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GAMES WITH IDENTICAL SHAPLEY VALUES

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ABSTRACT. We discuss several sets of cooperative games in which the Shapley value assigns zero payoffs to all players. Each set spans the kernel of the Shapley value and leads to a different characterization of games with identical Shapley values. The special games we identify deliver intuitive axiomatizations of the Shapley value. We explain how each basis of the kernel of the Shapley value can be augmented to construct a basis of the space of all games.

Keywords: Shapley value, kernel, axiomatization, factious oligarchies, paper tigers.

1. INTRODUCTION

In this chapter, we survey the research studying cooperative games with transferable utility that induce the same Shapley values. The problem of identifying all games that generate a given vector of Shapley values has been first considered by Kleinberg and Weiss (1985) and became known as the "inverse problem" in the literature. Since the Shapley value is a linear operator on the space of games, the inverse problem is equivalent to characterizing its kernel—the space of games in which the Shapley value assigns zero payoffs to all players. We discuss several sets of games that reflect a clear balance of power among players and coalitions and constitute bases for the kernel of the Shapley value. We show how these games can be used to develop new axiomatizations of the Shapley value.

The chapter is organized as follows. Section 2 provides basic definitions related to the Shapley value. In Section 3, we investigate the kernel of the Shapley value. We present three bases for this kernel as well as an intuitive characterization of games in the kernel. These

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classes of games lead to natural axiomatizations of the Shapley value, which we present in Section 4. In Section 5, we discuss how the bases for the kernel of the Shapley value can be completed to construct bases for the space of all games. Section 6 surveys alternative bases for the kernel of the Shapley value from the literature. Section 7 explores other interesting games that belong to the kernel of the Shapley value. Finally, Section 8 provides proofs of the new results and Section 9 concludes.

2. The Shapley Value

Fix a set N of $n \geq 2$ *players*. A *coalition* is any subset of players $S \subseteq N$. A game v with transferable payoffs, simply called a *game* henceforth, associates a real number $v(S)$ to any coalition S, which represents the *value* coalition S can create and share among its members ($v(\emptyset) = 0$). A *solution* ψ assigns a *payoff* $\psi_i(v)$ to each player $i \in N$ for every game v. The *kernel* $\mathcal{K}(\psi)$ of a solution ψ is the space of games in which ψ assigns 0 payoffs to all players: $\mathcal{K}(\psi) = \{v | \psi_i(v) = 0, \forall i \in N\}$.

Shapley (1953) proposed the following solution ϕ:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!}(v(S \cup \{i\}) - v(S)), \forall i \in N. \tag{1}$$

This solution, now known as the *Shapley value*, has the following interpretation. If players are ordered randomly (all orderings being equally likely), then $\phi_i(v)$ represents the expected marginal contribution of player i to the coalition formed by his predecessors. The Shapley value has many elegant properties. For a comprehensive treatment, the reader may consult the monograph edited by Roth (1988) and the textbooks of Moulin (1988) and Osborne and Rubinstein (1994). Here we discuss only some of its properties—most of which Shapley introduced in his original paper—necessary for our analysis. Since these properties have been used in the context of axiomatic characterizations of the Shapley value, we refer to them as *axioms*.

Some preliminary definitions are necessary for stating the classic axioms. Player i is *null* in game v if $v(S \cup \{i\}) = v(S)$ for all coalitions S. Players i and j are *interchangeable* in v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all coalitions S disjoint from $\{i, j\}$. A game v is *inessential* if $v(S) = \sum_{i \in S} v(\{i\})$ for all coalitions S.

Given the assumption that the empty coalition has value 0, we view games as column vectors in the linear (vector) space $\mathbb{R}^{2^N \setminus \{\emptyset\}}$, which has dimension $2^n - 1$. Likewise, we represent solutions $(\psi_i(v))_{i \in N}$ for specific games v as column vectors in $\mathbb{R}^N$. Hence, for any pair of games v and w and real number α, $v + \alpha w$ is the game in which the value of coalition S is given by $v(S) + \alpha w(S)$; similarly, $\psi(v) + \alpha\psi(w)$ denotes the vector $(\psi_i(v) + \alpha\psi_i(w))_{i \in N}$. We use the notation $\mathbf{0}$ for the zero vector in either $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ or $\mathbb{R}^N$ (the dimension will be clear from the context).

It is well-known that the Shapley value ϕ satisfies the following axioms.

Axiom (Null). Solution ψ satisfies the *null axiom* if $\psi_i(v) = 0$ whenever player i is null in game v.

Axiom (Linearity). Solution ψ satisfies the *linearity axiom* (or is *linear*) if $\psi(v + \alpha w) = \psi(v) + \alpha\psi(w)$ for every pair of games v and w and real number α.

Axiom (Symmetry). Solution ψ satisfies the *symmetry axiom* if $\psi_i(v) = \psi_j(v)$ whenever players i and j are interchangeable in game v.

Axiom (Inessential). Solution ψ satisfies the *inessential axiom* if $\psi_i(v) = v(\{i\})$ for all $i \in N$ in every inessential game v.

In his original paper, Shapley identified a salient basis for the linear space of all games—unanimity games—which also plays an important role in our analysis. For every non-empty coalition T, the *unanimity game* u^T *with ruling coalition* T is specified as follows:

$$u^T(S) = \begin{cases} 1 & \text{if } S \supseteq T \\ 0 & \text{otherwise.} \end{cases}$$

Shapley proved that the $2^n - 1$ games $(u^T)_{T \in 2^N \setminus \{\emptyset\}}$ are linearly independent and thus $(u^T)_{T \in 2^N \setminus \{\emptyset\}}$ constitutes a basis for the $(2^n - 1)$-dimensional space of games $\mathbb{R}^{2^N \setminus \{\emptyset\}}$.

3. The Kernel of the Shapley Value

Given the natural embedding of games and solutions in the corresponding linear spaces, the Shapley value can be expressed as $\phi(v) = Av$, where A is an $n \times (2^n - 1)$ matrix that reflects the coefficients from formula (1). For inessential games v, we have $Av = \phi(v) = (v(\{i\}))_{i \in N}$ because the Shapley value satisfies the inessential axiom. Since the space of vectors $(v(\{i\}))_{i \in N}$ derived from inessential games v has dimension n, the matrix A must have full row rank equal to n. It follows that, as Kleinberg and Weiss (1985) noted, the set of games in which all players have Shapley value 0—the kernel $\mathcal{K}(\phi) = \{v | Av = \mathbf{0}\}$—is a linear subspace of $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ of dimension $2^n - n - 1$.

In what follows, we construct several sets of games, each spanning a space of dimension $2^n - n - 1$, in which all players have Shapley value 0. Since $\mathcal{K}(\phi)$ has dimension $2^n - n - 1$ and contains each set of games, we conclude that every set spans the full space $\mathcal{K}(\phi)$.

An *oligarchy* is any coalition that consists of at least two players. The members of an oligarchy are called *oligarchs*. Let $\mathcal{O}$ denote the set of oligarchies, $\mathcal{O} = \{O \subseteq N || O| \geq 2\}$. We define multiple games for every oligarchy O.

The *dog eat dog game* $\underline{w}^O$ for oligarchy O is specified by

$$\underline{w}^O(S) = \begin{cases} 1 & \text{if } |S \cap O| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This game has been introduced by Yokote (2015) and is called the *commander game* in the follow-up paper of Yokote et al. (2016).

The *scapegoat game* $\bar{w}^O$ for oligarchy O is specified by

$$\bar{w}^O(S) = \begin{cases} 1 & \text{if } |S \cap O| = |O| - 1 \\ 0 & \text{otherwise.} \end{cases}$$

This game first appears in the study of Béal et al. (2016).

In the games constructed above, oligarchs have some power and are instrumental for value creation but the oligarchy is factious and cannot cooperate effectively to realize any value. In dog eat dog games, a coalition creates value only if it includes a single oligarch—the fierce "dog." In scapegoat games, a coalition generates value only if it contains all but one oligarch—the "scapegoat."

Yokote and Funaki (2015) construct a more general set of games with disharmonious oligarchies as follows. The *factious oligarchic game* for oligarchy O with *parameter* k ($1 \leq k \leq |O| - 1$) is given by

$$w_k^O(S) = \begin{cases} 1 & \text{if } |S \cap O| = k \\ 0 & \text{otherwise.} \end{cases}$$

In order to generate a basis for the kernel of the Shapley value, we allow for any variation in the parameter k as a function of the oligarchy O. Let $f : \mathcal{O} \to \{1, 2, \ldots, n-1\}$ be a function such that $1 \leq f(O) \leq |O| - 1$ for all $O \in \mathcal{O}$. The *family of factious oligarchic games* $(w_f^O)_{O \in \mathcal{O}}$ with *power structure* f is specified by $w_f^O := w_{f(O)}^O$ (with a slight abuse of notation). Note that dog eat dog games and scapegoat games are families of factious oligarchic games with special power functions f—the former specified by $f(O) = 1$ for all $O \in \mathcal{O}$, and the latter by $f(O) = |O| - 1$ for all $O \in \mathcal{O}$.

As Yokote (2015), Yokote and Funaki (2015), Yokote et al. (2016), and Béal et al. (2016) show, every player has Shapley value 0 in all types of oligarchic games defined above. To see this, consider the factious oligarchic game w_k^O for oligarchy O with parameter $k \leq |O| - 1$. In w_k^O, all players in $N \setminus O$ are null and must obtain Shapley value 0 since ϕ satisfies the null axiom. All oligarchs are interchangeable in w_k^O and should obtain the same Shapley value because ϕ satisfies the symmetry axiom. The common Shapley value of the oligarchs must be 0 because $w_k^O(N) = 0$. Since ϕ is linear and $\phi(w) = \mathbf{0}$ for all games w defined above, the Shapley value satisfies the following axioms.

Axiom (Dog Eat Dog). Solution ψ satisfies the *dog eat dog axiom* if $\psi(v) = \psi(v + \alpha w)$ for every game v, any dog eat dog game w, and all real numbers α.

Axiom (Scapegoat). Solution ψ satisfies the *scapegoat axiom* if $\psi(v) = \psi(v + \alpha w)$ for every game v, any scapegoat game w, and all real numbers α.

Axiom (Factious Oligarchy). Solution ψ satisfies the *factious oligarchy axiom* if there exists a power structure f such that $\psi(v) = \psi(v + \alpha w)$ for every game v, any factious oligarchic game w with power structure f, and all real numbers α.

The intuition for each of the three axioms is that changing the cooperation structure by adding disharmonious oligarchies should not affect the division of payoffs. Note that a solution ψ satisfies the dog eat dog, scapegoat, or factious oligarchy axiom if and only if $\psi(v) = \psi(v + w)$ for every game v and all games w that are linear combinations of dog eat dog, scapegoat, or factious oligarchic games, respectively.

We next introduce a set of games inspired by Hamiache (2001) and Béal et al. (2016). A *synergy function* is a game π with the property that $\pi(\{i\}) = 0$ for all $i \in N$. The *paper tiger game* with synergy π is defined by

$$w^\pi(S) = \sum_{i \in N} (\pi(S \cup \{i\}) - \pi(S)) \ \ (= \sum_{i \in N \setminus S} (\pi(S \cup \{i\}) - \pi(S))).$$

The interpretation of this game is that every player i is by nature a solitary "tiger", which can add synergies to any group S that excludes him. However, the synergy of the expanded group $S \cup \{i\}$ supersedes the original synergy of S, rendering i a "paper tiger." Since only outsiders add value to coalitions, all synergies "wash out" for the grand coalition, $w^\pi(N) = 0$.

The set of paper tiger games constitutes a linear subspace of $\mathbb{R}^{2^N \setminus \{\emptyset\}}$ that has dimension at most $2^n - n - 1$ because each component of any element $(w^\pi(S))_{S \in 2^N \setminus \{\emptyset\}}$ is a linear function of the $2^n - n - 1$ variables $(\pi(S))_{S \in \mathcal{O}}$. Béal et al. (2016) remark that for any oligarchy O, the paper tiger game w^π derived from the synergy function

$$\pi(S) = \begin{cases} 1 & \text{if } O \subseteq S \\ 0 & \text{otherwise} \end{cases}$$

is identical to the scapegoat game $\bar{w}^O$. Thus, the space of paper tiger games contains the linear space spanned by scapegoat games. Béal et al. argue that the space of scapegoat games has dimension $2^n - n - 1$, which implies that the space of paper tiger games has dimension $2^n - n - 1$ and coincides with the space spanned by scapegoat games. Hence, every paper tiger game is a linear combination of scapegoat games. The linearity of the Shapley value, along with the fact that $\phi(\bar{w}^O) = \mathbf{0}$ for all scapegoat games $\bar{w}^O$, implies that $\phi(w^\pi) = \mathbf{0}$ for every paper tiger game w^π. Therefore, the Shapley value satisfies the following axiom, which captures the “paper tiger” metaphor.

Axiom (Paper Tiger). Solution ψ satisfies the *paper tiger axiom* if $\psi(v) = \psi(v + w)$ for every game v and any paper tiger game w.

Yokote (2015) established that the set of dog eat dog games forms a basis for the kernel of the Shapley value, and Béal et al. (2016) proved that the set of scapegoat games has the same property. Yokote and Funaki (2015) generalized these two results to families of factious oligarchic games $(w_f^O)_{O\in\mathcal{O}}$ with special power structures f. In the analysis of Yokote and Funaki, $f(O)$ depends only on the size of O, i.e., $f(O) = g(|O|)$ where g is a function from $\{2, \dots, n\}$ to $\{1, \dots, n-1\}$. Moreover, their main result imposes the following “continuity” restriction on g:

$$g(k-1) - 1 \leq g(k) \leq g(k-1) + 1 \text{ for } k \in \{3, \dots, n\}.$$

We show that neither of these restrictions is necessary for the result: the family of factious oligarchic games $(w_f^O)_{O\in\mathcal{O}}$ is linearly independent and spans the kernel of the Shapley value for every power structure f. The proof of this result relies on a new basis of the set of all games consisting of games with oligarchic structures we develop in Section 5 (see Theorem 3).

Theorem 1. *The set of dog eat dog games constitutes a basis for the linear space $\mathcal{K}(\phi)$, and the same is true about the set of scapegoat games. More generally, the family of factious oligarchic games with any power structure forms a basis for $\mathcal{K}(\phi)$. Furthermore, $\mathcal{K}(\phi)$ is given by the set of paper tiger games.*

Section 8 at the end of the chapter provides the proof of Theorem 1. We next present two corollaries that invoke paper tiger games. In light of Theorem 1, we can restate either corollary using a linear combination of each type of oligarchic game in lieu of the paper tiger game. The first corollary follows from the linearity of the Shapley value.

Corollary 1. *Games v and w yield identical Shapley values if and only if their difference $v - w$ is a paper tiger game.*

Fix a game v. The *Shapley inessential game* w *of* v is defined by $w(S) = \sum_{i \in S} \phi_i(v)$ for all coalitions S. Since the Shapley value satisfies the inessential axiom, we have that $\phi_i(w) = w(\{i\}) = \phi_i(v)$ for all $i \in N$. Then the linearity of the Shapley value implies that $\phi(v - w) = \mathbf{0}$. Thus, as Kleinberg and Weiss (1985) observed, the game v can be decomposed into its Shapley inessential game w and the game $v - w$, which is an element of $\mathcal{K}(\phi)$. This conclusion leads to another corollary of Theorem 1.

Corollary 2. *Every game is the sum of its Shapley inessential game and a paper tiger game.*

4. Axiomatizations of the Shapley Value Based on Its Kernel

If a solution ψ is pinned down for inessential games by the inessential axiom, and the addition of games in $\mathcal{K}(\phi)$ does not affect the solution as implied by any of the dog eat dog, scapegoat, factious oligarchy, or paper tiger axioms, then ψ must coincide with the Shapley value ϕ. This observation, along with Theorem 1 and Corollary 2, lead to four axiomatizations of the Shapley value.

Theorem 2. *A solution is the Shapley value if and only if it satisfies the inessential axiom and any one of the dog eat dog, scapegoat, factious oligarchy, and paper tiger axioms.*

We finally comment on a connection between our paper tiger axiom and an axiom due to Hamiache (2001). Derive a synergy function π_v from a game v as follows:

$$\pi_v(S) = v(S) - \sum_{i \in S} v(\{i\}). \tag{2}$$

Let w^{π_v} denote the paper tiger game with synergy π, and define the game $v_\lambda = v + \lambda w^{\pi_v}$, where λ is a positive real number. Algebra leads to

$$v_\lambda(S) = v(S) + \lambda \sum_{i \in N \setminus S} \left(v(S \cup \{i\}) - v(S) - v(\{i\}) \right), \forall S \subseteq N.$$

Since w^{π_v} is a paper tiger game, Theorem 2 implies that the games v and v_λ have the same Shapley value for every λ. Hamiache (2001) uses this property, coined *associated consistency*, to develop a characterization of the Shapley value. In addition to the inessential axiom, his characterization requires a continuity axiom because associated consistency is a weaker version of our paper tiger axiom that applies only to pairs of games $(v, \lambda w^{\pi_v})$ for which the synergy function π_v has the special relation to v described by formula (2).

To obtain an alternative proof of Hamiache's result, Béal et al. (2016) remark that the matrix associated with the linear transformation $v \to w^{\pi_v}$ is upper triangular when expressed in the basis of unanimity games. Its kernel is formed by the set of inessential games, and all its non-zero eigenvalues are negative. This ensures that, for sufficiently small λ, the sequence generated by the iteration of the transformation $v \to v_\lambda$ converges to an inessential game v^∞ for every first term v. The associated consistency and continuity of the solution ψ are used to conclude that $\psi(v) = \psi(v^\infty)$. If ψ satisfies the inessential axiom, then $\psi_i(v) = \psi_i(v^\infty) = v^\infty(\{i\})$ for all players i, which proves that $\psi(v)$ is uniquely determined.

Kleinberg (2017) extends the work of Hamiache (2001) by exploring linear and anonymous solutions (called *membership solutions*) other than the Shapley value that satisfy associated consistency. A solution is *anonymous* if a change in the label of the players has no effect on the solution. Note that the equal division solution, which divides the value of the grand coalition evenly among all players, is linear and anonymous and satisfies associated consistency. Kleinberg proves that a solution is linear and anonymous and satisfies associated consistency if and only if it is a linear combination of the Shapley value and the equal division solution. An equivalent statement of this result is that a linear and anonymous solution satisfies associated consistency if and only if its kernel contains the kernel of the Shapley value.

5. Bases for the Space of Games

Recall that Shapley (1953) showed that the set of unanimity games $(u^T)_{T \in 2^N \setminus \{\emptyset\}}$ constitutes a basis for the space of all games. We construct a rich class of new bases for the space of games by expanding the set of oligarchic games from Section 3. Specifically, we allow for "singleton oligarchies" $O = \{i\}$ and consider the possibility that oligarchies are functional, so parameter k in the specification of the corresponding game w_k^O can take the value $|O|$ (which is necessary for singleton oligarchies to generate a game different from $\mathbf{0}$). Therefore, we redefine an *oligarchy* to be any nonempty coalition $O \subseteq N$ and specify the *oligarchic game* for oligarchy O with *parameter* k as in Section 3,

$$w_k^O(S) = \begin{cases} 1 & \text{if } |S \cap O| = k \\ 0 & \text{otherwise} \end{cases}$$

for $1 \leq k \leq |O|$, with the novelty that $k = |O|$ is an admissible parameter. Power functions need to be adjusted accordingly—$f : 2^N \setminus \{\emptyset\} \to \{1, 2, \ldots, n\}$ is a *power function* if $1 \leq f(O) \leq |O|$ for all $O \in 2^N \setminus \{\emptyset\}$. The *family of oligarchic games* $(w_f^O)_{O \in 2^N \setminus \{\emptyset\}}$ with *power structure* f is specified as before, $w_f^O := w_{f(O)}^O$.

By definition, for singleton coalitions $O = \{i\}$, every power function f satisfies $f(\{i\}) = 1$ and $w_f^{\{i\}} = u^{\{i\}}$. In general, $w_{|O|}^O := u^O$ for all oligarchies O. Thus, the new oligarchic games added to the set of factious ones are exactly the unanimity games. Note that the Shapley value for the unanimity game u^O is given by

$$\phi_i(u^O) = \begin{cases} 1/|O| & \text{if } i \in O \\ 0 & \text{if } i \in N \setminus O. \end{cases}$$

Hence, the newly added games do not belong to the kernel of the Shapley value. We establish that the family of oligarchic games with any power structure constitues a basis of the space of games, which generalizes the main result of Yokote and Funaki (2015) as discussed in Section 3.

Theorem 3. *For any power structure f, the set of oligarchic games $(w_f^O)_{O \in 2^N \setminus \{\emptyset\}}$ forms a basis for the space of all games.*

The proof of the theorem can be found in Section 8. The key ingredient of the proof is a representation of the oligarchic game for oligarchy O with parameter k in the basis of unanimity games,

$$w_k^O = \sum_{S \subseteq O, |S| \geq k} (-1)^{|S|-k} \binom{|S|}{k} u^S.$$

The coefficient of the game u_S in the unique linear decomposition of any game v in the basis $(u^T)_{T \in 2^N \setminus \{\emptyset\}}$ is known as the *Harsanyi (1959) dividend* of coalition S in game v. Hence, the identity above shows that the Harsanyi dividend of coalition S in the oligarchic game w_k^O is $(-1)^{|S|-k}\binom{|S|}{k}$ for $S \subseteq O, |S| \geq k$ and 0 otherwise. We then reach the desired conclusion by noting that the linear transformation $(u^T)_{T \in 2^N \setminus \{\emptyset\}} \to (w_f^O)_{O \in 2^N \setminus \{\emptyset\}}$ derived from the identity above is captured by a lower-triangular matrix with non-zero diagonal elements.

We can build an alternative basis for the space of games by augmenting any basis of the $(2^n - n - 1)$-dimensional kernel $\mathcal{K}(\phi)$ of the Shapley value identified in Theorem 1 with any collection of n linearly independent games that span a space whose only intersection with

$\mathcal{K}(\phi)$ is game $\mathbf{0}$. One obvious selection for the n games is the set of degenerate unanimity games with singleton ruling coalitions, $(u^{\{i\}})_{i\in N}$, which we call *trivial* games. By Theorem 3, trivial games are linearly independent and span the space of inessential games. Since $\phi(v) = (v(\{i\}))_{i\in N}$ for every inessential game v, the intersection of the set of inessential games and the kernel of the Shapley value is $\{\mathbf{0}\}$. It follows that any basis of $\mathcal{K}(\phi)$ described in Theorem 1 along with the collection of trivial games forms a basis for the set of all games. In a result related to Corollary 2, Yokote et al. (2016) show that the coefficient of the trivial game $u^{\{i\}}$ in the decomposition of any game in each of these bases coincides with the Shapley value of player i. To see this, note that the discussion above implies that every game v can be uniquely decomposed as a linear combination of games in any basis of $\mathcal{K}(\phi)$ and trivial games $u^{\{j\}}$ for $j \in N$. Let α_j denote the coordinate of $u^{\{j\}}$ in the decomposition of v. Then, the linearity of the Shapley value ϕ leads to

$$\phi_i(v) = \sum_{j\in N} \alpha_j \phi_i(u^{\{j\}}), \forall i \in N.$$

For any $j \in N$, since $u^{\{j\}}$ is an inessential game, we have $\phi_j(u^{\{j\}}) = u^{\{j\}}(\{j\}) = 1$ and $\phi_i(u^{\{j\}}) = u^{\{j\}}(\{i\}) = 0$ for $i \in N \setminus \{j\}$. Therefore, $\phi_i(v) = \alpha_i$ for all $i \in N$, as asserted. The following theorem collects results from Yokote and Funaki (2015) and Yokote et al. (2016).

Theorem 4. *The collection of trivial games and each family of factious oligarchic games with any power structure constitutes a basis for the space of all games. In every such basis, the coefficient of each trivial game in the decomposition of any given game coincides with the Shapley value of the corresponding player in that game.*

6. Other Bases

Kleinberg and Weiss (1985) provided the first characterization of the kernel of the Shapley value as a direct sum decomposition of linear spaces. Each game in their decomposition assigns non-zero values only to singletons or coalitions of a fixed size. Their decomposition consists of three types of games:

$$\{v | v(S) = v(S') \text{ if } |S| = |S'| = k; v(S) = 0 \text{ if } |S| \neq k\} \text{ for } 1 \leq k \leq n-1,$$

$$\Big\{v | \sum_{i \in N} v(\{i\}) = 0; v(S) = -\sum_{i \in S} v(\{i\}) \text{ if } |S| = k; v(S) = 0 \text{ if } |S| \neq 1, k\Big\} \text{ for } 2 \leq k \leq n-1,$$

$$\Big\{v | v(S) = 0 \text{ if } |S| \neq k; \sum_{i \in S} v(S) = 0, \forall i \in N\Big\} \text{ for } 2 \leq k \leq n-1.$$

Dragan et al. (1989) develop a different basis for the space of games building on the potential value of Hart and Mas-Colell (1989). Recall that the potential $P(S,v)_{S \subseteq N}$ of a game v is defined recursively by

$$P(S,v) = \frac{1}{|S|}\left(v(S) + P(S \setminus \{i\}, v)\right)$$

with the initial condition $P(\emptyset, v) = 0$. Hart and Mas-Colell showed that the Shapley value can be computed as

$$\phi_i(v) = P(N,v) - P(N \setminus \{i\}, v), \forall i \in N.$$

Dragan et al. pointed out that the potential function P can be interpreted as a linear endomorphism on the space of games, and hence one can derive a basis for this space by identifying a game w^T for every nonempty coalition T with the property that $P(T, w^T) = 1$ and $P(S, w^T) = 0$ if $S \neq T$. They found that

$$w^T(S) = \begin{cases} |S| & \text{if } S = T \\ -1 & \text{if } S = T \cup \{j\} \text{ with } j \notin T \\ 0 & \text{otherwise.} \end{cases}$$

It can then be checked that the set of games $(w^T)_{1 \leq |T| \leq n-2}$ together with the game $w^N + \sum_{i \in N} w^{N \setminus \{i\}}$ forms a basis for the kernel of the Shapley value.

Another basis for the kernel of the Shapley value can be obtained by considering a generalization of the Shapley value. Recall that a solution ψ is *efficient* if the total payoffs it allocates equal to the value of the grand coalition, i.e., $\sum_{i \in N} \psi_i(v) = v(N)$ for all games v. The Shapley value is a prominent solution which is linear, anonymous, and efficient. Ruiz et al. (1998) show that any linear, anonymous, and efficient solution takes the form ϕ^b, where

$$\phi_i^b(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!}(b_{|S|+1} v(S \cup \{i\}) - b_{|S|} v(S)), \forall i \in N$$

for a collection of constants $b = (b_k)_{0 \le k \le n}$ with $b_n = 1$.

Rojas and Sanchez (2016) analyze the subset of linear, anonymous, and efficient solutions ϕ^b with $b_k \neq 0$ for all k, which they call *regular* solutions. They provide a basis for the kernel of each regular solution ϕ^b consisting of the games $(v_T^b)_{T \subseteq N, |T| \neq 1}$ defined as follows:

$$v_N^b(S) = \begin{cases} 1 & \text{if } |S| = 1 \\ 0 & \text{otherwise} \end{cases}$$

and for T such that $2 \le |T| \le |N| - 1$,

$$v_T^b(S) = \begin{cases} 1 & \text{if } |S| = 1 \text{ and } S \cap T = \emptyset \\ \dfrac{b_1}{b_{|T|}} \dbinom{|N|-2}{|T|-1} & \text{if } S = T \\ 0 & \text{otherwise.} \end{cases}$$

Since the Shapley value is obtained by setting $b_k = 1$ for all k, the collection of games $(v_T^{(1,\ldots,1)})_{T \subseteq N, |T| \neq 1}$ is a new basis for the kernel of the Shapley value. As in Section 5, the authors further provide a basis of the space of all games by augmenting their basis of the kernel of any regular solution ψ^b. Rojas and Sanchez (2016) first prove that the kernel $\mathcal{K}(\psi^b)$ of any such solution ψ^b has dimension $2^n - n - 1$. Then they need to add the following collection of n games $(v_{\{i\}}^b)_{i \in N}$ such that:

$$v_{\{i\}}^b(S) = \begin{cases} \dfrac{1}{b_{|S|}} & \text{if } i \in S \text{ and } S \neq N \\ |N| & \text{if } S = N \\ 0 & \text{otherwise.} \end{cases}$$

In another recent study, Faigle and Grabish (2016) employed the change of basis underlying isomorphic linear operators to construct new bases for the space of games and for the kernel of linear values from existing linear representations of games. Starting from the Shapley interaction transform of Grabisch (1997), Faigle and Grabish obtain the basis $(b^T)_{T \subseteq N, |T| \neq 1}$ for the kernel of the Shapley value specified by

$$b^T(S) = \sum_{j=0}^{|S \cap T|} \binom{|S \cap T|}{j} B_{|T|-j},$$

where $B_0, B_1, \ldots$ are the cumbersome Bernoulli numbers.

While conceptually interesting, the approaches discussed in this section provide less immediate game theoretic intuitions for the kernel of the Shapley value.

7. Other Games in the Kernel of the Shapley Value

For any oligarchy $O \in \mathcal{O}$ and every nonempty set $K \subseteq \{1, 2, \ldots, |O| - 1\}$, the game w_K^O defined by

$$w_K^O(S) = \begin{cases} 1 & \text{if } |S \cap O| \in K \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

delivers Shapley value 0 to all players. This follows from the linearity of the Shapley value and the observation that each such game can be decomposed into factious oligarchic games with parameters in K,

$$w_K^O = \sum_{k \in K} w_k^O.$$

In particular, note that dog eat dog, scapegoat, and fictitious oligarchic games are all special instances of this set of games in which K is a singleton.

One interesting subset of the games w_K^O is obtained by setting $K = \{1, 2, \ldots, |O| - 1\}$ for every $O \in \mathcal{O}$. Specifically, define the (dysfunctional) *wolf pack game* $\tilde{w}^O$ for oligarchy O as follows:

$$\tilde{w}^O(S) = \begin{cases} 1 & \text{if } 1 \le |S \cap O| \le |O| - 1 \\ 0 & \text{otherwise.} \end{cases}$$

In dysfunctional wolf pack games, a coalition is productive only if it involves some but not all oligarchs—the "wolf pack" cannot coordinate as a whole. In light of the rich set of bases identified by Theorem 1, it is worth pointing out that the $2^n - n - 1$ wolf pack games obtained by varying the composition of the oligarchy are not always linearly independent and hence do not span the kernel of the Shapley value. For instance, for $n = 4$, one can check that the sum of all wolf pack games with oligarchies of size two is identical to the sum evaluated for oligarchies of size three.

Wolf pack games lie at the opposite end on the spectrum of dissent among oligarchs from dog eat dog games: every subset of oligarchs except for the entire oligarchy operates effectively in wolf pack games, while no two oligarchs can cooperate successfully in dog eat dog games. Yokote and Funaki (2015) consider an intermediate level of power struggle among oligarchs whereby only coalitions formed by half of the oligarchs are effective. This corresponds to setting $K = \{|O|/2\}$ for $|O|$ even and $K = \{(|O|+1)/2\}$ for $|O|$ odd in (3). Theorem 4 implies that this set of games augmented with the set of trivial games constitutes a basis for the kernel of the Shapley value. Yokote and Funaki employ the decomposition of games in this basis to identify games for which the Shapley value coincides with the prenucleolus.

8. Proofs

Proof of Theorem 1. Since dog eat dog games and scapegoat games are families of factious oligarchic games with two special power functions f—the former specified by $f(O) = 1$ for all $O \in \mathcal{O}$, and the latter by $f(O) = |O| - 1$ for all $O \in \mathcal{O}$—the statements about dog eat dog games and scapegoat games are implied by the one about general factious oligarchic games.

To prove the statement regarding factious oligarchic games, fix a power structure f and consider the family of factious oligarchic games $(w_f^O)_{O\in\mathcal{O}}$ it generates. By Theorem 3, the elements of the family $(w_f^O)_{O\in\mathcal{O}}$ are linearly independent. Since this family contains exactly $2^n - n - 1$ games, it spans a linear space of dimension $2^n - n - 1$. In Section 3, we have argued that $\mathcal{K}(\phi) = \{v|\phi(v) = \mathbf{0}\}$ is a linear subspace of $\mathbb{R}^{2^N\setminus\{\emptyset\}}$ of dimension $2^n - n - 1$ that contains all factious oligarchic games, including the ones in $(w_f^O)_{O\in\mathcal{O}}$. Since the space spanned by $(w_f^O)_{O\in\mathcal{O}}$ has dimension $2^n - n - 1$, it must coincide with $\mathcal{K}(\phi)$. Therefore, $(w_f^O)_{O\in\mathcal{O}}$ constitutes a basis for $\mathcal{K}(\phi)$.

Finally, the conclusion that the space of paper tiger games is identical to $\mathcal{K}(\phi)$ follows from the finding that $\mathcal{K}(\phi)$ spans the set of scapegoat games and the arguments provided after the definition of paper tiger games. □

Proof of Theorem 3. Fix a power structure f and consider the family of $2^n - 1$ oligarchic games $(w_f^O)_{O\in 2^N\setminus\{\emptyset\}}$ it generates. To establish that the family $(w_f^O)_{O\in 2^N\setminus\{\emptyset\}}$ forms a basis for the $(2^n - 1)$-dimension space of all games, it is sufficient to show that the games in the family are linearly independent.

We first argue that the oligarchic game for oligarchy O with parameter k can be decomposed in the basis of unanimity games as follows:

$$w_k^O = \sum_{S\subseteq O, |S|\geq k} (-1)^{|S|-k} \binom{|S|}{k} u^S.$$

We need to show that for every coalition $T \subseteq N$,

$$w_k^O(T) = \sum_{S \subseteq O, |S| \geq k} (-1)^{|S|-k} \binom{|S|}{k} u^S(T). \tag{4}$$

Fix a coalition T, and let $T' = T \cap O$ and $t = |T'|$.

Clearly, if $t < k$, then $w_k^O(T) = w_k^O(T') = 0$ and $u^S(T) = u^S(T') = 0$ for $S \subseteq O$ such that $|S| \geq k$. Hence, for $t < k$, both sides of equation (4) equal zero.

Suppose now that $t \geq k$. We can rewrite the right-hand side term in equation (4) as follows:

$$\begin{aligned}
\sum_{S \subseteq O, |S| \geq k} (-1)^{|S|-k} \binom{|S|}{k} u^S(T) &= \sum_{S \subseteq O, |S| \geq k} (-1)^{|S|-k} \binom{|S|}{k} u^S(T') \\
&= \sum_{S \subseteq T', |S| \geq k} (-1)^{|S|-k} \binom{|S|}{k} \\
&= \sum_{s=k}^{t} (-1)^{s-k} \binom{s}{k} \binom{t}{s} \\
&= \sum_{s=k}^{t} (-1)^{s-k} \binom{t}{k} \binom{t-k}{s-k} \\
&= \binom{t}{k} \sum_{s=0}^{t-k} (-1)^{s} \binom{t-k}{s} \\
&= \binom{t}{k} (1-1)^{t-k}.
\end{aligned}$$

The first equality follows from $u^S(T) = u^S(T \cap S) = u^S(T \cap O) = u^S(T')$ for $S \subseteq O$, the second relies on the fact that $u^S(T') = 1$ if $S \subseteq T'$ and $u^S(T') = 0$ otherwise (along with $T' \subseteq O$), and the third accounts for the number $\binom{t}{s}$ of sets $S \subseteq T'$ with $|S| = s \geq k$ given that $|T'| = t$. The fourth equality uses the formulae

$$\begin{aligned}
\binom{s}{k}\binom{t}{s} = \frac{s!}{k!(s-k)!} \frac{t!}{s!(t-s)!} &= \frac{t!}{k!(s-k)!(t-s)!} \\
&= \frac{t!}{k!(t-k)!} \frac{(t-k)!}{(s-k)!(t-s)!} = \binom{t}{k}\binom{t-k}{s-k},
\end{aligned}$$

while the fifth one simply changes the variable $s - k$ to s. The final equality follows from the binomial formula.

For $t \geq k$, claim (4) then follows from noting that

- if $t = k$, then $\binom{t}{k}(1-1)^{t-k} = 1 = w_k^O(T)$;
- if $t > k$, then $\binom{t}{k}(1-1)^{t-k} = 0 = w_k^O(T)$.

We are now prepared to show that the games $(w_f^O)_{O\in 2^N\setminus\{\emptyset\}}$ are linearly independent. Consider any linear order $\succeq$ on $2^N \setminus \emptyset$ extending the partial order $(2^N \setminus \emptyset, \supseteq)$ and construct the $(2^n-1)\times(2^n-1)$ matrix of coordinates of the games of $(w_f^O)_{O\in 2^N\setminus\emptyset}$ in the basis of unanimity games $(u^T)_{T\in 2^N\setminus\{\emptyset\}}$. This matrix is lower-triangular since the coordinates of w_f^O associated with u^T are zero whenever $T \succ O$. Moreover, each diagonal element in the matrix takes the form

$$(-1)^{|O|-f(O)}\binom{|O|}{f(O)} \neq 0.$$

Consequently, the matrix has full rank, which delivers the result. □

9. Conclusion

We introduced several classes of cooperative games in which the Shapley value yields zero payoffs to all players. These games deliver a rich set of bases for the kernel of the Shapley value and lead to multiple characterizations of games with identical Shapley values. Building on these games, we were able to provide new intuitive axiomatizations of the Shapley value. We explained how each basis of the kernel of the Shapley value can be enlarged to create a basis for the space of all games. Many of the games we presented admit straightforward game theoretic interpretations. However, some of the games require a deeper understanding of the power structure they induce among coalitions. It would be useful to develop more connections between the various bases of the kernel of the Shapley value.

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