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A characterization of the family of Weighted priority values

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Abstract

We introduce a new family of values for TU-games with a priority structure. This family both contains the Priority value recently introduced by Béal et al. (2021) and the Weighted Shapley values (Kalai and Samet, 1987). Each value of this family is called a Weighted priority value and is constructed as follows. A strictly positive weight is associated with each agent and the agents are partially ordered according to a binary relation. An agent is a priority agent with respect to a coalition if it is maximal in this coalition with respect to the partial order. A Weighted priority value distributes the dividend of each coalition among the priority agents of this coalition in proportion to their weights. We provide an axiomatic characterization of the family of the Weighted Shapley values without the additivity axiom. To this end, we borrow the Priority agent out axiom from Béal et al. (2021), which is used to axiomatize the Priority value. We also reuse, in our domain, the principle of Super weak differential marginality introduced by Casajus (2018) to axiomatize the Positively weighted Shapley values (Shapley, 1953a). We add a new axiom of Independence of null agent position which indicates that the position of a null agent in the partial order does not affect the payoff of the other agents. Together with Efficiency, the above axioms characterize the Weighted Shapley values. Finally, we show that this axiomatic characterization holds on the subdomain where the partial order is structured by levels. This entails an alternative characterization of the Weighted Shapley values.

Keywords: Differential marginality – Priority value – Shapley value – Superweak differiential marginality – Weighted Shapley value.

JEL code: C71

1. Introduction

The *(symmetric) Shapley value* (Shapley, 1953b) is probably the most popular single-valued solution concept for cooperative games. It is well known that the Shapley value distributes the Harsanyi dividend of each coalition (Harsanyi, 1959) in a game equally among its members. In

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order to account for asymmetries among the agents beyond the game itself, Shapley (1953a) discusses weighted versions of his value, the *Positively weighted Shapley values*. Each of these values associates a positive weight with each agent. These weights represent the proportions in which the members of a coalition share the Harsanyi dividend.

Other popular nonsymmetric versions of the Shapley value are proposed by Kalai and Samet (1987). They generalize the notion of positive weights to enable some agents to have zero weight. A weight system is now given by a list of positive weights as above with in addition an ordered partition of the agent set. This ordered partition reflects the fact that the population is structured by levels: agents belonging to a cell of the partition have priority over all the agents belonging to a dominated cell with respect to the order over the cells of the partition, when distributing the Harsanyi dividend of a coalition among its members. Kalai and Samet (1987) define the *Weighted Shapley values* as the values that distribute the Harsanyi dividend of a coalition in a game to its priority members in proportion to their weights. This way, if some member of a coalition has priority over another member of this coalition, the latter is assigned a zero weight in the sharing process. If the ordered partition is the coarsest one, then the Weighted Shapley values coincide with the Positively Shapley values.

Faigle and Kern (1992) and Béal et al. (2021) define nonsymmetric versions of the Shapley value by endowing a game with a priority structure represented by a partially ordered set (a poset henceforth) over the agent set. The poset reflects the asymmetries that may exist between agents, e.g., different needs, unequal merits, beyond the parameters of the game. Note that an ordered partition over the agent set is a special case of a poset: it corresponds to a specific ranked poset where each agent with the same rank dominates, and so has priority over, each other agent with a lower rank, and agents with the same rank are incomparable with respect to the poset. As Kalai and Samet (1987), Béal et al. (2021) consider that the priority structure does not influence the coalition formation process but the distribution of payoffs. They propose the *Priority value* which shares the Harsanyi dividend of a coalition in a game equally to its priority members. On the contrary, Faigle and Kern (1992) consider that the priority structure influences the coalition formation process. Only the coalitions formed by their priority members (i.e., the maximal agents in the subposet induced by this coalition) and all their subordinates are feasible. They define a *Shapley-like value* which distributes the Harsanyi dividend of a feasible coalition in a game among its priority agents in proportion to the number of times they appear last in a feasible ordering over the agent set. An ordering over an agent set is feasible if it is consistent with the underlying poset.

In this article, we generalize the Priority value by considering situations where the agents are part of a priority structure and each of them is assigned a positive weight. We then define the *Weighted priority values* as the values that shares the Harsanyi dividend of a coalition in a game among its priority agents in proportion to their weights. In the particular case where the priority structure is organized by levels, the Weighted priority values coincide with the Weighted Shapley values. If, moreover, each agent is assigned the same weight, then the Weighted priority values boil down to the Priority value. Finally, in the special case where each level of the poset contains exactly one agent (the induced ordered partition is the finest one or equivalently the poset is a linear order), then the Priority value coincides with the so-called downward marginal vector (van den Brink et al., 2007).

There are many axiomatic foundations for the Shapley value, the Positively weighted Shapley values and the Weighted Shapley, and Béal et al. (2021) propose two axiomatic characterizations of the Priority value. Recently, Casajus and Yokote (2017) provide a characterization of the Shapley value by three properties: Efficiency, Null agent, and Weak differential marginality. Efficiency and Null agent are standard axioms for single-valued solutions in cooperative games. The first one says that the worth of the grand coalition is fully redistributed among its members. The second one requires that if the marginal contribution of an agent to each coalition is null, then it receives a null payoff. Weak differential marginality is a relaxation of Differential marginality introduced by Casajus (2011). The latter axiom says that the payoffs of two agents change by the same amount whenever these two agents' marginal contributions to coalitions not containing either of them change by the same amount between two games.[1] In other words, equal change of agents' marginal contributions to coalitions containing neither of them should translate into equal payoff differentials. Weak differential marginality relaxes the conclusion of Differential marginality by requiring that their payoffs change in the same direction, i.e., the signs of the changes must coincide. This characterization, however, does not work for games with exactly two agents. In order to obtain a characterization for all games, it suffices to replace the Null agent axiom by the Null agent out axiom. The latter expresses an invariance principle: whenever a null agent is removed from the game, then the payoff of each other remaining agent is not impacted. In general, the Positively weighted Shapley values fail Weak differential marginality. Instead, they obey a relaxation of Weak differential marginality called Superweak differential marginality: whenever two agents' marginal contributions to coalitions not containing either of them do not change, then their payoffs should change in the same direction. Casajus (2018) provides a characterization of the Positively weighted Shapley values in terms of Efficiency, Null agent out and Superweak differential marginality.

The Weighted priority values obey to Efficiency and Null agent out but they violate Superweak differential marginality. In order to characterize this class of values, we weaken Superweak differential marginality and invoke two axioms whose combination is stronger that Null agent out. Firstly, we introduce a relaxation of Superweak differential marginality called Superweak differential marginality for agents with the same priority group: whenever two agents' marginal contributions to coalitions not containing either of them do not change, then their payoffs should change in the same direction provided these agents have the same priority group, where the priority group of an agent is the set of agents having priority over it. The Weighted priority values satisfy this axiom. Secondly, Null agent out is replaced by the Priority agent out axiom (Béal et al., 2021), which indicates that removing an agent does not affect the payoffs of the agents over which it has priority, and a new axiom called Independence of null agent position. The latter axiom states that a change in the position of a null agent in the priority structure should not affect the payoffs of the other agents. It turns out the combination of Priority agent out and Independence of null agent position entails Null agent out. As our main result, we show that the Weighted priority values are

[1] Differential marginality is itself a weak version of Strong differential monotonicity introduced by Casajus and Huettner (2013).

characterized by Efficiency, Superweak differential marginality for agents with the same priority group, Priority agent out and Independence of null agent position. This characterization result still holds when the priority structures are organized by levels, which gives a new characterization of the Weighted Shapley values on the domain where the population and the priority structure are variable.

This way, the position of the Positively weighted Shapley values within the larger classes of the Weighted Shapley values and the Weighted priority values respectively can be essentially pinpointed to the treatment of both specific agents and priority relationships between the agents. First, the hypothesis of Superweak differential marginality is only satisfied if the two agents are mutually dependent in this difference. Recall that two agents are mutually dependent when their marginal contributions to coalitions not containing either of them is zero (Nowak and Radzik, 1995). In contrast, the hypothesis of Superweak differential marginality with the same priority group is satisfied only if these two mutually dependent agents in this difference have the same priority group. Second, to take into account the asymmetries between the agents reflected by the priority structure, Null agent out is replaced by two axioms that imply it. The first one implies that an agent can get a positive part of the dividend of a coalition containing it only if this coalition does not contain an agent having priority over it; the second one states that a null agent does not affect the payoff of the other agents whatever its location in the priority structure. Thus, the axiomatic path with (weak) differential marginality and then Superweak differential marginality opened by Casajus (2011), Casajus and Yokote (2017) and Casajus (2018) to characterize the Shapley values and the Positively weighted Shapley values shows its potential to characterize larger classes of nonsymmetric versions of the Shapley value such as the Weighted priority values and Weighted Shapley values.

The remainder of this article is organized as follows. In Section 2, we provide basic definitions and notation. In Section 3, we define the Weighted priority values for cooperative games endowed with a priority structure. In Section 4, we introduce the axioms, state some intermediary results and provide our characterization of the class of Weighted priority values. The last section concludes.

2. Basic definitions and notation

2.1. Cooperative games with transferable utility

Let $\mathfrak{U}$ be a countably infinite universe of agents and let $\mathcal{N}$ denote the set of all finite (nonempty) subsets of $\mathfrak{U}$. A cooperative game with transferable utility, or simply a **TU-game**, is a pair (N, v) where $N \in \mathcal{N}$ is the set of agents and $v : 2^N \to \mathbb{R}$, with the convention $v(\varnothing) = 0$, is the **coalition function** on N. A subset S of N is a **coalition** and $v(S)$ is the worth that members of S can obtain if they cooperate. Henceforth, the singleton $\{i\}$ is denoted by i, and, for any nonempty coalition S, s denotes its cardinality $|S|$. Denote by G_N the class of all TU-games on $N \in \mathcal{N}$ and let

$$G = \bigcup_{N \in \mathcal{N}} G_N$$

the set of all TU-games.

The **subgame** (S, v^S) of (N, v) induced by coalition $S \neq \varnothing$ is such that v^S is the restriction of v to 2^S. When no confusion arises, (S, v^S) will be denoted by (S, v). The **null TU-game** on N is the TU-game $(N, \mathbf{0})$ such that $\mathbf{0}(S) = 0$, for each $S \subseteq N$. For $(N, v), (N, w) \in G_N$ and $c \in \mathbb{R}$, the TU-games $(N, v + w), (N, cv) \in G_N$ are defined as follows: for each $S \subseteq N$, $(v + w)(S) = v(S) + w(S)$, and $(cv)(S) = cv(S)$.

An agent $i \in N$ is a **null agent** in (N, v) if $v(S) = v(S\backslash i)$, for each $S \subseteq N$ such that $S \ni i$. Two agents $i, j \in N$, $i \neq j$, are called **equal** in (N, v) if

$$\forall S \subseteq N\backslash\{i, j\}, \quad v(S \cup i) = v(S \cup j),$$

and **mutually dependent** in (N, v) if

$$\forall S \subseteq N\backslash\{i, j\}, \quad v(S \cup i) = v(S) = v(S \cup j),$$

i.e., if they are only jointly (hence, equally) productive. Note that mutually dependent agents constitute a special case of equal agents.

For each nonempty coalition $S \subseteq N$, the **unanimity TU-game** induced by S is the TU-game (N, u_S) defined as: $u_S(T) = 1$ if $T \supseteq S$, and $u_S(T) = 0$ otherwise. It is well-known that any TU-game (N, v) admits a unique linear decomposition in terms of unanimity TU-games:

$$v = \sum_{\varnothing \subsetneq S \subseteq N} \Delta_S(v) u_S, \tag{1}$$

where each coordinate $\Delta_S(v) \in \mathbb{R}$ is called the **Harsanyi dividend** (Harsanyi, 1959) of S in (N, v), and is computed from the following recursive formula:

$$\Delta_S(v) = v(S) - \sum_{T \subsetneq S} \Delta_T(v).$$

Remark 1. *The following properties of the Harsanyi dividends hold:*

(i) *if $i \in N$ is a null agent in (N, v), then $\Delta_T(v) = 0$, for each $T \subseteq N$ such that $i \in T$;*

(ii) *if i and j are mutually dependent in (N, v), then $\Delta_T(v) = 0$, for each $T \subseteq N$ such that $|T \cap \{i, j\}| = 1$.*

(iii) *$\Delta_T(cv + w) = c \cdot \Delta_T(v) + \Delta_T(w)$, for each $\varnothing \neq T \subseteq N$.* □

The function sign : $\mathbb{R} \to \{-1, 0, 1\}$ is defined as: $\text{sign}(x) = 1$ if $x > 0$, $\text{sign}(x) = -1$ if $x < 0$, and $\text{sign}(x) = 0$ if $x = 0$.

2.2. TU-games with priority structure

A TU-game with a priority structure describes a situation where some agents in the TU-game have priority over some other agents. Formally, a **priority structure** on N is a **partially ordered set**, also called a **poset**, $\succcurlyeq$, on the agent set N. Recall that a poset $(N, \succcurlyeq)$ is a reflexive, antisymmetric and transitive binary relation. The relation $i \succcurlyeq j$ means that i has priority over j. Denote by P_N the set of all posets $(N, \succcurlyeq)$, where $N \in \mathcal{N}$ and $\succcurlyeq$ is a poset on N.

A poset $(N, \succcurlyeq)$ gives rise to the asymmetric binary relation $(N, \succ)$: $i \succ j$ if $i \succcurlyeq j$ and $i \neq j$. For an agent $i \in N$, define the **priority group on** i, denoted by $\uparrow_{\succ} i$, as the set of agents having priority over i in $(N, \succcurlyeq)$:

$$\uparrow_{\succ} i = \{j \in N : j \succ i\},$$

and the **set of agents over whom** i **has priority** in $(N, \succcurlyeq)$ as

$$\downarrow_{\succ} i = \{j \in N : i \succ j\}.$$

Two distinct agents i and j are **incomparable** in $(N, \succcurlyeq)$ if neither $i \succ j$ nor $j \succ i$.

The poset $(N, \succcurlyeq^0) \in P_N$ containing no priority relations among any pair of distinct agents is called the **trivial poset**. A poset $(N, \succcurlyeq) \in P_N$ is a **linear order** if, for any pair of agents $\{i, j\} \subseteq N$, either $i \succcurlyeq j$ or $j \succcurlyeq i$, that is, if $(N, \succcurlyeq)$ is complete. A poset $(N, \succcurlyeq) \in P_N$ is **structured by levels** if the agent set N is partitioned into $p \leqslant |N|$ ordered classes $(N_1, \ldots, N_p)$ representing p priority levels in the following sense:

$$\forall i \in N, \quad [i \succ j] \Longleftrightarrow [(i \in N_k, j \in N_\ell) \Rightarrow k > \ell].$$

Consequently, in each class N_ℓ, $\ell \in \{1, \ldots, p\}$, each pair of distinct agent are incomparable and have the same priority group. Figure 1 represents a priority relation structured by levels.

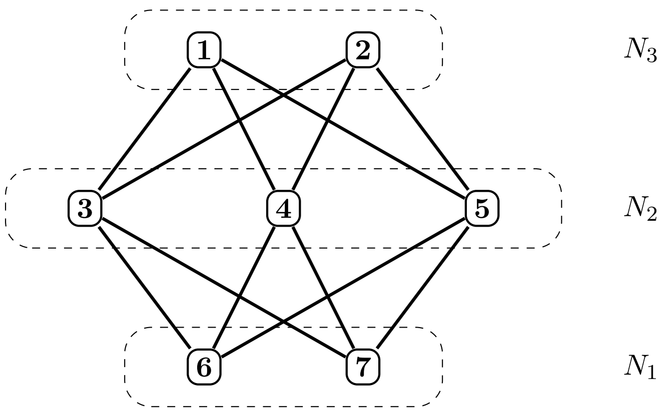

Figure 1: A poset structured by levels

Note that if $p = 1$, we have a trivial poset, and if $p = |N|$, we have a linear order. Denote by $P_N^L \subseteq P_N$ the set of all posets structured by levels on N.

For each nonempty coalition S, the **subposet** $(S, \succcurlyeq^S)$ of $(N, \succcurlyeq)$ induced by S is defined as follows: for each $i \in S$ and $j \in S$, $i \succcurlyeq^S j$ if $i \succcurlyeq j$. We will also use the notation $(S, \succcurlyeq)$ instead of $(S, \succcurlyeq^S)$ when no confusion arises. An agent i is a **priority agent** in $(S, \succcurlyeq)$ if, for $j \in S$, the relation $j \succcurlyeq i$ implies $i = j$. Denote by $M(S, \succcurlyeq)$ the nonempty subset of priority agents in $(S, \succcurlyeq)$.

The triple $(N, v, \succcurlyeq)$ where $N \in \mathcal{N}$, $(N, v) \in G_N$ and $(N, \succcurlyeq) \in P_N$ is called a **TU-game with priority structure** on N. Denote by GP_N the class of TU-games with priority structure that we can construct from G_N and P_N. In the same way, GP^L_N stands for the subclass of TU-games with priority structure that results from G_N and P^L_N. Finally set,

$$GP = \bigcup_{N\subseteq\mathcal{N}} GP_N \text{ and } GP^L = \bigcup_{N\subseteq\mathcal{N}} GP^L_N.$$

A **payoff vector** for a TU-game with priority structure $(N, v, \succcurlyeq) \in GP_N$ is a vector $x = (x_i)_{i\in N}$ assigning a payoff $x_i \in \mathbb{R}$ to each agent $i \in N$. We are interested in solutions defined either on GP or GP^L. For $D \in \{GP, GP^L\}$, a **solution** φ on D is a function that assigns a payoff vector $\varphi(N, v, \succcurlyeq) \in \mathbb{R}^N$ to any $(N, v, \succcurlyeq) \in D$.

3. Solutions for TU-games with priority structure and their relationships

This section introduces the family of Weighted Priority values on GP and shows that it generalizes several well-known (family of) solutions. Let $\mathbb{R}^{\mathfrak{U}}_{++} := \{f : \mathfrak{U} \to \mathbb{R}_{++}\}$ and $\omega_i := \omega(i)$ for all $\omega \in \mathbb{R}^{\mathfrak{U}}_{++}$ and $i \in \mathfrak{U}$. $\mathbb{R}^{\mathfrak{U}}_{++}$ is the collection of all **positive weight systems** on $\mathfrak{U}$.

For $\omega \in \mathbb{R}^{\mathfrak{U}}_{++}$, the **$\omega$-weighted priority value** in $(N, v, \succcurlyeq)$ is defined as:

$$\forall i \in N, \quad \mathrm{P}^\omega_i(N, v, \succcurlyeq) = \sum_{\substack{S\subseteq N:\\ i\in M(S,\succcurlyeq)}} \frac{\omega_i}{\sum\limits_{j\in M(S,\succcurlyeq)} \omega_j} \cdot \Delta_S(v). \tag{2}$$

When all weights are the same, then P^ω is the priority value P, i.e., when $\omega_i = c$ for some $c \in \mathbb{R}_{++}$, P^ω rewrites:

$$\begin{aligned}\forall i \in N, \quad \mathrm{P}^\omega_i(N, v, \succcurlyeq) &= \sum_{\substack{S\subseteq N:\\ i\in M(S,\succcurlyeq)}} \frac{1}{|M(S, \succcurlyeq)|} \cdot \Delta_S(v)\\ &= \mathrm{P}_i(N, v, \succcurlyeq).\end{aligned}$$

The ω-weighted priority values, $\omega \in \mathbb{R}^{\mathfrak{U}}_{++}$, constitutes the family of **Weighted priority values**.

Assume now that $(N, \succcurlyeq^L)$ is organized by levels. Let $(N_1, \ldots, N_p)$ be the ordered partition on N induced by $(N, \succcurlyeq^L)$ so that

$$\forall i \in N, \quad [i \succ^L j] \Longleftrightarrow [(i \in N_k, j \in N_\ell) \Rightarrow k > \ell].$$

For each nonempty coalition $S \subseteq N$, $i \in M(S, \succcurlyeq^L)$ if and only if $i \in \bar{S}$ where $\bar{S} := S \cap N_k$, and $k := \max\{\ell \in \{1, \ldots, p\} : S \cap N_\ell \neq \varnothing\}$, as defined in Kalai and Samet (1987). Therefore,

$$\begin{aligned}\forall i \in N, \qquad \mathrm{P}^\omega_i(N, v, \succcurlyeq^L) &= \sum_{\substack{S\subseteq N:\\ i\in\bar{S}}} \frac{\omega_i}{\sum\limits_{j\in\bar{S}} \omega_j} \cdot \Delta_S(v)\\ &= \mathrm{Sh}^{(\omega,\succcurlyeq^L)}_i(N, v),\end{aligned}$$

where $\mathrm{Sh}^{(\omega,\succcurlyeq^L)}$ denotes the $(\omega, \succcurlyeq^L)$**-weighted Shapley value**. The family of $(\omega, \succcurlyeq^L)$-weighted Shapley values, $\omega \in \mathfrak{U}$ and $(N, \succcurlyeq^L) \in P^L_N$, forms the **Weighted Shapley values**. Hence a weighted priority value, defined on GP, coincides with a weighted Shapley value, defined on GP^L.

If $p = 1$, i.e., $\succcurlyeq^L = \succcurlyeq^0$, then the induced ordered partition is $\{N\}$ and

$$\forall i \in N, \qquad \mathrm{P}^{\omega}_i(N, v, \succcurlyeq^L) = \sum_{\substack{S \subseteq N:\\ i \in S}} \frac{\omega_i}{\sum\limits_{j \in S} \omega_j} \cdot \Delta_S(v) = \mathrm{Sh}^{\omega}_i(N, v),$$

where Sh^{ω} stands for the **ω-positively weighted Shapley value**. If, moreover, all weights are the same, then Sh^{ω} coincides with the Shapley value Sh,

$$\forall i \in N, \qquad \mathrm{P}^{\omega}_i(N, v, \succcurlyeq^L) = \sum_{\substack{S \subseteq N:\\ i \in S}} \frac{1}{|S|} \cdot \Delta_S(v) = \mathrm{Sh}_i(N, v).$$

Finally, if $p = n$ then $\succcurlyeq^L$ is a linear order, which implies that each coalition contains a single priority agent. Without loss of generality, assume that $N = \{1, \ldots, n\}$ and that the induced ordered partition is such that $N_k = \{k\}$ for each $k \in N$. Then $\mathrm{P}^{\omega}(N, v, \succcurlyeq^L)$ coincides with the **downward marginal vector** $\mathrm{m}^{\pi}(N, v)$ where π is the permutation on N such that $\pi(i) = n-i+1$ for each $i \in \{1, \ldots, n\}$. Formally,

$$\begin{aligned}\forall i \in N, \qquad \mathrm{P}^{\omega}_i(N, v, \succcurlyeq) &= \sum_{\substack{S \subseteq N:\\ i \in M(S, \succcurlyeq)}} \frac{\omega_i}{\sum\limits_{j \in M(S, \succcurlyeq)} \omega_j} \cdot \Delta_S(v)\\ &= \sum_{\substack{S \subseteq N:\\ i \in M(S, \succcurlyeq)}} \Delta_S\\ &= v(\{i, \ldots, n\}) - v(\{i+1, \ldots, n\})\\ &= \mathrm{m}^{\pi}_i(N, v).\end{aligned}$$

The family of ω-positively weighted Shapley values, $\omega \in \mathfrak{U}$, forms the **Positively weighted Shapley values**.

Figure 2 represents the relationships between the different solutions for a fixed TU-game (N, v) when the poset $\succcurlyeq$ on N and the weights $\omega_i, i \in N$, vary.

4. Axioms for solutions on games with priority structure

We formulate the axioms for solutions on GP, state and prove the main characterization result. We also show that the axioms are logically independent. Furthermore, we point out that the main characterization result holds on GP^L as well. Before proving the main characterization result, we provide two intermediary results on the logical consequences of the combination of some axioms. The axioms invoked to design solutions can be divided up into punctual and relational axioms. A punctual axiom applies to each TU-game with priority structure separately and a relational axiom relates payoff vectors of TU-games with priority structure that are related in a certain way.

Consider a solution φ on GP. The first two axioms are punctual and straightforwardly generalize axioms for solutions on TU-games.

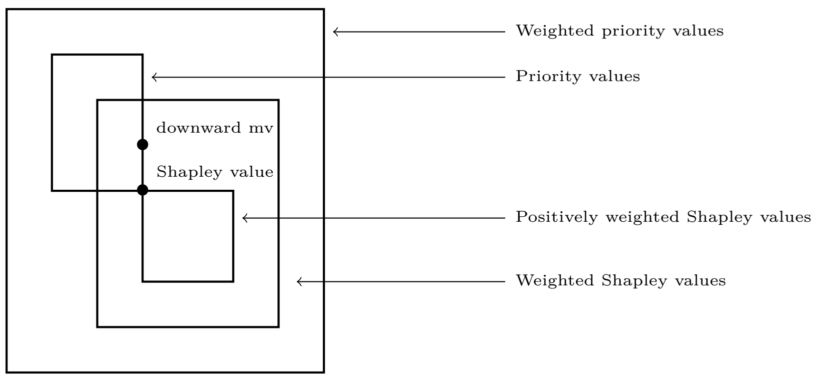

Figure 2: Relationships between the families of solutions.

Efficiency (E). For each $(N, v, \succcurlyeq) \in GP$, it holds that $\sum_{i \in N} \varphi_i(N, v, \succcurlyeq) = v(N)$.

Null agent (N). For each $(N, v, \succcurlyeq) \in GP$ and each null agent $i \in N$ in (N, v), it holds that $\varphi_i(N, v, \succcurlyeq) = 0$.

Next, we introduce relational properties of solutions on GP. The first two relational axioms consider a change of the agent set. In both axioms, an agent is removed from both the TU-game and the priority structure. Both axioms are invariance axioms that specify the same payoff vector for some specific agents across TU-games with priority structure that are somehow linked. The first axiom states that removing a null agent from a TU-game with a priority structure does not affect the payoffs of the remaining agents. This axiom is a straigthforward generalization of the well-known Null agent out axiom used for solutions in TU-games (see Derks and Haller, 1999). The second axiom states that removing an agent does not affect the payoffs of the agents over which it has priority: an agent i's payoff only relies on the subgame defined on the set of agents over whom i has priority or to which i is incomparable. Hence this axiom somewhat gives precedence to the poset structure before considering the coalition function.

Null agent out (NAO). For each $(N, v, \succcurlyeq) \in GP$ and each null agent $j \in N$ in (N, v), it holds that $\varphi_i(N, v, \succcurlyeq) = \varphi_i(N \backslash j, v, \succcurlyeq)$, for all $i \in N \backslash j$.

Priority agent out (PAO). For each $(N, v, \succcurlyeq) \in GP$ and each agent $j \in N$, it holds that $\varphi_i(N, v, \succcurlyeq) = \varphi_i(N \backslash j, v, \succcurlyeq)$, for all $i \in \downarrow_{\succ} j$.

Remark 2. *It is well-known that the combination of Efficiency* **(E)** *and Null agent out* **(NAO)** *on the domain of TU-games implies Null agent* **(N)**. *The Null agent axiom states that a null agent in a TU-game obtains a zero payoff. This implication remains trivially true on GP.*

The following relational axiom is new. It states that a change in the position of a null agent in the priority structure should not affect the payoffs of the other agents: payoffs of non-null agents only rely on the relative priority between non-null agents, independently of the positions of the null agents. Hence this axiom somewhat singles out the set of non-null agents before considering the poset structure.

Independence of null agent position (INAP). For each $(N, v, \succcurlyeq), (N, v, \succcurlyeq') \in GP$ and each null agent $j \in N$ in (N, v) such that $(N\backslash j, v, \succcurlyeq) = (N\backslash j, v, \succcurlyeq')$, it holds that $\varphi_i(N, v, \succcurlyeq) = \varphi_i(N, v, \succcurlyeq')$, for all $i \in N\backslash j$.

Finally, we introduce two relational axioms that consider variations on the coalitional function. The first axiom conceptualizes the idea that agents' payoffs should depend only on the marginal contributions they make to coalitions. The last axiom requires that the payoffs of two agents with the same priority group change in the same direction when their marginal contributions to coalitions not containing either of them do not change.

Marginality (M). For each $(N, v, \succcurlyeq), (N, w, \succcurlyeq) \in GP$ and each $i \in N$ such that $v(S \cup i) - v(S) = w(S \cup i) - w(S)$ for all $S \subseteq N\backslash i$, it holds that $\varphi_i(N, v, \succcurlyeq) = \varphi_i(N, w, \succcurlyeq)$.

Superweak differential marginality for agents with the same priority group (SWDM-SPG). For each $(N, v, \succcurlyeq), (N, w, \succcurlyeq) \in GP$ and each $i, j \in N$ such that $\uparrow_{\succ} i = \uparrow_{\succ} j$ and

$$\begin{aligned} &v(S \cup i) - v(S) = w(S \cup i) - w(S) \quad \text{and} \\ &v(S \cup j) - v(S) = w(S \cup j) - w(S), \quad \forall S \subseteq N\backslash\{i, j\}, \end{aligned}$$

it holds that $\text{sign}\,(\varphi_i(N, v, \succcurlyeq) - \varphi_i(N, w, \succcurlyeq)) = \text{sign}\,(\varphi_j(N, v, \succcurlyeq) - \varphi_j(N, w, \succcurlyeq))$.

Note that the hypothesis of Superweak differential marginality for agents with the same priority group is satisfied if and only if i and j are mutually dependent in $(N, v - w)$ and have the same priority group. The above two axioms are inspired by two axioms for solutions in TU-games. Marginality is a straightforward extension of the axiom introduced by Young (1985) to provide a characterization of the Shapley value in terms of Efficiency, Marginality, and Equal treatment of equals. Recall that the latter axiom states that two equal agents in a TU-game receive the same payoff. Superweak differential marginality for agents with the same priority group is an adaptation of Superweak differential marginality used by Casajus (2018) to characterize the Positively weighted Shapley values in terms of Efficiency, Superweak differential marginality and Null agent out. Superweak differential marginality indicates that the payoffs *of any two* mutually dependent agents in $(N, v - w)$ change in the same direction. The straightforward translation of this axiom in the context of TU-games with a priority structure is not satisfied by a weighted priority value in general: we propose a weakening of this axiom by imposing that these two mutually dependent agents must have the same priority group. Of course, in case the priority structure is the trivial one $(N, \succcurlyeq^0)$ both axioms apply the above principle for any pair of mutually dependent agents in $(N, v - w)$.

To sum up, the following results hold.

Theorem 1. *(Young (1985) and Casajus (2018))*

- *The Shapley value Sh is the only solution on G satisfying Efficiency, Marginality, and Equal treatment of equals;*
- *The Positively weighted Shapley values are the only solutions on G satisfying Efficiency, Superweak differential marginality and Null agent out.*

Remark 3. *All of the axioms introduced above are trivially compatible with GP^L, except Independence of null agent position* **(INAP)**. *Since this axiom is the only one that considers a change in the priority structure, we need to strengthening its hypothesis by requiring that $(N, \succcurlyeq')$, obtained from $(N, \succcurlyeq)$, is a priority structure organized by levels. In other words, the relocation of the null agent must be such that the induced priority structure remains a priority structure organized by levels. Despite this slight difference, we continue to use the expression Independence of null agent position to describe this axiom on GP^L.*

Proposition 1. *Priority agent out* **(PAO)** *and Independence of null agent position* **(INAP)** *on GP imply Null agent out* **(NAO)**.

Proof. Let φ be a solution on GP satisfying **PAO** and **INAP** and $j \in N$ a null agent in (N, v). Consider the poset $\succcurlyeq'$ on N defined as follows:

$$\forall \ell, m \in N, \quad \ell \succcurlyeq' m :\Longleftrightarrow \begin{cases} (a) \ \ell = j \text{ and } m \in N\backslash j, \\ (b) \ \ell \succcurlyeq m, \text{ and } \ell, m \in N\backslash j. \end{cases}$$

Notice that $\downarrow_{\succ'} j = N\backslash j$ and $(N\backslash j, v, \succcurlyeq) = (N\backslash j, v, \succcurlyeq')$. Then for each $i \in N\backslash j$, we have

$$\varphi_i(N, v, \succcurlyeq) \overset{\textbf{INAP}}{=} \varphi_i(N, v, \succcurlyeq') \overset{\textbf{PAO}}{=} \varphi_i(N\backslash j, v, \succcurlyeq).$$

Therefore, φ satisfies **NAO**. ∎

Proposition 2. *Efficiency* **(E)**, *Priority agent out* **(PAO)**, *Independence of null agent position* **(INAP)** *and Superweak differential marginality for agents with the same priority group* **(SWDM-SPG)** *on GP implies Marginality* **(M)**.

Proof. Let φ be a solution on GP satisfying **E**, **PAO**, **INAP** and **SWDMSPG**. Then, by Proposition 1 and Remark 2, φ also satisfies **N** and **NAO**. Let $(N, v, \succcurlyeq), (N, w, \succcurlyeq) \in GP$ and $i \in N$ such that $\forall S \subseteq N\backslash i$, $v(S \cup i) - v(S) = w(S \cup i) - w(S)$. Hence i is a null agent in $(N, v - w)$ so that, by points (i) and (iii) in Remark 1:

$$\forall T \ni i, \ \Delta_T(v) = \Delta_T(w). \tag{3}$$

Let $\overset{*}{i} \in \mathfrak{U}\backslash i$, $N^* = N \cup \overset{*}{i}$ and define the poset $\succcurlyeq'$ on N^* as follows:

$$\forall \ell, m \in N^*, \quad \ell \succcurlyeq' m :\Longleftrightarrow \begin{cases} (a) \ \ell \in \uparrow_{\succ} i \text{ and } m = \overset{*}{i}, \\ (b) \ \ell \succcurlyeq m \text{ otherwise.} \end{cases}$$

Note that $\succcurlyeq'^N = \succcurlyeq$.

Now, let $(N^*, \bar{v}, \succcurlyeq'), (N^*, \bar{w}, \succcurlyeq'), (N^*, \hat{v}, \succcurlyeq'), (N^*, \hat{w}, \succcurlyeq') \in GP$ given by

$$\bar{v} := \sum_{T \subseteq N : T \neq \emptyset} \Delta_T(v) \cdot u_T, \qquad \bar{w} := \sum_{T \subseteq N : T \neq \emptyset} \Delta_T(w) \cdot u_T,$$

and

$$\hat{v} := \sum_{T \subseteq N : i \in T} \Delta_T(v) \cdot u_T, \qquad \hat{w} := \sum_{T \subseteq N : i \in T} \Delta_T(w) \cdot u_T.$$

Note that by definition, $\hat{v} \overset{(3)}{=} \hat{w}$, $\bar{v}^N = v$, and $\bar{w}^N = w$. Moreover, i, $\overset{*}{i}$, $\bar{v}$ and $\hat{v}$ fulfill the conditions of **SWDMSPG**, as well as i, $\overset{*}{i}$, $\bar{w}$ and $\hat{w}$. Finally, notice that $\overset{*}{i}$ is a null agent in $(N^*, \bar{v})$, $(N^*, \bar{w})$, $(N^*, \hat{v})$, and $(N^*, \hat{w})$. Hence, we have

$$\text{sign}\left(\varphi_i(N^*, \bar{v}, \succcurlyeq') - \varphi_i(N^*, \hat{v}, \succcurlyeq')\right) \overset{\mathbf{SWDMSPG}}{=} \text{sign}\left(\varphi_{\overset{*}{i}}(N^*, \bar{v}, \succcurlyeq') - \varphi_{\overset{*}{i}}(N^*, \hat{v}, \succcurlyeq')\right) \overset{\mathbf{N}}{=} 0.$$

Then,

$$\varphi_i(N^*, \bar{v}, \succcurlyeq') = \varphi_i(N^*, \hat{v}, \succcurlyeq') \text{ and, analogously, } \varphi_i(N^*, \bar{w}, \succcurlyeq') = \varphi_i(N^*, \hat{w}, \succcurlyeq') \tag{4}$$

Finally, we obtain

$$\begin{aligned} \varphi_i(N, v, \succcurlyeq) &= \varphi_i(N, v, \succcurlyeq') \\ &\overset{\mathbf{NAO}}{=} \varphi_i(N^*, \bar{v}, \succcurlyeq') \\ &\overset{(4)}{=} \varphi_i(N^*, \hat{v}, \succcurlyeq') \\ &\overset{(3)}{=} \varphi_i(N^*, \hat{w}, \succcurlyeq') \\ &\overset{(4)}{=} \varphi_i(N^*, \bar{w}, \succcurlyeq') \\ &\overset{\mathbf{NAO}}{=} \varphi_i(N, w, \succcurlyeq') \\ &= \varphi_i(N, w, \succcurlyeq). \end{aligned}$$

Therefore, φ satisfies **M**. ■

Remark 4. *The implications mentioned in Remark 2, Proposition 1 and Proposition 2 also hold on GP^L. First, the axioms mentioned in Remark 2 do not depend on the priority structure. Thus the implication mentioned in this remark remains true on GP^L. Second, the proof of Proposition 1 relies on Independence of null agent position so as to put the null agent at the top of the poset: this is also possible in a poset organized by levels. Finally, the proof of Proposition 2 can also be adapted: the null agent $\overset{*}{i}$ has to be added at the same level as agent i so that they have the same priority group and the poset structure with or without $\overset{*}{i}$ is organized by levels.*

We have the material to prove the main characterization result.

Theorem 2. *The family of Weighted Priority values on GP is characterized by Efficiency (**E**), Priority agent out (**PAO**), Independence of null agent position (**INAP**) and Superweak differential marginality for agents with the same priority group (**SWDMSPG**).*

Proof. Fix $\omega \in \mathbb{R}^{\mathfrak{U}}_{++}$, let show that P^{ω} satisfies the four axioms.

- Let $N \in \mathcal{N}$ and $(N, v, \succcurlyeq) \in GP$, one has

$$\begin{aligned}\sum_{i \in N} \mathrm{P}^{\omega}_i(N, v, \succcurlyeq) &= \sum_{i \in N} \sum_{\substack{S \subseteq N: \\ i \in M(S,\succcurlyeq)}} \frac{\omega_i}{\sum\limits_{j \in M(S,\succcurlyeq)} \omega_j} \cdot \Delta_S(v) \\ &= \sum_{S \subseteq N} \sum_{i \in M(S,\succcurlyeq)} \frac{\omega_i}{\sum\limits_{j \in M(S,\succcurlyeq)} \omega_j} \cdot \Delta_S(v) \\ &= \sum_{S \subseteq N} \Delta_S(v) \\ &= v(N).\end{aligned}$$

  Therefore, P^{ω} satisfies **E**.

- Let $N \in \mathcal{N}$ and $(N, v, \succcurlyeq) \in GP$. For any $k \in N$, notice that for each $i \in \downarrow_{\succ} k$ and $S \subseteq N$, $i \in M(S, \succcurlyeq) \Rightarrow k \notin S$ so that

$$\begin{aligned}\mathrm{P}^{\omega}_i(N, v, \succcurlyeq) &= \sum_{\substack{S \subseteq N: \\ i \in M(S,\succcurlyeq)}} \frac{\omega_i}{\sum\limits_{j \in M(S,\succcurlyeq)} \omega_j} \cdot \Delta_S(v) \\ &= \sum_{\substack{S \subseteq N \backslash k: \\ i \in M(S,\succcurlyeq)}} \frac{\omega_i}{\sum\limits_{j \in M(S,\succcurlyeq)} \omega_j} \cdot \Delta_S(v) \\ &= \mathrm{P}^{\omega}_i(N \backslash k, v, \succcurlyeq),\end{aligned}$$

  which shows that P^{ω} satisfies **PAO**.

- Let $(N, v, \succcurlyeq), (N, v, \succcurlyeq') \in GP$ and $k \in N$ a null agent in (N, v) such that $\succcurlyeq^{N \backslash k} = \succcurlyeq'^{N \backslash k}$ or, with our notation, $(N \backslash k, \succcurlyeq) = (N \backslash k, \succcurlyeq')$. Because k is a null agent, it holds that $\Delta_S(v) = 0$ for each coalition $S \subseteq N$ such that $S \ni k$. Now consider $i \in N \backslash k$. Then, for each $S \subseteq N \backslash k$, $M(S, \succcurlyeq) = M(S, \succcurlyeq')$. Hence

$$\begin{aligned}\mathrm{P}^{\omega}_i(N, v, \succcurlyeq) &= \sum_{\substack{S \subseteq N: \\ i \in M(S,\succcurlyeq)}} \frac{\omega_i}{\sum\limits_{j \in M(S,\succcurlyeq)} \omega_j} \cdot \Delta_S(v) \\ &= \sum_{\substack{S \subseteq N \backslash k: \\ i \in M(S,\succcurlyeq)}} \frac{\omega_i}{\sum\limits_{j \in M(S,\succcurlyeq)} \omega_j} \cdot \Delta_S(v) \\ &= \sum_{\substack{S \subseteq N \backslash k: \\ i \in M(S,\succcurlyeq')}} \frac{\omega_i}{\sum\limits_{j \in M(S,\succcurlyeq')} \omega_j} \cdot \Delta_S(v) \\ &= \sum_{\substack{S \subseteq N: \\ i \in M(S,\succcurlyeq')}} \frac{\omega_i}{\sum\limits_{j \in M(S,\succcurlyeq')} \omega_j} \cdot \Delta_S(v) \\ &= \mathrm{P}^{\omega}_i(N, v, \succcurlyeq'),\end{aligned}$$

  so that P^{ω} satisfies **INAP**.

- Let $(N, v, \succcurlyeq), (N, w, \succcurlyeq) \in GP$ and $i, j \in N$ such that $\uparrow_{\succ} i = \uparrow_{\succ} j$, $v(S \cup i) - v(S) = w(S \cup i) - w(S)$ and $v(S \cup j) - v(S) = w(S \cup j) - w(S)$ for each $S \subseteq N \backslash \{i, j\}$. Note that i and j are mutually dependent in $(N, v - w)$. By points (ii) and (iii) in Remark 1, $\Delta_S(v - w) = 0$ for each $S \subseteq N$ such that $|S \cap \{i, j\}| = 1$. Furthermore, for each $S \supseteq \{i, j\}$, $i \in M(S, \succcurlyeq) \Leftrightarrow j \in M(S, \succcurlyeq)$. Using the linearity of P^ω, we get,

$$\begin{aligned}
\mathrm{P}_i^\omega(N, v, \succcurlyeq) - \mathrm{P}_i^\omega(N, w, \succcurlyeq) &= \mathrm{P}_i^\omega(N, v - w, \succcurlyeq) \\
&= \sum_{\substack{S \subseteq N: \\ i \in M(S, \succcurlyeq)}} \frac{\omega_i}{\sum\limits_{k \in M(S, \succcurlyeq)} \omega_k} \cdot \Delta_S(v - w) \\
&= \sum_{\substack{S \subseteq N: \\ i,j \in M(S, \succcurlyeq)}} \frac{\omega_i}{\sum\limits_{k \in M(S, \succcurlyeq)} \omega_k} \cdot \Delta_S(v - w) \\
&= \omega_i \cdot \sum_{\substack{S \subseteq N: \\ i,j \in M(S, \succcurlyeq)}} \frac{\Delta_S(v - w)}{\sum\limits_{k \in M(S, \succcurlyeq)} \omega_k} \qquad (5)
\end{aligned}$$

where $\omega_i > 0$.

Then, $\text{sign}\left(\mathrm{P}_i^\omega(N, v, \succcurlyeq) - \mathrm{P}_i^\omega(N, w, \succcurlyeq)\right)$ only depends on the sign of the expression

$$\sum_{\substack{S \subseteq N: \\ i,j \in M(S, \succcurlyeq)}} \frac{\Delta_S(v - w)}{\sum\limits_{k \in M(S, \succcurlyeq)} \omega_k},$$

in which i and j play symmetrical roles. Finally, it holds that

$$\text{sign}\left(\mathrm{P}_i^\omega(N, v, \succcurlyeq) - \mathrm{P}_i^\omega(N, w, \succcurlyeq)\right) = \text{sign}\left(\mathrm{P}_j^\omega(N, v, \succcurlyeq) - \mathrm{P}_j^\omega(N, w, \succcurlyeq)\right),$$

so that P^ω satisfies **SWDMSPG**. Moreover, from (5), it will be useful to note that:

$$\mathrm{P}_i^\omega(N, v - w, \succcurlyeq)\omega_j = \mathrm{P}_j^\omega(N, v - w, \succcurlyeq)\omega_i \qquad (6)$$

Regarding the uniqueness part of the proof, note first that, for $\lambda > 0$, one has $\mathrm{P}^{\lambda \cdot \omega} = \mathrm{P}^\omega$ meaning that the weights of a ω-weighted Priority value are defined up to a constant factor. Furthermore,

$$\mathrm{P}_i^\omega(\{i, j\}, u_{\{i,j\}}, \succcurlyeq^0) = \frac{\omega_i}{\omega_i + \omega_j} > 0 \text{ so that } \frac{\omega_i}{\omega_j} = \frac{\mathrm{P}_i^\omega(\{i, j\}, u_{\{i,j\}}, \succcurlyeq^0)}{\mathrm{P}_j^\omega(\{i, j\}, u_{\{i,j\}}, \succcurlyeq^0)}.$$

This expression will be used to define the weights in the uniqueness part of the proof that proceeds as follows. Consider φ satisfying the four abovementioned axioms. The proof is based on a series of claims contained in Appendix 5. Claim 1 and **E** imply that $\varphi_{\hat{\imath}}(\{j, \hat{\imath}\}, u_{\{j,\hat{\imath}\}}, \succcurlyeq^0) > 0$ so that it is possible to define the following weights $\omega^\varphi \in \mathbb{R}_{++}^{\mathfrak{U}}$ on $\mathfrak{U}$ as follows. Fix $\hat{\imath} \in \mathfrak{U}$ and set

$$\omega_{\hat{\imath}}^\varphi = 1 \qquad \text{and} \qquad \omega_j^\varphi = \frac{\varphi_j(\{j, \hat{\imath}\}, u_{\{j,\hat{\imath}\}}, \succcurlyeq^0)}{\varphi_{\hat{\imath}}(\{j, \hat{\imath}\}, u_{\{j,\hat{\imath}\}}, \succcurlyeq^0)} \quad \text{for each } j \in \mathfrak{U} \backslash \hat{\imath}. \qquad (7)$$

Let us show that φ coincides with $\mathrm{P}^{\omega^\varphi}$.

First note the following consistency property: for each $\omega \in \mathbb{R}^{\mathfrak{U}}_{++}$, there exists $\lambda = \frac{1}{\omega_{\hat{\imath}}} > 0$, which depends on the arbitrary choice of agent $\hat{\imath}$, such that $\omega^{\mathrm{P}^\omega} = \lambda\omega$. This ensures that the payoffs computed with $\mathrm{P}^{\omega^\varphi}$ are independent of the arbitrarily chosen agent $\hat{\imath}$ in the particular representation of the weights ω^φ.

Given a TU-game with a priority structure $(N, v, \succcurlyeq) \in GP$, define the subset of coalitions

$$\mathcal{T}(v) = \{T \subseteq 2^N \backslash \{\varnothing\} : \Delta_T(v) \neq 0\}.$$

The proof that $\varphi(N, v, \succcurlyeq) = \mathrm{P}^{\omega^\varphi}(N, v, \succcurlyeq)$ is done by induction on the cardinality of $\mathcal{T}(v)$.
Induction basis: if $\mathcal{T}(v) = \varnothing$, then $(N, v, \succcurlyeq) = (N, \mathbf{0}, \succcurlyeq)$ and by **N**, we have $\varphi_j(N, v, \succcurlyeq) = 0 = \mathrm{P}^{\omega^\varphi}(N, v, \succcurlyeq)$ for each $j \in N$ and each poset $\succcurlyeq$. The use of **N** follows from Proposition 1, Remark 2 and **E**.
Induction hypothesis: Assume that $\varphi(N, v, \succcurlyeq) = \mathrm{P}^{\omega^\varphi}(N, v, \succcurlyeq)$ for each $(N, v, \succcurlyeq) \in GP$ such that $|\mathcal{T}(v)| \leqslant t$, for $t \in \mathbb{N}$ and $t < |2^N \backslash \{\varnothing\}|$.
Induction step: Consider $(N, v, \succcurlyeq)$ such that $|\mathcal{T}(v)| = t + 1$. Set

$$Y(N, v, \succcurlyeq) = \{j \in N : j \in M(T, \succcurlyeq), \forall T \in \mathcal{T}(v)\}.$$

For $i \in N \backslash Y(N, v, \succcurlyeq)$, there is $T \in \mathcal{T}(v)$ such that $j \notin M(T, \succcurlyeq)$. Two cases arise:

- $j \notin T$. Then, using Proposition 2 and the induction hypothesis,

$$\varphi_j(N, v, \succcurlyeq) \overset{\mathbf{M}}{=} \varphi_j(N, v - \Delta_T(v) \cdot u_T), \succcurlyeq) \overset{\mathbf{IH}}{=} \mathrm{P}^{\omega^\varphi}_j(N, v - \Delta_T(v) \cdot u_T), \succcurlyeq) \overset{\mathbf{M}}{=} \mathrm{P}^{\omega^\varphi}_j(N, v, \succcurlyeq).$$

- $j \in T$. Then, there exists $i \in M(T, \succcurlyeq)$ such that $i \succ j$ and

$$\varphi_j(N, v, \succcurlyeq) \overset{\mathbf{PAO}}{=} \varphi_j(N \backslash i, v, \succcurlyeq) \overset{\mathbf{IH}}{=} \mathrm{P}^{\omega^\varphi}_j(N \backslash i, v, \succcurlyeq) \overset{\mathbf{PAO}}{=} \mathrm{P}^{\omega^\varphi}_j(N, v, \succcurlyeq).$$

Hence we have

$$\sum_{j \in Y(N, v, \succcurlyeq)} \varphi_j(N, v, \succcurlyeq) \overset{\mathbf{E}}{=} \sum_{j \in Y(N, v, \succcurlyeq)} \mathrm{P}^{\omega^\varphi}_j(N, v, \succcurlyeq). \tag{8}$$

Now if $|Y(N, v, \succcurlyeq)| \leqslant 1$, we are done. Otherwise, fix $\overset{*}{i} \in Y(N, v \succcurlyeq)$. Any two agents in $Y(N, v, \succcurlyeq)$ are mutually dependent in (N, v) and incomparable with respect to $\succcurlyeq$.
Let us define

$$Y^+(N, v, \succcurlyeq) = \{k \in N \backslash Y(N, v, \succcurlyeq), \exists j \in Y(N, v, \succcurlyeq), k \succ j\}.$$

By definition of $Y(N, v, \succcurlyeq)$, for each $T \in \mathcal{T}(v)$ we have $T \cap Y^+(N, v, \succcurlyeq) = \varnothing$. Hence all agents in $Y^+(N, v, \succcurlyeq)$ are null agents in (N, v) and by **NAO**, $\varphi_j(N, v, \succcurlyeq) = \varphi_j(N \backslash Y^+(N, v, \succcurlyeq), v, \succcurlyeq)$ for each $j \in Y(N, v, \succcurlyeq)$.

Now, for each $j \in Y(N, v, \succcurlyeq)$, we have $j \in M(N \backslash Y^+(N, v, \succcurlyeq), \succcurlyeq)$ so that $\overset{*}{i}$ and j have the same (empty) priority group. Hence we can apply Claim 9 in the appendix and we have

$$\forall j \in Y(N, v, \succcurlyeq) \backslash \overset{*}{i}, \qquad \varphi_j(N, v, \succcurlyeq) \cdot \omega^{\varphi}_{\overset{*}{i}} = \varphi_{\overset{*}{i}}(N, v, \succcurlyeq) \cdot \omega^{\varphi}_j. \tag{9}$$

The system formed by equations (8) and (9) has a unique solution. Furthermore, the payoffs $\mathrm{P}^{\omega^\varphi}_j(N, v, \succcurlyeq)$, for each $j \in Y(N, v, \succcurlyeq)$ form a solution for this system of equations: clearly, they meet (8); since every two agents in $Y(N, v, \succcurlyeq)$ are mutually dependent in (N, v) by Remark 1, they also satisfy (9) thanks to (6). Therefore, $\varphi(N, v, \succcurlyeq) = \mathrm{P}^{\omega^\varphi}(N, v, \succcurlyeq)$. ■

Remark 5. *The axioms invoked in Theorem 2 are logically independent:*

- *The null value $\varphi_i(N, v, \succcurlyeq) = 0$, for each $(N, v, \succcurlyeq) \in GP$ and $i \in N$ satisfies all the axioms except* **E**.

- *The Shapley value satisfies all the axioms except* **PAO**. *Indeed, this solution does not depend on the poset structure and so trivially satisfies* **INAP**. *Casajus (2018) has shown that the Shapley value satisfies the superweak differential marginality axiom which implies that it satisfies* **SWDMSPG**.

- *The Priority Equal Division value $PED(N, v, \succcurlyeq)$ defined below satisfies all the axioms except* **INAP**. *Given $(N, v, \succcurlyeq) \in GP$, define $\rho_{N,\succcurlyeq}(i) = \max\{|S| : i \in M(S, \succcurlyeq)\}$ for $i \in N$ and the partition $(\rho^{-1}_{N,\succcurlyeq}(k))_{k \in \rho_{N,\succcurlyeq}(N)}$ of N. Note that $i \succ j \Rightarrow \rho_{N,\succcurlyeq}(i) > \rho_{N,\succcurlyeq}(j)$. Finally, define*

$$\forall i \in N, \qquad PED_i(N, v, \succcurlyeq) = \frac{v(\rho^{-1}_{N,\succcurlyeq}([\![1, \rho_{N,\succcurlyeq}(i)]\!])) - v(\rho^{-1}_{N,\succcurlyeq}([\![1, \rho_{N,\succcurlyeq}(i) - 1]\!]))}{|\rho^{-1}_{N,\succcurlyeq}(\rho_{N,\succcurlyeq}(i))|}.$$

  This solution satisfies **E** *and* **PAO** *by construction. Note also that for $i, j \in N$, $\uparrow_{\succ} i = \uparrow_{\succ} j \Rightarrow \rho_{N,\succcurlyeq}(i) = \rho_{N,\succcurlyeq}(j)$ so that $PED_i(N, v, \succcurlyeq) = PED_j(N, v, \succcurlyeq)$ and* **SWDMSPG** *is satisfied.*

- *The Priority Equal Division value for non-null agents $PED^0(N, v, \succcurlyeq)$, defined as*

$$\forall i \in N, \qquad PED^0_i(N, v, \succcurlyeq) = \begin{cases} PED_i(N \backslash N_0, v, \succcurlyeq), & \text{if } i \notin N_0 \\ 0, & \text{otherwise,} \end{cases}$$

  where N_0 is the set of null agents in (N, v), satisfies all the axioms except **SWDMSPG**. *Indeed, it satisfies* **E**, **INAP** *and* **PAO** *by construction.* □

It can be verified that the proof of each claim stated in the appendix is independent of the domain $D \in \{GP, GP^L\}$. Therefore, by Remarks 3 and 4, the proof of Theorem 2 holds on the restricted domain GP^L where the Weighted priority values coincide with the Weighted Shapley values. The above remark leads to a new characterization of the family of Weighted Shapley values on the whole domain GP^L which, unlike the one in Kalai and Samet (1987), involves variable agent sets and variable ordered partitions, corresponding to the variable poset structures organized by levels.

Corollary 1. *The family of Weighted Shapley values on GP^L is characterized by Efficiency (**E**), Priority agent out (**PAO**), Independence of null agent position (**INAP**) and Superweak differential marginality for agents with the same priority group (**SWDMSPG**).*

5. Conclusion

Recently, Casajus (2021) provides a characterization of the class of Positively weighted Shapley values through Efficiency, Marginality, and a relaxation of Balanced contributions (Myerson, 1980) called Weak balanced contributions. Balanced contributions requires that the amount one agent gains or loses when another agent leaves the game equals the amount the latter agent gains or loses when the former agent leaves the game. Weakly balanced contributions requires that the direction (sign) of the change of one agent's payoff when another agent leaves the game equals the direction (sign) of the change of the latter agent's payoff when the former agent leaves the game. It turns out that the Priority value satisfies the Balanced contributions principle for each pair of agents having the same priority group. In view of the above results, the question naturally arises whether the classes of Weighted Priority values can be characterized by using Weak balanced contributions for pairs of agents with the same priority group, Efficiency, Marginality, and possibly Priority agent out or Independence of null agent position.

Appendix: detailed claims used in Theorem 2's proofs

The following claims are borrowed from the claims defined by Casajus (2018) for the domain G of TU-games. To extend these claims to our context of TU-games with priority structure, certain statements require that some pair of agents have the same priority group. Because the proofs can be adapted in a straightforward way from G to both domains GP and GP^L, we omit them and present only their statement. For illustrative purpose only, we reproduce the proof of Claims 1 and 9.

Consider φ satisfying the four axioms in Theorem 2.

Claim 0 (C0). *For each $i, j, k \in \mathfrak{U}$ such that $i \neq j \neq k \neq i$, the mapping $r_i^{ijk} : \mathbb{R} \to \mathbb{R}$ given by*

$$r_i^{ijk}(\rho) = \varphi_i\left(\{i, j, k\}, \rho \cdot u_{\{i,j,k\}}, \succcurlyeq^0\right), \qquad \forall \rho \in \mathbb{R},$$

is continuous.

Claim 1 (C1). *For each $i, j \in N$, $i \neq j$, such that i and j are both mutually dependent and have the same priority group in $(N, v, \succcurlyeq) \in GP$, we have* $\operatorname{sign}(\varphi_i(N, v, \succcurlyeq)) = \operatorname{sign}(\varphi_j(N, v, \succcurlyeq))$.

Proof. By Proposition 1 and Remark 2, φ satisfies **(N)** so that:

$$\begin{aligned}
\operatorname{sign}(\varphi_i(N, v, \succcurlyeq)) &\overset{\mathbf{N}}{=} \operatorname{sign}(\varphi_i(N, v, \succcurlyeq) - \varphi_i(N, \mathbf{0}, \succcurlyeq)) \\
&\overset{\mathbf{SWDMSPG}}{=} \operatorname{sign}(\varphi_j(N, v, \succcurlyeq) - \varphi_j(N, \mathbf{0}, \succcurlyeq)) \\
&\overset{\mathbf{N}}{=} \operatorname{sign}(\varphi_j(N, v, \succcurlyeq))
\end{aligned}$$

■

Claim 2 (C2). *For each $i, j \in N$, $i \neq j$, such that i and j are both mutually dependent and have the same priority group in $(N, v, \succcurlyeq) \in GP$, we have*

$$\varphi_i \left(N, v - (\varphi_i(N, v, \succcurlyeq) + \varphi_j(N, v, \succcurlyeq)) \cdot u_{\{i,j\}}, \succcurlyeq\right) = 0$$

and

$$\varphi_j \left(N, v - (\varphi_i(N, v, \succcurlyeq) + \varphi_j(N, v, \succcurlyeq)) \cdot u_{\{i,j\}}, \succcurlyeq\right) = 0.$$

Claim 3 (C3). *For each $i, j \in N$, $i \neq j$, such that i and j are both mutually dependent and have the same priority group in $(N, v, \succcurlyeq) \in GP$, we have*

$$\varphi_i(N, v, \succcurlyeq) = \varphi_i \left(\{i, j\}, (\varphi_i(N, v, \succcurlyeq) + \varphi_j(N, v, \succcurlyeq)) \cdot u_{\{i,j\}}, \succcurlyeq^0\right)$$

and

$$\varphi_j(N, v, \succcurlyeq) = \varphi_j \left(\{i, j\}, (\varphi_i(N, v, \succcurlyeq) + \varphi_j(N, v, \succcurlyeq)) \cdot u_{\{i,j\}}, \succcurlyeq^0\right).$$

Claim 4 (C4). *Let $i, j, k \in \mathfrak{U}$ be such that $i \neq j \neq k \neq i$ and $N = \{i, j, k\}$. For each $\Delta_{\{i,j\}}, \Delta_{\{i,k\}}, \Delta_N \in \mathbb{R}$, we have*

$$\begin{aligned} &\varphi_i \left(N, \Delta_N \cdot u_N + \Delta_{\{i,j\}} \cdot u_{\{i,j\}} + \Delta_{\{i,k\}} \cdot u_{\{i,k\}}, \succcurlyeq\right) \\ &\quad = \varphi_i \left(N, \Delta_N \cdot u_N + \Delta_{\{i,j\}} \cdot u_{\{i,j\}}, \succcurlyeq\right) + \varphi_i \left(N, \Delta_N \cdot u_N + \Delta_{\{i,k\}} \cdot u_{\{i,k\}}, \succcurlyeq\right) - \varphi_i \left(N, \Delta_N \cdot u_N, \succcurlyeq\right). \end{aligned}$$

Claim 5 (C5). *Let $i, j, k \in \mathfrak{U}$ be such that $i \neq j \neq k \neq i$ and $N = \{i, j, k\}$. Consider any poset $(N, \succcurlyeq) \in P_N$ such that $\uparrow_{\succ} i = \uparrow_{\succ} k$. For each $\Delta_{\{i,j\}}, \Delta_N \in \mathbb{R}$, we have*

$$\varphi_i \left(N, \Delta_N \cdot u_N + \Delta_{\{i,j\}} \cdot u_{\{i,j\}}, \succcurlyeq\right) = \varphi_i \left(N, \Delta_N \cdot u_N, \succcurlyeq\right) + \varphi_i \left(N, \Delta_{\{i,j\}} \cdot u_{\{i,j\}}, \succcurlyeq\right).$$

For each $i, j, k \in \mathfrak{U}$ such that $i \neq j \neq k \neq i$, let $r_{ij}^k : \mathbb{R} \to \mathbb{R}$ be the mapping given by

$$r_{ij}^k(\lambda) = \varphi_i \left(\{i, j, k\}, \lambda \cdot u_{\{i,j,k\}}, \succcurlyeq^0\right) + \varphi_j \left(\{i, j, k\}, \lambda \cdot u_{\{i,j,k\}}, \succcurlyeq^0\right), \quad \forall \lambda \in \mathbb{R}.$$

Claim 6 (C6). *Let $i, j \in \mathfrak{U}$ with $i \neq j$, such that there exists $k \in \mathfrak{U} \backslash \{i, j\}$ and r_{ij}^k is unbounded above or below. Then, we have*

$$\varphi_i \left(\{i, j\}, -\lambda \cdot u_{\{i,j\}}, \succcurlyeq^0\right) = -\varphi_i \left(\{i, j\}, \lambda \cdot u_{\{i,j\}}, \succcurlyeq^0\right), \qquad \forall \lambda \in \mathbb{R}.$$

Claim 7 (C7). *Let $i, j \in \mathfrak{U}$ with $i \neq j$, such that there exists $k \in \mathfrak{U} \backslash \{i, j\}$ such that r_{ij}^k is unbounded above or below. Then, we have*

$$\varphi \left(\{i, j\}, \lambda \cdot u_{\{i,j\}}, \succcurlyeq^0\right) = \lambda \varphi \left(\{i, j\}, u_{\{i,j\}}, \succcurlyeq^0\right), \qquad \forall \lambda \in \mathbb{R}.$$

Claim 8 (C8). *Let $i, j \in \mathfrak{U}$ such that $i \neq j$, we have*

$$\varphi \left(\{i, j\}, \lambda \cdot u_{\{i,j\}}, \succcurlyeq^0\right) = \lambda \varphi \left(\{i, j\}, u_{\{i,j\}}, \succcurlyeq^0\right), \qquad \forall \lambda \in \mathbb{R}.$$

Claim 9 (C9). *For each $i, j \in N$ with $i \neq j$, such that i and j are both mutually dependent and have the same priority group in $(N, v, \succcurlyeq) \in GP$, we have*

$$\varphi_i \left(N, v, \succcurlyeq\right) \omega_j^\varphi = \varphi_j \left(N, v, \succcurlyeq\right) \omega_i^\varphi,$$

where ω_i^φ and ω_j^φ are given by (7).

Proof. By **C1**, the claim holds true if $\varphi_i(N, v, \succcurlyeq) = 0$. If $\varphi_i(N, v, \succcurlyeq) \neq 0$, then, by **C1**, $\varphi_j(N, v, \succcurlyeq) \neq 0$ and $\varphi_i(N, v, \succcurlyeq) + \varphi_j(N, v, \succcurlyeq) \neq 0$. Hence we obtain:

$$\frac{\varphi_i(N, v, \succcurlyeq)}{\varphi_j(N, v, \succcurlyeq)} \overset{(\mathbf{C3})}{=} \frac{\varphi_i\left(\{i,j\}, (\varphi_i(N, v, \succcurlyeq) + \varphi_j(N, v, \succcurlyeq)) \cdot u_{\{i,j\}}, \succcurlyeq^0\right)}{\varphi_j\left(\{i,j\}, (\varphi_i(N, v, \succcurlyeq) + \varphi_j(N, v, \succcurlyeq)) \cdot u_{\{i,j\}}, \succcurlyeq^0\right)} \overset{(\mathbf{C8})}{=} \frac{\varphi_i\left(\{i,j\}, u_{\{i,j\}}, \succcurlyeq^0\right)}{\varphi_j\left(\{i,j\}, u_{\{i,j\}}, \succcurlyeq^0\right)}. \quad (10)$$

If $i = \hat{\imath}$ or $j = \hat{\imath}$, then definitions (7) and (10) entail the claim. For $i, j \in \mathfrak{U}\backslash\{\hat{\imath}\}$, observe that i, j and $\hat{\imath}$ are two by two mutually dependent and have the same priority group in $(\{i, j, \hat{\imath}\}, u_{\{i,j,\hat{\imath}\}}, \succcurlyeq^0)$ so that, by **C1** and **E**, $\varphi_k(\{i, j, \hat{\imath}\}, u_{\{i,j,\hat{\imath}\}}, \succcurlyeq^0) > 0$ for each $k \in \{i, j, \hat{\imath}\}$. It follows that:

$$\begin{aligned}
\frac{\varphi_i(N, v, \succcurlyeq)}{\varphi_j(N, v, \succcurlyeq)} &\overset{(10)}{=} \frac{\varphi_i\left(\{i,j\}, u_{\{i,j\}}, \succcurlyeq^0\right)}{\varphi_j\left(\{i,j\}, u_{\{i,j\}}, \succcurlyeq^0\right)} \\
&\overset{(10)}{=} \frac{\varphi_i(\{i,j,\hat{\imath}\}, u_{\{i,j,\hat{\imath}\}}, \succcurlyeq^0)}{\varphi_j(\{i,j,\hat{\imath}\}, u_{\{i,j,\hat{\imath}\}}, \succcurlyeq^0)} \\
&= \frac{\dfrac{\varphi_i(\{i,j,\hat{\imath}\}, u_{\{i,j,\hat{\imath}\}}, \succcurlyeq^0)}{\varphi_{\hat{\imath}}(\{i,j,\hat{\imath}\}, u_{\{i,j,\hat{\imath}\}}, \succcurlyeq^0)}}{\dfrac{\varphi_j(\{i,j,\hat{\imath}\}, u_{\{i,j,\hat{\imath}\}}, \succcurlyeq^0)}{\varphi_{\hat{\imath}}(\{i,j,\hat{\imath}\}, u_{\{i,j,\hat{\imath}\}}, \succcurlyeq^0)}} \\
&\overset{(10)}{=} \frac{\dfrac{\varphi_i(\{i,\hat{\imath}\}, u_{\{i,\hat{\imath}\}}, \succcurlyeq^0)}{\varphi_{\hat{\imath}}(\{i,\hat{\imath}\}, u_{\{i,\hat{\imath}\}}, \succcurlyeq^0)}}{\dfrac{\varphi_j(\{j,\hat{\imath}\}, u_{\{j,\hat{\imath}\}}, \succcurlyeq^0)}{\varphi_{\hat{\imath}}(\{j,\hat{\imath}\}, u_{\{j,\hat{\imath}\}}, \succcurlyeq^0)}} \\
&\overset{(7)}{=} \frac{\omega_i^{\varphi}}{\omega_j^{\varphi}}.
\end{aligned}$$

This concludes the proof. ■

Figure 3 allows to visualize the relationships between the axioms and the claims. The graph is formed by the claims (circled vertices) and the axioms (boxed vertices). An arc between two vertices v_1 and v_2 is drawn if v_1 is used to prove v_2. For instance, to represent that **N** is implied by **NAO** and **E**, an arc is drawn from the box labeled **NAO** to the box labeled **N**, and another arc is drawn from the box labeled **E** to the box labeled **N**. The vertices associated with **NAO** and **E** and the arcs emanating from them are in blue. A vertex is in red if a pair of agents with the same priority group is required in the statement of the associated axiom or claim. All the arcs emanating from a red vertex are depicted in red. A vertex is in black if either the axiom or the claim does not assume that some agents have the same priority group. All the arcs emanating from a black vertex are depicted in black. An arc is dotted only if it starts from the axioms **N** and **M** to indicate that these axioms are implied by some combination of the axioms in the statement of Theorem 2.

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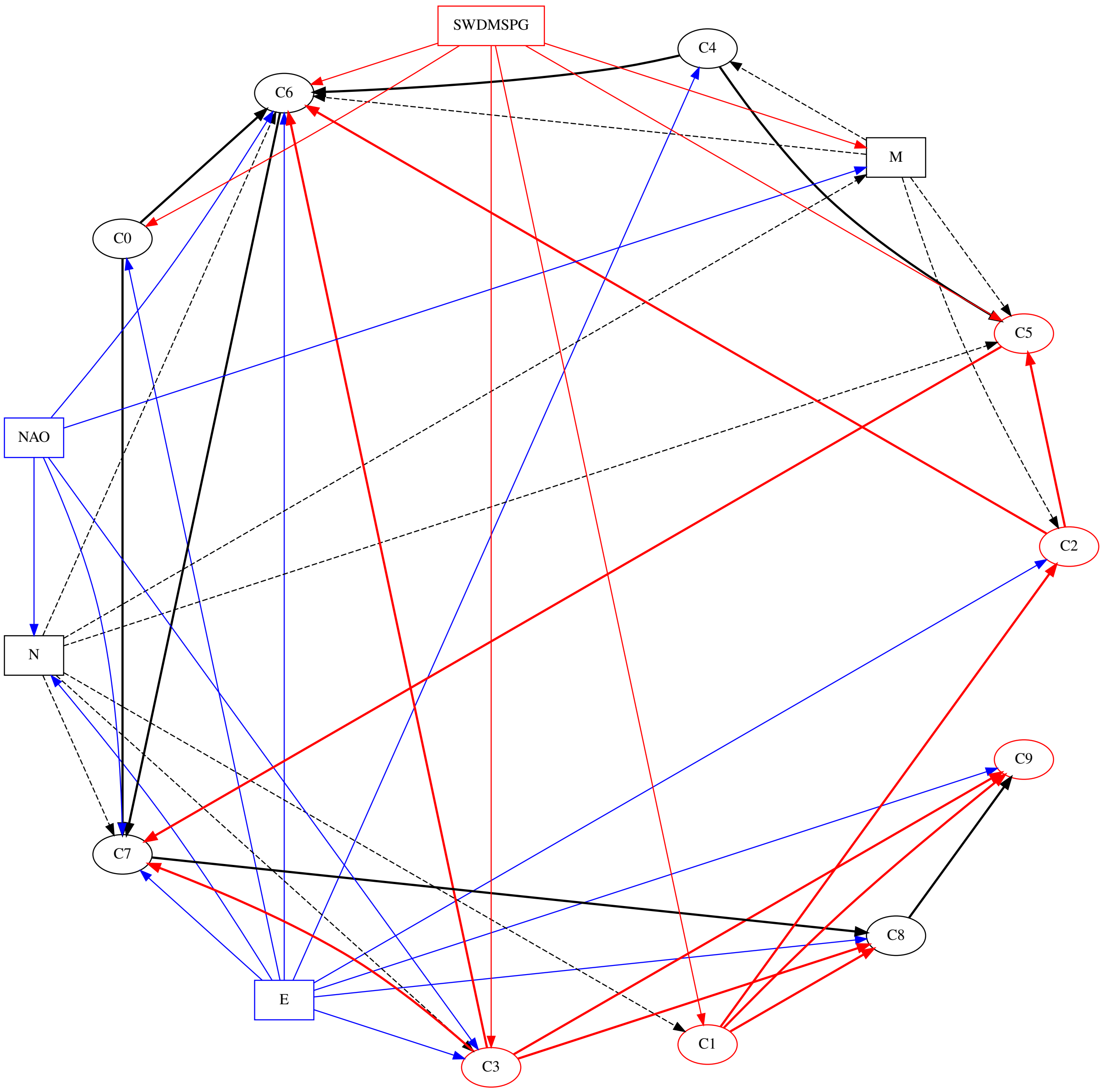

Figure 3: relationships between the axioms and the claims

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