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September 2019

Working paper No. 2019 – 06

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Cooperative games on intersection closed systems and the Shapley value

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Abstract

A situation in which a finite set of agents can obtain certain payoffs by cooperation can be described by a cooperative game with transferable utility, or simply a TU-game. In the literature, various models of games with restricted cooperation can be found, in which only certain subsets of the agent set are allowed to form. In this article, we consider such sets of feasible coalitions that are closed under intersection, i.e., for any two feasible coalitions, their intersection is also feasible. Such set systems, called intersection closed systems, are a generalization of the convex geometries. We use the concept of closure operator for intersection closed systems and we define the restricted TU-game taking into account the limited possibilities of cooperation determined by the intersection closed system. Next, we study the properties of this restricted TU-game. Finally, we introduce and axiomatically characterize a family of allocation rules for games TU-games on intersection closed systems, which contains a natural extension of the Shapley value.

Keywords: Cooperative game, linear basis, Intersection closed system, Shapley value.

1. Introduction

In its classical interpretation, a cooperative TU-game describes a situation in which the members of each coalition can cooperate to form a feasible coalition and earn its worth. Cooperative games on set systems are cooperative games in which the agents have restricted cooperation possibilities, which are defined by a set system. The first model in which the restrictions are defined by the connected subgraphs of a graph is due to Myerson (1977). Since then, many other set systems have been used to model restricted cooperation: Algaba et al. (2000, 2001) study cooperative games on union stable systems, Algaba et

*Corresponding author. This research has benefited from the financial support of IDEXLYON from Université de Lyon (project INDEPTH) within the Programme Investissements d'Avenir (ANR-16-IDEX-0005) and "Mathématiques de la décision pour l'ingénierie physique et sociale" (MODMAD).

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al. (2004) consider TU-games on antimatroids, Bilbao (1998) and Bilbao and Edelman (2000) introduce TU-games on convex geometries, Bilbao (2003) studies TU-games on augmenting set systems, and Lange and Grabisch (2009) consider TU-games on regular set systems. More recently, van den Brink et al. (2011) consider TU-games on union closed systems and Algaba et al. (2018) introduce TU-games on accessible union stable systems. All these models are relevant to apprehend many applications as underlined by Moretti and Patrone (2008) and van den Brink (2017), among others.

The purpose of this article is to study TU-games on intersection closed systems. A set system is intersection closed if the intersection of two feasible coalitions is still feasible. Every convex geometry is intersection closed, but an intersection closed system is not necessarily a convex geometry. Intersection closed systems appear naturally in various contexts: in a matching problem, the set of coalitions for which each member also has her partner in the coalition is intersection closed, the set of connected coalitions of nodes in a cycle-free graph is intersection closed, and under the conjunctive approach the so-called set of autonomous coalitions of a permission structure is intersection closed. To the best of our knowledge, Bilbao et al. (2000) is the unique work involving intersection closed systems. They focus on set-valued solution concepts while we investigate (single-valued) allocation rules.

Section 2 is a preliminary section in which we consider a new basis for the set of TU-games. In most cases, the basis of upper TU-games, also called unanimity TU-games, is used to analyze cooperative games on set systems. Here, the lower TU-games prove much more appealing. Section 3 presents TU-games on intersection closed systems. Section 4 introduces the restricted TU-games on intersection closed systems through a closure operator. We determine the image and the kernel of its operator. In particular, we provide a basis for the image of this operator in terms of lower games and a basis for the kernel of this operator in terms of Dirac TU-games. We then provide a formula to compute the coordinates of the restricted TU-games in the basis of lower TU-games. We also derive an expression of the closure operator in the basis of lower TU-games. Section 5 provides an axiomatic study of allocation rules for TU-games on intersection closed systems. We focus on allocation rules that are obtained by applying to the restricted TU-games an allocation rule for classical TU-games. An instance of such allocation rules is the allocation rule defined as the Shapley value (1953) on the restricted TU-game. We call it the Intersection rule. We combine three types of axioms. Some axioms describe the effect of the intersection closed system on the allocation rule. Some axioms describe the impact of the coalition function on the allocation rule. Finally, we invoke an axiom of consistency, which is specific to the targeted allocation rule. The first axiomatic result is a characterization of the Intersection rule that can be considered as close as possible to the classical characterization of the Shapley value. The second axiomatic result characterizes any allocation rule obtained by applying to the restricted TU-games an efficient and additive allocation rule for classical TU-games. As a particular case, we thus obtain a second characterization of the Intersection rule. These results exploit the properties found in section 4.

2. Preliminaries

For each finite set of elements S, the letter s denotes the cardinality of S. For a set S and $i \in S$, we use the notation $S \setminus i$ instead of $S \setminus \{i\}$. In the same way, $S \cup i$ stands for $S \cup \{i\}$. We use the notation $\subseteq$ to denote weak set inclusion and $\subset$ to denote proper set inclusion. For a poset $(P, \leq)$ and $x, y \in P$ such that $x \leq y$, $[x, y]$ denotes the set of elements $z \in P$ such that $x \leq z \leq y$. Given a finite poset, consider a linear extension of P and define the $p \times p$ upper triangular matrix ζ by $\zeta(x, y) = 1$ if $x \leq y$ and 0 otherwise. Since $\zeta(x, x) = 1$ for each $x \in P$, ζ has an inverse, say μ, called the *Möbius function*. In particular, if P is the collection 2^N of all subsets of a finite set N and is endowed with set inclusion, $\mu(S, T) = (-1)^{t-s}$ for $S \subseteq T$. Moreover, if $g : (P, \leq) \longrightarrow \mathbb{R}$ and $f : (P, \leq) \longrightarrow \mathbb{R}$ are two functions on a finite poset $(P, \leq)$ such that $g(x) = \sum_{y \geq x} f(y)$, then $f(x) = \sum_{y \geq x} g(y)\mu(x, y)$. This operation is known as the *Möbius inversion formula*.

A situation in which a finite set of agents can obtain certain payoffs by cooperating can be described by a cooperative game with transferable utility, or simply a *TU-game*, being a pair (N, v), where $N = \{1, \ldots, n\}$ is a finite set of $n \in \mathbb{N}$ agents and $v : 2^N \longrightarrow \mathbb{R}$ is a *coalition function* on N such that $v(\emptyset) = 0$. For any coalition $S \in 2^N$, $v(S)$ is the *worth of coalition* S, i.e., the total payoff $v(S)$ that the members of coalition S can obtain by agreeing to cooperate. Since we take the agent set N to be fixed, we denote the game (N, v) by its coalition function v. We denote the collection of all coalition functions on N by G_N. The *null TU-game* 0_{G_N} is the TU-game such that, for each $S \in 2^N$, $0_{G_N}(S) = 0$. For each coalition $S \in 2^N \setminus \emptyset$, define the *Dirac game* 1_S as $1_S(T) = 1$ if $T = S$, and $1_S(T) = 0$ otherwise. Clearly, the collection $\{1_S : S \in 2^N \setminus \emptyset\}$ of Dirac games forms a basis for G_N. It is often more convenient to use an alternative basis. The most well-known one is the collection $\{u_S : S \in 2^N \setminus \emptyset\}$ of *unanimity games*, where $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise.

In this article, we introduce another basis. To this end, for each $S \in 2^N \setminus \emptyset$, define the *lower game* ℓ_S as $\ell_S(T) = 1$ if $T \subseteq S$ and $T \neq \emptyset$, and $\ell_S(T) = 0$ otherwise. Lower games can be interpreted by a comparison with unanimity games. In a unanimity game u_S, the coalition of players S is productive as soon as all its members cooperate, possibly within larger coalitions. Hence the players in $N \setminus S$ are null. In a lower game ℓ_S, each member of S is productive but only if no other player in $N \setminus S$ cooperates. Thus, the players in $N \setminus S$ are *nullifying*, *i.e.* each coalition containing such a player achieves a zero worth. In van den Brink (2007), nullifying players are the corner stone of an axiomatization of the equal division value that is comparable to the classical characterization of the Shapley value.

Lemma 1 *The collection $\{\ell_S : S \in 2^N \setminus \emptyset\}$ of lower games is a basis for the linear space of all TU-games.*

Proof. It is well-known that G_N can be viewed as $(2^n - 1)$-dimensional linear space. Let $S_1, S_2, \ldots, S_{2^n-1}$ be a fixed sequence of all non-empty coalitions such that $n = s_1 \geq s_2 \geq \ldots \geq s_{2^n-1} = 1$. Further, let $A = [a_{ij}]$ be the $(2^n - 1) \times (2^n - 1)$ matrix defined as $a_{ij} = \ell_{S_i}(S_j)$ for each i and each j taken in $\{1, 2, \ldots, 2^n - 1\}$. By definition of the lower games, it follows that A is an upper triangular matrix such that, for each $i \in N$, $a_{ii} = 1$. Hence, the determinant of A is equal to 1. This implies that the collection

$\{\ell_S : S \in 2^N \setminus \emptyset\}$ constitutes a set of $(2^n - 1)$ independent TU-games in the linear space G_N, and thus a basis for G_N. ■

From Lemma 1, we deduce that for each $v \in G^N$, there exist $2^n - 1$ real numbers ν_S^v, $S \subseteq N$, $S \neq \emptyset$, uniquely determined, such that:

$$v = \sum_{S \in 2^N \setminus \emptyset} \nu_S^v \ell_S \tag{1}$$

The numbers ν_S^v are the coordinates of the game v with respect to the basis of lower games. Thus for each coalition T, we have

$$v(T) = \sum_{S \in 2^N \setminus \emptyset} \nu_S^v \, \ell_S(T) = \sum_{S \supseteq T} \nu_S^v$$

Using the Möbius inversion formula for $(2^N, \supseteq)$ we get:

$$\forall S \in 2^N \setminus \emptyset, \quad \nu_S^v = \sum_{T \supseteq S} (-1)^{t-s} v(T) \tag{2}$$

A *payoff vector* for a game is a vector $x \in \mathbb{R}^n$ assigning a payoff x_i to each agent $i \in N$. An *allocation rule* Φ is a function that assigns to any $v \in G_N$ a unique payoff vector $\Phi(v) \in \mathbb{R}^n$. The most well-known allocation rule for G_N is the Shapley value (Shapley, 1953) denoted by Sh and defined as:

$$\forall i \in N, \quad \mathrm{Sh}_i(v) = \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} \big[v(S \cup i) - v(S)\big].$$

3. Cooperative games on intersection closed systems

A set system is a pair (N, Ω) where N represents a finite set of agents of $\mathbb{N}$, and $\Omega \subseteq 2^N$ is a collection of feasible coalitions. A set system (N, Ω), ordered by set inclusion, is an *intersection closed system* if:

1. $\emptyset \in \Omega$;
2. if $T \in \Omega$ and $S \in \Omega$, then $S \cap T \in \Omega$.

If, moreover, $N \in \Omega$, then (N, Ω) forms a complete lattice of sets where:

$$\forall S \in \Omega, \forall T \in \Omega, \quad \inf\{S, T\} = S \cap T \quad \sup\{S, T\} = \bigcap \Big\{ Q \in \Omega : S \cup T \subseteq Q \Big\}.$$

Throughout this article, we assume that the grand coalition N is feasible so that, for a fixed set of agents N, $\Omega = \{\emptyset, N\}$ is the smallest intersection closed system and $\Omega = 2^N$ is the largest one. Let C_N be the collection of all such intersection closed systems on 2^N. In case the empty set may not belong to a set system, an intersection closed system is sometimes called closure space or closure system, Moore family, intersection ring (of sets), protopology, topped intersection structure, intersection semilattice (see Caspard and Monjardet, 2003).

An example of intersection closed system is pictured in Figure 1.

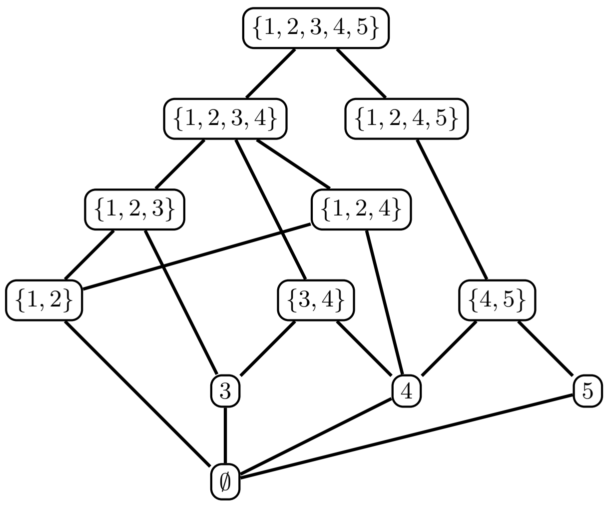

Figure 1.

A well-known instance of an intersection closed system is a *convex geometry* (Edelman, Jamison, 1985). A convex geometry is an intersection closed system which satisfies the following *extension property*:

3. For each $S \in \Omega$ such that $S \neq N$, there exists $i \in N \backslash S$ such that $S \cup i \in \Omega$.

The intersection closed system in Figure 1 is not a convex geometry: $\{3, 4\} \in \Omega$ but there is no $i \in \{1, 2, 5\}$ such that $\{3, 4\} \cup i \in \Omega$. It can be checked that this intersection closed system is neither an antimatroid, nor a union closed system, nor an accessible union stable system, nor an augmenting system and nor a regular system. Below we point out three relevant examples of intersection closed systems that are not convex geometries.

Example 1 (Matchings) Let M and W be two disjoint sets of agents of equal size such that $N = M \cup W$. A *matching* is a bijective map $\mu : M \longrightarrow W$. Given a matching μ, define the set system (N, Ω_μ) as:

$$\Omega_\mu = \Big\{ S \subseteq N : \mu(S \cap M) \subseteq S \text{ and } \mu^{-1}(S \cap W) \subseteq S \Big\}.$$

In words, $S \in \Omega_\mu$ if each agent in $M \cap S$ has his or her partner in S and each agent $W \cap S$ has his or her partner in S. The set system (N, Ω_μ) is intersection closed but does not satisfies the extension property since for each $S \in \Omega_\mu$ and each $i \in N \setminus S$, $S \cup i \notin \Omega_\mu$. So, (N, Ω_μ) is not a convex geometry. Interestingly enough, (N, Ω_μ) is also union closed (i.e. if $S, T \in \Omega_\mu$, then $T \cup S \in \Omega_\mu$) but it does not satisfies the accessibility property saying that if $S \in \Omega_\mu$, then there is $i \in S$ such that $S \setminus i \in \Omega_\mu$. Therefore, (N, Ω_μ) it is not an antimatroid either. $\square$

Example 2 (Forests) An *undirected graph* (or graph) (N, L) is given by a set N of nodes (the agents) and a set L of pairs of nodes, i.e. $L \subseteq \{\{i, j\} : i, j \in N, i \neq j\}$, these pairs being called *edges*. A *path* of (N, L) is a finite sequence $(i_0, i_1, ..., i_p)$ of nodes such that for $0 \le k < p$, $\{i_k, i_{k+1}\} \in L$ and, for $0 < k < p$, $i_{k-1} \neq i_{k+1}$. Two nodes i, j are *connected* in (N, L) if there exists a path (N, L) between i and j, and the graph (N, L) is *connected* if any two nodes are connected. The subgraph of (N, L) induced by a nonempty coalition of nodes $S \in 2^N$ is the graph $(S, L(S))$ with $L(S) = \{\{i, j\} \in L : i, j \in S\}$. A graph is *cycle-free* if there is at most one path connecting any pair of nodes. A coalition of nodes S is *connected* if $(S, L(S))$ is connected. By convention, $\emptyset$ is connected. Let Ω_L denote the set of all connected coalitions in (N, L), which are the relevant coalitions in Myerson (1977)'s model. If (N, L) is a cycle-free graph, then (N, Ω_L) is an intersection closed system, but it is a convex geometry only if (N, L) is also connected. $\square$

Example 3 (Permission structures) A *permission structure* on N is a function $P : N \longrightarrow 2^N$ which is asymmetric, that is, for each pair $i, j \in N$, if $j \in P(i)$, then $i \notin P(j)$. The transitive closure of P is denoted by $\hat{P}$, and the members of $\hat{P}^{-1}(i)$ are called the *superiors* of i. In the *conjunctive* approach developed by Gilles et al. (1992), a coalition of agents can form only if all superiors of each coalition's member also belong to the coalition.[1] Formally, a coalition S is called *autonomous* if $\cup_{i \in S} \hat{P}^{-1}(i) \subseteq S$. It is well-known from Gilles et al. (1992, Proposition 3.2) that the set system (N, Ω_P) where Ω_P is the set of autonomous coalitions of the permission structure P is an intersection closed system (and is also union closed). But, (N, Ω_P) is not always a convex geometry. As an example, set $N = \{1, 2, 3\}$, $P(1) = \{3\}$, $P(2) = \{3\}$ and $P(3) = \emptyset$. Then, $\Omega_P = \{\{1\}, \{2\}, \{1, 2, 3\}\}$ and (N, Ω_P) violates the extension property. $\square$

In a TU-game $v \in G_N$, any subset $S \subseteq N$ is assumed to be able to form a coalition and earn the worth $v(S)$. However, in most applications not every set of participants can form a feasible coalition. Therefore, cooperative game theory models have been developed that take into account restrictions on coalition formation. This can be modeled by considering a set of feasible coalitions $\Omega \subseteq 2^N$ that needs not contain all subsets of the agents set N.

Formally, a *TU-game on an intersection closed system* is a pair (v, Ω) where v is a TU-game in G_N and Ω in C_N is an intersection closed system representing the set of feasible coalitions.

An *allocation rule for TU-games on intersection closed systems* is a function $\Phi : G_N \times C_N \longrightarrow \mathbb{R}^n$ that assigns a payoff vector $\Phi(v, \Omega) \in \mathbb{R}^n$ to every pair $(v, \Omega) \in G_N \times C_N$.

The possible gains from cooperation as modeled by $v \in G_N$ and the restrictions on cooperation reflected by the intersection closed system Ω are incorporated in an Ω-*restricted game* (N, v^Ω). To introduce the Ω-restricted game, we need the following definition. Given $\Omega \in C_N$, define the operator c on 2^N that assigns to each coalition $S \in 2^N$ the coalition $c(S)$ defined as:

$$c(S) = \bigcap \Big\{ Q \in \Omega : S \subseteq Q \Big\}.$$

[1]See also van den Brink (1997) for the disjunctive approach and Faigle and Kern (1992) for a closely related model.

Since Ω is a finite lattice of sets, $c(S)$ is well-defined and corresponds to the smallest feasible coalition that contains S. For each $S \in 2^N$ and each $T \in 2^N$, we can easily verify that the following properties hold:

1. $S \subseteq c(S)$;
2. $S \subseteq T$ implies $c(S) \subseteq c(T)$;
3. $c(c(S)) = c(S)$.

From points 1-3, we conclude that the operator c is a *closure operator*. It follows that a coalition $S \in 2^N$ is feasible if and only if $c(S) = S$. Note also that:

$$\sup\{T, S\} = c(S \cup T).$$

Given a pair $(v, \Omega) \in G_N \times C_N$, we define the Ω*-restricted game* $v^\Omega \in G_N$ of (v, Ω) as:

$$\forall S \in 2^N, \quad v^\Omega(S) = v(c(S)),$$

i.e. v^Ω assigns to each coalition the worth of its closure, so that, for $S \in \Omega$, $v^\Omega(S) = v(S)$ for each coalition $S \in \Omega$.

4. Analysis

Fix an intersection closed system $\Omega \in C_N$. In this section, we address the properties of the mapping

$$L_\Omega : G_N \longrightarrow G_N, \quad v \longmapsto v^\Omega \tag{3}$$

Define the subspaces $G_{N,\Omega}$ and $G_{N,\overline{\Omega}}$ of G_N as:

$$G_{N,\Omega} = \big\{v \in G_N : \forall S \in N, v(S) = v(c(S))\big\} \text{ and } G_{N,\overline{\Omega}} = \big\{v \in G_N : \forall S \in \Omega, v(S) = 0\big\}.$$

Proposition 1 *The mapping L_Ω is the projection of G_N on the subspace $G_{N,\Omega}$ along the direction of $G_{N,\overline{\Omega}}$. Furthermore, the sub-collection of lower TU-games*

$$B_\Omega = \big\{\ell_S : S \in \Omega \setminus \emptyset\big\}$$

forms a basis for $G_{N,\Omega}$, and the sub-collection of Dirac TU-games

$$1_{\overline{\Omega}} = \big\{1_T : T \in 2^N \backslash \Omega\big\}$$

forms a basis for $G_{N,\overline{\Omega}}$.

Proof. First, the mapping L_Ω is a linear mapping such that for each $v \in G_{N,\Omega}$, $L_\Omega(v) = v$, and for each $v \in G_{N,\overline{\Omega}}$, $L_\Omega(v) = 0_{G_N}$. Next, consider any feasible coalition $S \in \Omega \setminus \emptyset$. We have $\ell_S \in G_{N,\Omega}$. Now, consider any Dirac game 1_T where $T \in 2^N \backslash \Omega$. Obviously, we have $1_T \in G_{N,\overline{\Omega}}$. Observe that $G_{N,\Omega} \cap G_{N,\overline{\Omega}} = \{0_{G_N}\}$. It follows that $B_\Omega \cup 1_{\overline{\Omega}}$ is a collection of $(\omega - 1) + (2^n - \omega) = 2^n - 1$ linearly independent elements of G_N (recall that, by Lemma 1, the lower TU-games are linearly independent, and any sub-collection of distinct Dirac TU-games are also linearly independent). Therefore $B_\Omega \cup 1_{\overline{\Omega}}$ is a basis for G_N, such that, for each $v \in B_\Omega$, $L_\Omega(v) = v$, and, for each $v \in 1_{\overline{\Omega}}$, $L_\Omega(v) = 0_G$, showing the result. ■

Given a coalition $S \in 2^N$, let $\max_\Omega S$ be the set of maximal sub-coalitions Q of S belonging to Ω. Note that $\max_\Omega S$ is non-empty since S contains the feasible set $\emptyset$. Given a non-empty collection of coalitions Θ chosen in $\max_\Omega S$, Q_Θ denotes the intersection of coalitions belonging to Θ, and θ stands for the number of elements of Θ. In the following, $\ell_\emptyset$ stands for the null game 0_{G_N}. Proposition 2 expresses each restricted lower TU-game either as a linear combination of lower games or as the supremum of lower games.

Proposition 2 *For each lower TU-game ℓ_S, $S \in 2^N \setminus \emptyset$, it holds that:*

$$\ell_S^\Omega = \sum_{\emptyset \subset \Theta \subseteq \max_\Omega S} (-1)^{\theta-1}\, \ell_{Q_\Theta} = \sup_{Q \in \max_\Omega S} \ell_Q \tag{4}$$

In particular, if $S \in \Omega$, then $\ell_S^\Omega = \ell_S$.

Proof. Pick any coalition $S \in 2^N \setminus \emptyset$ and consider the lower TU-game $\ell_S \in G_N$ and its projection $\ell_S^\Omega \in G_{N,\Omega}$. We first prove the first equality in (4). To this end, consider any coalition $T \in 2^N \setminus \emptyset$. We distinguish two cases:

(a) If $\max_\Omega S = \{\emptyset\}$, then the right-hand side of the first equality in (4) reduces to the null game. Moreover, $c(T)$ is not included in S, which implies that $\ell_S^\Omega(T) = \ell_S(c(T)) = 0$. Thus, both sides of the first equality are equal to zero.

(b) If $\max_\Omega S$ contains at least one non-empty coalition. Two subcases arise.

(b1) If T is not a subset of S, then $c(T)$ is not a subset of S since $c(T) \supseteq T$. On the one hand, by definition of the lower TU-game ℓ_S, we obtain $\ell_S^\Omega(T) = \ell_S(c(T)) = 0$. On the other hand, since $\max_\Omega S$ is a collection of subsets of S, we obtain $\ell_{Q_\Theta}(T) = 0$ for each non empty subset Θ of $\max_\Omega S$. Thus, if $c(T)$ is not a subset of S, then

$$\ell_S^\Omega(T) = \sum_{\emptyset \subset \Theta \subseteq \max_\Omega S} (-1)^{\theta-1}\, \ell_{Q_\Theta}(T) = 0.$$

(b2) If T is a subset of S, then there are two possibilities.

(b2.1) In case $T \subseteq Q$ for some $Q \in \max_\Omega S$, then $c(T) \subseteq Q \subseteq S$ since $c(T)$ is defined as the smallest feasible coalition that contains T. On the one hand, by definition of the lower TU-game ℓ_S, we obtain $\ell_S^\Omega(T) = \ell_S(c(T)) = 1$. On the other hand, for each $Q \in \max_\Omega S$ such that $T \subseteq Q$, we have $\ell_Q(T) = 1$. Consider the set Δ_{TS} formed with coalitions of $\max_\Omega S$ which contain T and so $c(T)$. For each non-empty collection of coalitions $\Theta \subseteq \Delta_{TS}$, we have $\ell_{Q_\Theta}(T) = 1$. In case $(\max_\Omega S) \setminus \Delta_{TS}$ is non-empty, we have $\ell_{Q_{\Theta'}}(T) = 0$ for each non-empty collection $\Theta' \subseteq (\max_\Omega S) \setminus \Delta_{TS}$. Thus,

$$\sum_{\emptyset \subset \Theta \subseteq \max_\Omega S} (-1)^{\theta-1}\, \ell_{Q_\Theta}(T) = \sum_{\emptyset \subset \Theta \subseteq \Delta_{TS}} (-1)^{\theta-1}\, \ell_{Q_\Theta}(T) = \sum_{\emptyset \subset \Theta \subseteq \Delta_{TS}} (-1)^{\theta-1}$$

Keeping in mind that for each non-empty finite set S, $\sum_{T \subseteq S} (-1)^t = 0$, we deduce that $\sum_{\emptyset \subset \Theta \subseteq \Delta_{TS}} (-1)^{\theta-1} = 1$, as desired.

(b2.2) In case there is no $Q \in \max_\Omega S$ such that $T \subseteq Q$, then, for each subset Θ of $\max_\Omega S$, $\ell_{Q_\Theta}(T) = 0$. Moreover, S does not contain $c(T)$ and so $\ell_S^\Omega(T) = \ell_S(c(T)) = 0$, as desired.

In the particular case when $S \in \Omega$, then $\max_\Omega S = \{S\}$ and the right-hand side of the first equality in (4) reduces to ℓ_S.

It remains to prove that $\ell_S^\Omega = \sup_{Q\in\max_\Omega S} \ell_Q$ in equality in (4). It suffices to note that, for each $T \in 2^N \setminus \emptyset$, $c(T) \subseteq S$ if and only if there is $Q \in \max_\Omega S$ such that $Q \supseteq T$. It follows that

$$\ell_S(T) = \sup_{Q\in\max_\Omega S} \ell_Q(T) = 1$$

if and only if there is $Q \in \max_\Omega S$ such that $Q \supseteq T$. ■

Proposition 3 *Let (v, Ω) be a game with restricted cooperation and let $v^\Omega \in G_{N,\Omega}$ be the associated Ω-restricted game given by:*

$$v^\Omega = \sum_{S\in\Omega\setminus\emptyset} \nu_S^{v^\Omega} \ell_S.$$

Then, for each $S \in \Omega \setminus \emptyset$, it holds that:

$$\nu_S^{v^\Omega} = \sum_{T\in\Omega:T\supseteq S} \left[\sum_{H\supseteq S:c(H)=T} (-1)^{h-s} \right] v(T)$$

Proof.

From equality (2), we have:

$$\forall S \in 2^N \setminus \emptyset, \quad \nu_S^{v^\Omega} = \sum_{H\supseteq S} (-1)^{h-s}\, v^\Omega(H).$$

The above expression can be decomposed as follows:

$$\sum_{H\supseteq S} (-1)^{h-s}\, v^\Omega(H) = \sum_{T\supseteq S:T\in\Omega} \left[\sum_{H\supseteq S:c(H)=T} (-1)^{h-s}\, v^\Omega(H) \right].$$

For $H \supseteq S$ such that $c(H) = T$, we have $v^\Omega(H) = v(c(H)) = v(T)$. Thus, we obtain:

$$\nu_S^{v^\Omega} = \sum_{T\supseteq S:T\in\Omega} \left[\sum_{H\supseteq S:c(H)=T} (-1)^{h-s} \right] v(T),$$

showing the result. ■

Corollary 1 *For each Ω-restricted lower game*

$$\ell_S^\Omega = \sum_{T\in\Omega\setminus\emptyset} \nu_T^{l_S^\Omega} \ell_T,$$

we have:

$$\forall T \in \Omega \setminus \emptyset, \quad \nu_T^{\ell_S^\Omega} = \sum_{\{H\supseteq T:\exists Q\in\max_\Omega S,\, Q\supseteq H\}} (-1)^{h-t} \tag{5}$$

Proof. Consider any $S \in 2^N \setminus \emptyset$. First note that the decomposition

$$\ell_S^\Omega = \sum_{T \in \Omega \setminus \emptyset} \nu_T^{\ell_S^\Omega} \ell_T$$

follows from Proposition 1. Next, by Proposition 3, for each $T \in \Omega \setminus \emptyset$, we have:

$$\begin{aligned}
\nu_T^{\ell_S^\Omega} &= \sum_{R \in \Omega : R \supseteq T} \left[\sum_{H \supseteq T : c(H) = R} (-1)^{h-t} \right] \ell_S(R) \\
&= \sum_{R \in \Omega \cap [T,S]} \left[\sum_{H \supseteq T : c(H) = R} (-1)^{h-t} \right] \\
&= \sum_{H \supseteq T} \left[\sum_{R \in \Omega \cap [T,S] : c(H) = R} (-1)^{h-t} \right].
\end{aligned}$$

For each $H \supseteq T$, there is at most one $R \in \Omega \cap [T, S]$ such that $c(H) = R$ and, moreover, such a coalition R exists if and only if there exists $Q \in \max_\Omega S$ such that $Q \supseteq H$. Thus, the results follows. ■

Remark 1 *Note that equation (5) implies that $\nu_T^{\ell_S^\Omega} = 0$ whenever T is not contained in S since, in this case, there is no $H \supseteq T$ contained in some $Q \in \max_\Omega S$. Equation (5) also implies that $\nu_T^{\ell_S^\Omega} = 1$ whenever $T \in \max_\Omega S$.*

Remark 2 *We now have two expressions of the decomposition of ℓ_S^Ω in the basis B_Ω of $G_{N,\Omega}$:*

$$\ell_S^\Omega = \sum_{T \in \Omega, \emptyset \subset T \subseteq S} \; \sum_{\{H \supseteq T : \exists Q \in \max_\Omega S, Q \supseteq H\}} (-1)^{h-t} \ell_T, \text{ and } \ell_S^\Omega = \sum_{\emptyset \subset \Theta \subseteq \max_\Omega S} (-1)^{\theta - 1} \ell_{Q_\Theta}$$

To understand this coincidence, notice that each $T \in \Omega$ can be such that either $T = Q_\Theta$ for different sets $\Theta \subseteq \max_\Omega S$ or for none of them.

The next result provides an expression of the linear operator L^Ω in the basis of lower TU-games.

Theorem 1 *Let (v, Ω) be a game with restricted cooperation in $G_N \times C_N$. The coordinates of v^Ω with respect to the basis of lower games are given by:*

$$\forall S \in \Omega \setminus \emptyset, \quad \nu_S^{v^\Omega} = \sum_{T \in \Omega : T \supseteq S} \nu_S^{l_T^\Omega} \nu_T^v.$$

Proof. The mapping which associates to each coalition function $v \in G_N$ the coordinates $\nu_S^{v^\Omega}$, $S \in \Omega \setminus \emptyset$ is linear. Because the collection $\{\ell_H : H \in 2^N \setminus \emptyset\}$ is a basis for G_N, it suffices to prove the above equality for any lower game ℓ_H. Obviously, we have $\nu_T^{\ell_H} = 1$

for $H = T$, and $\nu_T^{\ell_H} = 0$ otherwise. So, consider any ℓ_H, $H \in 2^N \setminus \emptyset$, and any $S \in \Omega \setminus \emptyset$. Two cases arise:
(a) For $H \supseteq S$, we have:

$$\sum_{T \in \Omega : T \supseteq S} \nu_S^{\ell_T^\Omega} \nu_T^{\ell_H} = \nu_S^{\ell_H^\Omega}.$$

(b) For $H \not\supseteq S$, we have

$$\sum_{T \in \Omega : T \supseteq S} \nu_S^{\ell_T^\Omega} \nu_T^{\ell_H} = 0 = \nu_S^{\ell_H^\Omega},$$

where the last equality comes from (5) and Remark 1. Thus, in both cases, we have the desired result. ■

5. The Intersection rule

There exist two main approaches to define a Shapley for TU-games on set systems. In the first approach, the Shapley value consists in considering equally likely compatible orderings of agents. An ordering is compatible with a set system (N, Ω) if it corresponds to a maximal chain of Ω, that is, an ordered sequence of feasible coalitions $\emptyset \subset \cdots S_t \subset S_{t+1} \cdots \subset N$, where $S_{t+1} \setminus S_t$ contains exactly one element for each t. For each compatible ordering, the agents enter a bargaining room one by one, and upon entering each agent is paid his marginal contribution, and the Shapley value of (v, Ω) is the average, over all such compatible orderings, of the marginal contribution vectors. This is the approach chosen, for instance, by Bilbao (1998) and Bilbao and Edelman (2000), to define and axiomatize a Shapley value on convex geometries. In case the set system does not contain a compatible ordering of agents, which may be the case for intersection closed systems, this approach is not very compelling. That is the reason why the second approach defines the Shapley value for TU-games on a set system as the Shapley value of a restricted TU-game. This approach is followed, for instance, by van den Brink et al. (2011) who define a Shapley value for union closed systems. A set system (N, Ω) is *union closed* if:

1. $\emptyset, N \in \Omega$;
2. if $T \in \Omega$ and $S \in \Omega$, then $S \cup T \in \Omega$.

The *Union rule* for TU-games on union closed systems is defined as:

$$U(v, \Omega) = Sh(v_\Omega).$$

The Ω-restricted game v_Ω is as follows:

$$\forall S \in N, \quad v_\Omega(S) = v(int(S)),$$

where $int(S)$ stands for the *interior* of S defined as the largest feasible coalition contained in S.

In this spirit, we introduce and axiomatize an allocation rule for TU-games on intersection closed systems. This allocation rule applies the Shapley value to the Ω-restricted game. Formally the *Intersection rule*, denoted by I, is defined as:

$$\forall (v, \Omega) \in G_N \times C_N, \quad I(v, \Omega) = (\mathrm{Sh} \circ L_\Omega)(v) = \mathrm{Sh}(v^\Omega) \tag{6}$$

We present several axioms that can be satisfied by allocation rules Φ on $G_N \times C_N$. The first two axioms are straightforward generalizations of the well-known Efficiency and Additivity axioms for G_N.

Efficiency. For each $(v, \Omega) \in G_N \times C_N$, it holds that:

$$\sum_{i \in N} \Phi_i(v, \Omega) = v(N).$$

Additivity. For each $(v, u, \Omega) \in G_N \times G_N \times C_N$, it holds that:

$$\Phi(v, \Omega) + \Phi(u, \Omega) = \Phi(v + u, \Omega),$$

where the TU-game $v + u \in G_N$ is defined as $(v + u)(S) = v(S) + u(S)$ for each $S \subseteq N$.

The next axiom states that in a TU-game on an intersection closed system where the worth of each feasible coalition is null, then agents receive identical payoffs.

Equality. For each $(v, \Omega) \in G_N \times C_N$ such that $v \in G_{N,\bar{\Omega}}$, it holds that:

$$\forall i, j \in N, \quad \Phi_i(v, \Omega) = \Phi_j(v, \Omega).$$

We can prove the following preliminary result, which characterizes the structure of an allocation rule satisfying Efficiency, Additivity and Equality.

Proposition 4 *If an allocation rule Φ on $G_N \times C_N$ satisfies Efficiency, Additivity and Equality, then, for each $(v, \Omega) \in G_N \times C_N$, it holds that $\Phi(v, \Omega) = \Phi(v^\Omega, \Omega)$.*

Proof. Consider any allocation rule Φ satisfying Efficiency, Additivity and Equality. Pick any $(v, \Omega) \in G_N \times C_N$, and consider the TU-game $v - v^\Omega$. We have, $v - v^\Omega \in G_{N,\bar{\Omega}}$ and, in particular, $(v - v^\Omega)(N) = 0$. By Efficiency and Equality, we obtain:

$$\Phi(v - v^\Omega, \Omega) = (0, \dots, 0) \tag{7}$$

On the other hand, since $v = v - v^\Omega + v^\Omega$, by Additivity, we have:

$$\Phi(v, \Omega) = \Phi(v - v^\Omega + v^\Omega, \Omega) = \Phi(v - v^\Omega, \Omega) + \Phi(v^\Omega, \Omega).$$

Applying (7), we get:

$$\Phi(v, \Omega) = \Phi(v^\Omega, \Omega),$$

as desired. ■

In order to single out specific allocation rules, we introduce two axioms relying on TU-games in which the worth of any coalition equals the worth of its closure. In a TU-game $v \in G_N$, two agents i and j are *equal* if, for each $S \subseteq N \setminus \{i, j\}$, $v(S \cup i) = v(S \cup j)$. It is well-known that the Shapley value on G_N treats equally equal agents. The Intersection rule I does not satisfies this axiom on $G_N \times C_N$. However, it satisfies the weaker axiom requiring the same payoff for equal agents in TU-games where the worth of any coalition equals the worth of its closure.

Equal treatment of equal agents for intersection closed systems. For each $(v, \Omega) \in G_N \times C_N$ where $v \in G_{N,\Omega}$, and for each pair of equal agents in v, it holds that:

$$\Phi_i(v, \Omega) = \Phi_j(v, \Omega).$$

An agent i is null in a TU-game $v \in G_N$ if, for each $S \subseteq N\backslash\{i\}$, $v(S \cup i) = v(S)$. Similarly as before, the Shapley value assigns a null payoff to null agents on G_N. This is not the case of the Intersection rule, unless the domain is restricted to TU-games where the worth of any coalition equals the worth of its closure.

Null agent for intersection closed systems. For each $(v, \Omega) \in G_N \times C_N$ where $v \in G_{N,\Omega}$, and for each null agent in v, it holds that:

$$\Phi_i(v, \Omega) = 0.$$

Theorem 2 *The allocation rule I on $G_N \times C_N$ is the unique allocation which satisfies Efficiency, Additivity, Equal treatment of equal agents for intersection closed systems, Null agent for intersection closed systems and Equality.*

Proof. (*Uniqueness*) Consider any allocation rule Φ satisfying Efficiency, Additivity, Equal treatment of equal agents for intersection closed systems, Null agent for intersection closed systems and Equality. We have to show that Φ is uniquely determined on $G_N \times C_N$. Consider any $(v, \Omega) \in G_N \times C_N$. By Proposition 4, we know that $\Phi(v, \Omega) = \Phi(v^\Omega, \Omega)$. By Lemma 1, we know that v^Ω admits the following linear decomposition:

$$v^\Omega = \sum_{S \in \Omega} \nu_S^{v^\Omega} \ell_S,$$

where the coordinates $\nu_S^{v^\Omega}$, $S \in \Omega$, are uniquely determined. Thus, by Additivity,

$$\Phi(v, \Omega) = \Phi(v^\Omega, \Omega) = \sum_{S \in \Omega} \Phi(\nu_S^{v^\Omega} \ell_S, \Omega).$$

It remains to show that, for each $S \in \Omega$, $\Phi(\nu_S^{v^\Omega} \ell_S, \Omega)$ is uniquely determined. By Proposition 2, for each constant $b \in \mathbb{R}$ and each $S \in \Omega$, we know that $b\ell_S \in G_{N,\Omega}$. Now, for $S = N$, since all agents are equal in $b\ell_N$, Efficiency and Equal treatment of equal agents for intersection closed systems yield

$$\forall b \in \mathbb{R}, \forall i \in N, \quad \Phi_i(b\ell_N, \Omega) = b/n. \tag{8}$$

Next, for each $S \in \Omega\backslash N$, define the game $v_S \in G_N$ as

$$v_S = \nu_S^{v^\Omega}(\ell_S - \ell_N),$$

and note that $v_S(N) = -\nu_S^{v^\Omega}$. Since $G_{N,\Omega}$ is a vector space, $v_S \in G_{(N,\Omega)}$. Moreover, it is easy to check that the agents in S are null in v_S and that the agents in $N \setminus S$ are equal in v_S. Thus, an application of Null agent for intersection closed systems yields that $\Phi_i(v_S, \Omega) = 0$ for each $i \in S$. In addition, by Equal treatment of equal agents for intersection closed systems and Efficiency, we obtain that $\Phi_i(v_S, \Omega) = -\nu_S^{v^\Omega}/(n-s)$ for each $i \in N \setminus S$. Hence, by Additivity, we obtain $\Phi(\nu_S^{v^\Omega}\ell_S, \Omega) = \Phi(\nu_S^{v^\Omega}\ell_N, \Omega) + \Phi(v_S, \Omega)$. Using (8), we get $\Phi_i(\nu_S^{v^\Omega}\ell_S, \Omega) = \nu_S^{v^\Omega}/n$ if $i \in S$ and $\Phi_i(\nu_S^{v^\Omega}\ell_S, \Omega) = \nu_S^{v^\Omega}(1/n - 1/(n-s))$ if $i \in N \setminus S$. This completes the proof of uniqueness.

(*Existence*) We show that the Intersection rule satisfies all axioms in the statement of Theorem 2.
Additivity and Efficiency. The allocation rule I inherits these axioms from the Shapley value and the fact that L^Ω is a linear operator such that $L_\Omega(v)(N) = v(c(N)) = v(N)$ for each $(v, \Omega) \in G_N \times C_N$.
Equal treatment of equal agents for intersection closed systems. Consider any situation $(v, \Omega) \in G_N \times C_N$ such that $v \in G_{N,\Omega}$. Therefore, we have:

$$I(v, \Omega) = Sh(v^\Omega) = Sh(v).$$

The fact that I satisfies Equal treatment of equal agents for intersection closed systems follows from the fact that Sh satisfies Equal treatment of equal agents on G_N.
Null agent for intersection closed systems. Consider any situation $(v, \Omega) \in G_N \times C_N$ such that $v \in G_{N,\Omega}$. Therefore, we have:

$$I(v, \Omega) = Sh(v^\Omega) = Sh(v).$$

The fact that I satisfies Null agent for intersection closed systems follows from the fact that Sh satisfies Null agents on G_N.
Equality Consider any situation $(v, \Omega) \in G_N \times C_N$ such that $v \in G_{N,\bar{\Omega}}$. Then, the Ω-restricted game v^Ω coincides with the null game 0_{G_N}. Therefore,

$$I(v, \Omega) = Sh(v^\Omega) = Sh(0_{G_N}) = (0, \ldots, 0),$$

where the last equality follows from the additivity of Sh. We conclude that I satisfies Equality. ■

Logical independence of the axioms.

- The null allocation rule satisfies all axioms except Efficiency.
- The equal division value ED, which assigns to each $(v, \Omega) \in G_N \times C_N$ and each $i \in N$ the payoff $ED_i(v, \Omega) = v(N)/n$ satisfies all axioms except Null agent for intersection closed systems.
- The allocation rule Φ such that $\Phi(v, \Omega) = I(v, \Omega)$ if there is at least one null agent in v and $\Phi(v, \Omega) = ED(v, \Omega)$ otherwise satisfies all axioms except Additivity.

- For a given permutation π on N, denote by S_i^π the agents preceding i in π. The allocation rule Φ which assigns to each $(v, \Omega) \in G_N \times C_N$ and each $i \in N$ the payoff $\Phi_i(v, \Omega) = v^\Omega(S_i^\pi \cup \{i\}) - v^\Omega(S_i^\pi)$ satisfies all axioms except Equal treatment of equal agents for intersection closed systems.
- The allocation rule Φ such that $\Phi(v, \Omega) = Sh(v)$ satisfies all axioms except Equality.

In addition to this first characterization result, which is as close as possible to the classical axiomatization of the Shapley value, it is possible to provide a second characterization of the Intersection rule as a corollary of a more general result. More specifically, the principle behind the Intersection rule can be extended to any allocation rule f for classical TU-games, and not only to the Shapley value. So, for any allocation rule f on G_N, we can construct the allocation rule Φ^f on $G_N \times C_N$ such that

$$\forall (v, \Omega) \in G_N \times C_N, \quad \Phi^f(v, \Omega) = f(v^\Omega).$$

Obviously, $\Phi^{Sh} = I$. Our second axiomatic result characterizes a family of allocation rules Φ^f. As in Theorem 2, the axioms of Efficiency, Additivity and Equality are invoked in addition to two new axioms.

The first axiom also considers a TU-game where the worth of any coalition equals the worth of its closure. As soon as the allocation rule depends on an exogenous structure, it is natural to wonder about structural changes that preserve the payoff allocation. The axiom below belongs to this category. It states that if a modification in the intersection closed system has no impact on the worth of all coalitions, then the resulting allocation should not change.

Invariance from irrelevant changes in intersection closed systems. For any $(v, \Omega, \Omega') \in G_N \times C_N \times C_N$ such that $v \in G_{N,\Omega} \cap G_{N,\Omega'}$, it holds that :

$$\Phi(v, \Omega) = \Phi(v, \Omega').$$

Observe that the case $\Omega \subset \Omega'$ is possible. In this situation, the axiom specifies how an intersection closed system can be augmented without changing the payoff allocation.

The final axiom imposes that if the intersection closed system is complete, then the allocation rule on games on intersection closed systems should coincides with an allocation rule on classical TU-games. The rationale behind this axiom is the aim to extend an allocation rule on classical TU-games to the richer class of games on intersection closed systems.

f-consistency. For a given allocation rule f on G_N, it holds that

$$\forall v \in G_N \quad \Phi(v, 2^N) = f(v).$$

A similar axiom is invoked in Alonso-Meijide et al. (2007, 2014), in which an allocation rule for games with a coalition structure is called Coalitional f-value if it coincides with a value

f for TU-games when the coalition structure trivially contains all singletons. The next result delineates the family of allocation rules Φ^f which are characterized by Efficiency, Additivity, Equality, f-consistency and Invariance from irrelevant changes in intersection closed systems.

Theorem 3 *There is a unique allocation rule Φ on $G_N \times C_N$ satisfying Efficiency, Additivity, Equality, f-consistency and Invariance from irrelevant changes in intersection closed systems if and only if f is an allocation rule on G_N satisfying Efficiency and Additivity on G_N. Moreover, $\Phi = \Phi^f$.*

In particular, we have the following corollary.

Corollary 2 *The Intersection rule I is the unique allocation rule on $G_N \times C_N$ satisfying Efficiency, Additivity, Equality, Sh-consistency and Invariance from irrelevant changes in intersection closed systems.*

Proof. (Theorem 3) Suppose that there is an allocation rule Φ on $G_N \times C_N$ that satisfies Efficiency, Additivity, Equality, f-consistency and Invariance from irrelevant changes in intersection closed systems. We start by showing that f must satisfy Efficiency and Additivity on G_N. For each $v \in G_N$, Efficiency and f-consistency imply

$$\sum_{i\in N} f_i(v) = \sum_{i\in N} \Phi_i(v, 2^N) = v(N),$$

proving that f also satisfies Efficiency on G_N. Similarly, for each $(v, u) \in G_N \times G_N$, Additivity and f-consistency implies that

$$f(v + u) = \Phi(v + u, 2^N) = \Phi(v, 2^N) + \Phi(u, 2^N) = f(v) + f(u),$$

proving that f also satisfies Additivity. Hence, if f violates either Efficiency or Additivity on G_N, then there is no allocation rule Φ on $G_N \times C_N$ that satisfies Efficiency, Additivity, Equality, f-consistency and Invariance from irrelevant changes in intersection closed systems.

Next, let us show that if f satisfies Efficiency and Additivity, then there is a unique allocation rule Φ on $G_N \times C_N$ that satisfies Efficiency, Additivity, Equality, f-consistency and Invariance from irrelevant changes in intersection closed systems.

(*Uniqueness*) As in the proof of Theorem 2, we build on Proposition 4 and Lemma 1 in order to write that

$$\Phi(v, \Omega) = \sum_{S\in\Omega} \Phi(\nu_S^{v^\Omega} \ell_S, \Omega),$$

for each $(v, \Omega) \in G_N \times C_N$. By Additivity, it is enough to prove that $\Phi(\nu_S^{v^\Omega} \ell_S, \Omega)$ is uniquely determined for each $(v, \Omega) \in G_N \times C_N$ and each $S \in \Omega$. Since $\nu_S^{v^\Omega} \ell_S \in G_{N,\Omega}$, for each $T \in 2^N$, it holds that

$$\left(\nu_S^{v^\Omega} \ell_S\right)^{\Omega}(T) = \nu_S^{v^\Omega} \ell_S(T) = \left(\nu_S^{v^\Omega} \ell_S\right)^{2^N}(T),$$

which implies that $\nu_S^{v^\Omega}\ell_S \in G_{N,2^N}$ and thus $\nu_S^{v^\Omega}\ell_S \in G_{N,\Omega} \cap G_{N,2^N}$. By Invariance from irrelevant changes in intersection closed systems and f-consistency, we get

$$\Phi(\nu_S^{v^\Omega}\ell_S, \Omega) = \Phi(\nu_S^{v^\Omega}\ell_S, 2^N) = f(\nu_S^{v^\Omega}\ell_S).$$

Thus, $\Phi(\nu_S^{v^\Omega}\ell_S, \Omega)$ is uniquely determined for each $(v, \Omega) \in G_N \times C_N$ and each $S \in \Omega$. This completes the uniqueness part.

(*Existence*) Consider the allocation rule Φ^f where f is an allocation rule on G_N satisfying Efficiency and Additivity.
Additivity and Efficiency. It is obvious that Φ^f inherits Efficiency and Additivity on $G_N \times C_N$ from the fact that f satisfies these two axioms on G_N.
Equality. Suppose that $v \in G_{N,\bar{\Omega}}$, or equivalently $v^\Omega = 0_{G_N}$. Because f satisfies Additivity, we obtain

$$\forall i \in N, \quad \Phi_i^f(v, \Omega) = f_i(v^\Omega) = f_i(0_{G_N}) = 0,$$

which proves that Φ^f satisfies Equality.
f-consistency. For each $v \in G_N$, it holds that $\Phi^f(v, 2^N) = f(v^{2^N}) = f(v)$.
Invariance from irrelevant changes in intersection closed systems. Suppose that $v \in G_{N,\Omega} \cap G_{N,\Omega'}$ for some $(v, \Omega, \Omega') \in G_N \times C_N \times C_N$. Thus $v^\Omega = v^{\Omega'}$, and so $\Phi^f(v, \Omega) = f(v^\Omega) = f(v^{\Omega'}) = \Phi^f(v, \Omega')$, as desired. ■

Logical independence of the axioms. In what follows, f' is an allocation rule on G_N that satisfies Efficiency and Additivity, and which differs from the allocation rule f appearing in the axiom of f-consistency, which satisfies Efficiency and Additivity too.

- The allocation rule Φ such that, for each $i \in N$,

  $$\Phi_i(v, \Omega) = f(v^\Omega) + \sum_{S \in 2^N \setminus \Omega} \big(v(S) - v^\Omega(S)\big) \tag{9}$$

  satisfies all axioms except Efficiency.

- The allocation rule Φ such that, for each $i \in N$, $\Phi_i(v, \Omega) = f(v^\Omega)$ if $v \in G_{N,\Omega}$ and $\Phi_i(v, \Omega) = f'(v^\Omega)$ otherwise satisfies all axioms Additivity.

- The allocation rule Φ such that $\Phi(v, \Omega) = f(v)$ for each $(v, \Omega) \in G_N \times C_N$ satisfies all axioms except Equality.

- The allocation rule Φ such that $\Phi(v, \Omega) = f'(v^\Omega)$ for each $(v, \Omega) \in G_N \times C_N$ satisfies all axioms except f-consistency.

- From each $(v, \Omega) \in G_N \times C_N$, construct the game $z_{v,\Omega} \in G_N$ such that, for each $S \in 2^N$, $z_{v,\Omega}(S) = v(S) \times |\Omega|/2^n$ if $|S| = 1$ and $z_{v,\Omega(S)} = v(S)$ otherwise. The allocation rule Φ which assigns to each $(v, \Omega) \in G_N \times C_N$ the payoffs $\Phi(v, \Omega) = f((z_{v,\Omega})^\Omega)$ satisfies all axioms except Invariance from irrelevant changes in intersection closed systems.

6. Conclusion

We would like to conclude this article by pointing out that the axiom system invoked in Theorem 3 has the potential to be applicable to other set systems than the intersection closed systems. Similarly, our axiomatic results might be adapted to other specification of the restricted TU-game obtained from an intersection closed system. Finally, it might be interesting to design an allocation rule specifically for our framework, that is, not extending an allocation rule from classical TU-games to TU-games on intersection closed systems. These lines of research are left for future work.

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