \{i\}$, then player $i$'s final payoff in Mechanism (B) is $v(i)$;
2. Otherwise, pick the top link $ij$ of $(L, \preceq)$.
   (a) Members of $\{i, j\}$ are involved in Mechanism (A) to share the value

$$Q = v(N) - v(C^N_i) - v(C^N_j) \geq 0.$$

   As above, $m^i$ and $m^j$ denote the expected payoffs obtained by $i$ and $j$ in Mechanism (A);
   (b) For each $p \in \{i, j\}$, elements of $N_p = C^N_p$ are involved in Mechanism (B) applied to $(N_p, v^{m^p}_p, L_p, \preceq_p) \in G$. By construction, $(N_p, v^{m^p}_p)$ remains superadditive.
   (c) The final payoff obtained by each player in $C^N_p$ in Mechanism (B) applied to $(N, v, L \preceq)$ coincides with the final payoff obtained in Mechanism (B) applied to $(N_p, v^{m^p}_p, L_p, \preceq_p)$.

Note that in step 2(a), players $i$ and $j$ are the representatives of their component $C^N_i$ and $C^N_j$, respectively. Therefore, the payoffs $m^i$ and $m^j$ are not received by $i$ and $j$, except in case in which $i$ and $j$ are the only elements of their component. To figure out how Mechanism (B) runs, consider the following example.

**Example 2** Let the ordered tree $(N, v, L, \preceq)$ be given by $N = \{1, 2, 3, 4, 5\}$, for each $i \in N$, $v(i) = i/4$, for each other nonempty coalition $S \in 2^N$, $v(S) = s^2$, $L = \{12, 23, 34, 25\}$ and $12 \preceq 25 \preceq 34 \preceq 23$ as in Figure 1. Mechanism (B) applied to $(N, v, L, \preceq)$ runs as follows. Because $N$ is not a singleton, go to stage 2 in Mechanism (B). Stage 2(a) indicates that players in $\{2, 3\}$, the endpoints of the Top link, are involved in Mechanism (A) to share the worth

$$v(N) - v(C^N_3) - v(C^N_2) = 12.$$

In Mechanism (A), players 2 and 3 receive expected payoffs $m^2$ and $m^3$, respectively. Next go to stage 2(b). At this stage, for each $p \in \{2,3\}$, elements of $N_p = C_p^N$ are involved in Mechanism (B) applied on $(N_p, v_p^{m^p}, L_p) \preceq_p) \in G$. Consider first the ordered tree TU-game $(N_3, v_3^{m^3}, L_3, \preceq_3)$, where $N_3 = \{3,4\}$, $v_3^{m^3}(N_3) = 4+m^3$, $L_3 = \{34\}$, and $34 \preceq_3 34$. By stage 2(a) of Mechanism (B), players 3 and 4 are involved in Mechanism (A) to share the value $4 + m^3 - 3/4 - 4/4 = 9/4 + m^3$. In this Mechanism (A), players 3 and 4 receive expected payoffs $m^{3,3}$ and $m^{3,4}$, respectively. By stage 1 of Mechanism (B), their final payoff will be $m^{3,3} + 3/4$ and $m^{3,4} + 4/4$, respectively. Next, consider the ordered tree TU-game $(N_2, v_2^{m^2}, L_2, \preceq_2)$, where $N_2 = \{1,2,5\}$, $v_2^{m^2}(N_2) = 9 + m^2$, $L_2 = \{12, 25\}$, and $12 \preceq_2 25$. By stage 2(a) of Mechanism (B), players 2 and 5 are involved in Mechanism (A) to share the value $9 + m^2 - 5/4 - 4 = 15/4 + m^2$. In this Mechanism (A), players 2 and 5 receive expected payoffs $m^{2,2}$ and $m^{2,5}$, respectively. By stage 1 of Mechanism (B), the final payoff received by player 5 in Mechanism (B) applied to $(N, v, L, \preceq)$ is $m^{2,5} + 5/4$. The final payoffs obtained by players 1 and 2 are those obtained in Mechanism (B) applied to $((N_2)_2, (v_2^{m^2})_2^{m^{2,2}}, (L_2)_2, (\preceq_2)_2)$, where $(N_2)_2 = \{1,2\}$, $(v_2^{m^2})_2^{m^{2,2}}(\{1,2\}) = v_2^{m^2}(\{1,2\}) + m^{2,2} = v(\{1,2\}) + m^{2,2}$, $(L_2)_2 = \{12\}$, and $12(\preceq_2)_2 12$. By stage 2(a) of Mechanism (B) applied to this game, players 1 and 2 are involved in Mechanism (A) to share the value $4 + m^{2,2} - 1/4 - 2/4 = 13/4 + m^{2,2}$, and received expected payoffs $m_1$ and $m^{2,2,2}$, respectively. Finally, by stage 1 of Mechanism (B), the final payoff received by player 1 is $m^{2,2,1} + 1/4$, and the final payoff of player 2 is $m^{2,2,2} + 2/4$.

The following figure depicts the sequential bargaining process, where the notation Mechanism $A(\{i,j\}, Q)$ means that the neighbors $i$ and $j$ are involved in Mechanism $A$ to bargain over the value $Q$. From the total order $(L, \preceq)$, we construct a binary tree rooted at $N$. The direct successors of $N$ are the two components $\{1,2,5\}$ and $\{3,4\}$ of $(N, L)$ resulting from the deletion of the top link 23. This means that the top link connects these two components and generate a surplus equal to 12. Agents 2 and 3 are involved in Mechanism $A(\{2,3\}, 12)$ to share 12. Then, the process is repeated according to the total order $(L, \preceq)$ until there is no link to consider. At each step of the process, the payoffs obtained through the Mechanism (A) are incorporated in the next step as described by Mechanism (B).



$\square$
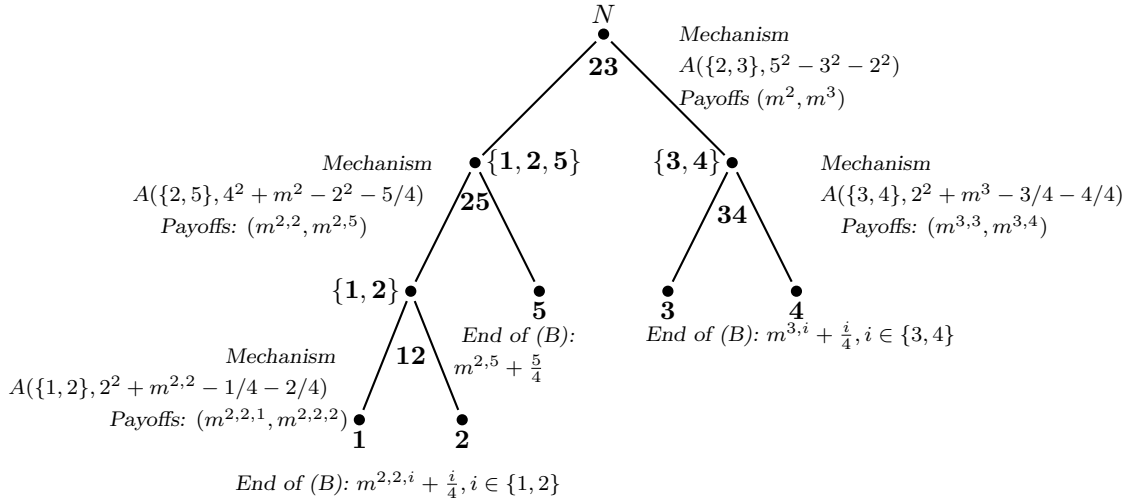
In the following result, we will use the fact that $\Phi^e$ satisfies the axiom of **(Strict) Aggregate**

**monotonicity** (Megiddo 1974) on $G$ saying that the allocation of each player strictly increases when the worth of the grand coalition $N$ increases while the worth of the other coalitions remains fixed. This axiom is well-known in problems of fair division. Formally, for $(N, v, L, \preceq) \in G$, each constant $c > 0$,

$$\forall i \in N, \quad \Phi_i^e(N, v^c, L, \preceq) > \Phi_i^e(N, v, L, \preceq), \tag{8}$$

where, here, $v^c(N) = v(N) + c$, and, for each $S \subset N$, $v^c(S) = v(S)$.

**Proposition 5** *For any $(N, v, L, \preceq) \in G$ such that $(N, v)$ is supperadditive, Mechanism (B) supports the payoff vector $\Phi^e(N, v, L, \preceq)$ in subgame Nash equilibrium.*

*Proof.* We proceed by induction on the number $n$ of players of $(N, v, L \preceq) \in G$.
*Initial step* If $(N, v, L \preceq) \in G^1$, say $N = \{i\}$, then, by definition of Mechanism (B), this player gets $v(i)$, which coincides with $\Phi_i^e(N, v, L \preceq)$ by point (a) of definition of $\Phi^e$.
*Induction hypothesis* Assume that the result holds for any element of $G^n$, $1 \leq n \leq r$, where $r \geq 1$.
*Induction step* Consider any $(N, v, L, \preceq) \in G^{r+1}$ such that $(N, v)$ is supperadditive. Let $ij$ denote the top link of $(L, \preceq)$. We proceed in two steps. In a first step, we show that on each subgame Nash equilibrium of Mechanism (B) applied to $(N, v, L, \preceq) \in G^{r+1}$, the payoff vector coincides with $\Phi^e(N, v, L, \preceq)$. In a second step, we prove that such an equilibrium exists.

*Uniqueness part* Pick any subgame Nash equilibrium of Mechanism (B) applied to $(N, v, L, \preceq) \in G^{r+1}$. Each player $p \in C_i^N = N_i$ receives the equilibrium payoff of Mechanism (B) applied to $(N_i, v_i^{m^i}, L_i, \preceq_i)$. By the induction hypothesis, the subgame equilibrium payoff obtained by each $p \in C_i^N$ coincides with $\Phi_p^e(N_i, v_i^{m^i}, L_i, \preceq_i)$. It remains to show that, on a subgame perfect equilibrium of Mechanism (B) applied to $(N, v, L, \preceq)$, $v_i^{m^i} = v_i$. By Aggregate monotonicity of $\Phi^e$ expressed by (8), player $i$ has an incentive to maximize $m^i$. The same argument holds for her neighbor $j$ in $(N_j, v_j^{m^j}, L_j, \preceq_j)$. By stage 2(a) of Mechanism (B), $(m^i, m^j)$ is obtained by playing the ultimatum game augmented by the bidding stage between $i$ and $j$ to share the value $v(N) - v(C_i^N) - v(C_j^N) \geq 0$, as described by Mechanism (A). The unique subgame equilibrium pair of payoffs of this Mechanism (A) is $(\hat{m}^i, \hat{m}^j)$. Assume, by way of contradiction, that there is one player, say $i$, that contemplates the possibility to deviate from her strategy at this stage. The resulting pair of payoffs is denoted by $(m^i, m^j) \neq (\hat{m}^i, \hat{m}^j)$ where $m^i \leq \hat{m}^i$. By inequality (8), we get $\Phi_i^e(N_i, v_i^{m^i}, L_i, \preceq_i) \leq \Phi_i^e(N_i, v_i^{\hat{m}^i}, L_i, \preceq_i)$. Therefore, we conclude that on any subgame Nash equilibrium of Mechanism (B), players $i$ and $j$ coordinate on the subgame Nash equilibrium of Mechanism (A). This proves the uniqueness part. In particular, on any subgame Nash equilibrium of Mechanism (B) applied to $(N, v, L, \preceq) \in G^{r+1}$, we have:

$$\hat{m}^i = \hat{m}^j = \frac{v(N) - v(C_i^N) - v(C_j^N)}{2},$$

i.e. $v_i^{m^i} = v_i$, which in turn ensures that the equilibrium payoff vector of Mechanism (B) is necessarily $\Phi^e(N, v, L, \preceq)$ by point (b) of definition of $\Phi^e$.

*Existence part* We propose the following strategy profile:
- In stage 2(a) of Mechanism (B), $i$ and $j$ (the endpoints of the top link) both play the equilibrium strategy of Mechanism (A) in order to share $v(N) - v(C_i^N) - v(C_j^N)$;

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- In stage 2(b) of Mechanism (B), each player in $C_p^N$, $p \in \{i, j\}$, plays an equilibrium strategy of Mechanism (B) applied to $(N_p, v_p^{m^p}, L_p, \preceq_p)$ whatever the payoff $m^p$ obtained in stage 2(a). The existence of such an equilibrium strategy is assumed by the induction hypothesis. Therefore, by definition of an equilibrium strategy, in each subgame of stage 2(b) of Mechanism (B), no player has an interest to deviate. Now, assume, without loss of generality, that player $i$ contemplates the possibility to deviate in stage 2(a) of Mechanism (B). In such a case, the payoff $m^i$ obtained by $i$ in Mechanism (A) is at most equal to the equilibrium payoff $\hat{m}^i$, and the final payoff obtained by the same player $i$ is, by the induction hypothesis, $\Phi_i^e(N_i, v_i^{m^i}, L_i, \preceq_i)$, while the final payoff he obtains by playing the proposed strategy is $\Phi_i^e(N_i, v_i^{\hat{m}^i}, L_i, \preceq_i) = \Phi_i^e(N, v, L, \preceq)$. By inequality (8), we have $\Phi_i^e(N_i, v_i^{m^i}, L_i, \preceq_i) \leq \Phi_i^e(N, v, L, \preceq)$, which proves that $i$ has no interest to deviate in stage 2(a). The same argument holds for $j$. This completes the existence part, and so the induction step. ∎

## 5. Conclusion

There are at least two natural extensions of our work. The first one is that our allocation rule can be easily adapted to account for an arbitrary graph. The adaptation is obvious if the graph is a forest (i.e. if it contains several trees) since our recursive procedure has just to be applied to each component. If the graph contains some cycle, the step procedure is the same whenever the deleted link is a bridge, i.e. if cutting this link creates two new components. Otherwise, deleting the link does not change the connectivity of its component (and of the other components), and it makes sense to assume that the temporary payoffs of the current components remain unchanged at that step.

The second extension takes the allocation rule presented in this article as a building block of a more sophisticated allocation rule for the classical Myerson's framework. More specifically, it is possible to compute the average over all total orders over the links of the payoff allocation specified by our allocation rule. The resulting allocation rule would provide an interesting mixture between the egalitarian and marginal aspects. There is a lack of such mixtures in the literature on cooperative games enriched by a graph. This extension can parallel the connection between the hierarchical vector proposed by Demange (2004) and the Average tree solution studied in Herings, van der Laan and Talman (2008), which is simply the average of the hierarchical outcomes induced by all the players. This second extension also share some similarities with the Position value (Meessen 1988, Borm et. al. 1992), which is obtained by averaging specific contribution vectors associated with all permutations of links of a graph, since there is an obvious bijection between the set of total orders and the set of all permutations of a set.

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