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Cohesive efficiency in TU-games: Two extensions of the Shapley value*

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Abstract

We relax the assumption that the grand coalition must form by imposing the axiom of Cohesive efficiency: the total payoffs that the players can share is equal to the maximal total worth generated by a coalition structure. We determine how the three main axiomatic characterizations of the Shapley value are affected when the classical axiom of Efficiency is replaced by Cohesive efficiency. We introduce and characterize two variants of the Shapley value that are compatible with Cohesive efficiency. We show that our approach is not limited to variants of the Shapley value.

Keywords: Cohesive efficiency, Shapley value, balanced contributions, potential, equal surplus division, consensus values, equal allocation of nonseparable costs, superadditive cover.

1. Introduction

Since the seminal work of Shapley (1953), much effort has been devoted to the problem of fair distribution of the surplus generated by a collection of players who are willing to cooperate with one another. The Shapley value and more egalitarian values have been extensively studied and axiomatized to answer this problem. The classical assumption underlying this approach is that the coalition of all players (the grand coalition) has formed, which implies that the question of formation of coalitions is not addressed. Another strand of the literature deals with the formation of coalitions by assuming that a value is chosen to reward the players when they strategically decide which coalition structure they can form. Hart and Kurz (1983) propose two pioneering such models.

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The aim of this article is to examine the problem of fair distribution of surplus as in Shapley (1953), but without neglecting the cooperation possibilities that coalitions other than the grand coalition may create. More specifically, we relax the assumption that the grand coalition must form by allowing the players to reorganize into coalitions if it benefits them, *i.e.* if a coalition structure (or partition) induces a total worth larger than the worth of the grand coalition. In the latter case, the game is not cohesive. Such situations happen frequently, including in some applications as pointed out in Section 2 where we consider shortest path and cost sharing problems. Contrary to Hart and Kurz (1983), the approach is non-strategic: the players do not decide which coalition structure they will form, but the impartial social planner in charge of the payoff allocation will take into account the optimal coalition structure. We materialize this requirement by the new axiom of cohesive efficiency, which simply imposes that the total distributed payoff equals the maximal total worth that a coalition structure can induce. In a sense, we take into account both the choice of an allocation and the formation of coalition, while leaving as much freedom as possible for the players to organize themselves.

Firstly, we determine the consequences of replacing the classical axiom of Efficiency by Cohesive efficiency. We focus on the main three axiomatic characterizations of the Shapley value. Myerson (1980) characterize the Shapley value by Efficiency and Balanced contributions. We show that there is still a unique value satisfying Cohesive efficiency and Balanced contributions, which can be formulated, for each cooperative game, as the Shapley value of its superadditive cover, *i.e.* a new game in which each coalition is associated with the maximal total worth generated by one of its partition. Non-cooperative foundations of this value can be found in Pérez-Castrillo and Wettstein (2001). To the contrary, the modern version of the classical characterization of the Shapley value by Shubik (1962) and the well-known characterization of the Shapley value by Young (1985) do not yield any value after replacing Efficiency by Cohesive efficiency. We even show that there is no value satisfying Cohesive Efficiency and either Additivity or Strong monotonicity.

Secondly, we introduce and characterize two variants of the Shapley value that are compatible with Cohesive Efficiency. One is the aforementioned Shapley value of the superadditive cover. The other also applies the Shapley value to a modified game, in which only the worth of the grand coalition is augmented to attain the maximal total worth that a coalition structure can induce, and hence assigns each player her Shapley value plus an equal share of the surplus generated by the optimal coalition structure. The former value coincides with the Shapley value on the class of superadditive games, while the latter value coincides with the Shapley value on the larger class of cohesive games. We invoke classical axioms such as Equal treatment of equals or the null player axiom. Some other axioms are new. As an example, the axiom of Invariance to inactive coalitions relies of the concept of strictly active coalitions, *i.e.* a coalition for which the worth is strictly larger than the total worth achieve by any other coalition structure of this coalition. The axiom imposes that the payoff allocation only depends on the worth of the strictly active coalitions.

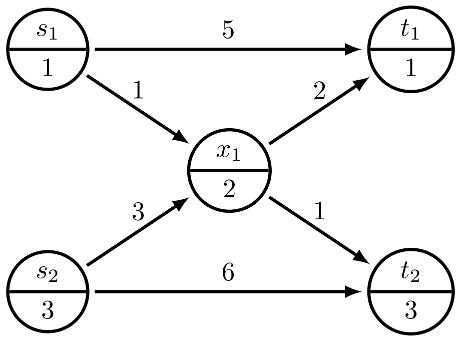
Thirdly, we introduce and characterize a variant of the equal surplus division value, of the class of consensus values (Ju et al., 2007) and of the equal allocation of non-separable costs in order to underline that our approach is not limited to the Shapley value and its variants. For two of these results, we invoke the natural axiom of Individual rationality, *i.e.* each player should obtain a payoff at least equal to her stand-alone worth.

Our example on shortest path problems possesses two optimal coalition structures. It is not possible, *a priori*, to select one over the other. As a consequence, it would not be satisfactory to apprehend our problem through a so-called game with a coalition structure (Aumann and Dreze, 1974). Finally, it should be mentioned that the counterpart of our study with respect to core allocations can be found in Casajus and Tutic (2007).

The rest of the article is organized as follows. Section 2 presents to examples. Section 3 provides the main definition. The key axiom of Cohesive efficiency is introduced in Section 4. Section 5 revisits the classical axiomatization of the Shapley value. The two variants of the Shapley value are examined in Section 6. Section 7 discusses the variant of other values.

2. Motivating examples

Example 1. ***(Shortest path problems)*** Fragnelli et al. (2000) study a class of TU-games induced by shortest path problems. A shortest path problem is described by a set of players, a set of nodes such that each node belongs to one and only one player and with at least of source and one sink, a set of arcs between nodes with a nonnegative length (a transportation cost for instance) and a positive value interpreted as the benefit obtained for transporting one unit of good from a source to a sink. The shortest path game associates with any coalition of players a worth equal to 0 if its members do not own a path from a source to the sink and otherwise the difference between the benefit value and the cost of the shortest path owned by the coalition. Fragnelli et al. (2000, Proposition 1) show that the class of shortest path games coincides with the class of monotonic games[1]. We consider below an example with player set $N = \{1, 2, 3\}$, set of nodes $\{s_1, s_2, x_1, t_1, t_2\}$ where s_k and t_k, $k \in \{1, 2\}$ are the sources and the sink, respectively. The next figure also specifies the player owning each node (in the lower part of the node) and the length of the arcs. Finally, assume that the benefit value is equal to 8.



The TU-game arising from this shortest path problem is described in the table below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	3	0	2	5	3	4	6

[1]A game is monotonic if $v(S) \leq v(T)$ whenever $S \subseteq T$.

If the goal is to find a fair distribution of the output that the grand coalition can produce, then it is disputable to distribute a total worth of 6. The reason is that the players can reorganize themselves so as to achieve a better output. For instance, player 1 alone induces a worth of 3 (by means of the path $s_1 \longrightarrow t_1$ she owns) and the coalition containing players 2 and 3 achieves a worth of 4 (by means of the path $s_2 \longrightarrow x_1 \longrightarrow t_2$ owned by $\{1,2\}$). In other words, two goods can be transported simultaneously by two disjoint subcoalitions of the grand coalition, yielding a total worth of 7. If a social planner is in charge of the organization of the players and of the payoff allocation, then she should distribute a total payoff of 7 because the players can produce this output by coordinating.

This becomes possible if the classical efficiency postulate is replaced by a stronger requirement of Cohesive efficiency: the total distributed worth is the best total worth induced by any reorganization of the grand coalition. It is useful to remark in our example that another partition of the grand coalition, $\{\{1,2\},\{3\}\}$, also induces the best total output of 7. The important implication is that it does not really make sense to rely on a called TU-game with coalition structure here. Which coalition structure, $\{\{1\},\{2,3\}\}$ or $\{\{1,2\},\{3\}\}$, should be used? □

Example 2. ***(Cost sharing problems)*** Each agent in a group announces a demand of output. The production cost function is denoted by C, so that the total cost of satisfying the aggregate demand has to be shared among the agents. As an example, set $N = \{1,2,3\}$, demands $q_i = i$ for each $i \in N$ and the cost function $C(q) = 3q^2/20 + 2q + 2$. We can associate to this cost sharing problem the classical stand-alone cost game (see Moulin, 1992, for instance), which assigns to each coalition of agents the cost of meeting the total demand of its members:

$$c(S) = C\Big(\sum_{i\in S} q_i\Big)$$

for each coalition S of agents. In our example, we obtain the table below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c(S)$	4.15	6.6	9.35	9.35	12.4	15.75	19.4

Here, the players have an incentive to produce at the minimal total cost. This means that we look for the partition yielding the minimal total cost.[2] It is easy to check that $c(\{1,2,3\}) < c(\{1\}) + c(\{2\}) + c(\{3\})$ or, equivalently, $C(q_1 + q_2 + q_3) < C(q_1) + C(q_2) + C(q_3)$. This comes from savings in fixed costs in spite of the increasing marginal cost (a common feature in the production of many raw materials). Here too, the agents can do better than the total cost 19.4 they achieve altogether. For instance, if agent 3 bears her stand-alone cost of 9.35 and if the remaining two agents cooperate to achieve a cost of 9.35 together, then a saving of 0.7 is obtained. This means that the agents benefit from split production at two plants instead of an inefficient production concentrated on a single plant. The partition $\{\{1,2\},\{3\}\}$ is indeed optimal, although the partition $\{\{1,3\},\{2\}\}$ also does better than the cost of the grand coalition. □

[2]It is possible to proceed as in Example 1 by considering the cost-saving game (N,v) such that, for each coalition $S \in 2^N$, $v(S) = \sum_{i\in S} c(\{i\}) - c(S)$.

3. Cooperative games with transferable utility

3.1. Cooperative games

Let $\mathcal{U} \subseteq \mathbb{N}$ be a fixed and infinite universe of players. Denote by U the set of all finite subsets of $\mathcal{U}$. A **cooperative game with transferable utility**, or simply a **game**, is a pair (N, v) where $N \in U$ and $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. For a game (N, v), we write (S, v) for the **subgame** of (N, v) induced by $S \in 2^N$ by restricting v to 2^S. Let s stands for the cardinality of S. Define $\mathcal{C}$ as the class of all games with a finite player set in U and $\mathcal{C}_N$ as the subclass of $\mathcal{C}$ containing the games with player set N. A game $(N, v) \in \mathcal{C}$ is **superadditive** if, for any pair of disjoint coalitions S and T, $v(S \cup T) \geq v(S) + v(T)$, and strictly superadditive if these inequalities are strict. A game $(N, v) \in \mathcal{C}$ is **zero-normalized** if, for each $i \in N$, $v(\{i\}) = 0$.

For each $b \in \mathbb{R}$, each $(N, v), (N, w) \in \mathcal{C}$, the game $(N, bv + w) \in \mathcal{C}$ is defined, for each $S \in 2^N$, as $(bv + w)(S) = bv(S) + w(S)$. The **unanimity game** on N induced by a nonempty coalition S, denoted by (N, u_S), is defined as $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise. Since Shapley (1953), it is well-known that each function v admits a unique decomposition into unanimity games:

$$v = \sum_{S \in 2^N \setminus \{\emptyset\}} \Delta_v(S) u_S \tag{1}$$

where $\Delta_v(S)$ is the **Harsanyi dividend** (Harsanyi, 1959) of S, defined recursively as $\Delta_v(S) = v(S) - \sum_{T \in 2^S \setminus \{\emptyset\}} \Delta_v(T)$. We also define the **opposite Dirac game** on N induced by a nonempty coalition S, denoted by $(N, \mathbf{1}_S^-)$, is defined as $\mathbf{1}_S^-(S) = -1$ and $\mathbf{1}_S^-(T) = 0$ if $T \in 2^N \setminus \{S\}$. Any function v admits an alternative unique decomposition into opposite Dirac games:

$$v = \sum_{S \in 2^N \setminus \{\emptyset\}} -v(S) \mathbf{1}_S^-. \tag{2}$$

The **null game** on N is denoted by $(N, \mathbf{0})$ and is defined, for each $S \in 2^N$, as $\mathbf{0}(S) = 0$. A player $i \in N$ is **null** in (N, v) if, for each $S \in 2^{N \setminus \{i\}}$ such that $S \ni i$, $v(S \cup \{i\}) = v(S)$. A player $i \in N$ is **at least as desirable as** another player $j \in N$ in a game (N, v) if, for each $S \in 2^{N \setminus \{i,j\}}$, $v(S \cup \{i\}) \geq v(S \cup \{j\})$. Two players that are at least as desirable as each other are called **equal**.

For each nonempty $N \in U$, let $P(N)$ be the set of all partitions of N. A coalition S is **(weakly) active** in a game (N, v) if, for each $P \in P(S) \setminus \{\{S\}\}$, $v(S) \geq \sum_{T \in P} v(T)$, and **strictly active** if the previous inequalities are strict. Note that any singleton $\{i\}$, $i \in N$, is strictly active in (N, v). Denote by $A(N, v)$ and $A^*(N, v)$ the nonempty sets of all active and strictly active coalitions in (N, v), respectively. A game (N, v) is called **cohesive** if $N \in A(N, v)$, *i.e.*, if N generates as least as much worth as any of its partitions. Any superadditive game is cohesive while the converse is not true. The classes of superadditive games and cohesive games form two convex cones, *i.e.* they are closed under linear combinations with positive coefficients. Hence, the addition of two superadditive (cohesive) games is a superadditive (cohesive) game, and the multiplication of a superadditive (cohesive) game by a positive scalar yields a superadditive (cohesive) game.

3.2. Values

A **value** on $\mathcal{C}$ (respectively on $\mathcal{C}_N$) is a function f that assigns a payoff vector $f(N,v) \in \mathbb{R}^N$ to any $(N,v) \in \mathcal{C}$ (respectively any $(N,v) \in \mathcal{C}_N$). Below we introduce well-known values for TU-games.

The **Shapley value** (Shapley, 1953) is the value Sh on $\mathcal{C}$ defined as:

$$\forall (N,v) \in \mathcal{C}, \forall i \in N, \quad Sh_i(N,v) = \sum_{S \in 2^N : S \ni i} \frac{(s-1)!(n-s)!}{n!}\big(v(S) - v(S \backslash \{i\})\big).$$

The **equal surplus division value** (Driessen and Funaki, 1991) is the value ESD on $\mathcal{C}$ defined as:

$$\forall (N,v) \in \mathcal{C}, \forall i \in N, \quad ESD_i(N,v) = v(\{i\}) + \frac{1}{n}\left(v(N) - \sum_{j \in N} v(\{j\})\right).$$

Fix any $N \in U$ and any $\lambda \in [0,1]$. The **λ-consensus value** (Ju et al., 2007) is the value CV^{λ} on $\mathcal{C}_N$ such that $CV^{\lambda} = \lambda Sh + (1-\lambda)ESD$.

The **equal division value** is the value ED on $\mathcal{C}$ defined as:

$$\forall (N,v) \in \mathcal{C}, \forall i \in N, \quad ED_i(N,v) = \frac{v(N)}{n}.$$

The **equal allocation of nonseparable costs** (Moulin, 1985) is the value $EANC$ on $\mathcal{C}$ defined as:

$$\forall (N,v) \in \mathcal{C}, \forall i \in N, \quad EANC_i(N,v) = v(N) - v(N \backslash \{i\}) + \frac{1}{n}\left(v(N) - \sum_{j \in N}\big(v(N) - v(N \backslash \{j\})\big)\right).$$

4. Cohesive efficiency

Suppose that for some partition $P \in P(N)$, it holds that $\sum_{T \in P} v(T) > v(N)$, i.e. (N,v) is not cohesive. Then, it is difficult to take for granted the traditional assumption that the grand coalition will form. However, in such a case, the aforementioned values will distribute the worth of N, letting aside a possible surplus. In this article, following Pérez-Castrillo and Wettstein (2001), we advocate that the total payoff that the player should get must be equal to the maximal total worth that they are able to achieve by organizing themselves into (mutually disjoint) coalitions, i.e. into partitions. We materialize this requirement by invoking the axiom of Cohesive efficiency.

Cohesive efficiency (CE) For each $(N,v) \in \mathcal{C}$,

$$\sum_{i \in N} f_i(N,v) = \max_{P \in P(N)} \sum_{T \in P} v(T).$$

This axiom is alternative to the classical axiom of Efficiency.

Efficiency (E) For each $(N,v) \in \mathcal{C}$, $\sum_{i \in N} f_i(N,v) = v(N)$.

The two axioms agree whenever the game under consideration is cohesive, but provide different recommendations otherwise. In a sense, we impose Cohesive efficiency in order to combine both objectives of an efficient coalition formation and a (forthcoming) fair payoff allocation. From a non-cooperative point of view, Pérez-Castrillo and Wettstein (2001) pursue the same two objectives by letting the players form an efficient equilibrium partition before paying them with a variant of the Shapley value satisfying Cohesive Efficiency (and called the superadditive Shapley value in the next section). The requirement in Cohesive efficiency also appears in Arnold and Schwalbe (2002) in a (non-axiomatic) model of coalition formation in TU-games. In the next sections, we study the consequences of imposing Cohesive efficiency in combination with well-known axioms and new axioms. For a fixed game (N, v), remark that if a partition $P^* \in P(N)$ is such that $\sum_{T \in P^*} v(T) = \max_{P \in P(N)} \sum_{T \in P} v(T)$, then it must be that each coalition in P^* is active. In the next section, we will highlight the key role of active coalitions by invoking new axioms.

5. Cohesive efficiency in classical characterizations of the Shapley value

We revisit three of the main characterizations of the Shapley value, one operating on $\mathcal{C}$, the other on $\mathcal{C}_N$. Myerson (1980) provides an elegant characterization of the Shapley value by Efficiency and the axiom of Balanced contributions below.

Balanced contributions (BC) (Myerson, 1980) For each $(N, v) \in \mathcal{C}$, each $i, j \in N$,

$$f_i(N, v) - f_i(N \backslash \{j\}, v) = f_j(N, v) - f_j(N \backslash \{i\}, v).$$

Replacing Efficiency in Myerson's result by any other condition on the total payoff distributed to the players yields a unique value as underlined by Hart and Mas-Colell (1989, section 3). Therefore, there is a unique value satisfying Cohesive efficiency and Balanced contributions. In the next result, we present the natural formulation of this value: it is non-linear but preserves in a sense the spirit of the Shapley value. To see this, define the **superadditive cover** of a game (N, v) as the game $(N, \overline{v})$ such that, for each $S \in 2^N$, $\overline{v}(S) = \max_{P \in P(S)} \sum_{T \in P} v(T)$.

Proposition 1. *There exists a unique value on $\mathcal{C}$ that satisfies Cohesive efficiency and Balanced contributions. It is the superadditive Shapley value SSh defined for each game as the Shapley value of its superadditive cover, i.e. for each $(N, v) \in \mathcal{C}$, $SSh(N, v) = Sh(N, \overline{v})$.*

Proof. As noted earlier, there is a unique value satisfying **CE** and **BC**. It remains to show that the superadditive Shapley value SSh satisfies the two axioms. By definition and the fact that Sh satisfies **E**, $SSh(N, v)$ satisfies **CE**. Next consider any game $(N, v) \in \mathcal{C}$ and any pair of players $i, j \in N$. For each nonempty $S \in 2^N$, note that the subgame of $(N, \overline{v})$ induced by S is equal to $(S, \overline{v})$ because, for each $T \in 2^S$, $\overline{v}(T)$ is only based on the subcoalitions in 2^T. So, since the Shapley value Sh satisfies **BC**, we can write that

$$\begin{aligned} & SSh_i(N, v) - SSh_i(N \backslash \{j\}, v) \\ = \; & Sh_i(N, \overline{v}) - Sh_i(N \backslash \{j\}, \overline{v}) \\ = \; & Sh_j(N, \overline{v}) - Sh_j(N \backslash \{i\}, \overline{v}) \\ = \; & SSh_j(N, v) - SSh_j(N \backslash \{i\}, v). \end{aligned}$$

as desired. ∎

The superadditive Shapley value is strategically implemented in Pérez-Castrillo and Wettstein (2001). Another characterization of the superadditive Shapley value can be easily obtained by adapting slightly the potential approach in Hart and Mas-Colell (1989). So define a **cohesive potential** as a function Q on $\mathcal{C}$ such that $Q(\varnothing, v) = 0$ and $\sum_{i\in N}\big(Q(N,v) - Q(N\backslash\{i\},v\big) = \overline{v}(N)$. Hence only the classical efficiency constraint is replaced by Cohesive efficiency. The obvious proof is omitted.

Proposition 2. *There exists a unique cohesive potential function Q on $\mathcal{C}$. For each game $(N,v) \in \mathcal{C}$ and each $i \in N$, $SSh_i(N,v) = Q(N,v) - Q(N\backslash\{i\},v)$.*

Another classical characterization of the Shapley value is provided by Young (1985), who combines Efficiency with Equal treatment and Strong monotonicity.

Equal treatment (ET) For each $(N,v) \in \mathcal{C}$, and each $i,j \in N$ who are equal in (N,v), $f_i(N,v) = f_j(N,v)$.

Strong monotonicity (SM) For each pair of games $(N,v),(N,w) \in \mathcal{C}$ and each player $i \in N$ such that, for each $S \subseteq N\backslash\{i\}$, $v(S\cup\{i\}) \geq w(S\cup\{i\})$, it holds that $f_i(N,v) \geq f_i(N,w)$.

This last axiom requires that a player's payoff is non decreasing with respect to her contributions to coalitions. The superadditive Shapley value does not satisfy this axiom as a corollary of the general impossibility result below.

Proposition 3. *For each $n \geq 3$, there exists no value on $\mathcal{C}_N$ satisfying Cohesive efficiency and Strong monotonicity.*

Proof. Let N be such that $\{1,2,3\} \subseteq N$. By **SM**, we get

$$\begin{array}{rrcl} & f_1(N, u_{\{1,2\}} + u_{\{2,3\}} - u_N) & = & f_1(N, u_{\{1,2\}} - u_N), \\ & f_2(N, u_{\{1,2\}} + u_{\{2,3\}} - u_N) & \geq & f_2(N, u_{\{1,2\}} - u_N), \\ & f_3(N, u_{\{1,2\}} + u_{\{2,3\}} - u_N) & \geq & f_3(N, u_{\{1,2\}} - u_N), \\ \forall i \in N\backslash\{1,2,3\}, & f_i(N, u_{\{1,2\}} + u_{\{2,3\}} - u_N) & = & f_i(N, u_{\{1,2\}} - u_N). \end{array} \tag{3}$$

Note that $\overline{(u_{\{1,2\}} + u_{\{2,3\}} - u_N)}(N) = (u_{\{1,2\}} + u_{\{2,3\}} - u_N)(N) = 1$ and $\overline{(u_{\{1,2\}} - u_N)}(N) = 1$, which implies that $f(u_{\{1,2\}} + u_{\{2,3\}} - u_N) = f(N, u_{\{1,2\}} - u_N)$ by **CE**. Considering $(N, u_{\{2,3\}} - u_N)$ instead of $(N, u_{\{1,2\}} - u_N)$ in the previous steps yields analogously $f(u_{\{1,2\}} + u_{\{2,3\}} - u_N) = f(N, u_{\{2,3\}} - u_N)$ and in turn that $f(N, u_{\{1,2\}} - u_N) = f(N, u_{\{2,3\}} - u_N)$. With the same arguments, it is easy to show that

$$f(N, u_{\{1,2\}} - u_N) = f(N, u_{\{2,3\}} - u_N) = f(N, u_{\{1,3\}} - u_N). \tag{4}$$

Next, **SM** also implies that

$$\begin{array}{rrcl} & f_3(N, u_{\{1,2\}} - u_N) & = & f_3(N, -u_N), \\ & f_2(N, u_{\{1,3\}} - u_N) & = & f_2(N, -u_N), \\ & f_1(N, u_{\{2,3\}} - u_N) & = & f_1(N, -u_N), \\ \forall i \in N\backslash\{1,2,3\}, & f_i(N, u_{\{2,3\}} - u_N) & = & f_i(N, -u_N). \end{array} \tag{5}$$

These equalities together with (4) entail the contradiction

$$1 = \sum_{i \in N} f_i(N, u_{\{1,2\}} - u_N) = \sum_{i \in N} f_i(N, -u_N) = 0, \tag{6}$$

where the first and last equalities come from **CE**. ■

It is obvious that this impossibility result fails when $n = 1$ or $n = 2$. For the case $n = 2$, let $N = \{i, j\}$ and consider the value f on $\mathcal{C}_{\{i,j\}}$ such that, for each $(\{i,j\}, v)$ and each $k \in \{i, j\}$, $f_k(\{i,j\}, v) = \max\{v(\{i\}), Sh_i(N, v)\}$. This value satisfies both Cohesive efficiency and Strong monotonicity.

The most classical characterization of the Shapley value is perhaps the one in Shubik (1962), in which the axioms of additivity, null player, equal treatment and efficiency are combined.

Additivity (A) For each $(N, v), (N, w) \in \mathcal{C}$, $f(N, v + w) = f(N, v) + f(N, w)$.

Null player (NP) For each $(N, v) \in \mathcal{C}$, and each player $i \in N$ null in (N, v), $f_i(N, v) = 0$.

The superadditive Shapley value satisfies the null player axiom (as we will demonstrate later), and the equal treatment property, but cannot be obtained by simply replacing efficiency by cohesive efficiency in Shubik's result. This observation can be deduced from the following more general result: Cohesive efficiency and Additivity are incompatible.

Proposition 4. *For each $n \geq 2$, there exists no value on $\mathcal{C}_N$ satisfying Cohesive efficiency and Additivity.*

Proof. The proof is by contradiction. Let f be a value on $\mathcal{C}$ satisfying **CE** and **A**. For any two games $(N, v), (N, w) \in \mathcal{C}$, **A** implies that

$$\sum_{i \in N} f_i(N, v + w) = \sum_{i \in N} f_i(N, v) + \sum_{i \in N} f_i(N, w). \tag{7}$$

Furthermore, an application of **CE** yields that $\sum_{i \in N} f_i(N, v + w) = \overline{v + w}(N)$, $\sum_{i \in N} f_i(N, v) = \overline{v}(N)$ and $\sum_{i \in N} f_i(N, w) = \overline{w}(N)$, so that (7) can be rewritten as $\overline{v + w}(N) = \overline{v}(N) + \overline{w}(N)$. Now, pick any zero-normalized strictly superadditive game (N, v), and recall that $\overline{v}(N) = v(N) > 0$. Note that $\overline{-v}(N) = 0$. However, $\overline{v + (-v)}(N) = \overline{\mathbf{0}}(N) = 0$, which is different from $\overline{v}(N) + \overline{-v}(N) = v(N)$, proving the result. ■

6. More possibility results on variants of the Shapley value

We present two possibility results in which we replace (or even decompose) the classical axiom of additivity by two axioms operating on the structure of the class $\mathcal{C}_N$. These new axioms point out the role of active coalitions in a game.

Active additivity (AA) For each $(N,v),(N,w) \in \mathcal{C}$ such that $A(N,v) = A(N,w) = 2^N \backslash \{\varnothing\}$, $f(N,v+w) = f(N,v) + f(N,w)$.

Active additivity is weaker than additivity since it only applies to games in which each nonempty coalition is active.

Invariance to inactive coalitions (IIC) For each $(N,v),(N,w) \in \mathcal{C}$ such that $A^*(N,v) = A^*(N,w)$ and $v(S) = w(S)$ for each $S \in A^*(N,v)$, $f(N,v) = f(N,w)$.

Invariance to inactive coalitions means that the strictly active coalitions are sufficient to specify the payoff allocations: in two games in which the strictly active coalitions are identical and enjoy the same worth, the payoff recommendation is the same. The core satisfies this principle (provided it is adapted to set-valued solutions): two games possess the same core if they have the same set of strictly active coalitions, and if the worth of any such coalition is the same across the two games. The same principle is often used in the literature on cooperative games with restriction on cooperation. For instance, it is called Independence of irrelevant coalitions in van den Brink et al. (2011).

Proposition 5. *The superadditive Shapley value SSh is the unique value on $\mathcal{C}$ that satisfies Cohesive efficiency, Null player, Equal treatment, Active additivity and Invariance to inactive coalitions.*

The proof relies on the next Lemma.

Lemma 1. *If a player i is at least as desirable as a player j in a game (N,v), then i is at least as desirable as j in $(N,\overline{v})$. If player $i \in N$ is null in (N,v) then player i is also null in $(N,\overline{v})$.*

Proof. Fix any game (N,v) and consider two players $i,j \in N$ such that i is at least as desirable as j in (N,v). By contradiction, assume that i is not as desirable as j in $(N,\overline{v})$. This means that there is $S \in 2^{N\backslash\{i,j\}}$ such that $\overline{v}(S \cup \{i\}) < \overline{v}(S \cup \{j\})$. Let P be any partition in $P(S \cup \{j\})$ such that $\overline{v}(S \cup \{j\}) = \sum_{T \in P} v(T)$. Denote by $T(j)$ the element of P containing j. Since, for each $R \in 2^S$, $v(R \cup \{i\}) \geq v(R \cup \{j\})$, it holds that $\sum_{T \in P} v(T) \leq \sum_{T \in P\backslash\{T(j)\}} v(T) + v(T(j)\backslash\{j\} \cup \{i\})$. Since $\sum_{T \in P\backslash\{T(j)\}} v(T)$ is independent of both i and j, $\overline{v}(S \cup \{i\}) \geq \sum_{T \in P\backslash\{T(j)\}} v(T) + v(T(j)\backslash\{j\} \cup \{i\}) \geq \overline{v}(S \cup \{j\})$, a contradiction with our initial assumption. This proves that i is at least as desirable as j in $(N,\overline{v})$.

Next, consider any null player $i \in N$ in (N,v). Consider any $S \subseteq N\backslash\{i\}$. Let $P \in P(S)$ be such that $\overline{v}(S) = \sum_{T \in P} v(T)$. Since i is null in (N,v), for each $T \in 2^S$, $v(T \cup \{i\}) = v(T)$. Thus, for each $T \in P$, similarly as in the first part,

$$\sum_{R \in P} v(R) = \sum_{R \in P\backslash\{T\}} v(R) + v(T \cup \{i\}) = \sum_{R \in P} v(R) + v(\{i\}) = \overline{v}(S \cup \{i\}),$$

which proves that i is null in $(N,\overline{v})$ too. ∎

Obviously, Lemma 1 implies that two equal players in a game (N,v) are still equal in $(N,\overline{v})$.

Proof. (Proposition 5) We already know that SSh satisfies **CE**, and it satisfies **NP** and **ET** as a consequence of Lemma 1 and the fact that Sh satisfies **NP** and **ET**. Regarding **AA**, note that any game (N,v) such that $A(N,v) = 2^N\backslash\{\varnothing\}$ is superadditive, and thus $v = \overline{v}$. This entails that $SSh(N,v) = Sh(N,v)$. Now, let $(N,v),(N,w) \in \mathcal{C}$ be such that $A(N,v) = A(N,w) = 2^N\backslash\{\varnothing\}$. By the previous remark and the fact that the Shapley value is additive, we immediately get that $SSh(N,v+w) = SSh(N,v) + SSh(N,w)$. Regarding **IIC**, remark that for any game $(N,v) \in \mathcal{C}$, $\overline{v}$ depends only on $A^*(N,v)$. Therefore, for two games $(N,v),(N,w) \in \mathcal{C}$ such that $A^*(N,v) = A^*(N,w)$ and, for each $S \in A^*(N,v)$, $v(S) = w(S)$, this means that $\overline{v} = \overline{w}$. Hence $SSh(N,v) = SSh(N,w)$.

For the uniqueness part, consider any value f satisfying the five axioms. Pick any game $(N,v) \in \mathcal{C}$ and consider its superadditive cover $(N,\overline{v})$. By definition $A^*(N,v) = A^*(N,\overline{v})$, and obviously, for each $S \in A^*(N,v)$, $v(S) = \overline{v}(S)$. Thus **IIC** implies that $f(N,v) = f(N,\overline{v})$. It remains to show that f is uniquely determined in $(N,\overline{v})$. From (1), we can write that

$$\overline{v} = \sum_{S\in 2^N:\Delta_{\overline{v}}(S)>0} \Delta_{\overline{v}}(S)u_S - \sum_{S\in 2^N:\Delta_{\overline{v}}(S)<0} -\Delta_{\overline{v}}(S)u_S,$$

or equivalently that

$$\overline{v} + \sum_{S\in 2^N:\Delta_{\overline{v}}(S)<0} -\Delta_{\overline{v}}(S)u_S = \sum_{S\in 2^N:\Delta_{\overline{v}}(S)>0} \Delta_{\overline{v}}(S)u_S. \tag{8}$$

In this last expression, for each $S \in 2^N\backslash\{\varnothing\}$, it clearly holds that $A(N,u_S) = 2^N\backslash\{\varnothing\}$, and each nonempty coalition in any positive combination of such games is also active. Then, successive applications of **AA** yield that both

$$f\left(N, \sum_{S\in 2^N:\Delta_{\overline{v}}(S)>0} \Delta_{\overline{v}}(S)u_S\right) = \sum_{S\in 2^N:\Delta_{\overline{v}}(S)>0} f(N,\Delta_{\overline{v}}(S)u_S), \tag{9}$$

and

$$f\left(N, \sum_{S\in 2^N:\Delta_{\overline{v}}(S)<0} -\Delta_{\overline{v}}(S)u_S\right) = \sum_{S\in 2^N:\Delta_{\overline{v}}(S)<0} f(N,-\Delta_{\overline{v}}(S)u_S), \tag{10}$$

so that another application of **AA** in (8) yields that

$$f(N,\overline{v}) + \sum_{S\in 2^N:\Delta_{\overline{v}}(S)<0} f(N,-\Delta_{\overline{v}}(S)u_S) = \sum_{S\in 2^N:\Delta_{\overline{v}}(S)>0} f(N,\Delta_{\overline{v}}(S)u_S).$$

Equivalently

$$f(N,\overline{v}) = \sum_{S\in 2^N:\Delta_{\overline{v}}(S)>0} f(N,\Delta_{\overline{v}}(S)u_S) - \sum_{S\in 2^N:\Delta_{\overline{v}}(S)<0} f(N,-\Delta_{\overline{v}}(S)u_S)$$

The last step is therefore to prove that f is uniquely determined in each game $(N, \alpha u_S)$ where $S \in 2^N \backslash \{\varnothing\}$ and $\alpha \in \mathbb{R}_{++}$. The proof is similar to the classical characterization of the Shapley value in Shubik (1962). In the game $(N, \alpha u_S)$, each player $i \in N \backslash S$ is null, so that $f_i(N, \alpha u_S) = 0$ for each such player by **NP**. Note also that $\alpha u_S(N) = \overline{\alpha u_S}(N) = \alpha$, so that **CE** and the previous remark imply that $\alpha = \sum_{i \in N} f_i(N, \alpha u_S) = \sum_{i \in S} f_i(N, \alpha u_S)$. Since all players in S are equal in $(N, \alpha u_S)$, it follows from **ET** that, for each $i \in S$, $f_i(N, \alpha u_S) = \alpha / s$, which completes the proof. ■

The logical independence of the axioms is demonstrated as follows.

- The null value satisfies each axiom except **CE**.
- The value that assigns to each game $(N, v) \in \mathcal{C}$ the payoffs $f(N, v) = ED(N, \overline{v})$ satisfies each axiom except **NP**.
- Let π be any permutation on $\mathcal{U}$. For a fixed $N \in U$, denote by $P_i^{\pi, N}$ the set of predecessors of i in N according to π, including i. The value that assigns to each game $(N, v) \in \mathcal{C}$ and each player $i \in N$ the payoff $f_i(N, v) = \overline{v}(P_i^{\pi, N}) - \overline{v}(P_i^{\pi, N} \backslash \{i\})$ satisfies each axiom except **ET**.
- The value that assigns to each game $(N, v) \in \mathcal{C}$ and each player $i \in N$ the payoff $f_i(N, v) = 0$ if $v = \mathbf{0}$ and $f_i(N, v) = Sh_i(N, \overline{v}^2) / \overline{v}(N)$ otherwise satisfies each axiom except **AA**.
- Consider the game $(N, w) \in \mathcal{C}$ such that, for each $i \in N$, $w(\{i\}) = i$, $w(N) = -n$ and $w(S) = 0$ otherwise. The value f that assigns the payoffs $f(N, w) = ED(N, \overline{w})$ and $f(N, v) = SSh(N, v)$ otherwise satisfies each axiom except **IIC**.

The axiom of Active additivity can perhaps be considered as too distant from the classical axiom of additivity, and similarly, the axiom of Invariance to inactive coalitions can also be judged as too demanding since the invariance principle can be applied to pairs of games that are very different. A reason is that these axioms mobilize all active coalitions and all strictly active coalitions, respectively, while our objective of a value satisfying Cohesive Efficiency explicitly relates to only a small number of active coalitions, those in the efficient partitions of the grand coalition. Below, we adopt another axiomatic view in which the grand coalition is the only coalition concerned with the principle of these two axioms. Firstly, this gives rise to a weak version of active additivity.

Cohesive additivity (CA) For each $(N, v), (N, w) \in \mathcal{C}$ such that $N \in A(N, v) \cap A(N, w)$, $f(N, v + w) = f(N, v) + f(N, w)$.

So the principle of Active additivity is no longer restricted to games in which each nonempty coalition is active, but is invoked for pairs of games that have in common the grand coalition as an active coalition. Secondly, in the same spirit as Invariance to inactive coalitions, we impose that the grand coalition is weakly active in a game, then the payoff allocation is the same in the game obtained by diminishing its worth.

Invariance to deactivating the grand coalition (IDGC) For each $(N,v),(N,w) \in \mathcal{C}$ such that $N \in A(N,v)\backslash A^*(N,v)$, for each $S \in 2^N\backslash\{N\}$, $v(S) = w(S)$ and $v(N) > w(N)$, $f(N,v) = f(N,w)$.

In the characterization below, we also invoke a weak version of Null player as an alternative of Null player used in Proposition 5. This axiom requires that a null player gets a zero payoff only if the game is cohesive.

Null player in a cohesive environment (NPCE). For each $(N,v) \in \mathcal{C}$ such that $N \in A(N,v)$, if $i \in N$ is a null player in (N,v), $f_i(N,v) = 0$.

Finally, in order to state the next result, for each game $(N,v) \in \mathcal{C}$, define its **efficient egalitarian extension** $(N,\hat{v}) \in \mathcal{C}$ such that, for each $S \in 2^N\backslash\{N\}$, $\hat{v}(S) = v(S)$, and $\hat{v}(N) = \overline{v}(N)$.

Proposition 6. *There exists a unique value on $\mathcal{C}$ that satisfies Cohesive efficiency, Equal treatment, Null player in a cohesive environment, Cohesive additivity and Invariance to deactivating the grand coalition. It is called the Efficient egalitarian Shapley value $EESh$ defined, for each $(N,v) \in \mathcal{C}$, as $EESh(N,v) = Sh(N,\hat{v})$.*

Proof. In order to show that $EESh$ satisfies the five axioms on $\mathcal{C}$, note that for each $(N,v) \in \mathcal{C}$ and each $i \in N$, we have

$$EESh_i(N,v) = Sh_i(N,v) + \frac{1}{n}\big(\overline{v}(N) - v(N)\big). \tag{11}$$

From (11) and the fact that Sh satisfies **E** and **ET**, it is obvious that $EESh$ satisfies **CE** and **ET**. Regarding **NPCE**, note that if $N \in A(N,v)$, then $\overline{v}(N) = v(N)$ implies that $EESh(N,v) = Sh(N,v)$ by (11). So $EESh$ satisfies **NPCE** since Sh satisfies **NP**. Next, pick any two games $(N,v),(N,w) \in \mathcal{C}$ such that $N \in A(N,v) \cap A(N,w)$. Note that $N \in A(N,v) \cap A(N,w)$ implies that $N \in A(N,v+w)$. So, the fraction in (11) vanishes in games (N,v), (N,w) and $(N,v+w)$, which means that $EESh$ satisfies **CA** because Sh satisfies **A**. Finally, regarding **IDGC**, consider two games $(N,v),(N,w) \in \mathcal{C}$ such that $N \in A(N,v)\backslash A^*(N,v)$, for each $S \in 2^N\backslash\{N\}$, $v(S) = w(S)$ and $v(N) > w(N)$. Since $N \in A(N,v)\backslash A^*(N,v)$, this means that there exists $P \in P(N)\backslash\{\{N\}\}$ such that $\overline{v}(N) = \sum_{T\in P} v(T) = v(N)$. In turn, this yields that $\overline{w}(N) = \sum_{T\in P} w(T)$ as well, and so that $v(N) = \overline{w}(N)$. Furthermore, $N \in A(N,v)$ implies that $EESh(N,v) = Sh(N,v)$. These last two properties yield that

$$\begin{aligned}
EESh_i(N,w) &= Sh_i(N,w) + \frac{1}{n}\big(\overline{w}(N) - w(N)\big)\\
&= \left(Sh_i(N,v) - \frac{1}{n}\big(v(N) - w(N)\big)\right) + \frac{1}{n}\big(\overline{w}(N) - w(N)\big)\\
&= Sh_i(N,v)\\
&= EESh_i(N,v),
\end{aligned}$$

as wanted.

For the uniqueness part, choose any $(N,v) \in \mathcal{C}$. We distinguish two cases. In the first case, assume that $N \in A(N,v)$. The proof uses the decomposition of v exploited in the proof of Proposition 5 because **CA** implies **AA** (except that we deal with v instead of $\overline{v}$ as in Proposition 5). It is therefore sufficient to show f is uniquely determined in each game $(N,\alpha u_S)$ where $S \in 2^N \backslash \{\varnothing\}$ and $\alpha \in \mathbb{R}_{++}$. In such a game, recall that **CE** reduces to **E** and note that **NPCE** can be applied since $(N,\alpha u_S)$ is cohesive. Hence as in the proof of Proposition 5, we get from **CE**, **NPCE** and **ET** that $f_i(N,\alpha u_S) = \alpha/s$ if $i \in S$ and $f_i(N,\alpha u_S) = 0$ if $i \in N\backslash S$. Using **CA**, we obtain, for each $i \in (N,v)$,

$$f_i(N,v) = \sum_{S \in 2^N : \Delta_v(S)>0} f(N,\Delta_v(S)u_S) - \sum_{S \in 2^N : \Delta_v(S)<0} f(N,-\Delta_v(S)u_S)$$

and so f is uniquely determined in (N,v).

In the second case, assume that $N \in 2^N \backslash A(N,v)$. Consider the efficient egalitarian extension $(N,\hat{v})$ of (N,v) as defined prior to the statement of Proposition 6. Since $N \in 2^N \backslash A(N,v)$, remark that $N \in A(N,\hat{v}) \backslash A^*(N,\hat{v})$. Moreover, for each $S \in 2^N \backslash \{N\}$, $\hat{v}(S) = v(S)$, and $\hat{v}(N) > v(N)$. As a consequence, we can apply **IDGC** to games $(N,\hat{v})$ and (N,v): $f(N,\hat{v}) = f(N,v)$, and so $f(N,v)$ is uniquely determined since $f(N,\hat{v})$ is uniquely determined by the first case. This completes the proof. ■

Remark 1. *From* (11)*, it is easy to figure out that among all values satisfying Cohesive efficiency, $EESh$ is the unique value that minimizes the Euclidean distance to the Shapley value. In this sense, $EESh$ can be considered as a very close variant of the Shapley value. This formulation of $EESh$ has the same flavor as the Efficient Egalitarian Myerson value for TU-games enriched by a communication graph introduced in van den Brink et al. (2012) and further studied in Béal et al.* (2015a)*.* □

Remark 2. *There is another natural reevaluation of the function v from which $EESh$ can be define. In order to see this, consider the game (N,v^*) such that, for each $S \in 2^N$,*

$$v^*(S) = v(S) + \frac{s}{n}\big(\overline{v}(N) - v(N)\big).$$

Each coalition receives its worth plus a share of the surplus created by the efficient reorganization of the player set that is proportional to its size. Hence, v^* is the sum of v and a symmetric game, which implies that $EESh(N,v) = Sh(N,v^*)$. □

Note also that $EESh$ violates the axiom of Null player. This means that the axiom of Null player in a cohesive environment cannot be replaced by Null player in Proposition 6. The logical independence of the axioms in Proposition 6 is demonstrated as follows.

- The null value satisfies each axiom except **CE**.
- The value that assigns to each game $(N,v) \in \mathcal{C}$ the payoffs $f(N,v) = ED(N,\overline{v})$ satisfies each axiom except **NPCE**.

- For each $N \in U$, let π be any permutation on N and denote by P_i^π the set of predecessors of i according to π, including i. The value that assigns to each game $(N, v) \in \mathcal{C}$ and each player $i \in N$ the payoff $f_i(N, v) = \overline{v}(P_i^\pi) - \overline{v}(P_i^\pi \backslash \{i\})$ satisfies each axiom except **ET**.

- The value that assigns to each game $(N, v) \in \mathcal{C}$ and each player $i \in N$ the payoff $f_i(N, v) = 0$ if $v = \mathbf{0}$ and $f_i(N, v) = Sh_i(N, \hat{v}^2)/\hat{v}(N)$ otherwise satisfies each axiom except **CA**.

- Consider the value that assigns to each game $(N, v) \in \mathcal{C}$ and each player $i \in N$ the payoff $f_i(N, v) = Sh_i(N, v)$ if $N \in A(N, v)$ and $v(N) \neq 0$, and $f_i(N, v) = Sh_i(N, v) \times \overline{v}(N)/v(N)$ otherwise. This value satisfies each axiom except **IDGC**.

7. Other allocation rules

7.1. Equal surplus division

We define below the natural axiom of individual rationality, which has been widely invoked to characterize the core (see Peleg, 1989; Tadenuma, 1992; Voorneveld and van den Nouweland, 1998; Hwang and Sudhölter, 2001). This axiom rules out situations in which some player is paid less than her stand-alone worth.

Individual rationality (IR) For each $(N, v) \in \mathcal{C}$ and each $i \in N$, $f_i(N, v) \geq v(\{i\})$.

It is worth to note that this axiom is satisfied by the superadditive Shapley value but not by the efficient egalitarian Shapley value. It is also satisfied by a more egalitarian value as pointed out in the next result.

Proposition 7. *There exists a unique value on $\mathcal{C}$ that satisfies Cohesive efficiency, Equal treatment, Individual rationality, Cohesive additivity and Invariance to deactivating the grand coalition. It is called the Efficient egalitarian equal surplus division $EEESD$ defined, for each $(N, v) \in \mathcal{C}$, as $EEESD(N, v) = ESD(N, \hat{v})$.*

Proof. It is obvious that $EEESD$ satisfies **CE** and **ET**. It also satisfies **IR**. In fact, for any $(N, v) \in \mathcal{C}$, $\overline{v}(N) \geq \sum_{i \in N} v(\{i\})$, $\overline{v}(N) = \hat{v}(N)$ and, for each $i \in N$, $v(\{i\}) = \hat{v}(\{i\})$ imply, for each $i \in N$, that

$$EEESD_i(N, v) = v(\{i\}) + \frac{1}{n}\left(\overline{v}(N) - \sum_{j \in N} v(\{j\})\right) \geq v(\{i\}).$$

Next, $EESD$ inherits **CA** from ESD since the latter value is linear. Finally, consider two games $(N, v), (N, w) \in \mathcal{C}$ such that $N \in A(N, v) \backslash A^*(N, v)$, for each $S \in 2^N \backslash \{N\}$, $v(S) = w(S)$ and $v(N) > w(N)$. Thus, there is $P \in P(N) \backslash \{\{N\}\}$ such that $\overline{v}(N) = \sum_{T \in P} v(T) = v(N)$. Since, for each $S \in 2^N \backslash \{N\}$, $v(S) = w(S)$ and $v(N) > w(N)$, we also get that $\overline{w}(N) = \overline{v}(N)$. Hence, for each

$i \in N$,

$$
\begin{aligned}
EEESD_i(N,v) &= \hat{v}(\{i\}) + \frac{1}{n}\Big(\hat{v}(N) - \sum_{j\in N}\hat{v}(\{j\})\Big)\\
&= v(\{i\}) + \frac{1}{n}\Big(\overline{v}(N) - \sum_{j\in N} v(\{j\})\Big)\\
&= w(\{i\}) + \frac{1}{n}\Big(\overline{w}(N) - \sum_{j\in N} w(\{j\})\Big)\\
&= EESD_i(N,w),
\end{aligned}
$$

which proves that $EEESD$ satisfies **IDGC**.

For the uniqueness part, consider any value f satisfying **CE**, **ET**, **IR**, **CA** and **IDGC**. Choose any $(N,v) \in \mathcal{C}$. We distinguish two cases.

CASE 1. Assume that $N \in A(N,v)$. As a start, remark that the collection of games $\{(N,u_N),(N,u_{\{i\}})_{i\in N},(N,\mathbf{1}_S^-)_{S\in 2^N:1<s<n}\}$ forms a basis of $\mathcal{C}_N$. Furthermore, all these games are cohesive. So, there exist unique coefficients $\alpha_v(S)$, $S \in 2^N\backslash\{\varnothing\}$, such that, for any $(N,v) \in \mathcal{C}_N$,

$$
v = \sum_{i\in N}\alpha_v(\{i\})u_{\{i\}} + \sum_{S\in 2^N:1<s<n}\alpha_v(S)\mathbf{1}_S^- + \alpha_v(N)u_N.
$$

Rearranging:

$$
\begin{aligned}
& v + \sum_{i\in N:\alpha_v(\{i\})<0} -\alpha_v(\{i\})u_{\{i\}} + \sum_{S\in 2^N:1<s<n,\alpha_v(S)<0} -\alpha_v(S)\mathbf{1}_S^- + q\times -\alpha_v(N)u_N\\
= & \sum_{i\in N:\alpha_v(\{i\})>0}\alpha_v(\{i\})u_{\{i\}} + \sum_{S\in 2^N:1<s<n,\alpha_v(S)>0}\alpha_v(S)\mathbf{1}_S^- + (1-q)\times\alpha_v(N)u_N,
\end{aligned}
\tag{12}
$$

where $q = 1$ if $\alpha_v(N) < 0$ and $q = 0$ if $\alpha_v(N) \geq 0$. The left part of (12) is a cohesive game as a linear combination of cohesive games with positive coefficients. The right part of (12) is a cohesive for the same reason. Therefore, from (12) and successive applications of **CA**, we obtain

$$
\begin{aligned}
& f(N,v) + \sum_{i\in N:\alpha_v(\{i\})<0} f(N,-\alpha_v(\{i\})u_{\{i\}}) + \sum_{S\in 2^N:1<s<n,\alpha_v(S)<0} f(N,-\alpha_v(S)\mathbf{1}_S^-) + f(N,q\times -\alpha_v(N)u_N)\\
= & \sum_{i\in N:\alpha_v(\{i\})>0} f(N,\alpha_v(\{i\})u_{\{i\}}) + \sum_{S\in 2^N:1<s<n,\alpha_v(S)>0} f(N,\alpha_v(S)\mathbf{1}_S^-) + f(N,(1-q)\times\alpha_v(N)u_N).
\end{aligned}
$$

As a consequence, the uniqueness of $f(N,v)$ can be shown by proving that, for any $\alpha \in \mathbb{R}_{++}$, for each $i \in N$, $f(N,\alpha u_{\{i\}})$ is uniquely determined, for each $S \in 2^N$ such that $1 < s < n$, $f(N,\alpha\mathbf{1}_S^-)$ is uniquely determined and $f(N,\alpha u_N)$ is uniquely determined. Firstly, consider the game $(N,\alpha u_N)$. We immediately get $f_i(N,\alpha u_N) = \alpha/n$ by **CE** and **ET**. Secondly, consider the game $(N,\alpha\mathbf{1}_S^-)$, with $1 < s < n$. For each $i \in N$, it holds that $\alpha\mathbf{1}_S^-(\{i\}) = 0$, so that **IR** implies that $f_i(N,\alpha\mathbf{1}_S^-(\{i\})) \geq 0$. Moreover, since $\overline{\alpha\mathbf{1}_S^-}(N) = 0$, **CE** implies that $\sum_{i\in N} f_i(N,\alpha\mathbf{1}_S^-(\{i\})) = 0$ and so, for each $i \in N$, $f_i(N,\alpha\mathbf{1}_S^-(\{i\})) = 0$. Thirdly, consider the game $(N,\alpha u_{\{i\}})$ for some $i \in N$ and some $\alpha \in \mathbb{R}_{++}$. The latter game is additive, which implies that $\overline{-\alpha u_{\{i\}}}(N) = -\alpha u_{\{i\}}(N) = \sum_{j\in N} -\alpha u_{\{i\}}(\{j\})$. As a consequence, a combination of **IR** and **CE** yields that $f_i(N,-\alpha u_{\{i\}}) = -\alpha u_{\{i\}}(\{i\}) = -\alpha$ and, for each $j \in N\backslash\{i\}$, $f_j(N,-\alpha u_{\{i\}}) = -\alpha u_{\{i\}}(\{j\}) = 0$. We conclude that $f(N,v)$ is uniquely determined.

Case 2. Assume that $N \notin A(N,v)$. Then the proof of this part is identical to the end of the proof of Proposition 6 and is omitted. ■

Remark 3. *It should be noted that $EEESD(N,v) = ESD(N,\overline{v})$ holds as well. It is also not difficult to figure out that dropping the axiom of Equal treatment yields a characterization of the following family of values: for each $N \in U$, there is some n-dimensional vector $\lambda^N = (\lambda_i^N)_{i\in N}$ satisfying $\lambda_i^N \in [0,1]$ for each $i \in N$ and $\sum_{i\in N}\lambda_i^N = 1$ such that, for each $(N,v) \in \mathcal{C}$ and each $i \in N$:*

$$f_i(N,v) = v(\{i\}) + \lambda_i^N\left(\overline{v}(N) - \sum_{j\in N} v(\{j\})\right).$$

Hence, the resulting values induce an exogenously weighted division of the surplus. Obviously, imposing Equal treatment leads, for each $N \in U$ and each $i \in N$, to $\lambda_i^N = 1/n$. Such values are similar to the Weighted Surplus Division values studied in Béal et al. (2015b). □

Another asymmetric example of the aforementioned family is provided below for the demonstration of the logical independence of the axioms in Proposition 7.

- The value that assigns to each game $(N,v) \in \mathcal{C}$ and each player $i \in N$ the payoff $f_i(N,v) = v(\{i\})$ satisfies each axiom except **CE**.
- The value that assigns to each game $(N,v) \in \mathcal{C}$ and each player $i \in N$ the payoff

$$f_i(N,v) = v(\{i\}) + \frac{2i}{n(n+1)}\left(\overline{v}(N) - \sum_{j\in N} v(\{j\})\right)$$

  satisfies each axiom except **ET**.
- The efficient egalitarian Shapley value $EESH$ satisfies each axiom except **IR**.
- The superadditive Shapley value SSh satisfies each axiom except **CA**.
- For each $N \in U$, pick a game (N,w) such that $w(\{i\}) = i$ for each $i \in N$ and $\overline{w}(N) > \sum_{i\in N} w(\{i\}) > w(N) > 0$. Now, consider the value that assigns to each game $(N,v) \in \mathcal{C}$ and each player $i \in N$ the payoff

$$f_i(N,w) = w(\{i\}) + \frac{w(\{i\}) + w(N)}{\sum_{j\in N} w(\{j\}) + nw(N)}\left(\overline{w}(N) - \sum_{j\in N} w(\{j\})\right)$$

  and $f_i(N,v) = EEESD_i(N,v)$ if $(N,v) \neq (N,w)$. This value satisfies each axiom except **IDGC**.

7.2. Consensus values

For a given $\lambda \in [0,1]$, the λ-superadditive consensus value is the value SCV^λ on $\mathcal{C}_N$ defined as $SCV^\lambda = \lambda SSh + (1-\lambda)EEESD$. Equivalently, for each $(N,v) \in \mathcal{C}_N$, $SCV^\lambda(N,v) = CV^\lambda(N,\overline{v})$. A characterization of the family of superadditive consensus values can be obtained by mean of the well-known axiom of Desirability and an extra weak axiom.

Desirability (D) (Maschler and Peleg, 1966) For each $(N,v) \in \mathcal{C}$ and each $i,j \in N$ such that i is at least as desirable as j in (N,v), $f_i(N,v) \geq f_j(N,v)$.

Null player in a zero-normalized cohesive environment (NP0) For each zero-normalized games $(N,v),(N,w) \in \mathcal{C}$ such that $N \in A(N,v) \cap A(N,w)$ and $v(N) = w(N)$, if i is a null player in both (N,v) and (N,w), then $f_i(N,v) = f_i(N,w)$.

This last axiom requires a null player obtains the same payoff in two zero-normalized cohesive games where the worth of the grand coalition is the same.

Proposition 8. *For each $N \in U$, a value on $\mathcal{C}_N$ satisfies Cohesive Efficiency, Active additivity, Invariance to inactive coalitions, Desirability, Individual rationality and Null player in a zero-normalized cohesive environment if and only it is a superadditive consensus value.*

This result will use the fact that Active additivity can be be strengthened into Active Linearity in presence of Cohesive Efficiency and Desirability.

Active linearity (AL) For each $(N,v),(N,w) \in \mathcal{C}$ such that $A(N,v) = A(N,w) = 2^N \backslash \{\varnothing\}$ and each $\alpha \in \mathbb{R}_+$, $f(N,\alpha v + w) = \alpha f(N,v) + f(N,w)$.

Condition $\alpha \geq 0$ ensures that $A(N,\alpha v + w) = 2^N \backslash \{\varnothing\}$. The following Lemma is analog to Lemma 5 in Casajus and Huettner (2013).

Lemma 2. *If a value f on $\mathcal{C}_N$ satisfies Cohesive Efficiency, Active additivity and Desirability, then it satisfies Active Linearity.*

Proof. Since **AA** implies homogeneity for rational scalars, it is enough to show that, for each $(N,v) \in \mathcal{C}$ such that $A(N,v) \in 2^N \backslash \{\varnothing\}$ and each $\alpha \in \mathbb{R}_+ \backslash \mathbb{Q}$, $f(N,\alpha v) = \alpha f(N,v)$. For such a game, using the decomposition (8) and **AA**, we shall only consider games of the form $(N,\alpha u_T)$ for a nonempty $T \in 2^N$, and since the result is trivial when $\alpha = 0$, let us assume that $\alpha > 0$. The rational numbers being dense in the reals, there are sequences $(\alpha_k^-)_{k\in\mathbb{N}}$ and $(\alpha_k^+)_{k\in\mathbb{N}}$ in $\mathbb{Q} \cap \mathbb{R}_+$ such that $0 < \alpha_k^- \leq \alpha \leq \alpha_k^+$ for each $k \in N$ and $\lim_{k\longrightarrow\infty} \alpha_k^- = \lim_{k\longrightarrow\infty} \alpha_k^+ = \alpha$. For each $i \in T$, it is clear that, for each $j \in N\backslash\{i\}$ and each $S \subseteq N\backslash\{i,j\}$,

$$\alpha_k^-\big(u_T(S\cup\{i\}) - u_T(S\cup\{j\})\big) \leq \alpha\big(u_T(S\cup\{i\}) - u_T(S\cup\{j\})\big) \leq \alpha_k^+\big(u_T(S\cup\{i\}) - u_T(S\cup\{j\})\big).$$

As a consequence, for each $k \in N$,

$$\begin{aligned} & f_i(N,\alpha u_T) - f_j(N,\alpha u_T) \\ = \; & f_i(N,\alpha_k^- u_T + (\alpha - \alpha_k^-)u_T) - f_j(N,\alpha_k^- u_T + (\alpha - \alpha_k^-)u_T) \\ = \; & f_i(N,\alpha_k^- u_T) - f_j(N,\alpha_k^- u_T) + f_i(N,(\alpha - \alpha_k^-)u_T) - f_j(N,(\alpha - \alpha_k^-)u_T) \\ \geq \; & f_i(N,\alpha_k^- u_T) - f_j(N,\alpha_k^- u_T), \end{aligned}$$

where the second equality comes from **AA** and the fact that $\alpha - \alpha_k^- \geq 0$, and the inequality comes from **D**. Proceeding in the same fashion, we also obtain that $f_i(N, \alpha u_T) - f_j(N, \alpha u_T) \leq f_i(N, \alpha_k^+ u_T) - f_j(N, \alpha_k^+ u_T)$. Combining these inequalities and using the fact that **AA** implies homogeneity for rational scalars, we get

$$\alpha_k^-(f_i(N, u_T) - f_j(N, u_T)) \leq f_i(N, \alpha u_T) - f_j(N, \alpha u_T) \leq \alpha_k^+(f_i(N, u_T) - f_j(N, u_T)).$$

Taking the limit and by assumption, we thus have

$$f_i(N, \alpha u_T) - f_j(N, \alpha u_T) = \alpha(f_i(N, u_T) - f_j(N, u_T)).$$

Summing over $j \in N$ yields

$$n f_i(N, \alpha u_T) - \sum_{j \in N} f_i(N, \alpha u_T) = n\alpha f_i(N, u_T) - \alpha \sum_{j \in N} f_i(N, u_T).$$

Using **CE**, we conclude that $f_i(N, \alpha u_T) = \alpha f_i(N, u_T)$. ■

Now, we can prove Proposition 8.

Proof. (Proposition 8) Fix any $N \in U$. For any $\lambda \in [0,1]$, SCV^λ satisfies **CE**, **AA**, **IIC** and **D** from Propositions 5 and 7 and Lemma 1. It satisfies **IR** by definition of the superadditive Shapley value and Proposition 7. Regarding **NP0**, consider two zero-normalized games $(N, v), (N, w) \in \mathcal{C}$ such that $N \in A(N, v) \cap A(N, w)$ and $v(N) = w(N)$ and a player i who is null in both (N, v) and (N, w). By Lemma 1, i is also null in $(N, \overline{v})$ so that $SSh_i(N, v) = 0$. Hence, for each $\lambda \in [0,1]$, $SCV_i^\lambda(N, v) = (1-\lambda)ESD(N, \overline{v})$ (recall that $EEESD(N, v)$ is equal to $ESD(N, \overline{v})$). Since (N, v) is zero-normalized, so is $(N, \overline{v})$, which implies that $ESD(N, \overline{v}) = ED(N, \overline{v}) = \overline{v}(N)/n$. Moreover, $N \in A(N, v)$ implies that $\overline{v}(N) = v(N)$. All in all, we get $SCV_i^\lambda(N, v) = (1-\lambda)v(N)/n$, and because $v(N) = w(N)$, we can conclude that $SCV_i^\lambda(N, v) = SCV_i^\lambda(N, w)$.

For the uniqueness part, consider a value f satisfying the six axioms. By Lemma 2, f satisfies **AL**. As in the proof of proposition 5, **IIC** implies that $f(N, v) = f(N, \overline{v})$ and we can use the decomposition (8) and **AL**. Therefore, it is enough to show that, for each nonempty $S \in 2^N$, $f(N, u_S)$ is a λ-superadditive consensus value for some fixed $\lambda \in [0,1]$. Pick such a game (N, u_S). If $S = N$, then $f_i(N, u_N) = 1/n$ for each $i \in N$ by **D** and **CE**. If $S = \{j\}$ for some $j \in N$, then **IR** implies that $f_i(N, u_{\{j\}}) \geq 0$ and $f_j(N, u_{\{j\}}) \geq 1$. Hence, **CE** forces $f_i(N, u_{\{j\}}) = 0$ and $f_j(N, u_{\{j\}}) = 1$. In these first two cases, the resulting payoff allocation corresponds to any convex combination between the Shapley value and the Equal Surplus division value since these values coincide in symmetric and additive games. Let us deal with the more general case where (N, u_S) is such that $1 < s < n$. Since the players in S are equal and the players in $N \backslash S$ are equal, **D** implies that there are real numbers a_S and b_S such that, for each $i \in S$, $f_i(N, u_S) = a_S$ and, for each $j \in N \backslash S$, $f_j(N, u_S) = b_S$. Using **CE**, we obtain that $b_S = (1 - s a_S)/(n - s)$. Now, define

$$\lambda_S = \frac{s(n a_S - 1)}{n - s}$$

and observe that $f(N,u_S) = \lambda_S Sh(N,u_S) + (1-\lambda_S)ESD(N,u_S)$. Remark also that (N,u_S) is zero-normalized whenever $s > 1$ and that a player j is null in two games (N,u_S) and (N,u_T) if $j \in N\backslash(S \cup T)$ (which forces that S and T different from N). An application of **NP0** yields, for each S,T such that $s,t \in \{2,\dots,n-1\}$ and $j \in N\backslash(S\cup T)$, that $f_j(N,u_S) = f_j(N,u_T)$. Since $Sh_j(N,u_S) = Sh_j(N,u_T) = 0$, conclude that $\lambda_S = \lambda_T$. Repeating this step, we get, for each S,T such that $s,t \in \{2,\dots,n-1\}$, that $\lambda_S = \lambda_T$. Denote by λ this quantity. All in all, we have proved that $f(N,u_S) = SCV^\lambda(N,u_S)$ for some $\lambda \in \mathbb{R}$ (including the cases where $s=1$ and $s=n$). It remains to show that $\lambda \in [0,1]$. So consider any game (N,u_S), $1 < s < n$, any $i \in S$ and any $j \in N\backslash S$. Then $f_i(N,u_S) = SCV_i^\lambda(N,u_S) = \lambda/s + (1-\lambda)/n$ if $i \in S$ and $f_j(N,u_S) = SCV_j^\lambda(N,u_S) = \lambda/s + (1-\lambda)/n$ if $j \in N\backslash S$. Firstly, **D** implies that $f_i(N,u_S) \geq f_j(N,u_S)$, which is equivalent to $\lambda \geq 0$. Secondly, **IR** implies that $f_j(N,u_S) \geq 0$, which is equivalent to $\lambda \leq 1$ and completes the proof. ■

The axioms in Proposition 7 are logically independent as demonstrated below.

- The value f that assigns to each game $(N,v) \in \mathcal{C}$ and each player $i \in N$ the payoff $f_i(N,v) = v(\{i\})$ satisfies each axiom except **CE**.
- Consider the game (N,w) such that $w(\{i\}) = i$ for each $i \in N$, $w(N) = n^2$ and $w(S)$=0 otherwise. Note that this game is cohesive but neither superadditive nor symmetric. Now, construct the value f which assigns the payoff $f(N,w) = ED(N,w)$ and $f(N,v) = SCV^\lambda$ for each $(N,v) \in \mathcal{C}\backslash\{(N,w)\}$ for some $\lambda \in [0,1]$. This value satisfies each axiom except **IIC**.
- Consider a vector or real numbers $(a_s)_{s\in\{1,\dots,n\}}$ such that $a_1 = a_n = 1$, $a_s \geq 0$ for each $s \in \{2,\dots,n-1\}$ and $a_s \neq a_t$ for some $s,t \in \{2,\dots,n-1\}$. Now, construct the value f that assigns to each game $(N,v) \in \mathcal{C}$ the payoffs $f(N,v) = SSh(N,av)$, where, for each nonempty $S \in 2^N$, $av(S) = a_s v(S)$. This value satisfies each axiom except **NP0**.
- Any value SCV^λ such that $\lambda < 0$ satisfies each axiom except **D**.
- The value that assigns to each game $(N,v) \in \mathcal{C}$ the payoffs $f(N,v) = ED(N,\overline{v})$ satisfies each axiom except **IR**.
- Consider any superadditive game (N,w) such that $w(N)/n \geq w(\{i\})$ for each $i \in N$. Now, construct the value f that assigns the payoffs $f(N,w) = ED(N,w)$ and $f(N,v) = SCV^\lambda$ for each $(N,v) \in \mathcal{C}\backslash\{(N,w)\}$ for some $\lambda \in [0,1]$. This value satisfies each axiom except **AA**.

7.3. Equal allocation of nonseparable costs

Béal et al. (2016) demonstrate that the equal allocation of nonseparable costs is the unique value on $\mathcal{C}$ satisfying Efficiency and the following axiom of Balanced collective contributions.

Balanced collective contributions (BCC) (Béal et al., 2016) For each $(N,v) \in \mathcal{C}$, each $i,j \in N$,

$$\frac{1}{n-1}\sum_{k\in N\backslash\{i\}}\big(f_k(N,v) - f_k(N\backslash\{i\},v)\big) = \frac{1}{n-1}\sum_{k\in N\backslash\{j\}}\big(f_k(N,v) - f_k(N\backslash\{j\},v)\big).$$

By the same arguments as those in the proof of Proposition 1, it is easy to prove the next result.

Proposition 9. *There exists a unique value on $\mathcal{C}$ that satisfies Cohesive efficiency and Balanced collective contributions. It is the superadditive equal allocation of nonseparable costs $SEANC$, that assigns to each game $(N, v) \in \mathcal{C}$ the payoffs $SEANC(N, v) = EANC(N, \overline{v})$.*

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