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A Core-partition solution for coalitional rankings with a variable population domain

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Abstract

A coalitional ranking problem is described by a weak order on the set of nonempty coalitions of a given agent set. A social ranking is a weak order on the set of agents. We consider social rankings that are consistent with stable/core partitions. A partition is stable if there is no coalition better ranked in the coalitional ranking than the rank of the cell of each of its members in the partition. The core-partition social ranking solution assigns to each coalitional ranking problem the set of social rankings such that there is a core-partition satisfying the following condition: a first agent gets a higer rank than a second agent if and only if the cell to which the first agent belongs is better ranked in the coalitional ranking than the cell to which the second agent belongs in the partition. We provide an axiomatic characterization of the core-partition social ranking and an algorithm to compute the associated social rankings.

Keywords: Coalitional ranking problem, social ranking, core partition, axiomatic characterization, hedonic games.

JEL code: C71

1. Introduction

Consider a social environment where a population of agents have the possibility to form coalitions in order to cooperate. A coalitional ranking, represented by a weak order (a transitive and complete binary relation) over the set of nonempty coalitions, ranks the coalitions according to their power. A pair formed by a finite set of agents and a coalitional ranking is a coalitional ranking problem.

A social ranking, represented by a weak order over the population of agents, determines the ranking of the agents on the basis of the information contained in a coalitional ranking. A social ranking solution on a domain of coalitional ranking problems is defined as a mapping that assigns to each coalitional ranking problem of this domain a set of social rankings over the agent set.

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Thus, a social ranking solution describes the influence of individual and coalitional power in the determination of the position of the agents in a population. Such a social ranking solution can be viewed either as the ordinal counterpart of a solution for cooperative games with transferable utility – where for each payoff vector, agents are ranked according to the payoff they receive for their participation in the game – or as the inverse of the well-known problem of ranking groups of objects from a ranking over the individual objects (see, e.g., Barberà et al., 2004).

Coalitional ranking problems and their social ranking solutions have been recently investigated by Khani et al. (2019), Bernardi et al. (2019) and Algaba et al. (2021). These studies consider single-valued solutions, that is, social ranking solutions that assign to each coalitional ranking of the considered domain a unique social ranking. Khani et al. (2019) introduce and axiomatically characterize a social ranking solution that is inspired from the Banzhaf index for cooperative voting games (Banzhaf, 1964). Bernardi et al. (2019) and Algaba et al. (2021) study lexicographic solutions based on the idea that the most influential agents are those who belong to (small) coalitions ranked in the highest position in the coalitional ranking, and provide several axiomatic characterizations of these solutions.

In this article, we are interested in set-valued solutions, that is, social ranking solutions that assign to each coalitional ranking problem of the domain a set, possible empty, of social rankings. We propose a set-valued social ranking solution based on a no-blocking condition. The action of blocking is the driving force underlying the concept of the core in many social environments. Precisely, our solution is constructed from the idea that the population of agents can be socially organized into a partition. Given a coalitional ranking problem, a partition is blocked by a coalition if the latter has a strictly higher rank in the coalitional ranking than the cells of the partition to which its members belong. A partition that is not blocked by any coalition is a stable partition or a core-partition. A partition induces a natural social ranking: an agent is better ranked than a second agent if and only if the cell to which the first agent belongs is better ranked in the coalitional ranking than the cell to which the second agent belongs. A social ranking is a core-partition social ranking if it is induced by a core-partition. The Core-partition social ranking solution introduced in this article assigns to each coalitional ranking the set of all core-partition social rankings. In other words, each selected social ranking is induced by an organization of the population into a partition that cannot be blocked by any coalition.

We consider the domain of all coalitional ranking problems that can be constructed from any finite agent set, that is, a domain of coalitional ranking problems with a variable set of agents. It turns out that the Core-partition social ranking solution is nonempty-valued on this domain.

Our main results are as follows. The first result provides an axiomatic characterization of the Core-partition social ranking solution which invokes three axioms (Proposition 2).

The first axiom is based on a specific expansion of the population of agents. So consider an initial coalitional ranking problem. Assume now that the population is expanded in such a way that the coalition of all newly added agents wins unanimous support: this coalition is the unique maximal coalition in the larger coalitional ranking, and the relative ranking between any two original coalitions remains the same. Then, our axiom imposes that the solution set of this new coalitional ranking on a larger set of agents is computed from the solution set of the initial one by putting the set of newcomers at the top of each social ranking of this solution set. In a sense, if a set of newcomers wins unanimous support in a coalitional ranking, then they are socially top-ranked. This axiom both has the flavor of some consistency-type axioms (if one start from the larger problem and if the coalition of top-ranked agents is removed from it) and of the so-called bracing lemma which also relates the solution sets before and after a similar population expansion. On these points, we refer to Thomson (2011) for more details.

The second axiom is an invariance axiom built from a monotonicity condition. Imagine that in a coalitional ranking problem two disjoint coalitions have the same rank and their union lies in their lower counter set, i.e., the union of these coalitions has a lower ranked than their constituent. The axiom indicates that the solution set is invariant to any monotonic transformation of the coalitional ranking induced by an improvement of this union, provided that this union remains in the lower contour set of the two original coalitions.

The last axiom is based on a decomposition principle. It specifies the conditions under which the solution set of a coalitional ranking problem can be decomposed as the union of the solution sets of variants of this coalitional ranking when some specific top coalitions of the coalitional ranking are each promoted as the only top coalition, ceteris paribus. Such a decomposition is possible according to the axiom if the set of top coalitions is closed/stable by union of disjoint coalitions.

Our second result provides a non-deterministic algorithm that computes all core-partitions and so yields in turn all core-partition social rankings (Proposition 3). Farrell and Scotchmer (1988) allude informally to the general idea of this algorithm in the case where there is a unique core-partition. In such a case, the algorithm is deterministic and trivial. Our algorithm deals with the subtleties appearing in the case of multiple core-partitions.

Our work can be connected to the literature on hedonic games. In a hedonic game, each agent is endowed with a preference (a weak order) over the set of coalitions to which it can belong and a popular objective is to find core-partitions. In order to do so, a coalition blocks a partition if all its members prefer this coalition to their respective cell of the partition. A partition is stable if it cannot be blocked by any coalition. A coalitional ranking problem can be considered as a specific hedonic game in which the preferences of any pair of agents agree when they compare two coalitions containing these two agents. This property, introduced in Farrell and Scotchmer (1988)[1], is sufficient for the nonemptiness of the set of core-partitions. Karatay and Klaus (2017) consider the domain of hedonic games with strict preferences: an agent cannot be indifferent between two coalitions to which it can belong. They provide an axiomatic characterization of the set of core-partitions on the subdomain of hedonic games with a nonempty core.

Finally, the approach in Piccione and Razin (2009) shares some similarities to ours. The authors also investigate social rankings emerging from core-partitions associated with a coalitional ranking. Nonetheless, there are several differences between the two articles. Firstly, Piccione and Razin (2009) consider coalitional ranking which are total orders (without ties) while we consider weak orders. Secondly, they impose a separability condition on their coalitional rankings: for four disjoints coalitions, if a first coalition is ranked higher that a second coalition and if a third coalition is ranked higher than a fourth coalition, then the union of the first and third coalitions is ranked

[1] It is is called the common ranking property in Banerjee et al. (2001).

higher than the union of the second and fourth coalitions. Contrary to this rather narrow class of coalitional rankings, we allow for all weak orders over the set of nonempty coalitions. Thirdly, the ranking induced by a partition used in Piccione and Razin (2009) is consistent with the ranking of the cells of the partition but they further decide between two agents belonging to a same cell by comparing the corresponding singletons. As a consequence, a social ranking in Piccione and Razin (2009) is always a total order on the population of agents while we allow for weak orders. Fourthly, the stability concept adopted by Piccione and Razin (2009) is based on a blocking condition that is weaker than the classical one, so that the set of their stable social rankings is larger than the set of core-partitions. The authors' main results are a full description of the (nonempty) set of partitions that are stable under all the coalitional rankings in their specific class and an axiomatic characterization of this set.

The rest of the article is organized as follows. Section 2 gives basic notation and definitions. Section 3 presents core-partitions and the core-partition social ranking solution and provides some of their properties. The axiomatic study is contained in section 4. Section 5 proves that the three axioms invoked in our characterization are logically independent. Section 6 provides the algorithm which computes all core-partitions. Section 7 comes back to the link between our model and hedonic games. Section 8 concludes.

2. Notation and definitions

For any finite set A, denote by a its cardinality, by Ω_A the set of nonempty subsets of A and by $\mathcal{R}(A)$ the set of weak orders on A, i.e., the set of all reflexive, transitive and complete binary relations on A. Given a weak order $R \in \mathcal{R}(A)$, $E^{(A,R)}$ stands for its associated set of equivalence classes, and, when A is nonempty, $M(A,R)$ represents its nonempty set of maximal/top elements, i.e., $M(A,R)$ is the greatest/best equivalence class of $E^{(A,R)}$ with respect to the quotient order. For any subset B of A, R_B stands for the restriction of R to the subset B, i.e.,

$$\forall x, y \in B, \quad xR_By \Longleftrightarrow xRy.$$

Let $\mathbb{N}$ be the universe of **agents** and $\mathcal{F}$ be the collection of all finite sets of $\mathbb{N}$. For any set of agents $N \in \mathcal{F}$, any element of Ω_N is a called a **coalition**. A **coalitional ranking problem** is a pair $(\Omega_N, \succsim)$ where $N \in \mathcal{F}$ is the agent set and $\succsim \in \mathcal{R}(\Omega_N)$ is the coalitional ranking. For two coalitions $S, T \in \Omega_N$, $S \succsim T$ means that S is at least as highly ranked as T in $\succsim$. The asymmetric and symmetric part of $\succsim$ are denoted by $\succ$ and $\sim$ respectively. For a coalitional ranking $\succsim \in \mathcal{R}(\Omega_N)$ and a coalition $S \in \Omega$, the **lower set** of $\succsim$ at S is the set

$$L(\succsim, S) = \{T \in \Omega_N : S \succsim T\}.$$

Denote by

$$\mathcal{R}_\Omega = \bigcup_{N \in \mathcal{F}} \mathcal{R}(\Omega_N),$$

the domain of coalitional rankings. A **social ranking** on N is a weak order $\geqslant$ in $\mathcal{R}(N)$. In a similar way as above, $>$ denote the asymmetric part of $\geqslant$ and $\cdot$ its symmetric part. Denote by

$$\mathcal{R}_\mathbb{N} = \bigcup_{N \in \mathcal{F}} \mathcal{R}(N)$$

the set of social rankings that one can construct from any finite set of agents of $\mathbb{N}$.

For any coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and any coalition $S \in \Omega_N$, the **coalitional ranking subproblem** induced by $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and S is the restricted coalitional ranking $\succsim_S$ to the set Ω_S.

A **social ranking (set-valued) solution** is a correspondence $f : \mathcal{R}_\Omega \rightrightarrows \mathcal{R}_\mathbb{N}$ which assigns to each coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega(N)$ a possibly empty set of social rankings $f(\Omega_N, \succsim) \subseteq \mathcal{R}(N)$. If $N = \varnothing$, then $\Omega_N = \varnothing$ and $\mathcal{R}(N) = \mathcal{R}(\Omega_N) = \{\varnothing\}$. In such a case, one uses the convention that $f(\Omega_\varnothing, \varnothing) = \{\varnothing\}$. The social ranking solution f is **nonempty-valued** on $\mathcal{R}_\Omega$, if it holds that $f(\Omega_N, \succsim) \neq \varnothing$ for each $(\Omega_N, \succsim) \in \mathcal{R}_\Omega(N)$.

3. Core-partitions and the core-partition ranking solution

Core partitions of a coalitional ranking problem

Given $N \in \mathcal{F}$, a **partition** on N is a set $P = \{P_1, \dots, P_k\}$ of mutually disjoint nonempty coalitions that covers N, i.e., P is such that

$$\bigcup_{q \in \{1,\dots,k\}} P_q = N \quad \text{and} \quad \forall q, r \in \{1,\dots,k\},\, q \neq r, \quad P_q \cap P_r = \varnothing.$$

For a given agent $i \in N$ and a partition P, $P(i)$ stands for the unique coalition/cell in P containing i. Denote by $\mathcal{P}(N)$ the set of partitions of N. For any two partitions $P = \{P_1, \dots P_k\}$ and $P' = \{P'_1, \dots, P_{k'}\}$ on N, P' is **coarser** than P if, for each $P'_q \in P'$, there is a nonempty subset $A_q \subseteq \{1, \dots, k\}$ such that

$$P'_q = \bigcup_{r \in A_q} P_r.$$

Given a coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}$ and a partition $P \in \mathcal{P}(N)$, one says that P is **blocked by coalition** $S \in \Omega_N$ if,

$$\forall i \in S, \quad S \succ P(i).$$

In words, P is blocked by S if, for each agent in S, coalition S is ranked higher than the coalition/cell in P containing this agent. The interpretation is that if agents prefer to be assigned to the best possible coalitions, then each one prefers to be in S than in the cell that the partition assigns to each of them. In this sense, P is not stable. A partition P is a **core-partition** if it is stable, i.e., if it cannot be blocked by any nonempty coalition. Equivalently, P is a core-partition if, for each $S \in \Omega_N$, there is $i \in S$ such that $P(i) \succsim S$. Denote by $\mathcal{CP}(\Omega_N, \succsim) \subseteq \mathcal{P}(N)$ the set of core-partitions of the coalitional ranking problem $(\Omega_N, \succsim)$.

Example 1. Assume that $N = \{1, 2, 3, 4, 5\}$. Consider the coalitional ranking problem $(\Omega_N, \succsim)$ such that

$$\{1, 2\} \sim \{2, 3\} \succ \{4, 5\} \sim \{3, 4\} \succ N \sim \{2\} \sim \{3\} \succ \{1\} \succ S \sim T,$$

for each other pair of coalitions $\{S, T\}$. This coalitional ranking admits five equivalence classes and

$$M(\Omega_N, \succsim) = \big\{\{1,2\},\{2,3\}\big\}.$$

Its set $\mathcal{CP}(\Omega_N, \succsim)$ of core-partitions contains three elements:

$$P = \big\{\{1,2\},\{4,5\},\{3\}\big\}, \quad P' = \big\{\{2,3\}\},\{4,5\},\{1\}\big\}, \quad P'' = \big\{\{1,2\},\{3,4\},\{5\}\big\}.$$

□

Example 2. Let $N = \{1, 2, \ldots, n\}$, and consider the coalitional ranking problem $(\Omega_N, \succsim)$ where coalitions are ranked according to the smallest index they contain:

$$\forall S, T, \in \Omega_N, \quad (S \succsim T) \Longleftrightarrow (\min(S) \leqslant \min(T)).$$

Hence S is ranked at the $\min(S)$-th equivalence class of $(\Omega_N, \succsim)$, where the best equivalence class contains all coalitions $S \ni 1$ and the worst equivalence class is the singleton $\{n\}$. For a nonempty coalition $T \subseteq N$, consider $i = \min(T)$. For any $S \ni i$, $S \succsim T$, from which one concludes that $\mathcal{CP}(\Omega_N, \succsim) = \mathcal{P}(N)$. □

Remark 1. *The notion of core-partition has been extensively used in coalition formation games. It is known that for $N \neq \varnothing$, $\mathcal{CP}(\Omega_N, \succsim) \neq \varnothing$ (see, e.g., Farrell and Scotchmer, 1988, Banerjee et al. 2001). The computation of $\mathcal{CP}(\Omega_N, \succsim)$ and the discussion on the nonemptiness of core partitions in the connected context of hedonic games are postponed to sections 6 and 7.*

The Core-partition social ranking solution

Let us now establish a link between partitions and social rankings. Each coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and each partition $P \in \mathcal{P}(N)$ induce a social ranking in $\mathcal{R}(N)$ denoted by $\succcurlyeq_{(\succsim,P)}$ such that

$$\forall i, j \in N, \quad i \succcurlyeq_{(\succsim,P)} j \Longleftrightarrow P(i) \succsim P(j).$$

In words, the agents are ranked consistently with the rank of their cell. We are interested in social rankings that emerge from core-partitions. Formally, a **core-partition social ranking** for a coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ is a social ranking $\succcurlyeq \in \mathcal{R}(N)$ such that there is a core-partition $P \in \mathcal{CP}(\Omega_N, \succsim)$ satisfying $\succcurlyeq = \succcurlyeq_{(\succsim,P)}$. Denote by $\mathcal{CSR}$ the social ranking solution which assigns to each coalitional ranking problem $(\Omega_N, \succsim)$ the set of its core-partition social rankings $\mathcal{CSR}(\Omega_N, \succsim)$. By Remark 1, we know that $\mathcal{CSR}$ is nonempty-valued on $\mathcal{R}_\Omega$.

Example 3. Consider again Example 1. Using $\mathcal{CP}(\Omega_N, \succsim)$, the set $\mathcal{CSR}(\Omega_N, \succsim)$ of core-partition social rankings is formed by the following social rankings:

$$1 \cdot_{(\succsim,P)} 2 >_{(\succsim,P)} 4 \cdot_{(\succsim,P)} 5 >_{(\succsim,P)} 3, \quad 2 \cdot_{(\succsim,P')} 3 >_{(\succsim,P')} 4 \cdot_{(\succsim,P')} 5 >_{(\succsim,P')} 1,$$

and

$$1 \cdot_{(\succsim,P'')} 2 >_{(\succsim,P'')} 4 \cdot_{(\succsim,P'')} 3 >_{(\succsim,P'')} 5.$$

□

For a coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and a partition $P \in \mathcal{P}(N)$, define

$$M_P(\Omega_N, \succsim) = \{S \in \Omega_N : S \in P \cap M(\Omega_N, \succsim)\},$$

as the set of cells of P that belong to the best equivalence class of $(\Omega_N, \succsim)$.

In the following, we establish properties for $M_P(\Omega_N, \succsim)$ and $\mathcal{CP}(\Omega_N, \succsim)$ that will be useful for the rest of the article. The first part of Proposition 1 indicates that $M_P(\Omega_N, \succsim)$ is nonempty; the second part of Proposition 1 indicates that a core-partition is consistent in the sense that by removing the cells of $M_P(\Omega_N, \succsim)$ from P, we get a core-partition of the coalitional ranking restricted to the set of remaining agents.

Proposition 1. *For each $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and each $P \in \mathcal{CP}(\Omega_N, \succsim)$, it holds that:*

(i) *for each $S \in M(\Omega_N, \succsim)$, there is $T \in M_P(\Omega_N, \succsim)$ such that $S \cap T \neq \varnothing$. Thus, $M_P(\Omega_N, \succsim) \neq \varnothing$;*

(ii) $P \backslash M_P(\Omega_N, \succsim) \in \mathcal{CP}\big(N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S), \succsim_{N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S)}\big)$.

Proof. Fix any $(N, \succsim) \in \mathcal{R}_\Omega$ and any $P \in \mathcal{CP}(\Omega_N, \succsim)$.

Part *(i)*. Pick any nonempty $S \in M(\Omega_N, \succsim)$ and $P \in \mathcal{CP}(\Omega_N, \succsim)$. Assume that, for each $T \in P \cap M(\Omega_N, \succsim)$, $S \cap T = \varnothing$. It results that, for each $i \in S$, $P(i)$ does not belong to $M(\Omega_N, \succsim)$ so that $S \succ P(i)$. This implies that $P \notin \mathcal{CP}(\Omega_N, \succsim)$, a contradiction. Thus, if $P \in \mathcal{CP}(\Omega_N, \succsim)$, then, for each $S \in M(\Omega_N, \succsim)$, there is $T \in M_P(\Omega_N, \succsim)$ such that $S \cap T \neq \varnothing$. It follows that $M_P(\Omega_N, \succsim) \neq \varnothing$ for $P \in \mathcal{CP}(\Omega_N, \succsim)$.

Part *(ii)*. First observe that $P \backslash M_P(\Omega_N, \succsim)$ is a partition of $N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S)$, where the set $\cup_{S \in M_P(\Omega_N, \succsim)} S \neq \varnothing$ by point (i) of Proposition 1. The case in which $N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S)$ is empty is trivial. So, assume that there is a nonempty coalition $T \subseteq N \backslash (\cup_{S \in M_P(\omega_N, \succsim)} S)$ such that, for each $i \in T$, $T \succ_{N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S)} P(i)$, where $P(i) \in P \backslash M_P(\Omega_N, \succsim)$. Obviously, $T, P(i) \subseteq N$, and, for each $i \in T$, $T \succ P(i)$, which yields that P is not a core-partition of $(\Omega_N, \succsim)$, a contradiction. Conclude that $P \backslash M_P(\Omega_N, \succsim)$ is a core-partition of $(N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S), \succsim_{N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S)})$. ■

Remark 2. *Given $(\Omega_N, \succsim)$ and $\trianglerighteq \in \mathcal{CSR}(\Omega_N, \succsim)$, let $P_\trianglerighteq$ be the partition of N into equivalent classes according to $\trianglerighteq$, and consider a partition $P \in \mathcal{CP}(\Omega_N, \succsim)$ such that $\trianglerighteq = \trianglerighteq_{(\succsim, P)}$. A priori, $P_\trianglerighteq$ and P can be different. For each $S \in P$ and each $i, j \in S$, $P(i) = P(j)$ so that $i \cdot j$. By definition of $P_\trianglerighteq$, there is $T \in P_\trianglerighteq$ such that $i, j \in T$. Therefore, $S \subseteq T$, which implies that $P_\trianglerighteq$ is coarser than any $P \in \mathcal{CP}(\Omega_N, \succsim)$ such that $\trianglerighteq = \trianglerighteq_{(\succsim, P)}$. In particular, the best equivalence class $M(N, \trianglerighteq)$ of $\trianglerighteq$ is given by:*

$$M(N, \trianglerighteq) = \cup_{S \in M_P(\Omega_N \succsim)} S.$$

□

Example 4. Assume that $N = \{1, 2, 3, 4, 5\}$. Consider the coalitional ranking problem $(\Omega_N, \succsim)$ such that

$$\{1, 2\} \sim \{2, 3\} \sim \{4, 5\} \sim \{3, 4\} \succ N \sim \{2\} \sim \{3\} \succ \{1\} \succ S \sim T,$$

for each other pair of coalitions $\{S, T\}$. Obviously, the partition

$$P = \{\{1,2\},\{4,5\},\{3\}\} \in \mathcal{CP}(\Omega_N, \succsim),$$

and the associated core-partition social ranking $\succcurlyeq_{(\succsim,P)}$ is given by

$$1 \sim_{(\succsim,P)} 2 \sim_{(\succsim,P)} 4 \sim_{(\succsim,P)} 5 \succ_{(\succsim,P)} 3.$$

The partition $P_{\succcurlyeq_{(\succsim,P)}}$ of N induced by the equivalence classes of $\succcurlyeq_{(\succsim,P)}$ is given by

$$P_{\succcurlyeq_{(\succsim,P)}} = \{\{1,2,4,5\},\{3\}\},$$

and $P_{\succcurlyeq_{(\succsim,P)}}$ is coarser than P. □

4. Axiomatic study

This section introduces three new axioms for a (set-valued) social ranking solution on $\mathcal{R}_\Omega$. Let $f : \mathcal{R}_\Omega \rightrightarrows \mathcal{R}_\mathbb{N}$ be any such social ranking solution.

The first axiom is based on a specific expansion of the population of agents. So consider an initial coalitional ranking problem, say $(N \backslash S, \succsim')$. Assume now that the population is expanded in such a way that the coalition S of all newly added agents wins unanimous support: S is the unique maximal coalition in the larger coalitional ranking $(N, \succsim)$, and the relative ranking between any two original coalitions remains the same, i.e. the restriction $\succsim_{N\backslash S}$ of $\succsim$ to the original population $N \backslash S$ coincides with $\succsim'$. Then, our axiom imposes that the solution set of this new coalitional ranking problem $(N, \succsim)$ is computed from the solution set of the initial problem $(N \backslash S, \succsim')$ by adding to each selected social ranking for $(N \backslash S, \succsim')$ the members of S as the only maximal elements.

To state formally this axiom, a definition is needed. For each $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$, each $S \in \Omega_N$ and each $\succcurlyeq \in \mathcal{R}(N \backslash S)$, denote by $\succcurlyeq^{+S} \in \mathcal{R}(N)$ the social ranking on N obtained from $\succcurlyeq$ by adding S as the set of maximal elements, i.e. $M(N, \succcurlyeq^{+S}) = \{S\}$, and, for each $i, j \in N \backslash S$, $i \succcurlyeq^{+S} j$ if and only if $i \succcurlyeq j$.

Unanimous extension (UE). For each $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ such that $M(\Omega_N, \succsim) = \{S\}$, it holds that

$$f(\Omega_N, \succsim) = \{ \succcurlyeq^{+S} : \succcurlyeq \in f(\Omega_{N\backslash S}, \succsim_{N\backslash S})\}.$$

In order to state the next axiom, we introduce a new operation on a coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$. For a given coalition $S \in \Omega_N$, we say that an alternative coalitional ranking problem $(\Omega_N, \succsim') \in \mathcal{R}_\Omega$ is induced from $(\Omega_N, \succsim)$ by a S-**improvement** if the two following conditions hold:

- $L(\succsim, S) \subseteq L(\succsim', S)$;
- $\forall R, T \in \Omega_N \backslash \{S\}$, $R \succsim' T \iff R \succsim T$.

In words, the position of S (weakly) improves while the relative ranking of any other pair of coalitions is unchanged.

The next axiom relies on a principle of invariance of the solution set to some improving move. Consider a situation where two disjoint coalitions have the same rank and their union has a lower rank. Then, this principle of invariance indicates that improving the rank of this union, *ceteris paribus*, does not alter the solution set, provided than the new rank of the union has still a rank not higher than that of the two original coalitions. This principle is related in a certain way to the Maskin monotonicity principle (Maskin, 1999). Loosely speaking, this Maskin monotonicity indicates that if an alternative is selected in a certain problem, then it is also selected in a related problem obtained when some elements from which the alternative is constructed improved, ceteris paribus. In our case, this principle applies to specific problems where two disjoint coalitions have the same rank, their union is lower ranked, and the improving move is bounded from above by the equivalence class to which these two disjoint coalitions belong.

Invariance to merger upgrading (IMU). For each $(\Omega_N, \succsim) \in \mathcal{R}_{\Omega_N}$, each pair of coalitions $\{S, T\} \subseteq \Omega_N$ such that $S \cap T = \varnothing$, $S \sim T$ and $(S \cup T) \in L(\succsim, S)$, and each $(\Omega_N, \succsim') \in \mathcal{R}_{\Omega_N}$ induced from $(\Omega_N, \succsim)$ by a $(S \cup T)$-improvement where $(S \cup T) \in L(\succsim', S)$, it holds that $f(\Omega_N, \succsim) = f(\Omega_N, \succsim')$.

The last axiom implements a decomposition property. It specifies situations where the solution set of a coalitional ranking problem can be expressed as the union of the solution sets of coalitional ranking problems built from the original one and containing a unique maximal element. The situations where such a decomposition is possible is related to the structure of the set of maximal elements of the coalitional ranking problem. A subset of coalitions $\mathcal{C} \subseteq \Omega_N$ is **stable by union of disjoint elements** if for any pair of coalitions $\{S, T\} \subseteq \mathcal{C}$ such that $S \cap T = \varnothing$, then $S \cup T \in \mathcal{C}$.[2] Assume that the set of maximal coalitions of a coalitional ranking problem is stable by union of disjoint elements. In such a situation, the axiom indicates that the solution set coincides with the union of the solution sets of variants of the original coalitional ranking problem obtained, for each maximal coalition intersecting all other maximal coalitions, by putting this maximal coalition as the unique maximal coalition, *ceteris paribus*.

For each $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ define as

$$I_M(\Omega_N, \succsim) = \big\{S \in M(\Omega_N, \succsim) : S \cap T \neq \varnothing, \forall T \in M(\Omega_N, \succsim)\big\},$$

the **set of intersecting coalitions** in $M(\Omega_N, \succsim)$. Now, for each $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and each $S \in I_M(\Omega_N, \succsim)$, denote by $(\Omega_N, \succsim^S)$ the unique coalitional ranking problem induced from $(\Omega_N, \succsim)$ by a S-improvement such that $M(\Omega_N, \succsim^S) = \{S\}$.

[2]The difference with union stable set systems as studied in Algaba et al. (2000) is that only the union of pairs of intersecting coalitions is considered; and the difference with union closed systems as studied in van den Brink et al. (2011) is that the union of all pairs of coalitions is considered. Thus, each union closed system is union stable and stable by union of disjoint elements.

Remark 3. *If $M(\Omega_N, \succsim)$ is stable by union of disjoint coalitions, then $I_M(\Omega_N, \succsim) \neq \varnothing$.*

Decomposition by top upgrading (DTU). Consider any $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ such that $M(\Omega_N, \succsim)$ is stable by union of disjoint coalitions. Then, it holds that

$$f(\Omega_N, \succsim) = \bigcup_{T \in I_M(\Omega_N, \succsim)} f(\Omega_N, \succsim^T).$$

The next result states that the combination of Unanimous extension, Invariance to merger upgrading and Decomposition by top upgrading characterizes the core-partition social ranking solution $\mathcal{CSR}$.

Proposition 2. *The social ranking solution $\mathcal{CSR}$ is the unique solution on $\mathcal{R}_\Omega$ satisfying Unanimous extension (UE), Invariance to merger upgrading (IMU), and Decomposition by top upgrading (DTU).*

Proof. The proof is divided into two parts (a) and (b).
(a) One shows that $\mathcal{CSR}$ satisfies the above three axioms. Pick any $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$.

Unanimous extension. Assume that $M(\Omega_N, \succsim) = \{S\}$ for some $S \in \Omega_N$. To show:

$$\mathcal{CSR}(\Omega_N, \succsim) = \{\succ^{+S} : \succ \in \mathcal{CSR}(\Omega_{N\backslash S}, \succsim_{N\backslash S})\} \tag{1}$$

To this end, it is sufficient to show that $P \in \mathcal{CP}(\Omega_N, \succsim)$ if and only if P is of the form $P = \{S\} \cup P'$ where $P' \in \mathcal{CP}(\Omega_{N\backslash S}, \succsim_{N\backslash S})$. For the "if part", consider such a partition $P \in \mathcal{P}(N)$ and any $T \in \Omega_N$. Two cases can be distinguished:

- if $T \cap S \neq \varnothing$, pick $i \in T \cap S$. One obtains $P(i) = S \succsim T$ because $S \in M(\Omega_N, \succsim)$;
- if $T \cap S = \varnothing$, then $T \in \Omega_{N\backslash S}$. Because P' is a core-partition of $(\Omega_{N\backslash S}, \succsim_{N\backslash S})$, there is $i \in T$ such that $P'(i) \succsim_{N\backslash S} T$. Hence, $P(i) = P'(i) \succsim T$.

From the above two cases, conclude that P cannot be blocked by T, and so $P \in \mathcal{CP}(\Omega_N, \succsim)$.

For the "only if part", by part (i) of Proposition 1, $M_P(\Omega_N, \succsim) \cap M(\Omega_N, \succsim) \neq \varnothing$ so that $S \in P$ and $M_P(\Omega, \succsim) = \{S\}$. Part (ii) of Proposition 1 implies that $P' = P \backslash \{S\} \in \mathcal{CP}(\Omega_{N\backslash S}, \succsim_{N\backslash S})$, as desired.

Now, if $P = \{S\} \cup P' \in \mathcal{CP}(\Omega_N, \succsim)$ for some $P' \in \mathcal{CP}(\Omega_{N\backslash S}, \succsim_{N\backslash S})$, then its associated social ranking $\succ_{(\succsim, P)}$ belongs to $\mathcal{CSR}(\Omega_N, \succsim)$ and obviously $\succ_{(\succsim,P)} = \succ^{+S}_{(\succsim_{N\backslash S}, P')}$. Reciprocally, $P' \in \mathcal{CP}(N\backslash S, \succsim_{N\backslash S})$ gives rise to a unique social ranking $\succ_{(\succsim_{N\backslash S}, P')} \in \mathcal{CSR}(N\backslash S, \succsim_{N\backslash S})$ from which the social ranking $\succ^{+S}_{(\succsim_{N\backslash S}, P')} = \succ_{(\succsim, P)}$ is a core-partition social ranking of $\mathcal{CSR}(\Omega_N, \succsim)$. Thus, (1) holds, as desired.

Invariance to merger upgrading. Consider a pair $\{S, T\} \subseteq \Omega_N$ such that $S \cap T = \varnothing$, $S \sim T$ and $(S \cup T) \in L(\succsim, S)$, and any coalitional ranking problem $(\Omega_N, \succsim') \in \mathcal{R}_\Omega$ induced from $\succsim$ by a $(S \cup T)$-improvement with $(S \cup T) \in L(\succsim', S)$. To show: $\mathcal{CSR}(\Omega_N, \succsim) = \mathcal{CSR}(\Omega_N, \succsim')$. One proceeds by double inclusion.

• $\mathcal{CSR}(\Omega_N, \succsim) \subseteq \mathcal{CSR}(\Omega_N, \succsim')$. By definition of $\mathcal{CSR}$, it suffices to show that $\mathcal{CP}(\Omega_N, \succsim) \subseteq \mathcal{CP}(\Omega_N, \succsim')$. Pick any $P \in \mathcal{CP}(N, \succsim)$, which means that, for each $R \in \Omega_N$, there is $i \in R$ such that $P(i) \succsim R$. For each $R \in \Omega_N \setminus \{(S \cup T)\}$, note that $P(i) \succsim R$ implies $P(i) \succsim' R$ because $S \cup T$ is the only improved coalition in the coalitional ranking $\succsim'$. If $R = S \cup T$, then, since there is $i \in S$ such that $P(i) \succsim S$ and $(S \cup T) \in L(\succsim, S) \cap L(\succsim', S)$, one obtains $P(i) \succsim' S \succsim' (S \cup T)$, whether $P(i) = S \cup T$ or not. Hence, $P \in \mathcal{CP}(N, \succsim')$.

• $\mathcal{CSR}(\Omega_N, \succsim) \supseteq \mathcal{CSR}(\Omega_N, \succsim')$. Pick any $\rhd \in \mathcal{CSR}(\Omega, \succsim')$ and any $P' \in \mathcal{CP}(\Omega_N, \succsim')$ such that $\rhd = \rhd_{(\succsim', P')}$. If $S \cup T \notin P'$, then define $P = P'$; and if $S \cup T \in P'$, then define $P = (P' \setminus \{S \cup T\}) \cup \{S\} \cup \{T\}$. Because $S \cup T \in P'$ only happens when $S \sim' T \sim' S \cup T$, one necessarily has $\rhd_{(\succsim, P)} = \rhd_{(\succsim', P')}$. Let us show that $P \in \mathcal{CP}(\Omega_N, \succsim)$ so that $\rhd \in \mathcal{CSR}(\Omega_N, \succsim)$. For each $R \in \Omega_N$, there is $i \in R$ such that $P'(i) \succsim' R$. If $P'(i) \neq S \cup T$, then $P(i) = P'(i) \succsim R$, whether $R = S \cup T$ or not. If $P'(i) = S \cup T$, then either $i \in S$ or $i \in T$ so that $P(i) = S$ or $P(i) = T$. By definition of $\succsim'$, $S \cup T \in L(\succsim', S) = L(\succsim', T)$ so that, in both cases, $P(i) \succsim' S \cup T = P'(i) \succsim' R$, which implies that $P(i) \succsim R$. Hence, for each $R \in \Omega_N$, there is $i \in R$ such that $P(i) \succsim R$. Conclude that $P \in \mathcal{CP}(\Omega_N, \succsim)$ so that $\rhd = \rhd_{(\succsim', P')} = \rhd_{(\succsim, P)} \in \mathcal{CSR}(\Omega_N, \succsim)$.

Decomposition by top upgrading. Assume that $M(\Omega_N, \succsim)$ is stable by union of disjoint coalitions. To show:

$$\mathcal{CSR}(\Omega_N, \succsim) = \bigcup_{T \in I_M(\Omega_N, \succsim)} \mathcal{CSR}(\Omega_N, \succsim^T).$$

One proceeds by double inclusion.

• $\mathcal{CSR}(\Omega_N, \succsim) \subseteq \cup_{T \in I_M(\Omega_N, \succsim)} \mathcal{CSR}(\Omega_N, \succsim^T)$. Pick any $\rhd \in \mathcal{CSR}(\Omega_N, \succsim)$ and any $P \in \mathcal{CP}(\Omega_N, \succsim)$ associated with $\rhd \in \mathcal{CSR}(\Omega_N, \succsim)$, i.e., $\rhd = \rhd_{(\succsim, P)}$. It suffices to show that there is $T \in I_M(\Omega_N, \succsim)$ and $P' \in \mathcal{CP}(\Omega_N, \succsim^T)$ such that $\rhd_{(\succsim^T, P')} = \rhd_{(\succsim, P)}$. Denote by T the union of the possibly several elements in $M_P(\Omega_N, \succsim)$ and define P' as the partition obtained from P by replacing the elements of $M_P(\Omega_N, \succsim)$ by their union T. Because $M(\Omega_N, \succsim)$ is stable by union of disjoint coalitions, $T \in M(\Omega_N, \succsim)$. Clearly, $M_{P'}(\Omega_N, \succsim) = \{T\}$, $\rhd_{(\succsim, P')} = \rhd_{(\succsim, P)}$ and $P' \in \mathcal{CP}(\Omega_N, \succsim)$. It remains to show that $T \in I_M(\Omega, \succsim)$ and that $P' \in \mathcal{CP}(\Omega_N, \succsim^T)$. The first claim results from part (i) of Proposition 1 and the definition of $I_M(\Omega, \succsim)$. For the second claim, pick any $S \in \Omega_N$. Because $P \in \mathcal{CP}(\Omega_N, \succsim)$, there is $i \in S$ such that $P(i) \succsim S$. If $i \in T$, then $T = P'(i) \succsim^T S$. If $i \notin T$, then $P'(i) = P(i) \notin M(\Omega_N, \succsim)$. By definition of $\succsim^T$ and the fact that $P' \in \mathcal{CP}(\Omega_N, \succsim)$, one obtains $P'(i) \succsim^T S$. Hence, $P' \in \mathcal{CP}(\Omega_N, \succsim^T)$.

• $\mathcal{CSR}(\Omega_N, \succsim) \supseteq \cup_{T \in I_M(\Omega_N, \succsim)} \mathcal{CSR}(\Omega_N, \succsim^T)$. Precisely, one establishes that, for each $T \in I_M(\Omega_N, \succsim)$, $\mathcal{CP}(\Omega_N, \succsim^T) \subseteq \mathcal{CP}(\Omega_N, \succsim)$ so that $\mathcal{CSR}(\Omega_N, \succsim^T) \subseteq \mathcal{CSR}(\Omega_N, \succsim)$. For each $P' \in \mathcal{CP}(\Omega_N, \succsim^T)$, $M_{P'}(\Omega_N, \succsim^T) = \{T\} \subseteq M(\Omega_N, \succsim)$ where the inclusion follows from the fact that $T \in I_M(\Omega_N, \succsim)$. Furthermore, $T \in I_M(\Omega_N, \succsim)$ implies $M_{P'}(\Omega_N, \succsim) = \{T\}$. Because, $P' \in \mathcal{CP}(\Omega_N, \succsim^T)$, for each $S \in \Omega_N$, there is $i \in S$ such that $P'(i) \succsim^T S$. If $S \cap T \neq \varnothing$, one picks $i \in S \cap T$ and $P'(i) = T \in M(\Omega_N, \succsim)$ so that $P'(i) \succsim S$. If $S \cap T = \varnothing$, then $P'(i) \neq T$, so that $P'(i) \succsim S$ by definition of $\succsim^T$. This yields that $P' \in \mathcal{CP}(\Omega, \succsim)$, as desired.

(b) One shows that if a social ranking solution f satisfies Unanimous extension, Invariance to merger upgrading and Decomposition by top upgrading, then $f = \mathcal{CSR}$. So, consider such a social ranking solution f. To prove that $f = \mathcal{CSR}$, one proceeds by induction on the number $n \geqslant 0$ of agents in a coalitional ranking problem.

Initialization. For $N = \varnothing$, by convention $f(\Omega_\varnothing, \varnothing) = \mathcal{CSR}(\Omega_\varnothing, \varnothing) = \{\varnothing\}$. For $N = \{i\}$ for some $i \in \mathbb{N}$, and for the (unique) coalitional ranking $\succsim$ on $\Omega_{\{i\}}$, $\{i\} \succsim \{i\}$, one applies Unanimous extension to f:

$$f(\Omega_{\{i\}}, \succsim) = \left\{\varnothing^{+\{i\}} : \{\varnothing\} = f(\Omega_\varnothing, \varnothing)\right\} = \{\cdot\},$$

meaning that $i \cdot i$. Obviously, $\mathcal{CSR}(\Omega_{\{i\}}, \succsim) = \{\cdot\}$, so that $f(\Omega_{\{i\}}, \succsim) = \mathcal{CSR}(\Omega_{\{i\}}, \succsim)$.

Induction hypothesis. Assume that $f(\Omega_N, \succsim) = \mathcal{CSR}(\Omega_N, \succsim)$ holds for any $(\Omega, \succsim) \in \mathcal{R}_{\Omega_N}$ such that $n < k$ for some integer $k \geqslant 1$.

Induction step. Consider any coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ such that $n = k$. If $M(\Omega_N, \succsim)$ is not stable by union of disjoints coalitions, $M(\Omega_N, \succsim)$ contains disjoint coalitions S and T such that $S \cup T$ is not in $M(\Omega_N, \succsim)$. Then, construct the coalitional ranking problem $(\Omega_N, \succsim')$ induced from $(\Omega_N, \succsim)$ by the $(S \cup T)$-improvement such that $S \cup T \in M(\Omega_N, \succsim')$. Because f and $\mathcal{CSR}$ satisfy Invariance to merger upgrading, $f(\Omega_N, \succsim') = f(\Omega_N, \succsim)$ and $\mathcal{CSR}(\Omega_N, \succsim') = \mathcal{CSR}(\Omega_N, \succsim)$. Continue in this fashion to eventually construct a coalitional ranking problem $(\Omega_N, \succsim^*)$ such that, for each $S, T \in M(\Omega_N, \succsim^*)$ where $S \cap T = \varnothing$, $S \cup T \in M(\Omega_N, \succsim^*)$ so that $M(\Omega_N, \succsim^*)$ is stable by union of disjoint coalitions. Because Ω_N is a finite set, the (unique) coalitional ranking problem $(\Omega_N, \succsim^*)$ is reached after a finite number of steps. If $M(\Omega_N, \succsim)$ is stable by union of disjoint coalitions, no operation is required and $\succsim = \succsim^*$. Then, by successive applications of Invariance to merger upgrading, one obtains:

$$f(\Omega_N, \succsim^*) = f(\Omega_N, \succsim) \quad \text{and} \quad \mathcal{CSR}(\Omega_N, \succsim^*) = \mathcal{CSR}(\Omega_N, \succsim). \tag{2}$$

Furthermore, by Remark 3, $I_M(\Omega_N, \succsim^*) \neq \varnothing$. Thus, the conditions underlying Decomposition by top upgrading are met in $(\Omega_N, \succsim^*)$. Now, consider any $S \in I_M(\Omega_N, \succsim^*)$ and construct the coalitional ranking problem $(\Omega_N, \succsim^{*S})$ induced from $(\Omega_N, \succsim^*)$ by the S-improvement such that $M(\Omega_N, \succsim^{*S}) = \{S\}$. Applying Unanimous extension to both f and $\mathcal{CSR}$ and the induction hypothesis, one gets:

$$\begin{aligned} f(\Omega_N, \succsim^{*S}) &\overset{\text{UE}}{=} \left\{ \gtrdot^{+S} : \gtrdot \in f(\Omega_{N\setminus S}, \succsim^{*S}_{N\setminus S})\right\} \\ &\overset{\text{Ind. Hypo.}}{=} \left\{ \gtrdot^{+S} : \gtrdot \in \mathcal{CSR}(\Omega_{N\setminus S}, \succsim^{*S}_{N\setminus S})\right\} \\ &\overset{\text{UE}}{=} \mathcal{CSR}(\Omega_N, \succsim^{*S}). \end{aligned} \tag{3}$$

Because S was arbitrarily chosen in $I_M(\Omega_N, \succsim^*)$, applying Decomposition by top upgrading to both f and $\mathcal{CSR}$ and using the equalities (2)-(3) yield that

$$\begin{aligned} f(\Omega_N, \succsim) &\overset{(2)}{=} f(\Omega_N, \succsim^*) \\ &\overset{(\mathrm{DTU})}{=} \bigcup_{S \in I_M(\Omega_N, \succsim^*)} f(\Omega_N, \succsim^{*S}) \\ &\overset{(3)}{=} \bigcup_{S \in I_M(\Omega_N, \succsim^*)} \mathcal{CSR}(\Omega_N, \succsim^{*S}) \\ &\overset{(\mathrm{DTU})}{=} \mathcal{CSR}(\Omega_N, \succsim^*) \\ &\overset{(2)}{=} \mathcal{CSR}(\Omega_N, \succsim), \end{aligned}$$

which completes the induction step.

The statement of Proposition 2 follows from (a) and (b). ∎

The logical independence of the axioms used in Proposition 2 is shown in the following section.

5. Logical independence of the axioms

Unanimous extension is not satisfied. Consider the constant solution f^C which assigns to each coalitional ranking problem the equivalence social relation $\cdot$, that is all agents belong to the same equivalence class:

$$\forall (\Omega_N, \succsim) \in \mathcal{R}_\Omega, \quad f^C(\Omega_N, \succsim) = \{\cdot\}.$$

Because the image $f^C(\mathcal{R}_\Omega)$ is the equivalence relation $\cdot$, it obviously satisfies Invariance to merger upgrading, and Decomposition by top upgrading. But f^C violates Unanimous extension for the following reason: the social ranking $\gtrdot^{+S}$ used to define the solution set $f^C(\Omega_N, \succsim)$ whenever $M(\Omega_N, \succsim)$ is a singleton, contains at least two distinct equivalence classes when $S \neq N$.

Invariance to merger upgrading is not satisfied. Pick any $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ and define $I(\Omega_N, \succsim)$ as:

$$I(\Omega_N, \succsim) = \big\{S \in \Omega_N : \forall T \in \Omega_N, (S \sim T \Rightarrow S \cap T \neq \varnothing) \wedge (T \succ S \Rightarrow (\exists R \in \Omega_N, (R \sim T) \wedge (R \cap T) = \varnothing))\big\}.$$

Note that $I(\Omega_N, \succsim) \neq \varnothing$. Consider the social ranking solution f^I defined recursively as follows:

$$\forall (\Omega_N, \succsim) \in \mathcal{R}_\Omega, \quad f^I(\Omega_N, \succsim) = \bigcup_{S \in I(\Omega_N, \succsim)} \big\{\gtrdot^{+S} : \gtrdot \in f^I(\Omega_{N \setminus S}, \succsim_{N \setminus S})\big\}.$$

In case $M(\Omega_N, \succsim) = \{S\}$, then $I(\Omega_N, \succsim) = \{S\}$ and so

$$f^I(\Omega_N, \succsim) = \{\gtrdot^{+S} : \gtrdot \in f^I(\Omega_{N \setminus S}, \succsim_{N \setminus S})\},$$

which is the definition of Unanimous extension. Thus, f^I indeed satisfies this axiom. In case $M(\Omega_N, \succsim)$ is stable by union of disjoint coalitions, $I(\Omega_N, \succsim) = I_M(\Omega_N, \succsim)$. For each $S \in I_M(\Omega_N, \succsim)$, $(\Omega_N, \succsim^S)$ is such that $M(\Omega_N, \succsim^S) = \{S\} = I(\Omega_N, \succsim^S)$. Note also that $\succsim^S_{N\setminus S} = \succsim_{N\setminus S}$, so that

$$\begin{aligned} f^I(\Omega_N, \succsim^S) &= \{\rhd^{+S} : \rhd \in f^I(\Omega_{N\setminus S}, \succsim^S_{N\setminus S})\} \\ &= \{\rhd^{+S} : \rhd \in f^I(\Omega_{N\setminus S}, \succsim_{N\setminus S})\}. \end{aligned}$$

Consequently,

$$f^I(\Omega_N, \succsim) = \bigcup_{S \in I_M(\Omega_N, \succsim)} f^I(\Omega_N, \succsim^S),$$

which shows that f^I satisfies Decomposition by top upgrading. To show that f^I violates Invariance to merger upgrading, consider the coalitional ranking problem $(\Omega_N, \succsim)$ with $N = \{1, 2, 3\}$ and defined as:

$$\{2\} \sim \{3\} \succ \{2,3\} \sim \{1\} \succ \{1,2,3\} \succ \{1,3\} \sim \{1,2\}.$$

It follows that $I(\Omega_N, \succsim) = \{\{1, 2, 3\}\}$, and so $f^I(\Omega_N, \succsim) = \{\cdot\}$, i.e., $1 \cdot 2 \cdot 3$. Now, consider the coalitional ranking problem $(\Omega_N, \succsim')$ defined as:

$$\{2\} \sim' \{3\} \sim' \{2,3\} \succ' \{1\} \succ' \{1,2,3\} \succ' \{1,3\} \sim' \{1,2\}.$$

The coalitional ranking problem $(\Omega_N, \succsim')$ is an improvement from $(\Omega_N, \succsim)$ induced by $\{2, 3\}$ that satisfies the conditions of Invariance to merger upgrading. One has $I(\Omega_N, \succsim') = \{\{2, 3\}\}$. By definition of f^I and the fact that it satisfies Unanimous extension,

$$f^I(\Omega_N, \succsim') = \{2 \cdot 3 > 1\} \neq f^I(\Omega_N, \succsim),$$

a violation of Invariance to merger upgrading.

Decomposition by top upgrading is not satisfied. Consider the following solution

$$\forall (\Omega_N, \succsim) \in \mathcal{R}, \quad f^\ell(\Omega_N, \succsim) = \mathcal{CSR}(N, \ell(\succsim)),$$

where $\ell : \mathcal{R}(\Omega_N) \to \mathcal{R}(\Omega_N)$ is defined as

$$S\ell(\succsim)T \iff \big((S \succ T) \text{ or } (S \sim T \text{ and } s \leqslant t)\big).$$

The solution f^ℓ satisfies Unanimous extension and Invariance to merger upgrading but not Decomposition by top upgrading. Indeed, if $M(\Omega_N, \succsim) = \{S\}$, then $M(\Omega_N, \ell(\succsim)) = \{S\}$ and

$$\begin{aligned} f^\ell(\Omega_N, \succsim) &= \mathcal{CSR}(\Omega_N, \ell(\succsim)) \\ &= \{\rhd^{+S} : \rhd \in \mathcal{CSR}(\Omega_{N\setminus S}, \ell(\succsim)_{N\setminus S})\} \\ &= \{\rhd^{+S} : \rhd \in \mathcal{CSR}(\Omega_{N\setminus S}, \ell(\succsim_{N\setminus S}))\} \\ &= \{\rhd^{+S} : \rhd \in f^\ell(\Omega_{N\setminus S}, \succsim_{N\setminus S})\}, \end{aligned}$$

which shows that f^ℓ satisfies Unanimous extension. Next, if $S, T \in \Omega_N$ are such that $S \cap T = \varnothing$, $S \sim T$ and $(S \cup T) \in L(\succsim, S)$, then for any $(\Omega_N, \succsim') \in \mathcal{R}(\Omega_N)$ induced from $\succsim$ by a $(S \cup T)$-improvement with $(S \cup T) \in L(\succsim', S)$, it holds that $\ell(\succsim')$ is induced from $\ell(\succsim)$ by a $(S \cup T)$-improvement with $(S \cup T) \in L(\ell(\succsim'), S)$. Thus, f^ℓ satisfies Invariance to merger upgrading due to the fact that $\mathcal{CSR}$ satisfies it. Finally, consider the coalitional ranking problem $(\Omega_N, \succsim)$ with $N = \{1, 2\}$ and defined as $\{1\} \sim \{1, 2\} \succ \{2\}$, then $\succsim' = \ell(\succsim)$ is as follows: $\{1\} \succ' \{1, 2\} \succ' \{2\}$. On the one hand, by definition of f^ℓ, one has:

$$f^\ell(\Omega_N, \succsim) = \mathcal{CSR}(\Omega_N, \succsim') = \{\rhd\},$$

where $M(N, \rhd) = \{1\}$. On the other hand, $M(\Omega_N, \succsim) = \{\{1\}, \{1, 2\}\}$ is stable by union of disjoint coalitions and $I_M(\Omega_N, \succsim) = M(\Omega_N, \succsim)$, so that $\cup_{S \in I_M(\Omega_N, \succsim)} f^\ell(\Omega_N, \succsim^S) = \{\rhd, \rhd'\}$, where $\rhd'$ is such that $M(N, \rhd') = \{1, 2\}$. This contradicts Decomposition by top upgrading.

6. Algorithmic study

This section is devoted to the computation of the core-partitions of a coalitional ranking problem $(\Omega_N, \succsim)$ from which the set of social rankings $\mathcal{CSR}(\Omega_N, \succsim)$ is obtained. A non-deterministic algorithm is proposed, which computes all core-partitions of a coalitional ranking problem. In particular, this algorithm permits to conclude that the set of all core-partitions of a coalitional ranking problem is nonempty. As noted in Remark 1, this nonemptiness property has already been shown by Farrell and Scotchmer (1988) in another context (see Section 7). In turn, this property implies that $\mathcal{CSR}$ is nonempty-valued on $\mathcal{R}_\Omega$. The algorithm is as follows.

1 INITIALIZATION : $\eta \longleftarrow N$
2 WHILE $\eta \neq \varnothing$ DO :
3 INITIALIZATION : PICK ANY $T \in M(\Omega_\eta, \succsim_\eta)$, $P \longleftarrow \{\{T\}\}$
4 WHILE $\exists T \in M(\Omega_\eta, \succsim_\eta)$ SUCH THAT $\forall S \in P,\ S \cap T = \varnothing$ DO :
5 $P \longleftarrow P \cup \{\{T\}\}$
6 $\eta \longleftarrow \eta \backslash (\cup_{T \in P} T)$
7 OUTCOME : $P \in \mathcal{CP}(\Omega_N, \succsim)$

Proposition 3. *For each $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$, the set of outcomes of the above non-deterministic algorithm is exactly $\mathcal{CP}(\Omega_N, \succsim)$.*

Proof. Firstly, remark that the execution of the algorithm always terminates after a finite number of steps since N is a finite set.

Secondly, we prove that each $P = \{P_1, \ldots, P_k\} \in \mathcal{CP}(\Omega_N, \succsim)$ is obtained as some output of the algorithm. Denote by t the number of equivalent classes of $\succsim$ and order them according to the quotient order: $M(\Omega_N, \succsim)$ is the equivalent class 1 and so on. Define the mapping $\pi : \{1, \ldots, k\} \longrightarrow \{1, \ldots, t\}$ such that P_i belongs to equivalent class $\pi(i)$ according to $\succsim$. Note that $P_i \succsim P_j$ if and only if $\pi(i) \leqslant \pi(j)$. By Remark 2, the pre-image $\pi^{-1}(1)$ is nonempty and each cell P_i such that $i \in \pi^{-1}(1)$ can be chosen in the inner loop $3 - 5$ of the algorithm to form the outpout P while

$\eta = N$: these cells define the set $\cup_{i \in \pi^{-1}(1)} P_i = \cup_{S \in M_P(\Omega_N, \succsim,)} S$. Then, according to the second part of Proposition 1, we have

$$P \backslash M_P(\Omega_N, \succsim) \in \mathcal{CP}\big(N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S), \succsim_{N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S)}\big).$$

As a consequence, starting from $\eta = N \backslash (\cup_{S \in M_P(\Omega_N, \succsim)} S)$, each coalition $P_i \in P \backslash M_P(\Omega_N, \succsim)$ is such that $\pi(i) > 1$ and may be chosen recursively at step 3 or 4 by the algorithm.

Thirdly, we prove that each output of the algorithm belongs $\mathcal{CP}(\Omega_N, \succsim)$. So, assume $P = \{P_1, \ldots, P_k\} \in \mathcal{P}(N)$ is some output of the algorithm. By construction, P is a partition of N, whatever the choices made in steps 3 and 4. As above, consider $\pi : \{1, \ldots, k\} \to \{1, \ldots, t\}$ so that P_i belongs to equivalent class $\pi(i)$ according to $\succsim$. Pick any $S \in \Omega_N$. Consider $i_0 \in \{1, \ldots, k\}$ such that $\pi(i_0) = \min\{\pi(i) : P_i \cap S \neq \varnothing\}$; the minimizer i_0 is possibly not unique. Let us show that $P_{i_0} \succsim S$. By construction, P_{i_0} enters P at step 3 or 4 for a given $\eta \subseteq N$. By definition of i_0, for each $i \in \{1, \ldots, k\}$ such that $P_i \cap S \neq \varnothing$, P_i enters P later in the algorithm. Hence, each $j \in S$ belongs to a $P_i = P(j)$ such that $\pi(i) \geqslant \pi(i_0)$ so that $S \in \Omega_\eta$. By definition of the algorithm, $P_{i_0} \in M(\Omega_\eta, \succsim_\eta)$ in $(\Omega_\eta, \succsim_\eta)$. Together with $S \in \Omega_\eta$, this implies that $P_{i_0} \succsim_\eta S$, and so $P_{i_0} \succsim S$. Conclude that each $j \in P_{i_0} \cap S$ satisfies $P(j) \succsim S$, which means that S cannot block P, as desired. ■

Example 5. The key steps of the algorithm can be represented schematically by its decision tree. Each node, other than a leaf, is labelled by the set η corresponding to the steps 1 and 6 from which a choice is made in step 3 and 4. These choices correspond to the label of each arrow. Each leaf computes the output of the algorithm. Consider Example 1. Recall that $N = \{1, 2, 3, 4, 5\}$ and

$$\{1, 2\} \sim \{2, 3\} \succ \{4, 5\} \sim \{3, 4\} \succ N \sim \{2\} \sim \{3\} \succ \{1\} \succ S \sim T,$$

for each other pair of coalitions $\{S, T\}$. The set $\mathcal{CP}(\Omega_N, \succsim)$ of core-partitions contains three elements:

$$P = \big\{\{1, 2\}, \{4, 5\}, \{3\}\big\}, \quad P' = \big\{\{2, 3\}\}, \{4, 5\}, \{1\}\big\}, \quad P'' = \big\{\{1, 2\}, \{3, 4\}, \{5\}\big\}.$$

Figure 1 draws the associated decision tree. □

7. Relationships with the core of hedonic games

A core-partition P of a coalitional ranking problem $(\Omega_N, \succsim) \in \mathcal{R}_\Omega$ can be viewed as a core stable partition of a specific class of hedonic games. A hedonic game describes a situation where agents organize themselves into coalitions so that the resulting structure of the society is represented by a partition. A specific feature of hedonic games is that each agent's preference relation over the set of partitions depends solely on the set of coalitions that contain this agent. This means that agents do not care how agents in other coalitions are grouped together. Denote by $\mathcal{N}_i$ the set of coalitions that include $i \in N$ and by $\succsim_i$ a preference relation (a weak order) over $\mathcal{N}_i$. The situation

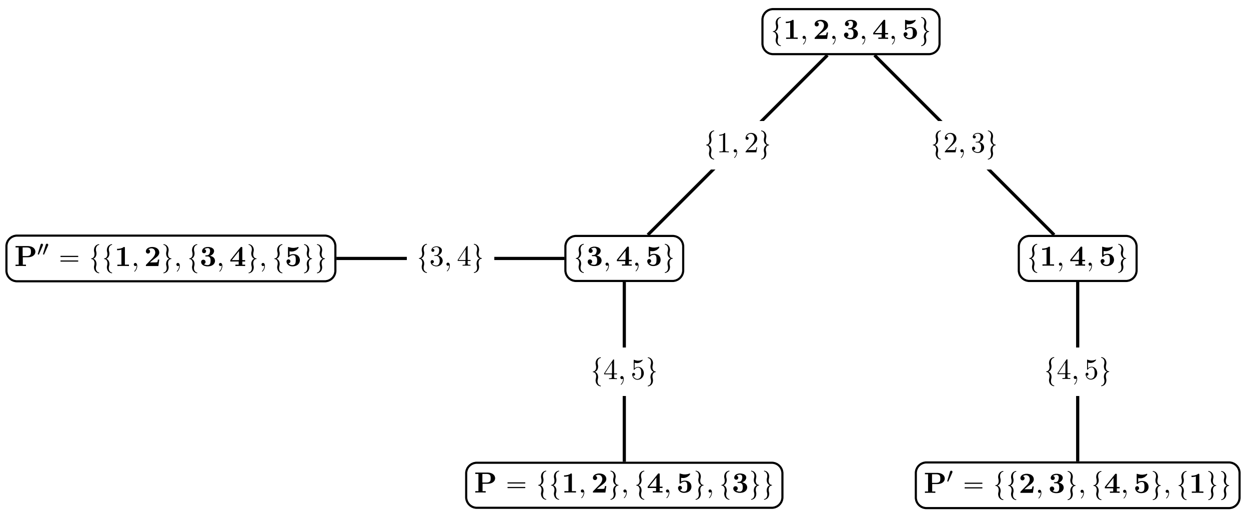

Figure 1: Decision tree of the algorithm applied to Example 1.

$(\Omega_N, (\succsim_i)_{i\in N})$ forms a **hedonic game**. In a similar way as above, a coalition $S \in \Omega_N$ blocks the partition P if

$$\forall i \in S, \quad S \succ_i P(i).$$

A core-partition of the hedonic game is a partition that is blocked by no coalition. The **core** of the hedonic game is the set, possibly empty, of all core partitions.

Assume that there is a coalitional ranking $\succsim$ on Ω_N such that the preference relations of $(\Omega_N, (\succsim_i)_{i\in N})$ satisfy the following condition:

$$\forall i \in N, \forall S, T \in \mathcal{N}_i, \quad S \succsim_i T \Longleftrightarrow S \succsim T.$$

In such a case, one says that the coalitional ranking problem $(\Omega_N, \succsim)$ is a **potential** for the hedonic game $(\Omega_N, (\succsim_i)_{i\in N})$ or has the **common ranking property** (see, Farrell and Scotchmer, 1988, Banerjee et al. 2001). Obviously, if $(\Omega_N, (\succsim_i)_{i\in N})$ admits as a potential the coalitional ranking problem $(\Omega_N, \succsim)$, the set of all core-partitions of $(\Omega_N, \succsim)$ constitutes the core of the hedonic game $(\Omega_N, (\succsim_i)_{i\in N})$. Farrell and Scotchmer (1988) show that this condition guarantees the nonemptiness of the core. The core of a hedonic game has received much attention in the literature. For instance, Banerjee et al. (2001) relax the common ranking property. They introduce two properties of top coalitions, which are sufficient to ensure the nonemptiness of the core of a hedonic game. Iehlé (2007) provides a necessary and sufficient condition under which the core of a hedonic is nonempty. This condition is based on a concept of pivotal balancedness.

Karatay and Klaus (2017) consider the domain of hedonic games with strict preferences. They offer, among other (im)possibilities results, an axiomatic characterization of the core on the sub-domain of hedonic games with a nonempty core in terms on Maskin monotonocity and Coalitional unanimity. In this context, Maskin monotonocity indicates that if a partition is selected by the solution set at some hedonic game, then it is also selected at a hedonic game where this partition improved in the preference ranking of the agents, ceteris paribus. Coalitional unanimity requires that a coalition which is unanimously best for all its members is always part of a partition of the solution set.

Despite the similarities between hedonic games and coalitional ranking problems, in the latter the objective is not to characterize stable partitions but social rankings emanating from coalitional rankings. In particular, $\mathcal{CSR}$ provides social rankings which are consistent with core-partitions.

8. Conclusion

Other stability concepts identifying subsets of "stable" partitions have been designed for hedonic games (see Aziz and Savani, 2016). A possible research agenda would be to replicate our approach for these alternative stability concepts, i.e. adapting the concept to coalitional ranking problems and finding an axiomatic characterization of the social ranking induced by the corresponding stable partitions.

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