Lexicographic solutions for coalitional rankings based on individual and collective performances

Sylvain Béal, Eric Rémila, Philippe Solal

October 2021

Working paper No. 2021 – 07

CRESE
30, avenue de l'Observatoire
25009 Besançon
France
http://crese.univ-fcomte.fr/

Lexicographic solutions for coalitional rankings based on individual and collective performances

Sylvain Béal[a,*], Eric Rémila[b], Philippe Solal[b]

[a] *CRESE EA3190, Univ. Bourgogne-Franche-Comté, F-25000 Besançon, France*
[b] *Université de Saint-Etienne, CNRS UMR 5824 GATE Lyon Saint-Etienne, France*

Abstract

A coalitional ranking describes a situation where a finite set of agents can form coalitions that are ranked according to a weak order. A social ranking solution on a domain of coalitional rankings assigns an individual ranking, that is a weak order over the agent set, to each coalitional ranking of this domain. We introduce two lexicographic solutions for a variable population domain of coalitional rankings. These solutions are computed from the individual performance of the agents, then, when this performance criterion does not allow to decide between two agents, a collective performance criterion is applied to the coalitions of higher size. We provide parallel axiomatic characterizations of these two solutions.

Keywords: Coalitional rankings - Converse consistency - Individual performance - Lexicographic criteria - Path monotonocity.
JEL classification: C71.

1. Introduction

In a large variety of social environments, a population of agents have the possibility to form coalitions in order to cooperate. For many real world applications however, it is not possible to evaluate precisely the worth of the coalitions (e.g. due to the lack of data, the complexity of the problem at hand, etc). In this case, one can be satisfied with qualitative information on the power of these coalitions, expressed by a *coalitional ranking*, which provides binary comparisons between coalitions. This binary relation is supposed to be a weak order, that is, it is a complete and transitive relation over the set of nonempty coalitions. Given this qualitative information on the power of the coalitions, the main objective is to design an individual ranking/weak order over the agent set. A *social ranking solution* on a class of coalitional rankings is defined as a mapping assigning to each coalitional ranking a unique individual ranking over the agent set.

Social ranking solutions have been recently investigated by Khani et al. (2019), Bernardi et al. (2019) and Algaba et al. (2021). Khani et al. (2019) introduce and axiomatically characterize a social ranking solution that is inspired from the Banzhaf index for cooperative voting games

*Corresponding author. Financial support from Université de Lyon (project INDEPTH) within the Programme Investissements d'Avenir (ANR-16-IDEX-0005) and "Mathématiques de la décision pour l'ingénierie physique et sociale" (MODMAD) is acknowledged.

Email addresses: `sylvain.beal@univ-fcomte.fr` (Sylvain Béal), `eric.remila@univ-st-etienne.fr` (Eric Rémila), `philippe.solal@univ-st-etienne.fr` (Philippe Solal)

Preprint submitted to Elsevier *Received: date / Accepted: date*

(Banzhaf, 1964). Bernardi et al. (2019) and Algaba et al. (2021) study lexicographic solutions based on the idea that the most influential agents are those who belong to (small) coalitions ranked in the highest position in the coalitional ranking, and provide several axiomatic characterizations of these solutions.

In this paper, we introduce two new lexicographic solutions for coalition rankings based on the individual and the collective performance of the agents in coalitions. First, the social ranking solution L^P ranks the agent according to the following procedure. If the individual performance of an agent is strictly better than the individual performance of another agent, then the rank of the former agent is strictly better than the one of the second agent. The *individual performance* of an agent is evaluated by the rank of its singleton coalition in the coalitional ranking. In case the individual performance of two agents is identical, that is, if their respective singleton coalitions belong to the same equivalence class of the coalitional ranking, the procedure examines the performance of these agents in coalitions of size two to which they belong. This *collective performance* is measured by the number of such coalitions of size two that are strictly better ranked than their singleton coalition. In this way, one measures the capacity of an agent to cooperate more efficiently with another agent than it would do alone. Given two agents with the same individual performance, if the collective performance of one of these two agents is strictly better than the collective performance of the other agent, then our social ranking solution ranks the first agent ahead the second agent. If these two agents have the same collective performance for coalitions of size two, then the procedure applies the same criterion to coalitions of size three, and so on for each coalition of higher size. Thus, the social ranking solution L^P proceeds lexicographically over the size of the coalitions to break the tie between two agents with identical individual performance.

A drawback of L^P is that it is insensitive to the quality of the performance of coalitions to which an agent may belong, provided this performance is strictly better than the individual performance of this agent. This point leads us to a second social ranking solution, denoted by L^{P^*}, designed to correct this bias: if two agents have identical individual performance, then the coalitions of size two whose performance is strictly better than this individual performance are explored, starting with the best equivalence class. If the number of coalitions of size two containing the first agent is strictly greater than the number of coalitions of size two containing the second agent in the best equivalence class, then the first agent is ranked ahead the second agent. Otherwise, that is, if these two numbers coincide, one moves to the second best equivalence class and proceeds in the same way. One continues to explore each equivalence class from the best equivalence to the equivalence class preceding the one measuring the individual performance of these two agents. If the procedure does not allow to break the tie between these two agents, then the procedure continues by exploring the collective performance of coalitions of size three and so on. Thus, the social ranking solution L^{P^*} proceeds with a double lexicographical criterion: the criterion on the size of the coalitions is applied first, then the criterion on the index of the equivalence classes (from the best equivalence to the equivalence class of the singleton coalition) is used.

We provide comparable axiomatic characterizations of L^P and L^{P^*}. To that end, several value judgments about social rankings solutions, expressed by the following list of principles, are introduced: a principle of neutrality, which indicates that each agent are impartially treated; a very weak principle of anonymity for coalitions, which indicates that only the size of the coalitions to which two agents may belong matters to rank these two agents; a principle of independence of irrelevant coalitions, which indicates that the positions of some coalitions are irrelevant to rank two agents; a principle of monotonicity which indicates how the ranking between two agents behave when the collective performance of one of the two agents change between two coalitional rankings; a principle of standardness for the two-agent case, which says that the first agent is better rank than the second agent if and only the individual performance of the first agent is strictly better than the one of the second agent; and a principle of converse consistency, which allows to deduce the ranking of two agents in a group from the knowledge of the ranking of these agents for the associated reduced problems that some subgroups face.

Our work and the literature on coalitional ranking problems have numerous connexions with other literatures and in particular with cooperative games with transferable utility (TU-games). Firstly, the social rankings that we are interested in are similar to the rankings that can be deduced from payoff vectors in TU-games, from richest to poorest. Thus, a social ranking solution can be viewed either as the ordinal counterpart of a value for TU-games or as the inverse problem of the well-known problem of ranking groups of objects from a ranking over the individual objects (see, e.g., Barberà et al., 2004).

Secondly, the two social ranking solutions that we introduce in this article have the same flavor as well-known and recent values for TU-games which mostly rely on the individual performance of the agents. For instance, the Center-of-gravity of the imputation-set (CIS) value (Driessen, Funaki, 1991) that first assigns to each agent its stand-alone worth and distributes the remainder of the worth of the grand coalition equally among all agents. Other examples are the Proportional value (Moriarity, 1975, Zou et al., 2021a) and the family of Proportional surplus division values introduced recently by Zou et al. (2020). The Proportional value distributes the worth of the grand coalition in proportion to the stand-alone worths of its members (Zou et al., 2021a). Each Proportional surplus division value assigns to each agent a compromise between its stand-alone worth and the average stand-alone worths over all agents, and then allocates the remaining worth among the agents in proportion to their stand-alone worth. For all these values, the ranking of agents in the population according to the payoffs they receive depends only on their individual performance, i.e., their stand-alone worth. Contrary to these aforementioned values, Zou et al. (2021b) design a new family of values, called the α-mollified values, that not only adopt the proportional and equal division principles, but also take the worths of all coalitions into account. This is close in spirit from the principles behind our solutions L^P and L^{P^*}.

Thirdly, our axioms are inspired by principles used in the axiomatic literature. Standardness is a well-accepted principle in TU-games which indicates that in two-agent TU-games, each one receives its stand-alone worth plus an equal share of the collective surplus (see, e.g., Hart and Mas Colell, 1989). Converse consistency is used to characterize solutions in several class of problems such that matching problems, fair division problems, cooperative game situations, non-cooperative game situations. Our axiom of neutrality is the counterpart of the axiom of anonymity used in numerous contexts and especially in TU-games. Our axioms of monotonicity are in line with other axioms of monotonicity which describe how a solution is influenced by modifications of the worth of coalitions or of the marginal contribution of the agents (for TU-games, see Megiddo, 1974, Young, 1985, and van den Brink et al., 2013, among others).

The rest of the article is organized as follows. Section 2 introduces the main notation and definitions. Section 3 presents the axioms and some preliminaries results. Section 4 is devoted to the axiomatic characterizations of L^P and L^{P*}. Section 5 provides the logical independence of the axioms used in the main characterization results.

2. Notation and preliminaries

For any finite set A, the notation $|A|$ stands for the cardinality of A. Let $\mathbb{N}$ be the set of potential active agents, and let $\mathcal{F}$ be the collection of all finite and nonempty subsets N of $\mathbb{N}$ containing at least two agents. Given $N \in \mathcal{F}$, containing $n \geq 2$ agents, a coalition of agents is any subset of N. Denote by Ω_N the collection of the $2^n - 1$ nonempty coalitions of N. A **coalitional ranking** is a pair $(N, \succsim)$, where $N \in \mathcal{F}$ and $\succsim$ is a **weak order** (a complete and transitive binary relation) on Ω_N. For any pair of coalitions T and S of Ω_N, $S \succsim T$ means that S is at least as highly ranked as coalition T. We denote by $\succ$ the asymmetric part of $\succsim$ and by $\sim$ its symmetric part. The **quotient set** is the set of all **equivalence classes** $E_1^{(N,\succsim)}, E_2^{(N,\succsim)}, \ldots, E_k^{(N,\succsim)}$, $k \in \{1, \ldots, 2^n - 1\}$, of $(N, \succsim)$. It is denoted by $\Omega_N/\sim$ and is totally ordered by the induced quotient relation $\succ^*$. Without loss of generality, assume that:

$$E_1^{(N,\succsim)} \succ^* E_2^{(N,\succsim)} \succ^* \cdots \succ^* E_k^{(N,\succsim)}.$$

Let $\mathcal{R}_{\Omega_N}$ be the set of coalitional rankings $(N, \succsim)$ that one can construct from the set of nonempty coalitions Ω_N. Let

$$\mathcal{R} = \bigcup_{N \in \mathcal{F}} \mathcal{R}_{\Omega_N},$$

and denote by $\mathcal{R}_N$ the set of **weak orders or individual rankings** on $N \in \mathcal{F}$.

A **social ranking solution** on $\mathcal{R}$ is a function f which assigns to each coalitional ranking $(N, \succsim) \in \mathcal{R}_{\Omega_N}$ a unique individual ranking $f(N, \succsim) \in \mathcal{R}_N$. For any pair of agents i and j of N,

$$i f(N, \succsim) j$$

means that i is at least as highly ranked as j for their participation to the coalition ranking $(N, \succsim)$. We denote by $\succ_{f(N,\succsim)}$ the asymmetric part of $f(N, \succsim)$ and by $\sim_{f(N,\succsim)}$ its symmetric part.

3. Axioms for social ranking solutions

In this section, we introduce a set of properties for a social ranking solution.

Consider any finite set of agents $N \in \mathcal{F}$. Let $\pi : \Omega_N \longrightarrow \Omega_N$ be a permutation on the elements of Ω_N, π^{-1} stands for its inverse, and Π_{Ω_N} denotes the set of such permutations. Given a permutation $\pi \in \Pi_{\Omega_N}$ and a coalitional ranking $(N, \succsim) \in \mathcal{R}_{\Omega_N}$, define the coalitional ranking $(N, \succsim_\pi) \in \mathcal{R}_{\Omega_N}$ as follows:

$$\forall S, T \in \Omega_N, \quad \big[S \succsim_\pi T\big] \Longleftrightarrow \big[\pi^{-1}(S) \succsim \pi^{-1}(T)\big].$$

Next, pick any agent $i \in N$. A permutation $\pi \in \Pi_{\Omega_N}$ is **agent i invariant** if:

$$\forall S \in \Omega_N, \quad [i \in S] \Longrightarrow [i \in \pi(S)].$$

A permutation $\pi \in \Pi_{\Omega_N}$ is **size invariant** if the following holds:

$$\forall S \in \Omega_N, \quad |\pi(S)| = |S|.$$

Denote by $\Pi^*_{\Omega_N}$ the subset of size invariant permutations in Π_{Ω_N}.

The first axiom, called Super Weak Coalitional Anonymity, has been introduced in Algaba et al. (2021). It indicates that the ranking between two distinct agents i and j in N is invariant under any permutation of the coalitions, which is agent i invariant, agent j invariant and size invariant.

Super Weak Coalitional Anonymity A social ranking solution f on $\mathcal{R}$ satisfies Super Weak Coalitional Anonymity if, for each $(N, \succsim) \in \mathcal{R}$, each pair $\{i, j\}$ of distinct agents in N, and each size invariant permutation $\pi \in \Pi^*_{\Omega_N}$ which is also agent i and agent j invariant, it holds that:

$$\forall i, j \in N, \quad \big[if(N, \succsim)j\big] \Longleftrightarrow \big[if(N, \succsim_\pi)j\big].$$

Example 1 *For any $N \in \mathcal{F}$ containing at least two agents, denote by i_m the lowest element of N and by j_m the lowest element of $N \setminus i_m$, so that $i_m < j_m$. Consider the social ranking solution f defined as follows: for any coalitional ranking $(N, \succsim)$ in $\mathcal{R}$:*

1. $\forall i > j_m, \quad i_m \succ_{f(N,\succsim)} i$ *and* $j_m \succ_{f(N,\succsim)} i$*;*
2. $\forall i, j > j_m, \quad i \sim_{f(N,\succsim)} j$*;*
3. (a) *if $N \succ N \setminus \{i_m, j_m\}$, then $i_m \succ_{f(N,\succsim)} j_m$;*
   (b) *if $N \setminus \{i_m, j_m\} \succ N$, then $j_m \succ_{f(N,\succsim)} i_m$;*
   (c) *if $N \setminus \{i_m, j_m\} \sim N$, then $j_m \sim_{f(N,\succsim)} i_m$.*

For each permutation π which is size invariant, agent i_m invariant and agent j_m invariant, we necessarily have $\pi(N) = N$ and $\pi(N \setminus \{i_m, j_m\}) = N \setminus \{i_m, j_m\}$. From this observation, we easily conclude that f satisfies Super Weak Coalitional Anonymity.

Let $\sigma : N \longrightarrow N$ be a **permutation** of the elements of the agent set $N \in \mathcal{F}$; σ^{-1} stands for its inverse, and Σ_N denotes the set of such permutations. For each $S \in \Omega_N$, $\sigma(S)$ denotes the subset of agents $\{\sigma(i) : i \in S\}$. Given a permutation of Σ_N and a coalitional ranking $(N, \succsim) \in \mathcal{R}_{\Omega_N}$, we define the coalitional ranking $(N, \succsim^\sigma) \in \mathcal{R}_{\Omega_N}$ in the following way:

$$\forall S, T \in \Omega_N, \quad \big[S \succsim^\sigma T\big] \Longleftrightarrow \big[\sigma^{-1}(S) \succsim \sigma^{-1}(T)\big].$$

The next axiom, introduced in Bernardi et al. (2019), indicates that the ranking of the agents does not depend on their label.

Neutrality A social ranking solution f satisfies Neutrality if, for each $(N, \succsim) \in \mathcal{R}$ and each permutation σ of Σ_N, it holds that:

$$\forall i, j \in N, \quad \big[if(N, \succsim)j\big] \Longleftrightarrow \big[\sigma(i)f(N, \succsim^\sigma)\sigma(j)\big].$$

Remark 1 *Note that a permutation σ can be viewed as a particular size invariant permutation in $\Pi^*_{\Omega_N}$ by considering the sets $\sigma(S)$, $S \in \Omega_N$. With this convention, we have $\succsim_\sigma = \succsim^\sigma$.*

The next axiom relies on an independence principle. It indicates that if an agent i has a higher individual performance than an another agent j, the relative individual ranking between i and j do not depend on the number of equivalence classes ranked strictly higher (with respect to the quotient weak order) than the one measuring the individual performance of agent i and the number of equivalence classes ranked lower than the one measuring the individual performance of agent j.

Independence of Irrelevant Equivalent Classes A social ranking solution f satisfies Independence of Irrelevant Equivalent Classes if, for each $N \in \mathcal{F}$, each pair of distinct agents $\{i,j\} \subseteq N$, and each pair of coalitional rankings $(N,\succsim),(N,\succsim') \in \mathcal{R}_{\Omega_N}$ such that $\{i\} \succsim \{j\}$, the following holds:

$$\Big[\forall S \in \Omega_N \setminus \{\{i\},\{j\}\},\ \big(S \succ \{i\} \Longleftrightarrow S \succ' \{i\}\big) \wedge \big(\{j\} \succsim S \Longleftrightarrow \{j\} \succsim' S\big)\Big]$$
$$\Longrightarrow \Big[\big(if(N,\succsim)j\big) \Longleftrightarrow \big(if(N,\succsim')j\big)\Big] \tag{1}$$

A weak version of Independence of Irrelevant Equivalent Classes only indicates that the relative individual ranking between i and j does not depend on the number of equivalence classes ranked lower than the one measuring the individual performance of agent j. But, contrary to Independence of Irrelevant Equivalent Classes, this weak version of independence is silent on the impact of the equivalence classes ranked strictly higher than the one measuring the individual performance of agent i on the relative individual ranking between i and j.

Notation: denote by q^i the index of the equivalence class $E_{q^i}^{(N,\succsim)}$ of $(N,\succsim)$ containing $\{i\}$.

Weak Independence of Irrelevant Equivalent Classes A social ranking solution f satisfies Weak Independence of Irrelevant Equivalent Classes if, for each $N \in \mathcal{F}$, each pair of distinct agents $\{i,j\} \subseteq N$, and each pair of coalitional rankings $(N,\succsim),(N,\succsim') \in \mathcal{R}_{\Omega_N}$ such that $\{i\} \succsim \{j\}$, $q^i = q'^i$ and, for $q < q^i$, $E_q^{(N,\succsim)} = E_q^{(N,\succsim')}$, the following holds:

$$\Big[\forall S \in \Omega_N \setminus \{\{i\},\{j\}\},\ (\{j\} \succsim S) \Longleftrightarrow (\{j\} \succsim' S)\Big] \Longrightarrow \Big[(if(N,\succsim)j) \Longleftrightarrow (if(N,\succsim')j)\Big] \tag{2}$$

Remark 2 *Obviously, if a social ranking solution f satisfies Independence of Irrelevant Equivalent Classes, then f satisfies Weak Independence of Irrelevant Equivalent Classes.*

Remark 3 *Note that the condition,*

$$\forall S \in \Omega_N \setminus \{\{i\},\{j\}\},\ (\{j\} \succsim S) \Longleftrightarrow (\{j\} \succsim' S)$$

implies that if $\{i\} \succ \{j\}$, then $\{i\} \succ' \{j\}$. And the more demanding condition,

$$\forall S \in \Omega_N \setminus \{\{i\},\{j\}\},\ \big(S \succ \{i\} \Longleftrightarrow S \succ' \{i\}\big) \wedge \big(\{j\} \succsim S \Longleftrightarrow \{j\} \succsim' S\big)$$

implies the following: for any coalition S such that $\{i\} \succsim S \succ \{j\}$, then $\{i\} \succsim' S \succ' \{j\}$.

The following axiom is an ordinal version of the standardness principle applied to solutions in TU-games. It indicates that in situations where there are only two agents, the ranking over these agents is determined by the ranking of their respective singleton coalitions, that is, the individual ranking does not depend on the coalition formed by these two agents.

Standardness. For each coalition ranking $(\{i,j\},\succsim) \in \mathcal{R}$, the following equivalence holds:

$$\big[\{i\} \succsim \{j\}\big] \iff \big[if(N,\succsim)j\big].$$

The next axiom is based on a converse consistency principle, which is a well-established principle for solutions in TU-games, bargaining situations and allocation problems. In our context, a solution is conversely consistent if, whenever, for some coalitional ranking, an individual ranking has the property that for certain proper subgroups of the agents it involves, the solution chooses the restriction of the individual ranking to the subgroup for the associated reduced problem this subgroup faces, then the individual should be the solution outcome for the problem.

Formally, for each coalitional ranking $(N, \succsim) \in \mathcal{R}$, where N contains at least two elements, and for each agent $k \in N$, define the coalitional ranking $(N \setminus k, \succsim_{-k})$ as:

$$\forall S, T \in \Omega_{N \setminus k}, \quad \big[S \succsim_{-k} T\big] \Longleftrightarrow \big[S \succsim T\big].$$

In words, $(N \setminus k, \succsim_{-k})$ is the restriction of $(N, \succsim)$ to the coalitions of $\Omega_{N \setminus k}$.

Converse Consistency A social ranking solution f satisfies Converse consistency if for each coalition ranking $(N, \succsim) \in \mathcal{R}$ such that N contains at least three elements, the following holds:

$$\forall i, j \in N, \forall k \in N \setminus \{i, j\}, \quad \big[i \succ_{f(N \setminus k, \succsim_{-k})} j\big] \implies \big[i \succ_{f(N, \succsim)} j\big].$$

Coalitional ranking solutions apply to different coalitional rankings in a coherent way. In this respect, it could be desirable to define axioms that impose a restriction on how the individual ranking change according to alternative monotonicity principles applied to coalitional rankings. To define such axioms, additional notions must be introduced.

Let $(N, \succsim)$ and $(N, \succsim')$ be two distinct coalitional rankings in $\mathcal{R}_{\Omega_N}$ and let S_0 be a coalition in Ω_N. The coalitional ranking $(N, \succsim')$ is **obtained from** $(N, \succsim)$ **and coalition** $S_0 \in \Omega_N$ if

$$\forall S, T \in \Omega_N \setminus S_0, \quad \big[S \succsim' T\big] \Longleftrightarrow \big[S \succsim T\big].$$

The transition from $(N, \succsim)$ to $(N, \succsim')$ is a **move** induced by S^0. If, furthermore,

$$\forall T \in \Omega_N \setminus S_0, \quad [S_0 \succsim T] \Longrightarrow [S_0 \succ' T],$$

then the move from $(N, \succsim)$ to $(N, \succsim')$ is S_0**-improving**. In words, a move is S_0-improving if the ranking between any two coalitions other than S_0 does not changed, and S_0 is strictly better ranked after the move. If, moreover,

$$\big[\{i\} \succsim S_0\big] \wedge \big[S_0 \succ' \{i\}\big],$$

this S_0-improving move from $(N, \succsim)$ to $(N, \succsim')$ is **individually rational for agent** i or simply (S_0, i)**-improving**. In such an S_0-improving move, $\{i\}$ is better ranked than S_0 before the move whereas S_0 becomes strictly better ranked than $\{i\}$ after the move.

Reciprocally, a move from $(N, \succsim)$ to $(N, \succsim')$ is S_0**-deteriorating** if

$$\forall T \in \Omega_N \setminus S_0, \quad \big[T \succsim S_0\big] \Longrightarrow \big[T \succ' S_0\big].$$

And, this deteriorating move from $(N, \succsim)$ to $(N, \succsim')$ is (S_0, i)**-deteriorating** if

$$\big[S_0 \succ \{i\}\big] \wedge \big[\{i\} \succsim' S_0\big].$$

In dynamic frameworks where the interactions and the performance of coalitions evolve rapidly in time, it could be meaningful to look at a sequence of moves either in favor or to the detriment of a single agent i and without affecting another agent j. To this end, we first introduce the notion of path and then the notion of ij-path. Consider two coalitional rankings $(N, \succsim)$ and $(N, \succsim')$ in $\mathcal{R}_{\Omega_N}$. A **path** between $(N, \succsim)$ and $(N, \succsim')$ is constituted by a sequence $(N, \succsim^{\ell})_{\ell=0}^{t} \subseteq \mathcal{R}_{\Omega_N}$ of pairwise distinct coalitional rankings and a sequence of pairwise distinct coalitions $(S^{\ell})_{\ell=0}^{t-1} \subseteq \Omega_N$ such that:

- $\succsim^0 = \succsim$ and $\succsim^t = \succsim'$;
- for each $\ell \in \{0, \ldots, t-1\}$, the transition from $(N, \succsim^\ell)$ to $(N, \succsim^{\ell+1})$ is a move induced by coalition S^ℓ from $(N, \succsim^\ell)$.

For two distinct agents i and j, a path is an ij-**path** if, moreover,

- for each $\ell \in \{0, \ldots, t-1\}$, $S^\ell \cap \{i, j\} = \{i\}$.

Thus an ij-path is formed by sequence of moves (improving or deteriorating) implying agent i and not agent j in each step. Recently, Moretti et al. (2021) introduce a similar notion of ij-path, which is more restrictive than the above one.

Further, an ij-path is **improving for agent** i if:

- all the moves induced by the coalitions S^ℓ, $\ell \in \{0, \ldots, t-1\}$, along the sequence are either (S^ℓ, i)-improving (that is, S^ℓ-improving and individually rational for agent i) or (S^ℓ, i)-deteriorating, and the first move of the sequence induced by coalition S^0 is (S^0, i)-improving.

Remark 4 *Note that along an improving ij-path for agent i, one necessarily has $S^\ell \neq \{i\}$ for each $\ell \in \{0, \ldots, t-1\}$. This follows from the definition of an (S^ℓ, i)- improving/deteriorating move.*

Among the ij-paths that are improving for agent i, we are interested in those which give the **priority to the smallest coalition**:

- the (first) (S^0, i)-improving move is such that, for each $\ell \in \{1, \ldots, t-1\}$, $|S^0| \leq |S^\ell|$ and $|S^0| = |S^\ell|$ implies $\{i\} \succsim^\ell S^\ell$.

In these paths, which are improving for agent i, the size of the coalition inducing the first improving and individually rational move is smaller than the size of any other coalition inducing a move along the sequence; and if another coalition of the same size induces a move at some step of the path, then it is ranked below the singleton coalition associated with the agent. This means that if a move along the path is induced by a coalition S^ℓ of size $|S^0|$, then it is necessarily (S^ℓ, i)-improving, and so $S^0 \succ^1 \{i\} \succsim^\ell S^\ell$.

Example 2 *Let $N = \{1, 2, 3, 4\}$, and consider the coalitional ranking $(N, \succsim)$ containing the following four equivalence classes:*

$$E_1^{(N,\succsim)} = \big\{\{2,3,4\}, \{2,4\}\big\}, \quad E_2^{(N,\succsim)} = \big\{\{1,3,4\}, \{1,2,3\}, \{1,2,4\}, \{2,3\}, \{1,4\}\big\},$$

$$E_3^{(N,\succsim)} = \big\{N, \{3,4\}, \{1,3\}, \{1\}, \{2\}, \{3\}\big\}, \quad E_4^{(N,\succsim)} = \big\{\{1,2\}, \{4\}\big\}.$$

One constructs the 14-path $(\succsim^0, \succsim^1, \succsim^2, \succsim^3)$, where $\succsim^0 = \succsim$, $S^0 = \{1,2\}$, $S^1 = \{1,2,3\}$, $S^2 = \{1,3\}$, and

$$E_1^{(N,\succsim^1)} = E_1^{(N,\succsim)},\ E_2^{(N,\succsim^1)} = E_2^{(N,\succsim)} \cup \big\{\{1,2\}\big\},\ E_3^{(N,\succsim^1)} = E_3^{(N,\succsim)},\ E_4^{(N,\succsim^1)} = E_4^{(N,\succsim)} \setminus \{\{1,2\}\},$$

$$E_1^{(N,\succsim^2)} = E_1^{(N,\succsim^1)},\ E_2^{(N,\succsim^2)} = E_2^{(N,\succsim^1)} \setminus \{\{1,2,3\}\},\ E_3^{(N,\succsim^2)} = E_3^{(N,\succsim^1)} \cup \{\{1,2,3\}\},\ E_4^{(N,\succsim^2)} = E_4^{(N,\succsim^1)},$$

$$E_1^{(N,\succsim^3)} = E_1^{(N,\succsim^2) \cup \big\{\{1,3\}\big\}},\quad E_2^{(N,\succsim^3)} = E_2^{(N,\succsim^2)},\ E_3^{(N,\succsim^3)} = E_3^{(N,\succsim^2)} \setminus \{\{1,3\}\},\ E_4^{(N,\succsim^3)} = E_4^{(N,\succsim^2)}.$$

Therefore, the move from $(N, \succsim^0) = (N, \succsim)$ to $(N, \succsim^1)$ is S^0-improving, the move from $(N, \succsim^1)$ to $(N, \succsim^2)$ is S^1-deteriorating, and the move from $(N, \succsim^2)$ to $(N, \succsim^3)$ is S^2-deteriorating. More specifically, the move from $(N, \succsim^0) = (N, \succsim)$ to $(N, \succsim^1)$ is individually rational for agent 1 since

$$\left[\{1\} \succsim^0 S^0\right] \wedge \left[S^0 \succ^1 \{1\}\right],$$

so that this move is $(S^0, 1)$-improving. The second move from $(N, \succsim^1)$ to $(N, \succsim^2)$ is $(S^1, 1)$-deteriorating since

$$\left[S^1 \succ^1 \{1\}\right] \wedge \left[\{1\} \sim^2 S^1\right],$$

and, the last move from $(N, \succsim^2)$ to $(N, \succsim^3)$ is $(S^2, 1)$-improving since

$$\left[\{1\} \sim^2 S^2\right] \wedge \left[S^2 \succ^3 \{1\}\right].$$

Therefore, this 14-path is improving for agent 1 and gives the priority to the smallest coalition since $|S^0| = |S^2| < |S^1|$ and $S^0 \succ^2 \{1\} \sim^2 S^2$.

■

We have the material to define two new axioms of monotonicity. The first axiom reflects the following principle. Suppose that an ij-path exists between two coalitional rankings $(N, \succsim)$ and $(N, \succsim')$. Assume further that this ij-path is improving for agent i and gives the priority to the smallest coalition. The principle indicates that in such a situation if agent i is at least highly ranked than agent j in $f(N, \succsim)$, then i becomes strictly better ranked than j in $f(N, \succsim')$. In this sense, this principle gives priority to the smallest coalition whatever the number of improving moves and deteriorating moves along this ij-path.

Individual Improving Path Monotonicity with Priority to the Smallest Coalition: A social ranking solution f satisfies Individual Improving Path Monotonicity with Priority to the Smallest Coalition if, for each coalition ranking $(N, \succsim) \in \mathcal{R}$, each pair $\{i, j\} \subseteq N$ of distinct agents, the following holds: for each ij-path from $(N, \succsim)$ to another coalitional ranking $(N, \succsim') \in \mathcal{R}$, which is improving for agent i and gives the priority to the smallest coalition, we have

$$\left[i f(N, \succsim) j\right] \Longrightarrow \left[i \succ_{f(N, \succsim')} j\right].$$

One also introduces a strong version of the above axiom. To this end, one considers a less restrictive class of ij-paths. This class contains the ij-paths that satisfy the following condition:

- all the moves induced by the coalitions S^ℓ, $\ell \in \{0, \ldots, t-1\}$ are such that $S^\ell \neq \{i\}$ and the first move of the sequence induced by coalition S^0 is S^0-improving and $S^0 \succ^1 \{i\}$.

If an ij-path satisfies the above condition, one says that it is **weakly improving for agent** i. Along such a path, one necessarily has either $S^\ell \succ^{\ell+1} \{i\}$ or $\{i\} \succsim^{\ell+1} S^\ell$ for each $\ell \in \{0, \ldots, t-1\}$ and $S^0 \succ^1 \{i\}$. But contrary to the notion of improving ij-path for agent i, it may be the case that $S^\ell \succ^\ell \{i\}$ and $S^\ell \succsim^{\ell+1} \{i\}$ or $\{i\} \succsim^\ell S^\ell$ and $\{i\} \succ^{\ell+1} S^\ell$. By definition and Remark 4, if an ij-path is improving for agent i, it is also weakly improving for agent i, but the converse is not true.

Finally, if such a weak improving ij-path for agent i satisfies the following additional condition:

- the first move induced by S^0 is such that, for each $\ell \in \{1, \ldots, t-1\}$, $|S^0| \leq |S^\ell|$, and $|S^0| = |S^\ell|$ implies $S^0 \succ^\ell S^\ell$,

then we say that this path gives the **priority to the smallest coalition**.

All in all, the class of ij-paths that are improving for agent i and give the priority to the smallest coalition formed a subset of the class of ij-paths that are weakly improving for agent i and give the priority to the smallest coalition. The following axiom is a strong version of Individual Improving Path Monotonicity with Priority to the Smallest Coalition, which uses the larger class of weak improving ij-paths for agent i.

Strong Individual Improving Path Monotonicity with Priority to the Smallest Coalition: A social ranking solution f satisfies Strong Individual Improving Path Monotonicity with Priority to the Smallest Coalition if, for each coalition ranking $(N, \succsim) \in \mathcal{R}$, each each pair $\{i, j\} \subseteq N$ of distinct agents, the following holds: for each ij-path from $(N, \succsim)$ to another coalitional ranking $(N, \succsim') \in \mathcal{R}$, which is weakly improving for agent i and gives the priority to the smallest coalition, we have

$$\big[i f(N, \succsim) j\big] \Longrightarrow \big[i \succ_{f(N, \succsim')} j\big].$$

Obviously, if a social ranking solution f satisfies Strong Individual Improving Path Monotonicity with Priority to the Smallest Coalition, then f satisfies Individual Improving Path Monotonicity with Priority to the Smallest Coalition, but not *vice et versa*.

4. Axiomatic study

Given a coalitional ranking $(N, \succsim) \in \mathcal{R}$, where $|N| = n$ and an agent $i \in N$, we construct the matrix $M^{(N,\succsim),i}$ of size (n, k) where each entry $M^{(N,\succsim),i}_{(p,q)}$ denotes the number of coalitions in the set

$$E^{(N,\succsim),i}_{(p,q)} = E^{(N,\succsim)}_q \cap \big\{S \in \Omega_N : S \ni i, |S| = p\big\} \tag{3}$$

that is, $M^{(N,\succsim),i}_{(p,q)}$ is the number of coalitions of size $p \leq n$ containing i and belonging to the equivalence class $E^{(N,\succsim)}_q$, where $q \leq k$. For each p, it holds that:

$$\sum_{q=1}^{k} M^{(N,\succsim),i}_{(p,q)} = \binom{n-1}{p-1}, \quad \text{and so} \quad \sum_{p=1}^{n}\sum_{q=1}^{k} M^{(N,\succsim),i}_{(p,q)} = 2^{n-1}. \tag{4}$$

The first result, already established in Algaba et al. (2021), shows that if a social ranking solution satisfies Super Weak Coalitional Anonymity and Neutrality, then two agents with the same matrix with respect to a coalitional ranking obtain the same individual ranking.

Proposition 1 *(Algaba et al., 2021) Let f be a social ranking solution on $\mathcal{R}$ satisfying Neutrality and Super Weak Coalitional Anonymity. For each $(N, \succsim) \in \mathcal{R}$, it holds that:*

$$\forall i, j \in N, \quad \big[M^{(N,\succsim),i} = M^{(N,\succsim),j}\big] \Longrightarrow \big[i \sim_{f(N,\succsim)} j\big].$$

The next result establishes that the combination of Standardness and Converse Consistency leads to a class of social ranking solutions in which the individual performance of the agents plays a key role to rank them. Precisely, if the individual performance of an agent, measured by the position of its singleton coalition, is strictly better than the individual performance of another agent, then the solution ranks the first agent ahead the second agent.

Proposition 2 *Let f be a social ranking solution on $\mathcal{R}$ satisfying Standardness and Converse Consistency, then, for each $(N, \succsim) \in \mathcal{R}$, it holds that:*

$$\forall i, j \in N, \quad \big[\{i\} \succ \{j\}\big] \Longrightarrow \big[i \succ_{f(N,\succsim)} j\big] \tag{5}$$

Proof. We proceed by induction on the number $n \geq 2$ of agents in a coalition ranking $(N, \succsim)$.
Initial step: If $n = 2$, assertion (5) holds by Standardness.
Induction hypothesis: Assume that $n \geq 2$ and that assertion (5) holds for all coalitional rankings with at most n agents.
Induction step: Pick any coalition ranking $(N, \succsim) \in \mathcal{R}$ containing $n+1$ agents. Assume that $\{i\} \succsim \{j\}$. Obviously, for $k \in N \setminus \{i, j\}$, we still have $\{i\} \succ_{-k} \{j\}$. By the induction hypothesis, we get $i \succ_{f(N \setminus k, \succsim_{-k})} j$. By Converse consistency, we obtain $i \succ_{f(N,\succsim)} j$. This completes the proof of the induction step. ■

Proposition 2 is silent on situations where two agents have the same individual performance, that is, for coalition rankings where the singleton coalitions of two agents belong to the same equivalence class. To complete the individual ranking, one introduces a new social ranking solution which takes into account both the individual performance of the agents and their performance in larger coalitions to discriminate between them when their individual performance is identical. In situations where two agents have the same individual performance, one examines the performance of these agents in coalitions of size two to which they may belong. This collective performance is measured by the number of such coalitions of size two that are strictly better ranked than the singleton coalitions. In this way, one measures the capacity of an agent to cooperate more efficiently with another agent than it would do alone. Given two agents with the same individual performance, if the collective performance of one of these two agents is strictly better than the collective performance of the other agent, then our social ranking solution ranks the first agent ahead the second agent. If these two agents have the same collective performance for coalitions of size two, then one applies the same procedure to coalitions of size three, and continues in this fashion for each coalition of higher size. Thus, this social ranking solution proceeds lexicographically to break the tie between two agents with identical individual performance: the procedure explores each row of the matrix starting with the second row (coalitions of size two) to the $(n-1)$th row. In particular, if the matrices of these two agents are identical with respect to a coalitional ranking, then our social ranking solution attributes to these agents the same individual rank. To formally define the above social ranking solution, a notation is needed.

Notation: given a coalition ranking $(N, \succsim) \in \mathcal{R}$ and two distinct agents $i, j \in N$ with the same individual performance, we denote by q^{ij} the index of the equivalence class $E_{q^{ij}}^{(N,\succsim)}$ to which $\{i\}$ and $\{j\}$ belong, i.e.,

$$M_{(1,q^{ij})}^{(N,\succsim),i} = M_{(1,q^{ij})}^{(N,\succsim),j} = 1.$$

Definition 1 The social ranking solution L^P on $\mathcal{R}$ is defined as follows: for each $(N, \succsim) \in \mathcal{R}$ and pair $\{i, j\}$ of distinct agents in N, $i \succ_{L^P(N,\succsim)} j$ if one of the following conditions holds:

1. $\{i\} \succ \{j\}$;
2. $\{i\} \sim \{j\}$ and there exists $p_0 \in \{2, \ldots, n-1\}$, such that:

- for $p \in \{2, \dots, p_0 - 1\}$, it holds that:

$$\sum_{q<q^{ij}} M^{(N,\succsim),i}_{(p,q)} = \sum_{q<q^{ij}} M^{(N,\succsim),j}_{(p,q)};$$

- and, for p_0, it holds that:

$$\sum_{q<q^{ij}} M^{(N,\succsim),i}_{(p_0,q)} > \sum_{q<q^{ij}} M^{(N,\succsim),j}_{(p_0,q)}.$$

Example 3 *Let $N = \{1, 2, 3, 4\}$, and consider the coalitional ranking $(N, \succsim)$ containing the following four equivalence classes:*

$$E^{(N,\succsim)}_1 = \{\{2,3,4\}, \{1,2\}, \{2,4\}\}, \quad E^{(N,\succsim)}_2 = \{\{1,3,4\}, \{1,2,3\}, \{1,3\}, \{2,3\}, \{1,4\}\},$$

$$E^{(N,\succsim)}_3 = \{N, \{3,4\}, \{1\}, \{2\}, \{3\}\}, \quad E^{(N,\succsim)}_4 = \{\{1,2,4\}, \{4\}\}.$$

So, for each $i \in N$, $M^{(N,\succsim),i}$ is an $(4, 4)$ matrix:

$$M^{(N,\succsim),1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad M^{(N,\succsim),2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad M^{(N,\succsim),3} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$M^{(N,\succsim),4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Here, $\{1\} \sim \{2\}$ and $q^{12} = 3$. And, for $p = \{2, 3, 4\}$, one has:

$$\sum_{q<3} M^{(N,\succsim),1}_{(p,q)} = \sum_{q<3} M^{(N,\succsim),2}_{(p,q)}.$$

Thus, there is no p_0 such that

$$\sum_{q<3} M^{(N,\succsim),1}_{(p_0,q)} > \sum_{q<3} M^{(N,\succsim),2}_{(p_0,q)}.$$

By Definition 1, one has $1 \sim_{L^P(N,\succsim)} 2$. One also has $\{3\} \succ \{4\}$ so that, by point 1 of Definition 1, $3 \succ_{L^P(N,\succsim)} 4$. Because $\{1\} \succ \{4\}$ and $\{2\} \succ \{4\}$, one gets $1 \succ_{L^P(N,\succsim)} 4$ and $2 \succ_{L^P(N,\succsim)} 4$. One sees that $\{1\} \sim \{2\} \sim \{3\}$ so that $q^{23} = q^{13} = q^{12} = 3$. Furthermore, for $p_0 = 2$, one has

$$\sum_{q<3} M^{(N,\succsim),2}_{(2,q)} = 3 > \sum_{q<3} M^{(N,\succsim),3}_{(2,q)} = 2,$$

from which one concludes that $2 \succ_{L^P(N,\succsim)} 3$. All in all, one obtains the following individual ranking,

$$1 \sim_{L^P(N,\succsim)} 2 \succ_{L^P(N,\succsim)} 3 \succ_{L^P(N,\succsim)} 4.$$

■

The following result provides a characterization of the L^P social ranking solution on $\mathcal{R}$.

Theorem 1 *The social ranking solution L^P is the unique social ranking solution on $\mathcal{R}$ satisfying Super Weak Coalitional Anonymity, Neutrality, Standardness, Converse Consistency, Independence of Irrelevant Equivalence Classes, and Individual Improving Path Monotonicity with Priority to the Smallest Coalition.*

Before turning to the proof of Theorem 1, we need a definition and a lemma.

Definition 2 Let $(N, \succsim) \in \mathcal{R}_{\Omega_N}$ and two distinct agents $i, j \in N$ with identical individual performance, that is, $\{i\} \sim \{j\}$. Define the coalitional ranking $(N, \succsim^{m,ij}) \in \mathcal{R}$ obtained from $(N, \succsim) \in \mathcal{R}_{\Omega_N}$ and $i, j \in N$ as follows: as above q^{ij} denotes the index of the equivalence class $E_{q^{ij}}^{(N,\succsim)}$ to which $\{i\}$ and $\{j\}$ belong. If there exists a coalition S such that $S \succ \{i\}$, then $(N, \succsim^{m,ij})$ contains two equivalent classes $E_1^{(N,\succsim^{m,ij})}$ and $E_2^{(N,\succsim^{m,ij})}$, such that:

$$E_1^{(N,\succsim^{m,ij})} = \bigcup_{q<q^{ij}} E_q^{(N,\succsim)} \text{ and } E_2^{(N,\succsim^{m,ij})} = \bigcup_{q \geq q_{ij}} E_q^{(N,\succsim)},$$

and

$$E_1^{(N,\succsim^{m,ij})} (\succ^{m,ij})^* E_2^{(N,\succsim^{m,ij})}.$$

Otherwise $(N, \succsim^{m,ij})$ contains only one equivalent class, i.e., all coalitions have the same rank in $(N, \succsim^{m,ij})$.

Example 4 *Consider Example 3. The coalitional ranking $(N, \succsim^{m,12})$ contains two equivalence classes:*

$$E_1^{(N,\succsim^{m,12})} = \big\{\{2,3,4\}, \{1,2\}, \{2,4\}, \{1,3,4\}, \{1,2,3\}, \{1,3\}, \{2,3\}, \{1,4\}\big\}$$

and

$$E_2^{(N,\succsim^{m,12})} = \big\{N, \{3,4\}, \{1\}, \{2\}, \{3\}, \{1,2,4\}, \{4\}\big\}.$$

■

Lemma 1 *Let f be a social ranking solution on $\mathcal{R}$ satisfying Independence of Irrelevant Equivalent Classes. Then, for each $(N, \succsim) \in \mathcal{R}$ and each pair of distinct agents $\{i, j\} \subseteq N$ such that $\{i\} \sim \{j\}$, the following equivalence holds:*

$$\big[i f(N, \succsim) j\big] \iff \big[i f(N, \succsim^{m,ij}) j\big].$$

Proof. It suffices to observe that, by construction, $(N, \succsim^{m,ij})$ satisfies condition (1) of Independence of Irrelevant Equivalent Classes. Therefore, the result follows by an application of Independence of Irrelevant Equivalent Classes. ■

Proof. (of Theorem 1). We first show that the social ranking solution L^P satisfies the axioms of the statement of Theorem 1.

Super Weak Coalitional Anonymity. Consider any coalitional ranking $(N, \succsim) \in \mathcal{R}$ and any permutation $\pi \in \Pi^*_{\Omega_N}$ satisfying the hypotheses of Super Weak Coalitional Anonymity for agent i and agent j. This permutation does not change the associated matrix of agent i and agent j, that is, $M^{(N,\succsim),i} = M^{(N,\succsim_\pi),i}$ and $M^{(N,\succsim),j} = M^{(N,\succsim_\pi),j}$. Thus, by Definition 1, L^P satisfies Super Weak Coalitional Anonymity.

Neutrality. Consider any coalitional ranking $(N,\succsim) \in \mathcal{R}$ and any permutation $\sigma \in \Sigma_N$. We have $M^{(N,\succsim),i} = M^{(N,\succsim^\sigma),\sigma(i)}$ and $M^{(N,\succsim),j} = M^{(N,\succsim^\sigma),\sigma(j)}$. Thus, by Definition 1, L^P satisfies Neutrality.

Standardness. If $N = \{i,j\}$, then Definition 1 of L^P indicates that, for each $(N,\succsim) \in \mathcal{R}$, $i \succ_{L^P} j$ if and only if $\{i\} \succ \{j\}$, which ensures that L^P satisfies Standardness.

Converse Consistency. Consider any coalitional ranking $(N,\succsim) \in \mathcal{R}$, where $|N| \geq 3$, and any two distinct agents $i,j \in N$. Two cases arise:

Case 1 If $\{i\} \succ \{j\}$, then, for each $k \in N \setminus \{i,j\}$, we obviously have $\{i\} \succ_{-k} \{j\}$ and so, by Definition 1 of L^P, $i \succ_{L^P(N\setminus k, \succsim_{-k})} j$. Because, $\{i\} \succ \{j\}$ we also have $i \succ_{L^P(N,\succsim)} j$, as desired.

Case 2 If $\{i\} \sim \{j\}$, consider, for each $p \geq 2$, the set, possibly empty, of coalitions of size p containing i but not j whose rank is strictly above $\{i\}$:

$$D_p^{(N,\succsim),i,\bar{j}} = \big\{S \in \Omega_N : S \ni i, S \not\ni j, |S| = p, S \succ \{i\}\big\}.$$

In a similar way, define $D_p^{(N,\succsim),i,j}$ as the set, possibly empty, of coalitions of size p containing both i and j and whose rank is strictly above $\{i\}$:

$$D_p^{(N,\succsim),i,j} = \big\{S \in \Omega_N : S \supseteq \{i,j\}, |S| = p, S \succ \{i\}\big\}.$$

By Definition 1, $i \succ_{L^P(N,\succsim)} j$ if and only if there exists an integer p_0 such that for $p < p_0$, $|D_p^{(N,\succsim)i,\bar{j}}| = |D_p^{(N,\succsim),\bar{i},j}|$ and $|D_{p_0}^{(N,\succsim),i,\bar{j}}| > |D_{p_0}^{(N,\succsim),\bar{i},j}|$. This assertion follows from the fact that

$$\sum_{q<q^{ij}} M_{pq}^{(N,\succsim),i} = |D_p^{(N,\succsim),i,\bar{j}}| + |D_p^{(N,\succsim),i,j}| \quad \text{and} \quad \sum_{q<q^{ij}} M_{pq}^{(N,\succsim),j} = |D_p^{(N,\succsim),\bar{i},j}| + |D_p^{(N,\succsim),i,j}|.$$

We have

$$(n-p-1)|D_p^{(N,\succsim),i,\bar{j}}| = \sum_{k \in N\setminus\{i,j\}} |D_p^{(N\setminus k,\succsim_{-k}),i,\bar{j}}| \tag{6}$$

To see why equality (6) holds, it suffices to note that each $S \in D_p^{(N,\succsim)i,\bar{j}}$ belongs to $D_p^{(N\setminus k,\succsim_{-k}),i,\bar{j}}$ if and only if $k \notin (S \cup j)$; and there are exactly $n-(p+1) = n-p-1$ such k. Thus, to show that L^P satisfies Converse Consistency, assume that

$$\forall k \in N \setminus \{i,j\}, \quad i \succ_{L^P(N\setminus k,\succsim_{-k})} j.$$

Then, there exists an integer p_0^k such that

$$\forall p < p_0^k, \quad |D_p^{(N\setminus k,\succsim_{-k}),i,\bar{j}}| = |D_p^{(N\setminus k,\succsim_{-k}),\bar{i},j}|, \quad \text{and} \quad |D_{p_0^k}^{(N\setminus k,\succsim_{-k}),i,\bar{j}}| > |D_{p_0^k}^{(N\setminus k,\succsim_{-k}),\bar{i},j}|.$$

Let $p_0 = \min\{p_0^k : k \in N \setminus \{i,j\}\}$. By definition of p_0, for each $p < p_0$, we have,

$$\forall k \in N \setminus \{i,j\}, \quad |D_p^{(N\setminus k,\succsim_{-k}),i,\bar{j}}| = |D_p^{(N\setminus k,\succsim_{-k}),\bar{i},j}|.$$

Thus, by equality (6), we obtain

$$\forall p < p_0, \quad |D_p^{(N,\succsim),i,\bar{j}}| = |D_p^{(N,\succsim),\bar{i},j}|.$$

And, by definition of p_0, we have,

$$\forall k \in N \setminus \{i,j\}, \quad |D_{p_0}^{(N\setminus k, \succsim_{-k}), i, \bar{j}}| \geq |D_{p_0}^{(N\setminus k, \succsim_{-k}), \bar{i}, j}|.$$

Furthermore, there exists $k_0 \in N \setminus \{i,j\}$ such that

$$|D_{p_0}^{(N\setminus k_0, \succsim_{-k_0}), i, \bar{j}}| > |D_{p_0}^{(N\setminus k_0, \succsim_{-k_0}), \bar{i}, j}|.$$

Thus, by equality (6), we have

$$|D_{p_0}^{(N,\succsim) i, \bar{j}}| > |D_{p_0}^{(N,\succsim), \bar{i}, j}|,$$

and so, by Definition 1, $i \succ_{L^P(N,\succsim)} j$. Conclude that L^P satisfies Converse Consistency.

Independence of Irrelevant Equivalent Classes is satisfied by L^P because, by Definition 1, for each $(N,\succsim) \in \mathcal{R}$ such that $\{i\} \sim \{j\}$, the relative individual ranking between i and j depends only on the number of coalitions ranked higher that $\{i\}$ and $\{j\}$ and not on the number and composition of the equivalent classes ranked higher than $E_{q^{ij}}^{(N,\succsim)}$. Thus, if $(N,\succsim)$ and $(N,\succsim') \in \mathcal{R}$ meet condition (1), then the relative individual ranking between i and j in the two coalitional rankings is the same under L^P.

Individual Improving Path Monotonicity with Priority to the Smallest Coalition follows directly from Definition 1.

We now show that L^P is the unique social ranking solution on $\mathcal{R}$ satisfying Super Weak Coalitional Anonymity, Neutrality, Standardness, Converse Consistency, Invariance of Independence of Irrelevant Equivalent Classes, and Individual Improving Path Monotonicity with Priority to the Smallest Coalition. Let f be a social ranking social satisfying these six axioms. We show that $f = L^P$. Pick any $N \in \mathcal{F}$ and any $i,j \in N$. We proceed in two steps.
Step 1 Assume that $i \succ_{L^P(N,\succsim)} j$. To show: $i \succ_{f(N,\succsim)} j$. We distinguish two exclusive cases:

(a) $\{i\} \succ \{j\}$. Then, by Proposition 2, we get $i \succ_{f(N,\succsim)} j$.
(b) $\{i\} \sim \{j\}$. Consider the coalition ranking $(N, \succsim^{m,ij})$ obtained from $(N,\succsim)$ as defined in Definition 2. Because $i \succ_{L^P(N,\succsim)} j$, by Lemma 1, we also have $i \succ_{L^P(N,\succsim^{m,ij})} j$. Because $i \succ_{L^P(N,\succsim^{m,ij})} j$, there exists a coalition $S \in \Omega_N$ such that $S \ni i$ and $S \succ^{m,ij} \{i\}$, meaning that $\succsim^{m,ij}$ contains two equivalence classes $E_1^{(N,\succsim^{m,ij})}$ and $E_2^{(N,\succsim^{m,ij})}$. This implies that both matrices $M^{(N,\succsim^{m,ij}),i}$ and $M^{(N,\succsim^{m,ij}),j}$ have two columns, and there exists an integer p_0 such that

$$\forall p < p_0, \quad M_{(p,1)}^{(N,\succsim^{m,ij}),i} = M_{(p,1)}^{(N,\succsim^{m,ij}),j} \quad \text{and} \quad M_{(p_0,1)}^{(N,\succsim^{m,ij}),i} > M_{(p_0,1)}^{(N,\succsim^{m,ij}),j}.$$

From the equivalence classes $E_1^{(N,\succsim^{m,ij})}$ and $E_2^{(N,\succsim^{m,ij})}$, construct another coalitional ranking $(N,\succsim')$ in the following way. For each size $p \in \{p_0, \ldots, n-1\}$,[1]

[1]Note that, for each coalitional ranking $(N,\succsim)$, we necessarily have $M_{(n,q)}^{(N,\succsim),i} = M_{(n,q)}^{(N,\succsim),j}$ whatever $q \in \{1,\ldots,k\}$ and the pair $\{i,j\} \subseteq N$.

1. if $M^{(N,\succsim^{m,ij}),i}_{(p,1)} > M^{(N,\succsim^{m,ij}),j}_{(p,1)}$, then there are at least $M^{(N,\succsim^{m,ij}),i}_{(p,1)} - M^{(N,\succsim^{m,ij}),j}_{(p,1)}$ coalitions of size p in $E_1^{(N,\succsim^{m,ij})}$ containing i and not j. These $M^{(N,\succsim^{m,ij}),i}_{(p,1)} - M^{(N,\succsim^{m,ij}),j}_{(p,1)}$ coalitions are moved towards the equivalence class $E_2^{(N,\succsim^{m,ij})}$;
2. if $M^{(N,\succsim^{m,ij}),i}_{(p,1)} < M^{(N,\succsim^{m,ij}),j}_{(p,1)}$, then $M^{(N,\succsim^{m,ij}),i}_{(p,2)} > M^{(N,\succsim^{m,ij}),j}_{(p,2)}$ due to the fact that

$$M^{(N,\succsim^{m,ij}),i}_{(p,1)} + M^{(N,\succsim^{m,ij}),i}_{(p,2)} = M^{(N,\succsim^{m,ij}),j}_{(p,1)} + M^{(N,\succsim^{m,ij}),j}_{(p,2)} = 2^{p-1}.$$

   Thus, there are at least $M^{(N,\succsim^{m,ij}),i}_{(p,2)} - M^{(N,\succsim^{m,ij}),j}_{(p,2)}$ coalitions of size p in $E_2^{\succsim(N,\succsim^{m,ij})}$ containing i and not j. These $M^{(N,\succsim^{m,ij})i}_{(p,2)} - M^{(N,\succsim^{m,ij}),j}_{(p,2)}$ coalitions are moved towards the equivalence class $E_1^{(N,\succsim^{m,ij})}$.

From points 1. and 2., one obtains a new coalitional ranking $(N,\succsim')$ such that $M^{(N,\succsim'),i} = M^{(N,\succsim'),j}$. By Proposition 1, $i \sim_{f(N,\succsim')} j$.

Consider now the path from $(N,\succsim')$ to $(N,\succsim^{m,ij})$. The key point is that $(N,\succsim^{m,ij})$ can be obtained from $(N,\succsim')$ by ij-path which is improving for agent i. Furthermore, this ij-path can be constructed is such way that that the first (improving) move is induced by a coalition of size p_0. Precisely, such an ij-path, formed by the sequences $(N,\succsim^\ell)_{\ell=0}^t \subseteq \mathcal{R}_{\Omega_N}$ and $(S^\ell)_{\ell=0}^{t-1}$, starts by an improving move through a coalition S^0 of size p_0 such that $S^0 \sim' \{i\}$ and $S^0 \succ^{m,ij} \{i\}$. For each p such that $M^{(N,\succsim^{m,ij}),i}_{(p,1)} > M^{(N,\succsim^{m,ij}),j}_{(p,1)}$ the ij-path contains $M^{(N,\succsim^{m,ij}),i}_{(p,1)} - M^{(N,\succsim^{m,ij}),j}_{(p,1)}$ improving moves and each coalition S^ℓ inducing such a move from $(N,\succsim^\ell)$ to $(N,\succsim^{\ell+1})$ is such that $S^\ell \sim^\ell \{i\}$, $S^\ell \succ^{m,ij} \{i\}$, that is, $S^0 \succ^\ell \{i\} \sim^\ell S^\ell$ and $S^0 \sim^{\ell+1} S^\ell$. Thus, such moves are (S^ℓ, i)-improving. For each p such that $M^{(N,\succsim^{m,ij}),i}_{(p,1)} < M^{(N,\succsim^{m,ij}),j}_{(p,1)}$ the path contains $M^{(N,\succsim^{m,ij}),i}_{(p,2)} - M^{(N,\succsim^{m,ij}),j}_{((p,2)}$ deteriorating moves. For each S^ℓ inducing such a move from $(N,\succsim^\ell)$ to $(N,\succsim^{\ell+1})$, one has $p > p_0$ by definition of p_0, $S^\ell \succ^\ell \{i\}$ and $S^\ell \sim^{\ell+1} \{i\}$. Thus, such moves are (S^ℓ, i)-deteriorating. There is no other move in such a path. Thus, this ij-path is improving for agent i and the first coalition S^0 along the path is such that $p_0 = |S^0| \leq |S^\ell|$ for each $\ell \in \{1,\ldots,t-1\}$, and $|S^0| = |S^\ell|$ implies $S^0 \succ^\ell \{i\} \sim^\ell S^\ell$.

By Individual Improving Path Monotonicity with Priority to the Smallest Coalition, one concludes that $i \succ_{f(N,\succsim^{m,ij})} j$. By Lemma 1, we get $i \succ_{f(N,\succsim)} j$. We have shown the implication

$$[i \succ_{L^P(N,\succsim)} j] \Longrightarrow [i \succ_{f(N,\succsim)} j], \tag{7}$$

which completes **Step 1**.

Step 2 Assume that $i \sim_{L^P(N,\succsim)} j$. To show: $i \sim_{f(N,\succsim)} j$. By Lemma 1, one has $i \sim_{L^P(N,\succsim^{m,ij})} j$. By definition of L^P, we have $M^{(N,\succsim^{m,ij}),i} = M^{(N,\succsim^{m,ij}),j}$. Thus, by Proposition 1, $i \sim_{f(N,\succsim^{m,ij})} j$. Using Lemma 1, one finally gets the desired result $i \sim_{f(N,\succsim)} j$.

From **Step 1** and **Step 2**, conclude that $L^P(N,\succsim) = f(N,\succsim)$, which completes the proof of Theorem 1. ■

Step 1 (b), which shows implication (7), is instructive. Consider a coalitional ranking with $k \geq 2$ equivalent classes and two distinct agents i and j such that i and j have the same individual performance but i is strictly better rank than j under L^P. The proof consists in applying Independence of Irrelevant Equivalent Classes to deduce that the relative ranking between i and

j is not modified under L^P when the equivalent classes of the coalitional ranking are merged in two equivalent classes, that is, when one restricts the problem to a coalitional ranking that lists the coalitions with a rank strictly better than the rank of the singletons $\{i\}$ and $\{j\}$ and places the other coalitions in the equivalence class of $\{i\}$ and $\{j\}$. In a second step, it is enough to make coalitional moves from this merged coalitional ranking to reach another coalitional ranking such that the matrices associated with i and j are identical. Here, one applies Neutrality and Super Weak Coalitional Anonymity to deduce that i and j have the same rank under f (see Proposition 1). In a third step, one goes back to the original merged coalitional ranking through an ij-path which is improving for agent i and gives the priority to the smallest coalition. From this, one deduces that i has a strictly better rank than j under f in this merged coalitional ranking. In a last step, one goes back to the original ranking with k equivalent classes by using Independence of Irrelevant Equivalent Classes to conclude that i has a strictly better rank than j under f in this original ranking. Figure 1 details the steps of this proof.

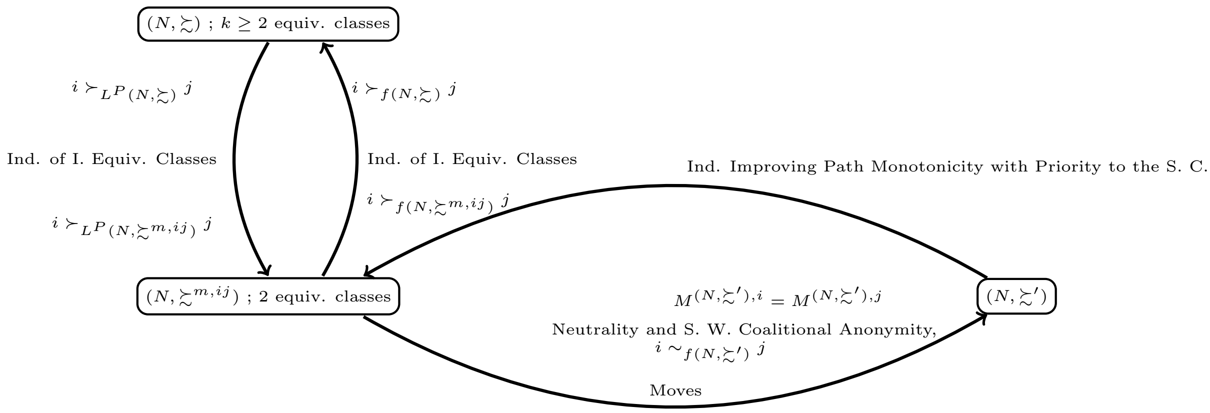

Figure 1: Given a coalitional ranking $(N,\succsim) \in \mathcal{R}$, proof of **Step 1** (b) consists in applying Independence of Irrelevant Equivalent Classes (see Lemma 1) to obtain $(N,\succsim^{m,ij})$ where certain equivalent classes of $(N,\succsim)$ have been merged, and where $(N,\succsim^{m,ij})$ contains two equivalence classes. From $(N,\succsim^{m,ij})$, one creates a sequence of moves from $(N,\succsim^{m,ij})$ to reach $(N,\succsim')$ where $(N,\succsim')$ is such that $M^{(N,\succsim'),i} = M^{(N,\succsim'),j}$. From this, one applies Neutrality and Super Weak Coalitional Anonymity (see Proposition 1) to deduce that i and j have the same individual rank in $(N,\succsim')$ under f. Next, one makes the reverse path using an ij-path, improving for agent i and which gives priority to the smallest coalition. This path connects $(N,\succsim')$ and $(N,\succsim^{m,ij})$. By Individual Improving Path Monotonicity with Priority to the Smallest Coalition, one deduces that $i \succ_{f(N,\succsim^{m,ij})} j$ and, by Independence of Irrelevant Equivalent Classes, one can split the equivalent classes of $(N,\succsim^{m,ij})$ to obtain the original coalition ranking $(N,\succsim)$ without modifying the relative individual ranking between i and j, that is, $i \succ_{f(N,\succsim)} j$.

The following example illustrates the procedure of the proof of Theorem 1.

Example 5 *Consider the coalitional ranking $(N,\succsim)$ introduced in Example 3. One has $\{3\} \sim \{2\}$. The coalitional ranking $(N,\succsim^{m,32})$ contains two equivalence classes:*

$$E_1^{(N,\succsim^{m,32})} = \big\{\{2,3,4\},\{2,4\},\{1,3,4\},\{1,2,3\},\{1,3\},\{2,3\},\{1,4\}\big\}$$

and

$$E_2^{(N,\succsim^{m,32})} = \big\{N,\{3,4\},\{1\},\{2\},\{2,4\},\{3\},\{1,2,4\},\{4\}\big\}.$$

The associated matrices are as follows:

$$M^{(N,\succsim^{m,32}),2} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 2 & 1 \\ 0 & 1 \end{pmatrix}, \quad M^{(N,\succsim^{m,32}),3} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 3 & 0 \\ 0 & 1 \end{pmatrix}.$$

Here $p_0 = 3$, and $M^{(N,\succsim^{m,32}),3}_{(p_0,1)} - M^{(N,\succsim^{m,32}),2}_{(p_0,1)} = 1$ so that $M^{(N,\succsim^{m,32}),2}_{(p_0,2)} - M^{(N,\succsim^{m,32}),3}_{(p_0,2)} = 1$. Pick the only coalition $S = \{1,3,4\}$ of size $p_0 = 3$ in $E_1^{(N,\succsim^{m,32})}$ that contains 3 but not 2. Move it towards the equivalence class $E_2^{(N,\succsim^{m,32})}$. Thanks to this move, one reaches the coalitional ranking $(N, \succsim')$ such that

$$M^{(N,\succsim'),2} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 2 & 1 \\ 0 & 1 \end{pmatrix} = M^{(N,\succsim'),3}.$$

By Proposition 1, one gets $3 \sim_{f(N,\succsim')} 2$. Now, from $(N, \succsim')$, consider the reverse move where $\{1,3,4\}$ is moved from $E_2^{(N,\succsim')}$ to $E_1^{(N,\succsim')}$. In this way, one comes back to $(N, \succsim^{m,23})$ through the $(\{1,3,4\}, 3)$-improving move, that is $\{1,3,4\} \sim' \{3\}$ and $\{1,2,3\} \succ^{m,32} \{3\}$. This move constitutes an 32-path, improving for agent 3 and which trivially gives priority to the smallest coalition. By Individual Improving Path Monotonicity with Priority to the Smallest Coalition, one deduces that $3 \succ_{f(N,\succsim^{m,32})} 2$ and, by Independence of Irrelevant Equivalent Classes, one finally gets the desired result, $3 \succ_{f(N,\succsim)} 2$.

■

The social ranking solution L^P breaks the tie between two agents with identical individual performance by comparing the number of coalitions of size two, then of size three and so on, whose performance is strictly better than the individual performance of these two agents. Therefore, L^P is insensitive to the quality of the performance of these coalitions, as soon as this performance is strictly better than the individual performance of these two agents. The following alternative social ranking solution, denoted by L^{P^*}, corrects this bias: if two agents have identical individual performance, then the coalitions of size two whose performance is strictly better than this individual performance are explored, starting with the best equivalence class. If the number of coalitions of size two containing the first agent is strictly greater than the number of coalitions of size two containing the second agent in the best equivalence class, then the first agent is ranked ahead the second agent. Otherwise, that is, if these two numbers coincide, one moves to the second best equivalence class and proceeds in the same way. One continues to explore each equivalence class from best equivalence to the equivalence class measuring the performance of these two agents. If the procedure does not allow to break the tie between these two agents, then the procedure continues by exploring the collective performance of coalitions of size three and so on. Thus, the social ranking solution L^{P^*} proceeds with a double lexicographical criterion: the criterion on the size of the coalitions is applied first, then the criterion on the index of the equivalence classes (from the index 1 of the best equivalence class to the index $q^{ij}-1$) is used. The formal definition of L^{P^*} is as follows.

Definition 3 The social ranking solution L^{P^*} on $\mathcal{R}$ is defined as follows: for each $(N, \succsim) \in \mathcal{R}$ and pair $\{i, j\}$ of distinct agents in N, $i \succ_{L^{P^*}(N,\succsim)} j$ if one of the following conditions holds:

1. $\{i\} \succ \{j\}$;
2. $\{i\} \sim \{j\}$ and there exists a pair of integers (p_0, q_0), where $2 \leq p_0 < n$ and $1 \leq q_0 < q^{ij}$, such that:
   - for each $p < p_0$, it holds that:
$$\forall q < q^{ij}, \quad M_{(p,q)}^{(N,\succsim),i} = M_{(p,q)}^{(N,\succsim),j};$$
   - and, for p_0, it holds that:
$$\forall q < q_0, \quad M_{(p_0,q)}^{(N,\succsim),i} = M_{(p_0,q)}^{(N,\succsim),j} \text{ and } M_{(p_0,q_0)}^{(N,\succsim),i} > M_{(p_0,q_0)}^{(N,\succsim),j}.$$

Example 6 *Consider again the social ranking of Example 3. Because $\{1\} \succ \{4\}$ and $\{2\} \succ \{4\}$, one gets $1 \succ_{L^{P^*}(N,\succsim)} 4$ and $2 \succ_{L^{P^*}(N,\succsim)} 4$, as it is the case for L^P. Consider now agent 1 and agent 2. We have $\{1\} \sim \{2\}$ and $q^{12} = 3$. For $p_0 = 2$ and $q_0 = 1 < q^{12}$, one has:*

$$2 = M_{21}^{(N,\succsim),2} > M_{21}^{(N,\succsim),1} = 1.$$

Thus, by point 2 of Definition 3, $2 \succ_{L^{P^*}(N,\succsim)} 1$. In the same way, $\{1\} \sim \{3\}$, and, for $p_0 = 2$ and $q_0 = 1 < q^{13} = q^{12}$, one has

$$1 = M_{21}^{(N,\succsim),1} > M_{21}^{(N,\succsim),3} = 0,$$

so that $1 \succ_{L^{P^*}(N,\succsim)} 3$. All in all, one obtains the following individual ranking,

$$2 \succ_{L^{P^*}(N,\succsim)} 1 \succ_{L^{P^*}(N,\succsim)} 3 \succ_{L^{P^*}(N,\succsim)} 4,$$

whereas L^P ranks the agents as follows,

$$1 \succ_{L^P(N,\succsim)} 2 \succ_{L^P(N,\succsim)} 3 \succ_{L^P(N,\succsim)} 4.$$

■

The social ranking L^{P^*} satisfies all axioms of the statement of Theorem 1 except Independence of Irrelevant Equivalence Classes and Individual Improving Path Monotonicity with Priority to the Smallest Coalition. Nevertheless, L^{P^*} satisfies the weak version of the axiom of independence and the strong version of the axiom of path monotonicity. Removing Independence of Irrelevant Equivalence Classes and Individual Improving Path Monotonicity with Priority to the Smallest Coalition from the statement of Theorem 1 and adding Weak Independence to Irrelevant Equivalence Classes, and Strong Individual Improving Path Monotonicity with Priority to the Smallest Coalition, one obtains a characterization of L^{P^*}.

Theorem 2 *The social ranking solution L^{P^*} is the unique social ranking solution on $\mathcal{R}$ satisfying Super Weak Coalitional Anonymity, Neutrality, Standardness, Converse Consistency, Weak Independence to Irrelevant Equivalence Classes, and Strong Individual Improving Path Monotonicity with Priority to the Smallest Coalition.*

As for the proof of Theorem 1, we need a definition and two lemmas.

Definition 4 Let $(N,\succsim) \in \mathcal{R}$ and two distinct agents $i,j \in N$ with identical individual performance, that is $\{i\} \sim \{j\}$. Define the coalitional ranking $(N,\succsim^{\underline{m},ij}) \in \mathcal{R}$ obtained from $(N,\succsim) \in \mathcal{R}_N$ and $i,j \in N$ as follows: as above denote by q^{ij} the index of the equivalence class $E_{q^{ij}}^{(N,\succsim)}$ to which $\{i\}$ and $\{j\}$ belong. If there exists a coalition S such that $S \succ \{i\}$, then $(N,\succsim^{\underline{m},ij})$ contains q^{ij} equivalent classes $E_1^{(N,\succsim^{\underline{m},ij})},\dots,E_{q^{ij}}^{(N,\succsim^{\underline{m},ij})}$, such that:

$$\forall q < q^{ij}, \quad E_q^{(N,\succsim^{\underline{m},ij})} = E_q^{(N,\succsim)} \text{ and } E_{q^{ij}}^{(N,\succsim^{\underline{m},ij})} = \bigcup_{q \geq q^{ij}} E_q^{(N,\succsim)},$$

and

$$E_1^{(N,\succsim^{\underline{m},ij})}(\succ^{\underline{m},ij})^* \dots (\succ^{\underline{m},ij})^* E_{q^{ij}}^{(N,\succsim^{\underline{m},ij})}.$$

Otherwise $(N,\succsim^{\underline{m},ij})$ contains only one equivalent class, i.e., all coalitions have the same rank in $(N,\succsim^{\underline{m},ij})$.

Lemma 2 *Let f be a social ranking solution on $\mathcal{R}$ satisfying Weak Independence of Irrelevant Equivalent Classes. Then, for each $(N,\succsim) \in \mathcal{R}$ and each pair of distinct agents $\{i,j\} \subseteq N$ such that $\{i\} \sim \{j\}$, the following equivalence holds:*

$$[if(N,\succsim)j] \iff [if(N,\succsim^{\underline{m},ij})j].$$

Proof. It suffices to observe that, by construction, $(N,\succsim^{\underline{m},ij})$ satisfies condition (2) of Weak Independence of Irrelevant Equivalent Classes. ■

Lemma 3 *Consider any coalitional ranking $(N,\succsim) \in \mathcal{R}$ and any pair of distinct agents $\{i,j\} \subseteq N$ such that $M^{(N,\succsim),i} \neq M^{(N,\succsim),j}$. Let (p_0,q_0) be the pair such that*

1. $M_{(p_0,q_0)}^{(N,\succsim),i} \neq M_{(p_0,q_0)}^{(N,\succsim),j}$;
2. *for each (p,q) such that either $p < p_0$ or $p = p_0$ and $q < q_0$, $M_{(p,q)}^{(N,\succsim),i} = M_{(p,q)}^{(N,\succsim),j}$.*

Assume that $M_{(p_0,q_0)}^{(N,\succsim),i} > M_{(p_0,q_0)}^{(N,\succsim),j}$. Then, there exists a coalitional ranking $(N,\succsim') \in \mathcal{R}$ such that $M^{(N,\succsim'),i} = M^{(N,\succsim'),j}$ and an ij-path from $(N,\succsim)$ to $(N,\succsim')$ such that, for each pair (p,q) where $M_{(p,q)}^{(N,\succsim),i} \leq M_{(p,q)}^{(N,\succsim),j}$, no coalition $S \in E_{(p,q)}^{(N,\succsim),i}$ induces a move along the ij-path. Futhermore, the ij-path can be chosen in such a way that the last move is induced by a coalition $S \in E_{(p_0,q_0)}^{(N,\succsim),i}$.[2]

The proof of this Lemma 3 is relegated to the Appendix. Thanks to Lemma 3, the steps in the proof of Theorem 2 are similar to those of Theorem 1. We therefore limit ourselves to providing the key points of this proof, some details being similar to those of the proof of Theorem 1.

Proof. (of Theorem 2). The fact the L^{P^*} satisfies Super Weak Coalitional Anonymity, Neutrality, Standardness, Weak Independence of Irrelevant Equivalence Classes, and Weak Individual Improving Path Monotonicity with Priority to the Smallest Coalition is straightforward. To see that it

[2]Recall from (3) that $E_{(p,q)}^{(N,\succsim),i} = E_q^{(N,\succsim)} \cap \{S \in \Omega_N : S \ni i, |S| = p\}$.

satisfies Converse Consistency, consider any coalitional ranking $(N, \succsim) \in \mathcal{R}$, where $|N| \geq 3$, and any two distinct agents $i, j \in N$. Two cases can be distinguished:
Case 1: If $\{i\} \succ \{j\}$, then, for each, $k \in N \setminus \{i, j\}$, we obviously have $\{i\} \succ_{-k} \{j\}$ and so, by Definition 3, $i \succ_{L^{P^*}(N \setminus k, \succsim_{-k})} j$. Because, $\{i\} \succ \{j\}$ one also has $i \succ_{L^{P^*}(N, \succsim)} j$, as desired.
Case 2: If $\{i\} \sim \{j\}$, for each size $p \geq 2$ and each $q < q^{ij}$ define the set, possibly empty, of coalitions of size p containing i but not j:

$$D_{pq}^{(N,\succsim),i,\bar{j}} = \left\{S \in E_q^{(N,\succsim)} : S \ni i, S \not\ni j, |S| = p\right\}.$$

In a similar way, define $D_{pq}^{(N,\succsim),i,j}$ as the set of coalitions of size p in the equivalence class $E_q^{(N,\succsim)}$ containing both i and j:

$$D_{pq}^{(N,\succsim),i,j} = \left\{S \in E_q^{(N,\succsim)} : S \supseteq \{i, j\}, |S| = p\right\}.$$

Remark that, by definition,

$$M_{pq}^{(N,\succsim),i} = |D_{pq}^{(N,\succsim),i,\bar{j}}| + |D_{pq}^{(N,\succsim),i,j}| \quad \text{and} \quad M_{pq}^{(N,\succsim),j} = |D_{pq}^{(N,\succsim),\bar{i},j}| + |D_{pq}^{(N,\succsim),i,j}|.$$

Thus, to show that L^{P^*} satisfies Converse Consistency, assume that

$$\forall k \in N \setminus \{i, j\}, \quad i \succ_{L^{P^*}(N \setminus k, \succsim_{-k})} j.$$

By Definition 3 and the above remark, $i \succ_{L^{P^*}(N \setminus k, \succsim_{-k})} j$ if and only if there exists a pair of integers (p_0^k, q_0^k), where $2 \leq p_0^k < n - 1$ and $1 \leq q_0^k < q^{ij}$, such that:

- for each $p < p_0^k$, it holds that:

$$\forall q < q^{ij}, \quad |D_{pq}^{(N \setminus k, \succsim_{-k}),i,\bar{j}}| = |D_{pq}^{(N \setminus k, \succsim_{-k}),\bar{i},j}|;$$

- and, for p_0^k, it holds that:

$$\forall q < q_0, \quad |D_{p_0^k q}^{(N \setminus k, \succsim_{-k}),i,\bar{j}}| = |D_{p_0^k q}^{(N \setminus k, \succsim_{-k}),\bar{i},j}| \quad \text{and} \quad |D_{p_0^k q_0^k}^{(N \setminus k, \succsim_{-k}),i,\bar{j}}| > |D_{p_0^k q_0^k}^{(N \setminus k, \succsim_{-k}),\bar{i},j}|.$$

Let $(p_0, q_0) = \min\{(p_0^k, q_0^k) : k \in N \setminus \{i, j\}\}$ where the minimum is taken over the pairs (p_0^k, q_0^k), $k \in N \setminus \{i, j\}$ ordered according to the following lexicographic relation:

$$(p, q) <_L (p', q') \text{ if } p < p' \text{ or } p = p' \text{ and } q < q'.$$

And, in a similar way as (6), one obtains:

$$\forall k \in N \setminus \{i, j\}, \quad (n - p - 1)|D_{pq}^{(N,\succsim),i,\bar{j}}| = \sum_{k \in N \setminus ij} |D_{pq}^{(N \setminus k, \succsim_{-k}),i,\bar{j}}| \tag{8}$$

By definition of (p_0, q_0), for each $k \in N \setminus \{i, j\}$, it holds that:

$$\forall p < p_0, \forall q < q^{ij}, \quad |D_{pq}^{(N \setminus k, \succsim_{-k}),i,\bar{j}}| = |D_{pq}^{(N \setminus k, \succsim_{-k}),\bar{i},j}|;$$

and, for p_0,

$$\forall q < q_0, \quad |D_{p_0 q}^{(N \setminus k, \succsim_{-k}),i,\bar{j}}| = |D_{p_0 q}^{(N \setminus k, \succsim_{-k}),\bar{i},j}| \quad \text{and} \quad |D_{p_0 q_0}^{(N \setminus k, \succsim_{-k}),i,\bar{j}}| \geq |D_{p_0 q_0}^{(N \setminus k, \succsim_{-k}),\bar{i},j}|.$$

Furthermore, there is $k^* \in N \setminus \{i, j\}$ such that:

$$|D^{(N\setminus k^*,\succsim_{-k^*}),i,\bar{j}}_{p_0^{k^*} q_0^{k^*}}| > |D^{(N\setminus k^*,\succsim_{-k^*}),\bar{i},j}_{p_0^{k^*} q_0^{k^*}}|.$$

Therefore, by (8), one obtains:

$$\forall p < p_0, \forall q < q^{ij}, \quad |D^{(N,\succsim),i,\bar{j}}_{pq}| = |D^{(N,\succsim),\bar{i},j}_{pq}|;$$

and for p_0,

$$\forall q < q_0, \quad |D^{(N,\succsim),i,\bar{j}}_{p_0 q}| = |D^{(N,\succsim),\bar{i},j}_{p_0 q}| \text{ and } |D^{(N,\succsim),i,\bar{j}}_{p_0 q_0}| > |D^{(N,\succsim),\bar{i},j}_{p_0 q_0}|,$$

which ensures that $i \succ_{L^{P^*}(N,\succsim)} j$, as desired.

To show that L^{P^*} is the unique social ranking solution on $\mathcal{R}$ satisfying Super Weak Coalitional Anonymity, Neutrality, Standardness, Converse Consistency, Weak Independence of Irrelevant Classes, and Strong Individual Improving Path Monotonicity with Priority to the Smallest Coalition, let f be a social ranking social satisfying these six axioms. We have to show that $f = L^{P^*}$. Pick any $N \in \mathcal{F}$, any $(N, \succsim) \in \mathcal{R}_N$ and any $i, j \in N$. We proceed in two steps.

Step 1: Assume that $i \succ_{L^{P^*}(N,\succsim)} j$. To show: $i \succ_{f(N,\succsim)} j$. We distinguish two exclusive cases:

(a) $\{i\} \succ \{j\}$. Then, by Proposition 2, we get $i \succ_{f(N,\succsim)} j$.

(b) $\{i\} \sim \{j\}$. Consider the coalition ranking $(N, \succsim^{\underline{m},ij})$ obtained from $(N, \succsim)$ as defined in Definition 4. Because $i \succ_{L^{P^*}(N,\succsim)} j$, by Lemma 2, we also have $i \succ_{L^{P^*}(N,\succsim^{\underline{m},ij})} j$. Because $i \succ_{L^P(N,\succsim^{\underline{m},ij})} j$, there exists a coalition $S \in \Omega_N$ such that $S \ni i$ and $S \succ^{\underline{m},ij} \{i\}$, meaning that $\succsim^{m,ij}$ contains $q_{ij} > 1$ equivalence classes $E_1^{(N,\succsim^{\underline{m},ij})}, \ldots, E_{q^{ij}}^{(N,\succsim^{\overline{m},ij})}$. This implies that both matrices $M^{(N,\succsim^{\underline{m},ij}),i}$ and $M^{(N,\succsim^{\underline{m},ij}),j}$ have q_{ij} columns, and there exists a pair (p_0, q_0) as defined in point 2 of Definition 4, such that $M^{(N,\succsim^{\underline{m}ij}),i}_{(p_0,q_0)} > M^{(N,\succsim^{\underline{m},ij}),j}_{(p_0,q_0)}$. By Lemma 3, there is an ij-path from $(N, \succsim^{\underline{m},ij})$ to a coalitional ranking $(N, \succsim')$ such that $M^{(N,\succsim'),i} = M^{(N,\succsim'),j}$ and the last move is induced by a coalition $S^0 \ni i$ of size p_0 belonging to $E_{q_0}^{(N\succsim^{m,ij})}$.

By Proposition 1, we have $i \sim_{f(N,\succsim')} j$. Next, starting from $(N, \succsim')$, one can construct the reverse ij-path ending with the coalitional ranking $(N, \succsim^{\underline{m},ij})$ where this first move is induced by a coalition belonging to $E^{(N,\succsim^{m,ij}),i}_{(p_0,q_0)}$. By definition of $(N, \succsim')$ (note that in the latter $\{i\}$ and $\{j\}$ belong to the worst equivalence class), each of these moves is weakly S^ℓ-improving for i and the first move induced by coalition $S^0 \in E_{q_0}^{(N,\succsim^{m,ij})}$ is such that $|S^0| = p_0$; and, for each other coalition S^ℓ along this path such that $|S^\ell| = p_0$, one necessarily has $S^0 \succ^\ell S^\ell$ by definition of the pair (p_0, q_0) as defined in point 2 of Definition 4 (see above). By Strong Individual Improving Path Monotonicity with Priority to the Smallest Coalition, one obtains $i \succ_{f(N,\succsim^{\underline{m},ij})} j$ and then Lemma 2 applies to obtain $i \succ_{f(N,\succsim)} j$. This concludes the proof of **Step 1**.

Step 2: Assume that $i \sim_{L^{P^*}(N,\succsim)} j$. In this case, the proof is identical to **Step 2** of the proof of Theorem 1, except that one uses Lemma 2 instead of Lemma 1.

This completes the proof of Theorem of 2. ■

5. Logical independence of the axioms

The following list of social ranking solutions shows that the axioms used in Theorem 1 are logically independent.

Neutrality is not satisfied. Consider the social ranking solution f on $\mathcal{R}$ defined as follows: for any $(N, \succsim) \in \mathcal{R}$ an any pair $\{i, j\} \subseteq N$ such that $\{i\} \succsim \{j\}$,

$$i \succ_{f(N,\succsim)} j \quad \text{if either } \big[\{i\} \succ \{j\}\big] \text{ or } \big[\{i\} \sim \{j\} \text{ and } i < j \text{ and } |N| \neq 2\big].$$

The social ranking solution f satisfies Super Weak Coalitional Anonymity, Standardness, Independence to Irrelevant Equivalent Classes, Converse Consistency, and Individual Improving Path Monotonicity with Priority to the Smallest Coalition, but obviously violates Neutrality.

Individual Improving Path Monotonicity with Priority to the Smallest Coalition is not satisfied. Consider the social ranking solution f on $\mathcal{R}$ defined as follows: for any $(N, \succsim) \in \mathcal{R}$,

$$i \succ_{f(N,\succsim)} j \quad \text{if } \{i\} \succ \{j\}.$$

The social ranking solution f satisfies Neutrality, Super Weak Coalitional Anonymity, Standardness, Independence to Irrelevant Equivalent Classes, Converse Consistency, but violates Individual Improving Path Monotonicity with Priority to the Smallest Coalition.

Standardness is not satisfied. Consider the social ranking solution f on $\mathcal{R}$ defined as follows: for any $(N, \succsim) \in \mathcal{R}$,

$$i \succ_{f(N,\succsim)} j \quad \text{if either } \{j\} \succ \{i\} \text{ or } \{i\} \sim \{j\} \text{ and } i \succ_{L^P(N,\succsim)} j.$$

The social ranking solution f satisfies Neutrality, Super Weak Coalitional Anonymity, Independence to Irrelevant Equivalent Classes, Converse Consistency, Individual Improving Path Monotonicity with Priority to the Smallest Coalition, but violates Standardness.

Converse Consistency is not satisfied. Consider the social ranking solution f on $\mathcal{R}$ defined as follows: for any $(N, \succsim) \in \mathcal{R}$,

$$i \succ_{f(N,\succsim)} j \text{ if } \begin{cases} \{i\} \succ \{j\} & \text{and } |N| = 2, \\ \{j\} \succ \{i\} & \text{and } |N| \neq 2, \\ \{i\} \sim \{j\} & i \succ_{L^P(N,\succsim)} j. \end{cases}$$

The social ranking solution f satisfies Neutrality, Super Weak Coalitional Anonymity, Independence to Irrelevant Equivalent Classes, Standardness, Individual Improving Path Monotonicity with Priority to the Smallest Coalition, but violates Converse Consistency.

Independence to Irrelevant Equivalent Classes is not satisfied. The social ranking L^{P^*} satisfies Neutrality, Super Weak Coalitional Anonymity, Standardness, Converse Consistency, (Strong) Individual Improving Path Monotonicity with Priority to the Smallest Coalition, but violates Independence to Irrelevant Equivalent Classes.

The logical independence of the axioms used in Theorem 2 can be shown in a similar way. The proof is omitted.

6. Conclusion

This paper proposed two new solutions for coalitional ranking problems. Both social ranking solutions use a lexicographic criterion to rank the agents. The main axiomatic results show that these two solutions satisfy similar principles. To switch from L^P to L^{P^*}, it is sufficient to weaken the independence axiom and to strengthen the path monotonicity axiom. It is possible to refine the solution L^{P^*} by exploring the equivalence classes that are ranked below the equivalence class containing the singleton coalitions of two indifferent agents with respect of L^{P^*}. Such a modification requires a revision of the independence and the path monotonicity axioms.

To construct the social ranking solutions L^P and L^{P^*}, one first considers singleton coalitions, then coalitions of size two, etc. Another option would be to start with coalitions of size $n-1$, then of size $n-2$, etc. Such a solution allows to put forward first the collective performance of the agents, then the individual performance. At this stage of the analysis, we have not succeeded in axiomatizing such a social ranking solution.

7. Appendix

Proof. (of Lemma 3) Let $\Delta_{ij}^{(N,\succsim)}$ be the number of pairs (p,q) such that $M_{(p,q)}^{(N,\succsim),i} \neq M_{(p,q)}^{(N,\succsim),j}$. Recall that, for each p,

$$\sum_{q=1}^{k} M_{(p,q)}^{(N,\succsim),i} = \sum_{q=1}^{k} M_{(p,q)}^{(N,\succsim),j}.$$

Thus, when there exists (p_1,q_1) such that $M_{(p_1,q_1)}^{(N,\succsim),i} > M_{(p_1,q_1)}^{(N,\succsim),j}$, then there exists q_2 such that $M_{(p_1,q_2)}^{(N,\succsim),i} < M_{(p_1,q_2)}^{(N,\succsim),j}$. This is turn implies that $\Delta_{ij}^{(N,\succsim)} \geq 2$. Moreover, when $\Delta_{ij}^{(N,\succsim)} = 2$, there exists a unique $q_2 > q_0$ such that $M_{(p_0,q_2)}^{(N,\succsim),j} - M_{(p_0,q_2)}^{(N,\succsim),i} = M_{(p_0,q_0)}^{(N,\succsim),i} - M_{(p_0,q_0)}^{(N,\succsim),j}$, and for any pair $(p,q) \notin \{(p_0,q_0),(p_0,q_2)\}$, $M_{(p,q)}^{(N,\succsim),i} = M_{(p,q)}^{(N,\succsim),j}$. The proof is done by induction on $\Delta_{ij}^{(N,\succsim)}$.

Initialization: $\Delta_{ij}^{(N,\succsim)} = 2$. It is possible to create an ij-path starting from $(N,\succsim)$ and formed by $M_{(p_0,q_0)}^{(N,\succsim),i} - M_{(p_0,q_0)}^{(N,\succsim),j}$ moves. Each of these moves is induced by a coalition $S \in E_{(p_0,q_0)}^{(N,\succsim),i}$ – that is, by (3), a coalition $S \in E_{q_0}^{(N,\succsim)}$ of size p_0 and containing i – and S is moved to $E_{(p_0,q_2)}^{(N,\succsim),i}$. The coalitonal ranking $(N,\succsim')$ obtained at this end of this path is such that $M^{(N,\succsim'),i} = M^{(N,\succsim'),j}$, and, obviously, the last move of the path is induced by coalition $S \in E_{q_0}^{(N,\succsim)}$ of size p_0. Thus, we are done.

Induction hypothesis: Assume that the result holds for any $(N,\succsim) \subseteq \mathcal{R}$ such that $\Delta_{ij}^{(N,\succsim)} \leq k$, for $k \geq 2$.

Induction step: Let $k \geq 2$ and $(N,\succsim) \subseteq \mathcal{R}$ such that $\Delta_{ij}^{(N,\succsim)} = k+1$. Let (p_1,q_1) and (p_1,q_2) such that

$$M_{(p_1,q_1)}^{(N,\succsim),i} > M_{(p_1,q_1)}^{(N,\succsim),j} \text{ and } M_{(p_1,q_2)}^{(N,\succsim),i} < M_{(p_1,q_2)}^{(N,\succsim),j}.$$

We distinguish two cases.

Case 1: Assume that (p_1, q_1) can be chosen in such way that $(p_1, q_1) \neq (p_0, q_0)$, that is, (p_0, q_0) is not the only pair such that

$$M^{(N,\succsim),i}_{(p_0,q_0)} > M^{(N,\succsim),j}_{(p_0,q_0)}.$$

It is possible to create an ij-path starting from $(N, \succsim)$ formed by

$$\min\left\{M^{(N,\succsim),i}_{(p_1,q_1)} - M^{(N,\succsim),j}_{(p_1,q_1)}, M^{(N,\succsim),j}_{(p_1,q_2)} - M^{(N,\succsim),i}_{(p_1,q_2)}\right\}$$

moves. Each of these moves is induced by a coalition $S \in E^{(N,\succsim),i}_{(p_1,q_1)}$; and S is moved from $E^{(N,\succsim),i}_{(p_1,q_1)}$ to $E^{(N,\succsim),i}_{(p_1,q_2)}$. The coalitonal ranking $(N, \succsim'')$ obtained at the last step of this path is such that $\Delta^{(N,\succsim'')}_{ij} < \Delta^{(N,\succsim)}_{ij}$ since $M^{(N,\succsim''),i}_{(p_1,q_1)} = M^{(N,\succsim''),j,}_{(p_1,q_1)}$ or $M^{(N,\succsim''),i}_{(p_1,q_2)} = M^{(N,\succsim''),j}_{(p_1,q_2)}$. Furthermore, because $(p_0, q_0) \neq (p_1, q_1)$, one also has

$$M^{(N,\succsim''),i}_{(p_0,q_0)} = M^{(N,\succsim),i}_{(p_0,q_0)} > M^{(N,\succsim),j}_{(p_0,q_0)} = M^{(N,\succsim''),j}_{(p_0,q_0)}.$$

Thus, the induction hypothesis applies to $(N, \succsim'')$: there exists an ij-path satisfying the condition of the Lemma 3 starting with $(N, \succsim'')$ and ending with a coalitional ranking $(N, \succsim')$ such that $M^{(N,\succsim'),i} = M^{(N,\succsim'),j}$. Concatenating the above two ij-paths, we get an ij-path from $(N, \succsim)$ to $(N, \succsim')$ satisfying the hypothesis of the Lemma 3. Notice that each coalition used in the first ij-path from $(N \succsim)$ to $(N, \succsim'')$ is not used in the second ij-path from $(N, \succsim'')$ to $(N, \succsim')$. Indeed, such a coalition belongs to the set $E^{(N,\succsim'')}_q$ such that $M^{(N,\succsim''),i}_{(p,q)} \leq M^{(N,\succsim''),j}_{(p,q)}$.

Case 2: One necessarily have $(p_0, q_0) = (p_1, q_1)$, that is, (p_0, q_0) is the only pair such that

$$M^{(N,\succsim),i}_{(p_0,q_0)} > M^{(N,\succsim),j}_{(p_0,q_0)}.$$

Thus, for each size $p \neq p_0$ and each $q \in \{1, \ldots, k\}$, $M^{(N,\succsim),i}_{(p,q)} = M^{(N,\succsim),j}_{(p,q)}$, and, for each $q \neq q_0$, $M^{(N,\succsim),i}_{(p_0,q)} \leq M^{(N,\succsim),j}_{(p_0,q)}$. Because

$$\sum_{q=1}^{k} M^{(N,\succsim),i}_{(p,q)} = \sum_{q=1}^{k} M^{(N,\succsim),j}_{(p,q)} \text{ and } k+1 \geq 3,$$

there exist q_2 and q_2' in $\{1, 2, \ldots, k\}$ such that $q_0 < q_2 < q_2'$ and

$$M^{(N,\succsim),i,}_{(p_0,q_2)} < M^{(N,\succsim),j}_{(p_0,q_2)} \text{ and } M^{(N,\succsim),i}_{(p_0,q_2')} < M^{(N,\succsim),j}_{(p_0,q_2')},$$

which in turn implies that

$$M^{(N,\succsim),j}_{(p_0,q_2)} - M^{(N,\succsim),i}_{(p_0,q_2)} < M^{(N,\succsim),i}_{(p_0,q_0)} - M^{(N,\succsim),j}_{(p_0,q_0)}.$$

Thus, it is possible to create an ij-path starting from $(N, \succsim)$ and formed by $M^{(N,\succsim),j}_{(p_0,q_2)} - M^{(N,\succsim),i}_{(p_0,q_2)}$ moves, each of them being induced by a coalition $S \in E^{(N,\succsim)}_{q_0}$ of size p_0, which is moved from $E^{(N,\succsim)}_{q_0}$

to $E_{q_2}^{(N,\succsim)}$. On the one hand, the coalitonal ranking $(N,\succsim'')$ obtained at the end of this path is such that $M_{(p_0,q_2)}^{(N,\succsim''),i} = M_{(p_0,q_2)}^{(N,\succsim''),j}$, so that $\Delta_{ij}^{(N,\succsim'')} < \Delta_{ij}^{(N,\succsim)}$. On the other hand,

$$\begin{aligned} M_{(p_0,q_0)}^{(N,\succsim''),i} &= M_{(p_0,q_0)}^{(N,\succsim),i} - \left(M_{(p_0,q_2)}^{(N,\succsim),j} - M_{(p_0,q_2)}^{(N,\succsim),i}\right) \\ &> M_{(p_0,q_0)}^{(N,\succsim),i} - \left(M_{(p_0,q_0)}^{(N,\succsim),i} - M_{(p_0,q_0)}^{(N,\succsim),j}\right) \\ &= M_{(p_0,q_0)}^{(N,\succsim),j} \\ &= M_{(p_0,q_0)}^{(N,\succsim''),j}. \end{aligned}$$

Thus, the induction hypothesis applies to $(N,\succsim'')$: there exists an ij-path satisfying the condition of the lemma starting with $(N,\succsim'')$ and ending with a coalitional ranking $(N,\succsim')$ such that $M^{(N,\succsim'),i} = M^{(N,\succsim'),j}$. In the same way as in **Case 1**, one concatenates the above two ij-paths from $(N,\succsim)$ to $(N,\succsim'')$ and from $(N,\succsim'')$ to $(N,\succsim')$ to get an ij-path from $(N,\succsim)$ to $(N,\succsim')$ satisfying the hypothesis of the Lemma 3. This completes the induction step. ■

References

[1] Algaba E., Moretti S., Rémila E, Solal P. (2021). Lexicographic solutions for coalitional rankings. *Social Choice and Welfare*, https://doi.org/10.1007/s00355-021-01340-z.

[2] Banzhaf J III (1964). Weighted voting doesn't work: a mathematical analysis. *Rutgers Law Review*, 19: 317-343.

[3] Barberà S., Bossert W., Pattanaik P. (2004). Ranking sets of objects. In Handbook of Utility Theory: Extensions 2: 893-977.

[4] Bernardi G., Lucchetti R., Moretti S. (2019). Ranking objects from a preference relation over their subsets. *Social Choice and Welfare*, 52: 589-606.

[5] Driessen T., Funaki Y. (1991). Coincidence of and collinearity between game theoretic solutions. *OR Spektrum*, 13:15-30.

[6] Hart S., Mas Colell (1989). Potential, Value, and Consistency. *Econometrica*, 57: 589-614.

[7] Khani H., Moretti S., M. Öztürk (2019). An ordinal Banzhaf index for social ranking. In : Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence (IJCAI-19), pp. 378-384.

[8] Moriarity S. (1975). Another approach to allocating joint costs. *Accounting Review*, 50:791-795.

[9] Meggido N. (1974). On the nonmonotonicity of the bargaining set, the kernel, and the nucleolus of a game. *SIAM Journal of Applied Mathematics*, 27:355-358.

[10] van den Brink R., Funaki Y., Ju Y. (2013). Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values. *Social Choice and Welfare*, 40:693-714.

[11] Young H.P. (1985). Monotonic Solutions of Cooperative Games. *International Journal of Game Theory*, 14:65-72.

[12] Zou Z., van den Brink R., Chun Y., Funaki Y. (2021a). Axiomatizations of the proportional division value. *Social Choice and Welfare*, https://doi.org/10.1007/s00355-020-01299-3.

[13] Zou Z., van den Brink R., Funaki Y. (2021b). Compromising between the proportional and equal division values. *Journal of Mathematical Economics*, https://doi.org/10.1016/j.jmateco. 2021.102539.

[14] Zou Z., van den Brink R., Chun Y., Funaki Y. (2020). Sharing the surplus and proportional values. Tinbergen Institute Discussion Paper, No. TI 2020-014/II.