Majority properties of positional social preference correspondences

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Abstract

We characterize the positional social preference correspondences (SPC) satisfying the qualified majority property for any given majority threshold. We also characterize the positional SPCs satisfying the minimal majority property. We next evaluate the probability that the Borda, the Plurality and the Antiplurality SPCs fulfil the two aforementioned properties under two assumptions on individuals' preferences in the presence of three and four alternatives for various sizes of the society. Our results show that the Borda SPC is the positional SPC which better behaves in relation with the qualified majority principle and the minimal majority principle. Finally, we propose some remarks on the concept of Condorcet consistency for social choice correspondences.

Keywords: social preference correspondence; social choice correspondence; positional rule; qualified majority; probability; Condorcet consistency.

JEL classification: D71

1 Introduction

Consider a society where $h \geq 2$ individuals have to select some elements in a given set of $n \geq 2$ alternatives or to determine a social preference on that set by means of their individual preferences. Assume that individual and social preferences are required to be linear orders (strict rankings) and call a preference profile any list of h linear orders each of them associated

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with a specific individual in the society. Any correspondence from the set of preference profiles to the set of alternatives [resp. the set of social preferences] represents then a particular decision process which determines a possibly empty set of alternatives [resp. social preferences] whatever preferences individuals express. Such correspondences are called social choice correspondences (SCCs) [resp. social preference correspondences (SPCs)].

Among all the conceivable SCCs and SPCs, the positional ones are frequently used in practical situations. A positional SCC [resp. SPC] is associated with a scoring vector $w = (w_r)_{r=1}^n \in \mathbb{R}^n$ such that $w_1 \geq w_2 \geq \ldots \geq w_n$ and $w_1 > w_n$. Given a preference profile, each time an alternative is ranked r-th by one individual it obtains w_r points; the SCC selects those alternatives realizing the greatest score; the SPC selects those social preferences that are consistent with the total score obtained by alternatives, namely if the total score of an alternative is greater than the one of another alternative then the former alternative has to be socially preferred to the latter one. Note that if a SCC and a SPC are both associated with the same scoring vector then the elements selected by the SCC are the ones that are first ranked by at least one of the linear orders selected by the SPC (see Proposition 20).

The notion of a Condorcet winner has received a considerable amount of attention in the social choice literature due to its intuitive appeal. Given a preference profile, a Condorcet winner is an alternative which is preferred to any other alternative by a simple majority of individuals. As it is well known however, except for special situations, the existence of a Condorcet winner is not guaranteed for all the conceivable preference profiles. Thus, the simple majority principle cannot be used in general to build nonempty valued SCCs. It is then surely an interesting problem to determine sensible weaker variants of the simple majority principle. A possible way to weaken that principle is to consider different level of majority than simple majority. The principle of Condorcet consistency,[1] basically introduced by Ferejonh and Grether (1974) and Greenberg (1979), is based on that idea. Given $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$, a SPC is said to satisfy the Condorcet consistency with respect to the majority threshold ν (briefly ν-Condorcet consistency) if, for every preference profile p, the fact that an alternative x_1 is preferred to another alternative x_2 by at least ν individuals implies that x_2 is not selected. A crucial result by Greenberg (1979) shows that there exists a nonempty valued ν-Condorcet Consistent SCC if and only if $\nu > \frac{n-1}{n}h$. As a consequence, every preference profile admits Condorcet winner if and only if $\lfloor \frac{h}{2} \rfloor + 1 > \frac{n-1}{n}h$, that is, $h = 2$ or $n = 2$ or $(h, n) = (4, 3)$.

The concept of Condorcet consistency can be immediately translated to the framework of SPCs. Given $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$, a SPC is said to satisfies the qualified majority property associated with the majority threshold ν (briefly ν-majority property) if, for every preference profile p, the fact that an alternative x_1 is preferred to another alternative x_2 by at least ν individuals implies that x_1 is ranked over x_2 in any social preference associated with p.

The analysis of Condorcet consistency of SCCs is developed for positional SCCs by Baharad and Nitzan (2003) and Courtin et al. (2015) (see Section 6 for further details). That analysis gives important information on the quality of positional SCCs. Similarly, the analysis of the majority properties of positional SPCs seems to be an interesting issue to address. Unexpectedly, at the best of our knowledge, such an issue has not been considered in the literature, yet. In this paper we fully characterize the positional SPCs which satisfy the ν-majority property (Propositions 5 and 7). As a particular instance, we get that the well-known Borda SPC fulfils the ν-majority property if and only if $\nu > \frac{n-1}{n}h$.

We focus then on a further property for SPCs, called minimal majority property, recently

[1] We use here the terminology used in Baharad and Nitzan (2003) and Courtin et al. (2015).

introduced by Bubboloni and Gori (2014, 2015). A SPC is said to satisfy the minimal majority property if it associates with every preference profile p those social preferences which are consistent with the ν-majority property for all majority thresholds ν which allow to get at least a social preference. We then characterize the positional SPCs which satisfy the minimal majority property (Propositions 9, 10, 12 and 13) showing, in particular, that the Borda SPC fulfils the minimal majority property if and only if $n = 2$ or $h = 2$ or $(h, n) = (4, 3)$ (corresponding to those situations where a Condorcet winner always exists).

The last part of the paper is devoted to the evaluation of the probability that some classic positional SPCs fulfil the two aforementioned properties by using two well-known assumptions on the individuals' preferences often used for such studies. We believe that studying the probability of the agreement between some well-known SPCs and the two considered variants of the majority principle is an important research direction. To the best of our knowledge, our paper can be considered the first to explore such a framework.

The paper is organized as follows. Section 2 describes the basic framework and some preliminary results. Sections 3 and 4 describe our main results regarding the conditions under which a positional SPC satisfies the ν-majority property and the minimal majority property. Section 5 presents our computational analysis related to the probability of some well-known positional SPCs to fulfil the two aforementioned conditions. Section 6 is devoted to some comments on the concept of Condorcet consistency of positional SCCs. Finally, the last section presents our conclusions.

2 Definitions and preliminary results

Let $h, n \in \mathbb{N}$ with $h, n \geq 2$ be fixed throughout the paper, $H = \{1, \ldots, h\}$ be the set of individuals and $N = \{1, \ldots, n\}$ be the set of alternatives. A preference relation on N is a linear order on N, namely a complete, transitive and antisymmetric binary relation on N. The set of linear orders on N is denoted by $\mathcal{L}(N)$. Given $q \in \mathcal{L}(N)$ and $x_1, x_2 \in N$, we usually write $x_1 \succ_q x_2$ instead of $(x_1, x_2) \in q$ and $x_1 \neq x_2$, and we define the rank of x_1 at q as $\mathrm{rank}_q(x_1) = |\{x \in N : x \succ_q x_1\}| + 1$. A preference profile is an element of the set $\mathcal{P} = \mathcal{L}(N)^h$. If $p \in \mathcal{P}$ and $i \in H$, the i-th component of p is denoted by p_i and represents the preferences of individual i. A social preference correspondence (SPC) is a function from $\mathcal{P}$ to the set of the subsets of $\mathcal{L}(N)$. A SPC F is nonempty valued if, for every $p \in \mathcal{P}$, $F(p) \neq \varnothing$. Given two SPCs F_1 and F_2, we say that F_1 is a refinement of F_2 and we write $F_1 \subseteq F_2$ if, for every $p \in \mathcal{P}$, $F_1(p) \subseteq F_2(p)$. We write $F_1 \not\subseteq F_2$ otherwise.

Let $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$ be a majority threshold. Given $p \in \mathcal{P}$ and $x_1, x_2 \in N$, we write $x_1 \succ_\nu^p x_2$ if $|\{i \in H : x_1 \succ_{p_i} x_2\}| \geq \nu$ and $x_1 \not\succ_\nu^p x_2$ otherwise. The ν-majority SPC, denoted by M_ν, is defined, for every $p \in \mathcal{P}$, as

$$M_\nu(p) = \{q \in \mathcal{L}(N) : \forall x_1, x_2 \in N,\ x_1 \succ_\nu^p x_2 \text{ implies } x_1 \succ_q x_2\}.$$

It is known that M_ν is nonempty valued if and only if $\nu > \frac{n-1}{n} h$ (see, for instance, Propositions 6 and 7 in Bubboloni and Gori, 2014). Given a SPC F, we say that F satisfies the ν-majority property if $F \subseteq M_\nu$. As a consequence, if $\nu \leq \frac{n-1}{n} h$ and F is nonempty valued, then $F \not\subseteq M_\nu$. Note also that if $\nu, \nu' \in \mathbb{N} \cap (\frac{h}{2}, h]$ with $\nu \leq \nu'$, then $M_\nu \subseteq M_{\nu'}$. Thus, if F satisfies the ν-majority property, then it satisfies the ν'-majority property as well.

For every $p \in \mathcal{P}$, it is well defined the number

$$\nu(p) = \min\{\nu \in \mathbb{N} \cap (\tfrac{h}{2}, h] : M_\nu(p) \neq \varnothing\},$$

called minimal majority threshold. The minimal majority SPC, denoted by M, is defined, for every $p \in \mathcal{P}$, as $M(p) = M_{\nu(p)}(p)$. Of course, M is nonempty valued and $M \subseteq M_\nu$ for all $\nu > \frac{n-1}{n}h$. Given a SPC F, we say that F satisfies the minimal majority property if $F \subseteq M$.

Consider now the set

$$\mathfrak{W} = \{w = (w_r)_{r=1}^n \in \mathbb{R}^n : w_1 \geq w_2 \ldots \geq w_n, w_1 > w_n\},$$

whose elements are called scoring vectors, and let $\Gamma : \mathfrak{W} \to \mathbb{R}$ and $\gamma : \mathfrak{W} \to \mathbb{R}$ be defined, for every $w \in \mathfrak{W}$, as

$$\Gamma(w) = w_1 - w_n, \quad \gamma(w) = \min_{r \in \{1,\ldots,n-1\}} w_r - w_{r+1}.$$

Fixed $w \in \mathfrak{W}$, the scoring function associated with w is the function $s_w : \mathcal{P} \times N \to \mathbb{R}$ defined, for every $p \in \mathcal{P}$ and $x \in N$, as

$$s_w(p, x) = \sum_{i \in H} w_{\mathrm{rank}_{p_i}(x)};$$

the positional SPC associated with w, denoted by F_w, is defined, for every $p \in \mathcal{P}$, as

$$F_w(p) = \{q \in \mathcal{L}(N) : \forall x_1, x_2 \in N,\ s_w(p, x_1) > s_w(p, x_2) \text{ implies } x_1 \succ_q x_2\}.$$

Consider now $w^b = (w_r^b)_{r=1}^n \in \mathfrak{W}$ defined, for every $r \in \{1, \ldots, n\}$, as $w_r^b = n - r$. The vector w^b is called Borda scoring vector and the corresponding positional SPC is called Borda SPC. Note that $\Gamma(w^b) = n - 1$ and $\gamma(w^b) = 1$.

The next proposition shows the simple argument needed to prove that, for every $w \in \mathfrak{W}$, F_w is nonempty valued.

Proposition 1. *Let $w \in \mathfrak{W}$ and $p \in \mathcal{P}$. Then $F_w(p) \neq \varnothing$.*

Proof. Let $s_w(p, N) = \{s_w(p, x) \in \mathbb{R} : x \in N\}$ and, for every $t \in s_w(p, N)$, $N_w(p, t) = \{x \in N : s_w(p, x) = t\}$. Note that $\{N_w(p, t)\}_{t \in s_w(p,N)}$ is a partition of N. For every $t \in s_w(p, N)$, pick any bijective function $\omega_t : N_w(p, t) \to \{1, \ldots, |N_w(p, t)|\}$. A simple check shows that

$$q^* = \big\{(x, y) \in N^2 : s_w(p, x) - s_w(p, y) > 0\big\} \cup \bigcup_{t \in s_w(p,N)} \big\{(x, y) \in N_w(p, t)^2 : \omega_t(x) \leq \omega_t(y)\big\} \quad (1)$$

is an element of $F_w(p)$. □

In fact, later on in the paper, we need the following more general version of Proposition 1.

Proposition 2. *Let $w \in \mathfrak{W}$ and $p \in \mathcal{P}$. Consider k distinct $x_1, \ldots, x_k \in N$ with $1 \leq k \leq n$ and assume that, for every $i_1, i_2 \in \{1, \ldots, k\}$ with $i_1 < i_2$, $s_w(p, x_{i_1}) \geq s_w(p, x_{i_2})$. Then there exists $q^* \in F_w(p)$ such that, for every $i_1, i_2 \in \{1, \ldots, k\}$ with $i_1 < i_2$, $x_{i_1} \succ_{q^*} x_{i_2}$.*

Proof. We refer to the notation introduced in the proof of Proposition 1. Let $r = |\{s(p, x_i) \in \mathbb{R} : i \in \{1, \ldots, k\}\}|$ and $\{t_j\}_{j=1}^r = \{s(p, x_i) \in \mathbb{R} : i \in \{1, \ldots, k\}\} \subseteq s_w(p, N)$. For every $t \in \{t_j\}_{j=1}^r$, consider the nonempty set $I(t) = \{i \in \{1, \ldots, k\} : s_w(p, x_i) = t\}$ and note that $\varnothing \neq \{x_i\}_{i \in I(t)} \subseteq N_w(p, t)$, so that there exists a bijective function $\omega_t : N_w(p, t) \to \{1, \ldots, |N_w(p, t)|\}$ such that $\omega_t(x_{i_1}) < \omega_t(x_{i_2})$ for all $i_1, i_2 \in I(t)$ with $i_1 < i_2$. For every $t \in s_w(p, N) \setminus \{t_j\}_{j=1}^r$, pick any bijective function $\omega_t : N_w(p, t) \to \{1, \ldots, |N_w(p, t)|\}$. A simple check shows that q^* built as in (1) fulfils the desired properties. □

We complete the section with two simple but useful lemmata.

Lemma 3. *Let $w \in \mathfrak{W}$. Then $\frac{\Gamma(w)}{n-1} \geq \gamma(w)$.*

Proof. Assume by contradiction that $\frac{\Gamma(w)}{n-1} < \gamma(w)$, that is,

$$\frac{w_1 - w_n}{n-1} < \min_{r \in \{1,\ldots,n-1\}} w_r - w_{r+1}.$$

Then, for every $r \in \{1, \ldots, n-1\}$, we have that $\frac{w_1 - w_n}{n-1} < w_r - w_{r+1}$. As a consequence, we get

$$w_1 - w_n = \sum_{r=1}^{n-1} \frac{w_1 - w_n}{n-1} < \sum_{r=1}^{n-1} w_r - w_{r+1} = w_1 - w_n$$

and the contradiction is found. □

Lemma 4. *Let $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$, $w \in \mathfrak{W}$, $p \in \mathcal{P}$ and $x_1, x_2 \in N$. If $x_1 \succ_\nu^p x_2$, then*

$$s_w(p, x_1) - s_w(p, x_2) \geq (\Gamma(w) + \gamma(w))\nu - \Gamma(w)h.$$

Proof. Assume $x_1 \succ_\nu^p x_2$ and define $H_1 = \{i \in H : x_1 \succ_{p_i} x_2\}$ and $H_2 = \{i \in H : x_2 \succ_{p_i} x_1\}$. Note that $H_1 \cap H_2 = \varnothing$, $H_1 \cup H_2 = H$, $|H_1| \geq \nu$ and $|H_2| \leq h - \nu$. Then we have that

$$s_w(p, x_1) - s_w(p, x_2) = \sum_{i \in H} \left(w_{\mathrm{rank}_{p_i}(x_1)} - w_{\mathrm{rank}_{p_i}(x_2)} \right)$$

$$= \sum_{i \in H_1} \left(w_{\mathrm{rank}_{p_i}(x_1)} - w_{\mathrm{rank}_{p_i}(x_2)} \right) - \sum_{i \in H_2} \left(w_{\mathrm{rank}_{p_i}(x_2)} - w_{\mathrm{rank}_{p_i}(x_1)} \right)$$

$$\geq |H_1|\gamma(w) - |H_2|\Gamma(w) \geq \nu\gamma(w) - (h - \nu)\Gamma(w) = \nu(\Gamma(w) + \gamma(w)) - h\Gamma(w)$$

and the proof is complete. □

3 Positional SPCs and the ν-majority property

The next proposition states conditions on the scoring vector w and the majority threshold ν that are sufficient to make the positional SPC associated with w satisfy the ν-majority property.

Proposition 5. *Let $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$ and $w \in \mathfrak{W}$. If $\nu > \frac{\Gamma(w)}{\Gamma(w)+\gamma(w)} h$, then $F_w \subseteq M_\nu$.*

Proof. Let $p \in \mathcal{P}$ and consider $q \in F_w(p)$. We have to show that $q \in M_\nu(p)$. In other words, we have to prove that, for every $x_1, x_2 \in N$, if $x_1 \succ_\nu^p x_2$, then $x_1 \succ_q x_2$. Consider then $x_1, x_2 \in N$ such that $x_1 \succ_\nu^p x_2$. Using Lemma 4 and since $\nu > \frac{\Gamma(w)}{\Gamma(w)+\gamma(w)} h$, we get

$$s_w(p, x_1) - s_w(p, x_2) \geq \nu(\Gamma(w) + \gamma(w)) - h\Gamma(w) > 0.$$

Then, as $q \in F_w(p)$, we deduce $x_1 \succ_q x_2$. □

Observe that if $n = 2$, then, for every $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$ and $w \in \mathfrak{W}$, we have that $F_w \subseteq M_\nu$ since $\frac{\Gamma(w)}{\Gamma(w)+\gamma(w)} = \frac{1}{2}$.

Proposition 5 implies the following interesting result.

Corollary 6. *Let $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$. Then the three following conditions are equivalent:*

(i) M_ν *is nonempty valued;*

(ii) $\nu > \frac{n-1}{n}h$;

(iii) $F_{w^b} \subseteq M_\nu$.

Proof. By Proposition 7 in Bubboloni and Gori (2014) we know that (i) implies (ii). Since

$$\frac{\Gamma(w^b)}{\Gamma(w^b)+\gamma(w^b)}h = \frac{n-1}{n}h, \tag{2}$$

by Proposition 5 we get that (ii) implies (iii). Finally, since Proposition 1 guarantees that F_{w^b} is nonempty valued, (iii) trivially implies (i). □

It is worth noting that Corollary 6 implies that, among all the positional SPCs, the Borda one satisfies the strongest possible version of the ν-majority property or, in other words, it satisfies the ν-majority property for the largest possible set of majority thresholds. Note also that the structure of the proof of Corollary 6 allows to deduce that the inequality $\nu > \frac{n-1}{n}h$ implies that M_ν is nonempty valued by means of some simple properties of the Borda SPC, that is, (2) and Propositions 1 and 5. That provides an alternative proof of Proposition 6 in Bubboloni and Gori (2014) (see also Can and Storcken, 2013, Example 4).

The next proposition allows to completely characterize those positional SPCs satisfying the ν-majority property. Indeed, it shows that the condition on w and ν found in Proposition 5 is also necessary to make the positional SPC associated with w fulfil the ν-majority property.

Proposition 7. *Let $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$ and $w \in \mathfrak{W}$. If $F_w \subseteq M_\nu$, then $\nu > \frac{\Gamma(w)}{\Gamma(w)+\gamma(w)}h$.*

Proof. Assume by contradiction $\nu \leq \frac{\Gamma(w)}{\Gamma(w)+\gamma(w)}h$. Consider $\hat{r} \in \{1, \ldots, n-1\}$ such that $w_{\hat{r}} - w_{\hat{r}+1} = \gamma(w)$; $q_\alpha \in \mathcal{L}(N)$ such that, for every $x, y \in N$, $x < y$ (as numbers) implies $x \succ_{q_\alpha} y$; $q_\beta \in \mathcal{L}(N)$ such that $\mathrm{rank}_{q_\beta}(\hat{r}) = n$ and $\mathrm{rank}_{q_\beta}(\hat{r}+1) = 1$; $p \in \mathcal{P}$ such that $p_i = q_\alpha$ for all $i \in \{1, \ldots, \nu\}$, and $p_i = q_\beta$ for all $i \in \{\nu+1, \ldots, h\}$.[2] Since, for every $x, y \in N$ with $x < y$, we have that $x \succ_\nu^p y$, then $M_\nu(p) = \{q_\alpha\}$. Note that

$$\begin{aligned} s_w(p, \hat{r}) - s_w(p, \hat{r}+1) &= \nu(w_{\hat{r}} - w_{\hat{r}+1}) - (h-\nu)(w_1 - w_n) \\ &= \nu\gamma(w) - (h-\nu)\Gamma(w) = (\Gamma(w)+\gamma(w))\nu - \Gamma(w)h. \end{aligned}$$

Since $\nu \leq \frac{\Gamma(w)}{\Gamma(w)+\gamma(w)}h$, we get $s_w(p, \hat{r}) - s_w(p, \hat{r}+1) \leq 0$. By Proposition 2 there exists $q^* \in F_w(p)$ such that $\hat{r}+1 \succ_{q^*} \hat{r}$. Then $q^* \neq q_\alpha$ so that $F_w \not\subseteq M_\nu$, a contradiction. □

Corollary 8 below shows that if two components of a scoring vector are equal, then the corresponding SPC does not fulfil any type of qualified majority property. That is the case, for instance, of the Plurality SPC, defined by the scoring vector $w = (1, 0, \ldots, 0)$, and of the Antiplurality SPC, defined by $w = (1, \ldots, 1, 0)$.

Corollary 8. *Let $w \in \mathfrak{W}$ be such that $\gamma(w) = 0$. Then, for every $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$, $F_w \not\subseteq M_\nu$.*

Proof. Assume by contradiction that there exists $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$ such that $F_w \subseteq M_\nu$. Then, by Proposition 7, we get $\nu > \frac{\Gamma(w)}{\Gamma(w)+\gamma(w)}h = h$, a contradiction. □

[2]If $\nu = h$, then we set $\{\nu+1, \ldots, h\} = \varnothing$.

4 Positional SPCs and the minimal majority property

The next propositions fully describe which positional SPCs fulfil the minimal majority property. In particular, they imply that the Borda SPC satisfies the minimal majority property if and only if $n = 2$ or $h = 2$ or $(h, n) = (4, 3)$. Recall that, as described in the introduction, those conditions on h and n are equivalent to say that the Condorcet winner exists for every preference profile. Moreover, if the Borda SPC does not satisfy the minimal majority property, then no positional SPC does.

Proposition 9. *Let $n = 2$ and let $\mathcal{L}(N) = \{q_\alpha, q_\beta\}$, where $1 \succ_{q_\alpha} 2$ and $2 \succ_{q_\beta} 1$. Then, for every $w \in \mathfrak{W}$,*

$$F_w(p) = M(p) = \begin{cases} \{q_\alpha\} & \text{if } |\{i \in H : 1 \succ_{p_i} 2\}| > \frac{h}{2}, \\ \mathcal{L}(N) & \text{if } |\{i \in H : 1 \succ_{p_i} 2\}| = \frac{h}{2}, \\ \{q_\beta\} & \text{if } |\{i \in H : 1 \succ_{p_i} 2\}| < \frac{h}{2}. \end{cases}$$

Proof. Straightforward. □

Proposition 10. *Let $h = 2$. Then, for every $w \in \mathfrak{W}$, the two following conditions are equivalent:*

(i) $F_w \subseteq M$;

(ii) $w_1 > w_2 > \ldots > w_n$.

Proof. First of all note that, since $h = 2$, $\mathbb{N} \cap (\frac{h}{2}, h] = \{2\}$. Moreover, due to Corollary 6, M_2 is nonempty valued. As a consequence, $M = M_2$. Given now $w \in \mathfrak{W}$, we have that *(ii)* is equivalent to $\nu > \frac{\Gamma(w)}{\Gamma(w)+\gamma(w)}h$ when $h = 2$ and $\nu = 2$. Then we conclude using Propositions 5 and 7. □

Lemma 11. *Let $h \geq 3$, $n \geq 3$ and $(h, n) \neq (4, 3)$. Then $\lceil \frac{h+1}{2} \rceil \leq \frac{n-1}{n}h$.*

Proof. Assume first h odd so that $\lceil \frac{h+1}{2} \rceil = \frac{h+1}{2}$. Then we have to show that $\frac{h+1}{2} \leq \frac{n-1}{n}h$, that is, $h \geq \frac{n}{n-2}$. Since $h \geq 3$ and $3 \geq \frac{n}{n-2}$ for all $n \geq 3$, $h \geq \frac{n}{n-2}$ follows. Assume now h even so that $\lceil \frac{h+1}{2} \rceil = \frac{h+2}{2}$. Then we have to show that $\frac{h+2}{2} \leq \frac{n-1}{n}h$, that is, $h \geq \frac{2n}{n-2}$. If $n = 3$, then $h \geq 6$ and $6 = \frac{2n}{n-2}$ which imply $h \geq \frac{2n}{n-2}$. If $n \geq 4$, then $h \geq 4$ and $4 \geq \frac{2n}{n-2}$ which imply $h \geq \frac{2n}{n-2}$. □

Proposition 12. *Let $h \geq 3$, $n \geq 3$ and $(h, n) \neq (4, 3)$. Then, for every $w \in \mathfrak{W}$, $F_w \not\subseteq M$.*

Proof. Fix $w \in \mathfrak{W}$ and define $\nu_0 = \lceil \frac{h+1}{2} \rceil$. Consider $\hat{r} \in \{1, \ldots, n-1\}$ such that $w_{\hat{r}} - w_{\hat{r}+1} = \gamma(w)$; $q_\alpha \in \mathcal{L}(N)$ such that, for every $x, y \in N$, if $x < y$ (as numbers), then $x \succ_{q_\alpha} y$; $q_\beta \in \mathcal{L}(N)$ such that $\mathrm{rank}_{q_\beta}(\hat{r}) = n$ and $\mathrm{rank}_{q_\beta}(\hat{r}+1) = 1$; $p \in \mathcal{P}$ such that $p_i = q_\alpha$ for all $i \in \{1, \ldots, \nu_0\}$, and $p_i = q_\beta$ for all $i \in \{\nu_0 + 1, \ldots, h\}$ (note that $\nu_0 < h$). Since, for every $x, y \in N$ with $x < y$, we have that $x \succ^p_{\nu_0} y$, then $M_{\nu_0}(p) = \{q_\alpha\}$. As a consequence, $\nu(p) = \nu_0$ and $M(p) = M_{\nu_0}(p)$. Note now that

$$s_w(p, \hat{r}) - s_w(p, \hat{r}+1) = \nu_0(w_{\hat{r}} - w_{\hat{r}+1}) - (h - \nu_0)(w_1 - w_n) = \nu_0\gamma(w) - (h - \nu_0)\Gamma(w).$$

By Lemma 11, we have that $\nu_0 \leq \frac{n-1}{n}h$, that is, $\frac{h-\nu_0}{\nu_0} \geq \frac{1}{n-1}$. Using now Lemma 3, we get

$$\Gamma(w)\frac{h-\nu_0}{\nu_0} \geq \frac{\Gamma(w)}{n-1} \geq \gamma(w),$$

which implies $\nu_0\gamma(w) - (h-\nu_0)\Gamma(w) \leq 0$. Thus, $s_w(p,\hat{r}) \leq s_w(p,\hat{r}+1)$ and Proposition 2 implies the existence of $q^* \in F_w(p)$ such that $\hat{r}+1 \succ_{q^*} \hat{r}$. Then $q^* \neq q_\alpha$ and $F_w(p) \not\subseteq M(p)$. Thus $F_w \not\subseteq M$. □

Proposition 13. *Let* $(h,n) = (4,3)$. *Then, for every* $w \in \mathfrak{W}$, *the two following conditions are equivalent:*

(i) $F_w \subseteq M$;

(ii) $3\min\{w_1 - w_2, w_2 - w_3\} > w_1 - w_3$.

Proof. First of all note that $\mathbb{N} \cap (\frac{h}{2}, h] = \{3,4\}$. Moreover, due to Corollary 6, M_3 is nonempty valued. As a consequence, $M = M_3$. Given now $w \in \mathfrak{W}$, we have that (ii) is equivalent to $\nu > \frac{\Gamma(w)}{\Gamma(w)+\gamma(w)}h$ when $h=4$ and $\nu = 3$. Then we conclude using Propositions 5 and 7. □

5 Computational results

From the previous results we understand that the Borda SPC is the scoring SPC which better behaves in relation with the qualified majority principle and the minimal majority principle. For instance, while Corollary 6 shows that the Borda SPC is a refinement of M_ν as soon as $\nu > \frac{n-1}{n}h$, there is no possible threshold ν for which this is true under other commonly studied SPCs in the literature. In the next sections, our purpose is to measure at which extend the Borda SPC performs better than those SPCs . Our attention will be restricted to the Plurality SPC defined by the scoring vector $w = (1,0,\ldots,0)$ and the Antiplurality SPC defined by $w = (1,\ldots,1,0)$. Our probabilistic results are based on the two major assumptions on the distribution of individual preferences. The first one is called the Impartial Culture (IC) condition which was introduced for the first time in the social choice literature by Guilbaud (1952); it assumes in our framework of linear orderings that each individual independently chooses, with equal likelihood, one of the linear orderings in $\mathcal{L}(N)$. Therefore, each preference profile $p \in \mathcal{L}(N)^h$ occurs with probability $1/(n!)^h$. The second assumption is called the Impartial Anonymous Culture (IAC) condition and it was introduced for the first time in the literature of social choice theory by Gehrlein and Fishburn (1976) and Kuga and Nagatani (1974); it assumes in our framework that every voting situation, a vector giving the numbers of individuals who each have a specific linear ordering, occurs with the same probability. Hence, each voting situation $\tilde{h} = (h_1, h_2, \ldots, h_{n!})$, where h_j represents the number of individuals who have the j^{th} linear ordering, occurs with the probability $1/\binom{h+n!-1}{n!-1}$. For more details on those and other probabilistic assumptions and their use in social choice theory, the reader may refer to Gehrlein and Lepelley (2011, 2017) and Diss and Merlin (2020).

As mentioned in the introduction, there has been a significant interest over the past three decades in developing representations for the probability of interesting voting events under the two aforementioned assumptions. The most significant part of the research on these probabilities make use of analytical and geometrical techniques in order to obtain exact theoretical probabilities of the studied voting events in the case of three-alternative elections and more recently in the case of four-alternative elections (see, for instance, Brandt et al., 2020a, 2020b; Bruns and Söger, 2015; Bruns et al., 2019; Bubboloni et al., 2020; Diss el al., 2020; vEl Ouafdi et al., 2020a, 2020b; Kamwa and Merlin, 2019). However, it turns out that in our framework the implementation of those techniques are difficult to manage. This is due to the fact that studying the probability that a given SPC is a refinement of M_ν (or M) leads to several cases and sub-cases which further complicate the calculations even in three-alternative elections. The codes that we use, and which are described below, give an overview on the complexity of obtaining exact results even in three-alternative elections. This is the reason why we resort to computer (Monte-Carlo) simulations. This also explains why we restrict our attention in this paper to three-alternative and four-alternative elections. We believe however that the considered values of n, h, and ν should give us enough information regarding the probability of the inclusion of the outcome of the studied positional SPCs in M_ν or M and how the Borda SPC behaves in comparison to the other SPCs. As mentioned in the introduction, to the best of our knowledge, the type of probability calculations we consider in this paper are new in the literature of social choice theory.

5.1 Some positional SPCs and the ν-majority property

Courtin et al. (2015) study the probability that six positional SCCs (Borda, Plurality, Antiplurality, Nanson, Coombs, and Hare)[3] satisfy the Condorcet consistency with respect to the majority threshold ν under the IAC model. Thus, the analysis that we conduct in this section can be seen as an extension of this paper from the framework of SCCs to SPCs. Our first concern is the probability of the inclusion of the outcome given by the studied positional SPCs in M_ν. Before presenting our results, we describe hereafter the methodology applied in calculating our estimated probabilities. We take the Borda SPC (denoted by F_{w^b}) as an example but the steps work similarly for the two other considered SPCs.

Step 1. Fix the number of individuals h, the number of alternatives n and the value of $\nu \in (\frac{h}{2}, h]$. List then all the possible elements of $\mathcal{L}(N)$, say $q_1, q_2, \ldots, q_{n!}$.

Step 2. Under the IC (resp. IAC), randomly generate a preference profile p (resp. anonymous preference profile p).

Step 3. Compute the two binary relations

$$R_{w^b} = \{(x,y) \in N^2 : s_{w^b}(p,x) > s_{w^b}(p,y)\} \quad \text{and} \quad R_\nu = \{(x,y) \in N^2 : x \succ_\nu^p y\}.$$

Step 4. Check the truth of the following statement:

$$\forall i \in \{1, \ldots n!\}, \text{ if } R_{w^b} \subseteq q_i \text{ then } R_\nu \subseteq q_i.$$

If the statement is true, then $F_{w^b}(p) \subseteq M_\nu(p)$; if the statement is false, then $F_{w^b}(p) \not\subseteq M_\nu(p)$.

Step 5. Iterate Steps 2, 3 and 4 one million times and count the number of cases for which $F_{w^b}(p) \subseteq M_\nu(p)$. The desired probability is calculated by dividing the previous value on the number of iterations.

[3]The Nanson, Coombs and Hare SCCs are three special cases of sequential positional SCCs based respectively on the scores of Borda, Antiplurality, and Plurality where alternatives are successively removed in a multi-stage process until a winner is found.

Our results are provided in Tables 1 to 12 (see the Appendix).[4] The results should be read following the parity of h. Indeed, differences between our results for odd numbers and even numbers of individuals are expected since all the considered SPCs encounter much more ties between alternatives with even numbers of individuals than with odd numbers. Note that the parity of h has significant consequences on the probability of voting events in general; this effect decreases for larger electorates since the probability that a tie is observed between two or more alternatives approaches zero when h tends to infinity. As noticed before, the considered cases of h and ν should give us enough information for comments. From Table 1, for instance, we can deduce that if there are 9 individuals and 3 alternatives, the Borda SPC selects an outcome consistent with the 5-majority rule with a probability of 0.7017, an outcome consistent with the 6-majority rule with a probability of 0.9922, and (from Corollary 6) an outcome consistent with the (7,8,9)-majority rule with a probability of 1. Recall that, as a consequence of Corollary 8, we know that for all the possible values of ν there is a profile p such that the outcome obtained under the Plurality/Antiplurality SPC is not a subset of $M_\nu(p)$. Then, it is interesting to consider all the values of ν in $(\frac{h}{2}, h]$ as the probability is theoretically less than 1, i.e., we do not consider only values smaller than $\frac{n-1}{n}h$ as it could be the case for the Borda SPC. However, it is found that the probabilities obtained under the three SPCs converge to 1.0000 for a value of ν roughly around $60\% - 65\%$ of h particularly when h increases. Our results also show very rapid convergence under the Borda SPC in comparison to the two other SPCs. Moreover, it is shown that our probabilities approach the value 1.0000 much faster with IC than with IAC.

Most importantly, our results clearly show that, over the entire range for all parameters n, h, and ν, the Borda SPC performs always better than the Plurality SPC and the Antiplurality SPC which both would be expected to behave in the same way under IC and IAC. Indeed, taking into account the small differences in terms of the estimated probabilities of the two SPCs, it is not clear which positional SPC does better than the other. Note finally that the difference between the performance of Borda SPC and the two other SPCs is generally more important under IC than IAC, particularly when h increases.

5.2 Some positional SPCs and the minimal majority property

Our second objective is the probability of the inclusion of the outcome obtained under some positional SPCs in M. We also describe the methodology that we use in estimating our probabilities since it is slightly different than the one that we used in Subsection 5.1. We again take the Borda SPC (denoted by F_{w^b}) as an example.

Step 1. Fix the number of individuals h and the number of alternatives n. List then all the possible elements of $\mathcal{L}(N)$, say $q_1, q_2, \ldots, q_{n!}$.

Step 2. Under the IC (resp. IAC), randomly generate a preference profile p (resp. anonymous preference profile p).

Step 3. Compute the binary relation $R_{w^b} = \{(x, y) \in N^2 : s_{w^b}(p, x) > s_{w^b}(p, y)\}$.

Step 4. Set $\nu = \lfloor \frac{h}{2} \rfloor + 1$ (that is, the smallest integer value in $(\frac{h}{2}, h]$) and operate as follows:

[4] Note that in those tables the probability values 1 and 1.0000 have different meanings. The first one corresponds to an exact value while the second one is obtained using our simulation method. Thus, the value 1 referred to a certain property means that one of our theoretical results can be applied and that such a property is then true.

Step 4.1. Compute $R_\nu = \{(x,y) \in N^2 : x \succ_\nu^p y\}$.

Step 4.2. If R_ν does not contain a cycle, then define $R = R_\nu$ and go to Step 5;

Step 4.3. If R_ν contains a cycle, then set $\nu = \nu + 1$ and go to Step 4.1.

Step 5. Check the truth of the following statement:

$$\forall i \in \{1, \dots n!\}, \text{ if } R_{w^b} \subseteq q_i \text{ then } R \subseteq q_i.$$

If the statement is true, then $F_{w^b}(p) \subseteq M(p)$; if the statement is false, then $F_{w^b}(p) \not\subseteq M(p)$.

Step 6. Iterate Steps 2 to 5 one million times and count the number of cases for which $F_{w^b}(p) \subseteq M(p)$. The desired probability is calculated by dividing the previous value on the number of iterations.

Tables 13 and 14 (see the Appendix) show computed values of the probability that $F_w \subseteq M$ for three-alternative and four-alternative elections under both the IC and IAC assumptions for various values of h and n under the three considered positional SPCs. Some remarks emerge from these tables.

First, our results should again be read following the parity of h which affects our results. When h increases with the same parity we observe that the probability that $F_w \subseteq M$ is generally vanishing when the number of individuals h increases, except some values of h where the probability could increase when h increases.

Second, our results also show very different behaviours of the three considered positional SPCs. The probability that $F_w(p) \subseteq M(p)$ strongly varies for different SPCs with particular emphasis on the superiority of Borda SPC. It clearly exhibits the best behaviour of the SPCs studied, with probabilities of up to 100% for some particular values of h, and with 79% under IAC and 77% under IC with 10^6 individuals and three alternatives. It also seems that the convergence of the probability to its limiting value is much faster for Borda SPC than the two other SPCs.

Third, when the number of alternatives increases from three to four, the relative behaviour of the three considered positional SPCs remains vastly unchanged with probabilities decreasing for the three SPCs. The Plurality and Antiplurality SPCs are quite non robust to the minimal majority property when h and n increases, with only about 9% occurrence probability for $h = 10^6$ and $n = 4$.

6 Some remarks on the Condorcet consistency

The important papers by Ferejonh and Grether (1974) and Greenberg (1979) suggest two different versions of the Condorcet consistency principle for SCCs as well as of the qualified majority principle for SPCs, depending on the nature of the majority threshold. According to Ferejonh and Grether (1974), the majority threshold is to be thought as an element of the set $(\frac{1}{2}, 1]$, interpreted as a percentage of individuals; according to Greenberg (1979), the majority threshold is instead an integer in the set $(\frac{h}{2}, h]$, representing a specific number of individuals. A careful discussion about the different definitions of Condorcet consistency is proposed in the next sections. That analysis allows to correct some errors present in the literature.

6.1 A comparison between two different definitions

A social choice correspondence SCC is a function from $\mathcal{P}$ to the set of subsets of N. A SCC f is nonempty valued if, for every $p \in \mathcal{P}$, $f(p) \neq \varnothing$. Given two SCCs f_1 and f_2 we say that f_1 is a refinement of f_2 and we write $f_1 \subseteq f_2$ if, for every $p \in \mathcal{P}$, $f_1(p) \subseteq f_2(p)$. For every $\alpha \in (\frac{1}{2}, 1]$ and $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$, let c_α and m_ν be SCCs defined, for every $p \in \mathcal{P}$, as

$$c_\alpha(p) = \{x \in N : \forall y \in N,\ |\{i \in H : y \succ_{p_i} x\}| < \alpha h\},$$

$$m_\nu(p) = \{x \in N : \forall y \in N,\ |\{i \in H : y \succ_{p_i} x\}| < \nu\}.$$

Given $\alpha \in (\frac{1}{2}, 1]$, we say that a SCC f is α-Condorcet consistent if $f \subseteq c_\alpha$; given $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$, we say that a SCC f is ν-Condorcet consistent if $f \subseteq m_\nu$. The definition of α-Condorcet consistency is in line with Ferejonh and Grether (1974) and Baharad and Nitzan (2003) and it is the one used by Courtin et al. (2015); the definition of ν-Condorcet consistency, which is the one presented in the introduction, is instead in line with Greenberg (1979). Those definitions are formally different but clearly strictly related. Our purpose is to well understand the link between them.

Proposition 14. *Let $\alpha \in (\frac{1}{2}, 1]$. Then $\lceil \alpha h \rceil \in \mathbb{N} \cap (\frac{h}{2}, h]$ and $c_\alpha = m_{\lceil \alpha h \rceil}$.*

Proof. Of course, $\lceil \alpha h \rceil \in (\frac{h}{2}, h]$. Observe first that if $\beta \in \mathbb{R}$ and $k \in \mathbb{Z}$, then $k < \beta$ if and only if $k < \lceil \beta \rceil$. Indeed, if $k < \beta$, then $k < \lceil \beta \rceil$ since $\beta \leq \lceil \beta \rceil$. If instead $k \geq \beta$, then $k \geq \min\{z \in \mathbb{Z} : z \geq \beta\} = \lceil \beta \rceil$. Considering now $p \in \mathcal{P}$, we get that, for every $x, y \in N$,

$$|\{i \in H : y \succ_{p_i} x\}| < \alpha h \quad \text{if and only if} \quad |\{i \in H : y \succ_{p_i} x\}| < \lceil \alpha h \rceil,$$

so that $c_\alpha(p) = m_{\lceil \alpha h \rceil}(p)$. □

Proposition 15. *Let $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$. Then $\frac{\nu}{h} \in (\frac{1}{2}, 1]$ and $m_\nu = c_{\frac{\nu}{h}}$.*

Proof. Of course, $\frac{\nu}{h} \in (\frac{1}{2}, 1]$. Since $\lceil \frac{\nu}{h} h \rceil = \nu$, we get $m_\nu = c_{\frac{\nu}{h}}$ due to Proposition 14. □

The problem of the existence of nonempty valued and ν-Condorcet consistent SCCs is completely solved by the next proposition which is an immediate consequence of Corollary 3 in Greenberg (1979) and its proof.[5]

Proposition 16. *Let $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$. Then m_ν is nonempty valued if and only if $\nu > \frac{n-1}{n} h$.*

By Propositions 16 and 14, it is possible to clarify under which conditions on α there are nonempty valued and α-Condorcet consistent SCCs.

Proposition 17. *Let $\alpha \in (\frac{1}{2}, 1]$. Then c_α is nonempty valued if and only if $\lceil \alpha h \rceil > \frac{n-1}{n} h$.*

Proof. Simply note that, by Proposition 14, c_α is nonempty valued if and only if $m_{\lceil \alpha h \rceil}$ is nonempty valued and that, by Proposition 16, $m_{\lceil \alpha h \rceil}$ is nonempty valued if and only if $\lceil \alpha h \rceil > \frac{n-1}{n} h$. □

[5] Proposition 16 can also be seen as a consequence of Proposition 6 in Bubboloni and Gori (2014) and the proof of Proposition 7 in Bubboloni and Gori (2014).

We stress that Courtin et al. (2015, p.234) propose a different version of Proposition 17. Indeed, they state that, as a consequence of the results in Ferejohn and Grether (1974), it is known that[6]

$$\text{for every } \alpha \in (\tfrac{1}{2}, 1],\ \alpha > \tfrac{n-1}{n} \text{ implies } c_\alpha \text{ is nonempty valued,} \tag{3}$$

$$\text{for every } \alpha \in (\tfrac{1}{2}, 1],\ c_\alpha \text{ nonempty valued implies } \alpha > \tfrac{n-1}{n}. \tag{4}$$

Unfortunately, while (3) is correct and can be seen as a consequence of Proposition 17, (4) is generally false. Indeed, consider the next lemmata.

Lemma 18. *If $n = 2$ or n divides h, then, for every $\alpha \in (\frac{1}{2}, 1]$, $\lceil \alpha h \rceil > \frac{n-1}{n} h$ implies $\alpha > \frac{n-1}{n}$.*

Proof. Let $\alpha \in (\frac{1}{2}, 1]$ such that $\lceil \alpha h \rceil > \frac{n-1}{n} h$. If $n = 2$, we immediately get $\alpha > \frac{n-1}{n}$. If n divides h, we get $\frac{n-1}{n} h \in \mathbb{Z}$. Then $\alpha h > \frac{n-1}{n} h$ which implies $\alpha > \frac{n-1}{n}$. □

Lemma 19. *If $n \geq 3$ and n does not divide h, then there exists $\alpha^* \in (\frac{1}{2}, 1]$ with $\alpha^* \leq \frac{n-1}{n}$ such that $\lceil \alpha^* h \rceil > \frac{n-1}{n} h$.*

Proof. Set $\alpha^* = \frac{n-1}{n}$ and observe that $\alpha^* \in (\frac{1}{2}, 1]$ since $n \geq 3$. Moreover, $\alpha^* h \notin \mathbb{Z}$ because $n-1$ and n are coprime and, by assumption, n does not divide h. Thus $\lceil \alpha^* h \rceil > \alpha^* h = \frac{n-1}{n} h$. □

Thus, by Lemmata 18 and 19 and Proposition 17, we get that (4) holds true if and only if $n = 2$ or n divides h.

6.2 Positional SCCs and the Condorcet consistency

Given $w \in \mathfrak{W}$, the positional SCC associated with w, denoted by f_w, is defined, for every $p \in \mathcal{P}$, as

$$f_w(p) = \{x \in N : \forall y \in N,\ s_w(p,x) \geq s_w(p,y)\}.$$

The positional SCC f_{w^b} is called the Borda SCC. It is obvious that, for every $w \in \mathfrak{W}$, f_w is nonempty valued. Moreover, the next proposition holds true.

Proposition 20. *Let $w \in \mathfrak{W}$. Then, for every $p \in \mathcal{P}$,*

$$f_w(p) = \{x \in N : \exists q \in F_w(p) \text{ such that } \mathrm{rank}_q(x) = 1\}.$$

Proof. Let $x \in f_w(p)$. Setting $F^1_w(p) = \{x \in N : \exists q \in F_w(p) \text{ such that } \mathrm{rank}_q(x) = 1\}$, we must show that $x \in F^1_w(p)$. Consider then $1 \leq k \leq n$ and distinct $x_1, \ldots, x_k \in N$ such that $x_1 = x$ and $f_w(p) = \{x_1, \ldots, x_k\}$. By Proposition 2, there exists $q \in F_w(p)$ such that, for every $i_1, i_2 \in \{1, \ldots, k\}$ with $i_1 < i_2$, $x_{i_1} \succ_q x_{i_2}$. In particular, $x \succ_q y$ for all $y \in f_w(p) \setminus \{x\}$. Moreover, for every $y \in N \setminus f_w(p)$, we also have $x \succ_q y$ since $s_w(p,y) < s_w(p,x)$. As a consequence, $\mathrm{rank}_q(x) = 1$ which implies $x \in F^1_w(p)$.

Let now $x \in F^1_w(p)$. If by contradiction $x \notin f_w(p)$, then there exists $y \in N$ such that $s_w(p,y) > s_w(x,p)$. Then, for every $q \in F_w(p)$, $y \succ_q x$ so that $\mathrm{rank}_q(x) \geq 2$. Then $x \notin F^1_w(p)$, a contradiction. □

From Propositions 5 and 20, we get the following proposition. Note that the second part of the statement agrees with Proposition 1 in Courtin et al. (2015).

[6] Essentially the same statement can be also found in Baharad and Nitzan (2003, p.688).

Proposition 21. *Let $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$, $\alpha \in (\frac{1}{2}, 1]$ and $w \in \mathfrak{W}$. Then*

(i) $\nu > \frac{\Gamma(w)}{\Gamma(w)+\gamma(w)} h$ implies $f_w \subseteq m_\nu$;

(ii) $\alpha > \frac{\Gamma(w)}{\Gamma(w)+\gamma(w)}$ implies $f_w \subseteq c_\alpha$.

Proof. (i) Let $p \in \mathcal{P}$ and $x \in f_w(p)$. Assume by contradiction that $x \notin m_\nu(p)$. Then there exists $y \in N$ such that $y \succ_\nu^p x$. By Proposition 5 we have $F_w \subseteq M_\nu$. As a consequence, for every $q \in F_w(p)$, we have that $y \succ_q x$ so that $\mathrm{rank}_q(x) \geq 2$. Applying Proposition 20, we finally get the contradiction.

(ii) Since $\alpha > \frac{\Gamma(w)}{\Gamma(w)+\gamma(w)}$, we get $\lceil \alpha h \rceil > \frac{\Gamma(w)}{\Gamma(w)+\gamma(w)} h$. Then, by (i) and Proposition 14, we conclude $f_w \subseteq m_{\lceil \alpha h \rceil} = c_\alpha$. □

We finally observe that Courtin et al. (2015, Theorem 2) propose an interesting result about the Borda (f_{w^b}), the Plurality (f_{pl}), the Nanson (f_{na}), and the Coombs (f_{co}) SCCs. Indeed, they state that,

$$\text{for every } \alpha \in (\tfrac{1}{2}, 1] \text{ and } f \in \{f_{w^b}, f_{pl}, f_{na}, f_{co}\},\ \alpha > \tfrac{n-1}{n} \text{ implies } f \subseteq c_\alpha, \tag{5}$$

$$\text{for every } \alpha \in (\tfrac{1}{2}, 1] \text{ and } f \in \{f_{w^b}, f_{pl}, f_{na}, f_{co}\},\ f \subseteq c_\alpha \text{ implies } \alpha > \tfrac{n-1}{n}. \tag{6}$$

While there is no problem with (5), their proof of (6) is based on (4) and then, due to the previous discussion about (4), it only works if $n = 2$ or n divides h. However, thing can be easily fixed as follows.

Theorem 22 (Courtin et al. 2015, Theorem 2 revised)**.** *Let $\nu \in \mathbb{N} \cap (\frac{h}{2}, h]$, $\alpha \in (\frac{1}{2}, 1]$ and $f \in \{f_{w^b}, f_{pl}, f_{na}, f_{co}\}$. Then*

(i) $f \subseteq m_\nu$ if and only if $\nu > \frac{n-1}{n} h$;

(ii) $f \subseteq c_\alpha$ if and only if $\lceil \alpha h \rceil > \frac{n-1}{n} h$.

Proof. (i) Assume $\nu > \frac{n-1}{n} h$. If $f = f_{w^b}$, then apply Proposition 21 and (2). If $f \in \{f_{pl}, f_{na}, f_{co}\}$, then note that $\frac{\nu}{h} \in (\frac{1}{2}, 1]$ and $\frac{\nu}{h} > \frac{n-1}{n}$. Then, applying (5) and Proposition 15, we get $f \subseteq c_{\frac{\nu}{h}} = m_\nu$. Assume now $\nu \leq \frac{n-1}{n} h$. Since f is nonempty valued, by Proposition 16, we get $f \nsubseteq m_\nu$.

(ii) Assume $\lceil \alpha h \rceil > \frac{n-1}{n} h$. Then, by (i) and Proposition 14, $f \subseteq m_{\lceil \alpha h \rceil} = c_\alpha$. Assume now $\lceil \alpha h \rceil \leq \frac{n-1}{n} h$. Since f is nonempty valued, by Proposition 17, we get $f \nsubseteq c_\alpha$ □

Note that, from Theorem 22(ii) and Lemma 19, we can understand that if $n \geq 3$ and n does not divide h, then (6) is in fact false.

7 Conclusion

In this paper we studied the conditions under which positional SPCs satisfy two variants of the majority principle called the qualified majority property and the minimal majority property. This study allows us to understand the extent to which positional SPCs respect the two properties. It turns out that, among all the positional SPCs, the Borda SPC satisfies the properties for the largest set of pairs (h, n). We also studied by a computational approach the probability that, given a preference profile, the Borda, the Plurality and the Antiplurality SPCs fulfil the two properties restricted to that preference profile, providing a broad comparison of these classic SPC. It comes from our probabilistic results that the Borda SPC performs better comparatively to the Plurality and the Antiplurality SPCs, which both seem to behave very similarly. Our results also give an understanding of how the probabilities change as the parameters h, n and ν vary.

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Appendix

Table 1: The probability that $F_w \subseteq M_\nu$ for $n = 3$ and $h \in [2, 9]$ under IAC

F_w	$h \backslash \nu$	$\lfloor \frac{h}{2} \rfloor + 1$	$\lfloor \frac{h}{2} \rfloor + 2$	$\lfloor \frac{h}{2} \rfloor + 3$	$\lfloor \frac{h}{2} \rfloor + 4$	$\lfloor \frac{h}{2} \rfloor + 5$
Borda Plurality Antiplurality	2	1 0.8226 0.8229				
Borda Plurality Antiplurality	3	0.7805 0.4875 0.4889	1 0.9677 0.9678			
Borda Plurality Antiplurality	4	1 0.6963 0.6983	1 0.9923 0.9923			
Borda Plurality Antiplurality	5	0.7114 0.3189 0.3156	1 0.8968 0.8969	1 0.9971 0.9974		
Borda Plurality Antiplurality	6	0.9673 0.6927 0.6918	1 0.9574 0.9580	1 0.9988 0.9991		
Borda Plurality Antiplurality	7	0.6959 0.3776 0.3751	1 0.8226 0.8218	1 0.9809 0.9803	1 0.9995 0.9995	
Borda Plurality Antilurality	8	0.9306 0.6117 0.6080	1 0.9097 0.9086	1 0.9906 0.9900	1 0.9997 0.9997	
Borda Plurality Antiplurality	9	0.7017 0.4183 0.4173	0.9922 0.7799 0.7782	1 0.9503 0.9483	1 0.9945 0.9947	1 0.9998 0.9998

Table 2: The probability that $F_w \subseteq M_\nu$ for $n = 3$ and $h \in [2, 9]$ under IC

F_w	$h \backslash \nu$	$\lfloor \frac{h}{2} \rfloor + 1$	$\lfloor \frac{h}{2} \rfloor + 2$	$\lfloor \frac{h}{2} \rfloor + 3$	$\lfloor \frac{h}{2} \rfloor + 4$	$\lfloor \frac{h}{2} \rfloor + 5$
Borda Plurality Antiplurality	2	1 0.8203 0.8201				
Borda Plurality Antiplurality	3	0.7840 0.4799 0.4803	1 0.9667 0.9656			
Borda Plurality Antiplurality	4	1 0.6946 0.6964	1 0.9935 0.9939			
Borda Plurality Antiplurality	5	0.7223 0.2974 0.2947	1 0.9103 0.9098	1 0.9988 0.9989		
Borda Plurality Antiplurality	6	0.9784 0.7071 0.7043	1 0.9749 0.9748	1 0.9998 0.9997		
Borda Plurality Antiplurality	7	0.6988 0.3425 0.3426	1 0.8572 0.8586	1 0.9929 0.9932	1 0.9999 0.9999	
Borda Plurality Antiplurality	8	0.9570 0.6205 0.6189	1 0.9497 0.9508	1 0.9983 0.9984	1 1.0000 1.0000	
Borda Plurality Antiplurality	9	0.6909 0.3867 0.3839	0.9975 0.8316 0.8305	1 0.9829 0.9839	1 0.9996 0.9995	1 1.0000 1.0000

Table 3: The probability that $F_w \subseteq M_\nu$ for $n = 3$ and $h = 50/51$

	F_w	$h\backslash\nu$	26	30	33	34	35	40	45	50	51
IAC	Borda		0.8014	0.9833	0.9999	1	1	1	1	1	
	Plurality	50	0.5376	0.7804	0.8951	0.9222	0.9425	0.9917	0.9997	1.0000	
	Antiplurality		0.5370	0.7810	0.8974	0.9227	0.9426	0.9916	0.9997	1.0000	
	Borda		0.7548	0.9717	0.9995	1.0000	1	1	1	1	1
	Plurality	51	0.5003	0.7510	0.8770	0.9073	0.9297	0.9881	0.9995	1.0000	1.0000
	Antiplurality		0.5016	0.7535	0.8777	0.9042	0.9310	0.9879	0.9992	0.9999	1.0000
IC	Borda		0.8276	1.0000	1.0000	1	1	1	1	1	
	Plurality	50	0.4327	0.9538	0.9992	0.9998	1.0000	1.0000	1.0000	1.0000	
	Antiplurality		0.4293	0.9528	0.9991	0.9999	1.0000	1.0000	1.0000	1.0000	
	Borda		0.6998	1.0000	1.0000	1.0000	1	1	1	1	1
	Plurality	51	0.3435	0.9223	0.9977	0.9995	0.9994	1.0000	1.0000	1.0000	1.0000
	Antiplurality		0.3462	0.9240	0.9977	0.9995	0.9999	1.0000	1.0000	1.0000	1.0000

Table 4: The probability that $F_w \subseteq M_\nu$ for $n = 3$ and $h = 100/101$

	F_w	$h\backslash\nu$	51	59	60	66	67	70	80	90	100	101
IAC	Borda		0.7834	0.9802	0.9872	0.9999	1	1	1	1	1	
	Plurality	100	0.5352	0.7752	0.8001	0.9070	0.9190	0.9492	0.9930	0.9997	1.0000	
	Antiplurality		0.5311	0.7756	0.7998	0.9091	0.9203	0.9503	0.9928	0.9996	1.0000	
	Borda		0.7600	0.9742	0.9824	0.9999	1.0000	1	1	1	1	1
	Plurality	101	0.5131	0.7626	0.7864	0.8983	0.9118	0.9445	0.9913	0.9996	1.0000	1.0000
	Antiplurality		0.5121	0.7623	0.7865	0.8985	0.9124	0.9445	0.9915	0.9996	1.0000	1.0000
IC	Borda		0.7997	1.0000	1.0000	1.0000	1	1	1	1	1	
	Plurality	100	0.3583	0.9807	0.9914	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
	Antiplurality		0.3582	0.9813	0.9916	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
	Borda		0.7082	1.0000	1.0000	1.0000	1.0000	1	1	1	1	1
	Plurality	101	0.3015	0.9708	0.9869	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Antiplurality		0.3000	0.9711	0.9868	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 5: The probability that $F_w \subseteq M_\nu$ for $n = 3$ and $h = 1000/1001$

	F_w	$h\backslash\nu$	501	584	600	666	667	700	800	900	1000	1001
IAC	Borda		0.7724	0.9802	0.9906	1.0000	1	1	1	1	1	
	Plurality	1000	0.5279	0.7804	0.8142	0.9241	0.9254	0.9562	0.9938	0.9999	1.0000	
	Antiplurality		0.5255	0.7797	0.8161	0.9222	0.9254	0.9543	0.9939	0.9998	1.0000	
	Borda		0.7693	0.9796	0.9903	1.0000	1.0000	1	1	1	1	1
	Plurality	1001	0.5216	0.7777	0.8133	0.9227	0.9244	0.9553	0.9935	0.9998	1.0000	1.0000
	Antiplurality		0.5249	0.7754	0.8145	0.9215	0.9249	0.9538	0.9933	0.9997	1.0000	1.0000
IC	Borda		0.7558	1.0000	1.0000	1.0000	1	1	1	1	1	
	Plurality	1000	0.2831	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
	Antiplurality		0.2835	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
	Borda		0.7286	1.0000	1.0000	1.0000	1.0000	1	1	1	1	1
	Plurality	1001	0.2670	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Antiplurality		0.2686	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 6: The probability that $F_w \subseteq M_\nu$ for $n = 3$ and $h = 10^6$

	$F_w \backslash \nu$	500,001	593,333	600,000	666,666	700,000	800,000	900,000	10^6
	Borda	0.7712	0.9800	0.9912	1.0000	1	1	1	1
IAC	Plurality	0.5257	0.8056	0.8147	0.9250	0.9557	0.9945	0.9999	1.0000
	Antiplurality	0.5282	0.8037	0.8188	0.9256	0.9569	0.9940	0.9998	1.0000
	Borda	0.7381	1.0000	1.0000	1.0000	1	1	1	1
IC	Plurality	0.2703	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Antiplurality	0.2725	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 7: The probability that $F_w \subseteq M_\nu$ for $n = 4$ and $h \in [2, 9]$ under IAC

F_w	$h \backslash \nu$	$\lfloor \frac{h}{2} \rfloor + 1$	$\lfloor \frac{h}{2} \rfloor + 2$	$\lfloor \frac{h}{2} \rfloor + 3$	$\lfloor \frac{h}{2} \rfloor + 4$	$\lfloor \frac{h}{2} \rfloor + 5$
Borda		1				
Plurality	2	0.4144				
Antiplurality		0.4158				
Borda		0.4984	1			
Plurality	3	0.0000	0.8256			
Antilurality		0.0000	0.8267			
Borda		0.9402	1			
Plurality	4	0.3399	0.9548			
Antilurality		0.3396	0.9534			
Borda		0.4206	1	1		
Plurality	5	0.0000	0.6962	0.9885		
Antilurality		0.0000	0.6951	0.9883		
Borda		0.8666	1	1		
Plurality	6	0.3309	0.8813	0.9971		
Antilurality		0.3299	0.8808	0.9971		
Borda		0.4037	0.9867	1	1	
Plurality	7	0.0576	0.6043	0.9597	0.9991	
Antilurality		0.0570	0.6032	0.9593	0.9992	
Borda		0.8084	0.9994	1	1	
Plurality	8	0.2751	0.8129	0.9864	0.9999	
Antilurality		0.2737	0.8124	0.9866	0.9998	
Borda		0.3987	0.9660	1	1	1
Plurality	9	0.0641	0.5455	0.9207	0.9957	0.9999
Antilurality		0.0630	0.5462	0.9203	0.9955	0.9999

Table 8: The probability that $F_w \subseteq M_\nu$ for $n = 4$ and $h \in [2, 9]$ under IC

F_w	$h\backslash\nu$	$\lfloor\frac{h}{2}\rfloor + 1$	$\lfloor\frac{h}{2}\rfloor + 2$	$\lfloor\frac{h}{2}\rfloor + 3$	$\lfloor\frac{h}{2}\rfloor + 4$	$\lfloor\frac{h}{2}\rfloor + 5$
Borda		1				
Plurality	2	0.4156				
Antiplurality		0.4184				
Borda		0.4989	1			
Plurality	3	0.0000	0.8256			
Antilurality		0.0000	0.8256			
Borda		0.9419	1			
Plurality	4	0.3383	0.9536			
Antilurality		0.3382	0.9548			
Borda		0.4201	1	1		
Plurality	5	0.0000	0.6966	0.9880		
Antilurality		0.0000	0.6956	0.9880		
Borda		0.8646	1	1		
Plurality	6	0.3294	0.8804	0.9970		
Antilurality		0.3302	0.8818	0.9969		
Borda		0.4051	0.9862	1	1	
Plurality	7	0.0576	0.6056	0.9584	0.9992	
Antilurality		0.0585	0.6051	0.9594	0.9992	
Borda		0.8100	0.9993	1	1	
Plurality	8	0.2763	0.8138	0.9868	0.9999	
Antilurality		0.2760	0.8139	0.9861	0.9998	
Borda		0.3980	0.9655	1	1	1
Plurality	9	0.0636	0.5486	0.9228	0.9955	1.0000
Antilurality		0.0626	0.5465	0.9232	0.9956	1.0000

Table 9: The probability that $F_w \subseteq M_\nu$ for $n = 4$ and $h = 50/51$

	F_w	$h\backslash\nu$	26	30	31	32	35	37	38	40	45	50	51
IAC	Borda		0.5484	0.9889	0.9975	0.9995	1.0000	1.0000	1	1	1	1	
	Plurality	50	0.2046	0.7532	0.8477	0.9136	0.9897	0.9984	0.9994	0.9999	1.0000	1.0000	
	Antiplurality		0.2032	0.7556	0.8469	0.9127	0.9895	0.9984	0.9993	0.9999	1.0000	1.0000	
	Borda		0.4392	0.9751	0.9935	0.9987	1.0000	1.0000	1.0000	1	1	1	1
	Plurality	51	0.1496	0.6903	0.7968	0.8795	0.9821	0.9964	0.9986	0.9998	1.0000	1.0000	1.0000
	Antiplurality		0.1527	0.6883	0.7949	0.8779	0.9822	0.9965	0.9990	0.9998	1.0000	1.0000	1.0000
IC	Borda		0.5847	0.9995	0.9999	1.0000	1.0000	1.0000	1	1	1	1	
	Plurality	50	0.2129	0.9192	0.9685	0.9910	0.9998	1.0000	1.0000	1.0000	1.0000	1.0000	
	Antiplurality		0.2160	0.9192	0.9703	0.9904	0.9999	1.0000	1.0000	1.0000	1.0000	1.0000	
	Borda		0.4180	0.9981	0.9998	1.0000	1.0000	1.0000	1.0000	1	1	1	1
	Plurality	51	0.1326	0.8723	0.9473	0.9813	0.9997	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Antiplurality		0.1331	0.8710	0.9476	0.9813	0.9996	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 10: The probability that $F_w \subseteq M_\nu$ for $n = 4$ and $h = 100/101$

	F_w	$h \backslash \nu$	51	60	63	70	75	80	90	100	101
IAC	Borda	100	0.5063	0.9939	0.9997	1.0000	1.0000	1	1	1	
	Plurality		0.1944	0.8083	0.9167	0.9940	0.9995	1.0000	1.0000	1.0000	
	Antiplurality		0.1973	0.8073	0.9155	0.9933	0.9996	1.0000	1.0000	1.0000	
	Borda	101	0.4520	0.9903	0.9993	1.0000	1.0000	1	1	1	1
	Plurality		0.1696	0.7755	0.8992	0.9913	0.9991	1.0000	1.0000	1.0000	1.0000
	Antiplurality		0.1702	0.7777	0.898	0.9918	0.9993	0.9999	1.0000	1.0000	1.0000
IC	Borda	100	0.5402	1.0000	1.0000	1.0000	1.0000	1	1	1	
	Plurality		0.1994	0.9918	0.9995	1.0000	1.0000	1.0000	1.0000	1.0000	
	Antiplurality		0.1989	0.9918	0.9996	1.0000	1.0000	1.0000	1.0000	1.0000	
	Borda	101	0.4256	1.0000	1.0000	1.0000	1.0000	1	1	1	1
	Plurality		0.1434	0.9869	0.9992	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Antiplurality		0.1431	0.9877	0.9991	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 11: The probability that $F_w \subseteq M_\nu$ for $n = 4$ and $h = 1000/1001$

	F_w	$h \backslash \nu$	501	600	625	700	750	800	900	1000	1001
IAC	Borda	1000	0.4698	0.9968	0.9998	1.0000	1.0000	1	1	1	
	Plurality		0.1908	0.8432	0.9241	0.9969	0.9998	1.0000	1.0000	1.0000	
	Antiplurality		0.1899	0.8435	0.9244	0.9962	0.9998	1.0000	1.0000	1.0000	
	Borda	1001	0.4616	0.9959	0.9997	1.0000	1.0000	1	1	1	1
	Plurality		0.1862	0.8425	0.9220	0.9961	0.9998	1.0000	1.0000	1.0000	1.0000
	Antiplurality		0.1875	0.8418	0.9242	0.9959	0.9998	1.0000	1.0000	1.0000	1.0000
IC	Borda	1000	0.4765	1.0000	1.0000	1.0000	1.0000	1	1	1	
	Plurality		0.1173	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
	Antiplurality		0.1173	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	
	Borda	1001	0.4423	1.0000	1.0000	1.0000	1.0000	1	1	1	1
	Plurality		0.1053	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Antiplurality		0.1069	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 12: The probability that $F_w \subseteq M_\nu$ for $n = 4$ and $h = 10^6$

	$F_w \backslash \nu$	500,001	600,000	625,000	700,000	750,000	800,000	900,000	10^6
IAC	Borda	0.4612	0.9969	0.9997	1.0000	1.0000	1	1	1
	Plurality	0.1909	0.8486	0.9256	0.9958	0.9997	1.0000	1.0000	1.0000
	Antiplurality	0.1896	0.8466	0.9245	0.9963	0.9997	0.9999	1.0000	1.0000
IC	Borda	0.4478	1.0000	1.0000	1.0000	1.0000	1	1	1
	Plurality	0.0655	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	Antiplurality	0.0651	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Table 13: The probability that $F_w \subseteq M$ for $n = 3$

h	IAC			IC		
	Borda	Plurality	Antiplurality	Borda	Plurality	Antiplurality
2	1	0.8221	0.8245	1	0.8202	0.8207
3	0.8305	0.5388	0.5393	0.8331	0.5280	0.5274
4	1	0.6962	0.6977	1	0.6951	0.6951
5	0.7758	0.3790	0.3804	0.7831	0.3607	0.3611
6	0.9716	0.6964	0.6968	0.9811	0.7097	0.7081
7	0.7623	0.4287	0.4305	0.7679	0.3967	0.3959
8	0.9394	0.6157	0.6213	0.9627	0.6249	0.6261
9	0.7643	0.4669	0.4667	0.7616	0.4447	0.4443
50	0.8191	0.5575	0.5519	0.8539	0.4487	0.4487
51	0.7826	0.5251	0.5274	0.7534	0.3805	0.3813
100	0.8050	0.5515	0.5501	0.8297	0.3785	0.3762
101	0.7863	0.5327	0.5336	0.7561	0.3330	0.3321
1000	0.7931	0.5435	0.5434	0.7870	0.3032	0.2998
1001	0.7921	0.5433	0.5424	0.7628	0.2898	0.2895
10^6	0.7930	0.5433	0.5430	0.7691	0.2894	0.2929

Table 14: The probability that $F_w \subseteq M$ for $n = 4$

h	IAC			IC		
	Borda	Plurality	Antiplurality	Borda	Plurality	Antiplurality
2	1	0.4169	0.4152	1	0.4171	0.4206
3	0.6664	0.1680	0.1692	0.6651	0.1691	0.1682
4	0.9424	0.3379	0.3394	0.9412	0.3398	0.3385
5	0.6260	0.1606	0.1607	0.6288	0.1621	0.1616
6	0.8812	0.3475	0.3468	0.8802	0.3442	0.3461
7	0.6198	0.1946	0.1949	0.6193	0.1940	0.1940
8	0.8406	0.3025	0.2996	0.8406	0.2994	0.3015
9	0.6108	0.1863	0.1882	0.6116	0.1856	0.1861
50	0.6234	0.2448	0.2449	0.6681	0.2595	0.2608
51	0.5535	0.2087	0.2052	0.5601	0.2023	0.2032
100	0.5827	0.2368	0.2337	0.6309	0.2412	0.2430
101	0.5438	0.2162	0.2143	0.5506	0.2022	0.2008
1000	0.5417	0.2270	0.2284	0.5617	0.1486	0.1496
1001	0.5364	0.2281	0.2263	0.5339	0.1399	0.1406
10^6	0.5340	0.2266	0.2260	0.5331	0.0870	0.0890