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July 2022

Working paper No. 2022 – 07

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Early contributors, cooperation and fair rewards in crowdfunding*

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Abstract

We address the issue of rewarding fairly contributors participating in a funded crowdfunding project. We develop a theoretical non-strategic model of crowdfuding and introduce on a new reward rule, which specifies the individual rewards obtained by the contributors as a function of both their financial contributions and the timing of these contributions. Our model share some similarities with other models of ressource sharing in which the axiomatic method is frequently used. Taking this approach, we characterize this new reward rule by a pair of natural axioms, and it turns out that the resulting rewards coincide with the Shapley value of a suitable cooperative game built from the crowdfunding project. This allocation rule conveys what we call the signaling effect: if two contributors make the same financial contribution, then the earlier of the two obtains a greater reward. In specific but relevant cases, we provide extra properties of this reward rule.

Keywords: Crowdfunding, signaling, early contributions, fairness, cooperative games, Shapley value, core.

1. Introduction

Crowdfunding is now an essential tool for financing small- and medium-sized businesses. Crowdfunding aims to attract a large number of contributors or funders (who can be called the "crowd"). To achieve this, most crowdfunding methods offer contributors compensation that can take the form of a reward or a share in the company. According to Miglo (2022), the theoretical literature has developed around five issues:

*Financial support from the Fintech Chair at Paris Dauphine University is gratefully acknowledged.

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- To what extent does crowdfunding allow a company to better understand market demand, especially through pre-orders? (see for instance Chemla and Tinn, 2020);

- To what extent does crowdfunding allow a company whose project quality is not known with certainty to attest its quality to contributors? (see for instance Chen et al., 2018);

- To what extent does crowdfunding allow the creation of network effects, in particular through increased information exchange between contributors or between contributors and the company? (see for instance Belleflamme et al., 2014);

- To what extent does crowdfunding mitigate moral hazard problems? (see Schwienbacher, 2018; Belavina et al., 2020, for the choice of an effort by the entrepreneur and the issue of funds diversion);

- What roles do behavioral biases play in crowdfunding? (see for instance Miglo, 2021);

In this article, we address the issue of rewarding fairly the participating contributors by designing a suitable reward rule. In particular, we focus on two aspects of crowfunding that deviate from current practices.

The first one is that if two contributors made identical financial contributions to the project but at different time, the earlier of the two contributors can obtain a greater reward. There are multiple reasons which can justify such a difference in treatment. The signal sent by these two contributions is different. According to the notion of information cascade, two identical signals do not have the same effect if they are not sent at the same time. Rewarding the earliest contributor allows us to emphasize that the impact of their contribution (a distinct externality in the sense of Hu et al., 2015) on the success of the crowdfunding campaign is not the same. An early contribution is more valuable to an entrepreneur who seeks to estimate the uncertain demand for his asset (on this issue, see Strausz, 2017; Ellman and Hurkens, 2019, among others). Furthermore, an early contribution allows the contributor to promote the project to her network and thus encourage new contributions. Cason et al. (2021) also insist on the critical role of the first contributions from the point of view of potential contributors who are waiting to evaluate the dynamics of the growth of the amount fundraised.

We collected data from over 1 694 reward-based projects from three platforms (Miimosa, Tudigo, Winefunding) between 03/02/2013 to 10/16/2021. These platforms were chosen because of the availability of data on the timing of contributions. In total, 1 279 projects have a fundraising time less than 50 days, 415 have a fundraising time more than 50 days and the average collected amount by project is 5 017.3 euros. Figures 1 and 2 report the daily amount collected and the daily number of contributors (on the vertical axis), respectively, as a function of the fundraising time in days (on the horizontal axis). The empirical findings support the results in the literature that the first days are critical: the daily number of contributors and the daily amount collected decreased over time. Based on these evidence, we design a reward rule which consecrates the aforementioned signaling effects.

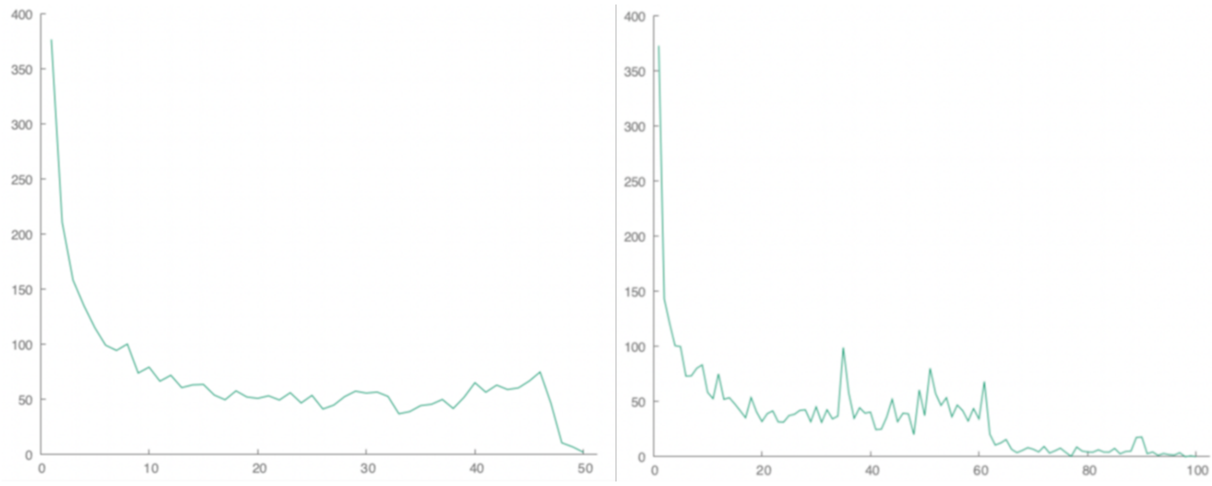

Figure 1: Daily amount collected in euros – left (< 50 days), right (> 50 days)

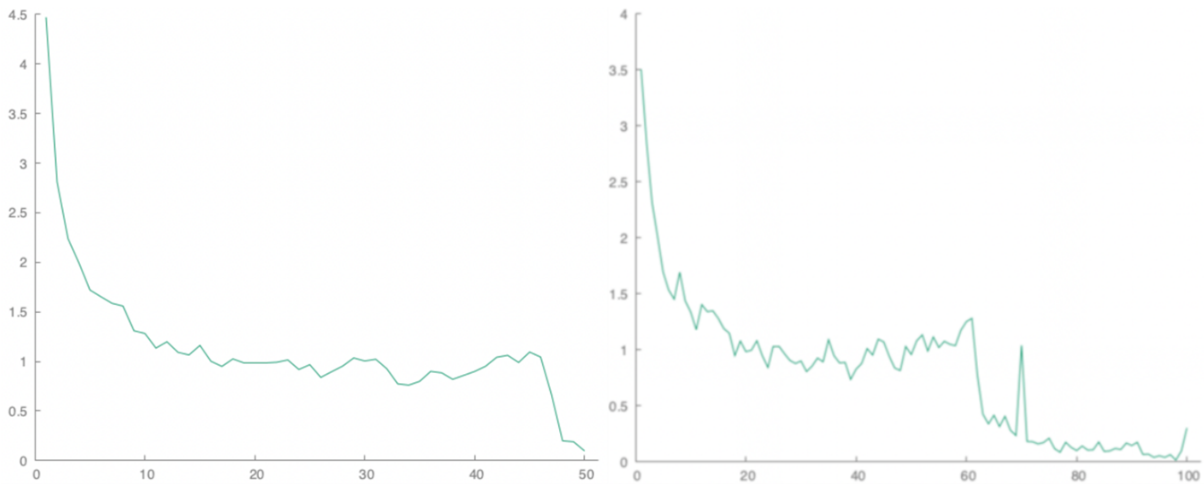

Figure 2: Daily contributors – left (< 50 days), right (> 50 days)

The second aspect on which we focus is a principle of collective reward generation. In practice, the reward level of a given contributor highly depends on the level of her contributions but little on the contribution level of the other contributors. To the contrary, we propose a new model in which the reward of each contributor can depend of all contributions to the project through a two-step procedure: (1) the total accumulated contribution yields the total reward that must be shared among the participating contributors and then (2) the previous amount of reward is distributed among the contributors according to our reward rule. We think that the principle of collective reward generation is relevant to take into account the potential synergies that distinct contributions can bring to the value of a project. At the very least, it seems fairer that these synergies should benefit contributors and not be captured solely by the entrepreneur.

Our results are based on an original non-strategic model of crowdfunding that incorporates the essential components of the problem: contributions, their timing, and a non-decreasing reward function that maps total contributions into total reward. An instance of our model should be seen as a snapshot of a funded crowdfunding project. Therefore, the reader should not expect to find in this article the reasons for the success of a campaign or the intensity with which contributors participate. Rather, the model is calibrated to determine what should be fair rewards for contributors. In order to achieve this objective, we rely on the theory of cooperative games, which has been extensively used in the past years to apprehend various applications in economics and finance (see Graham et al., 1990; Ambec and Sprumont, 2002; Maniquet, 2003; Baloga et al., 2017, among others), but not yet in the context of crowdfunding, to the best of our knowledge.

We rely on the two main tools of cooperative game theory, namely the Shapley value and the core. Our findings can be summarized as follows.

Firslty, we impose two desirable axioms for an arbitrary reward rule. The axiom of *Full distribution* states that the total available reward is fully distributed among the participating contributors. The axiom of *Fair rewarding* requires that the withdrawal of a first contributor must have the same impact on the reward of a second contributor as if we measure the impact of the withdrawal of the second contributor on the reward of the first contributor. It should be clear that this last axiom does not imply that two contributors are always equally important to the project. The withdrawal of a large contributor will result in a sharp decline in the value of the project and the total amount of reward to be distributed to the remaining contributors. The impact on a small contributor, however, will be moderate. Conversely, the withdrawal of a small contributor will result in a small decrease in the total amount of reward to be distributed to the remaining contributors. Hence, although a large contributor will absorb a significant portion of this change, the overall effect on that contributor will be moderate as well. We show that there is a unique reward rule satisfying the combination of the two axioms, and it turns out that this reward rule coincides with the Shapley value of a specific cooperative game built from the studied crowdfunding problem. In order to define this cooperative game, we rely on an assumption about the contributions that an contributor could make if some contributors withdraw – the so-called willingness to contribute. In this last situation, the contributor's contribution remains the same only if she still observes all signals relevant to, i.e. if all contributors preceding her indeed make their contributions.

Secondly, as we already underlined, the characterized reward rule conveys the signaling effect: an early contributor obtains a greater reward than a late contributor if they make the same financial contribution. In the final part of the article, we point out that this result remains valid for a wide range of assumptions about the contributors' willingness to contribute.

Thirdly, we obtain additional results on our reward rule when the reward function exhibits extra properties. If the reward function is convex, we show that the associated cooperative game is convex/supermodular, which means in the context of crowdfunding that the marginal contribution of each contributor in the game increases with the size of the group of contributors she joins. In this class of games, our reward rule is a core allocation. Therefore, if this cooperative game is used to describe the value of the project in terms of the participation of each conceivable group of contributors, then it means that none of these groups could provide its members with a higher reward than the one that our reward rule assigns them. In this sense, our reward rule can be considered as stable. If the reward function is a threshold function, i.e. the project is worth nothing until a certain threshold of contribution is reached, then the contributors can be partitioned into three groups: the earliest contributors (group 1), the intermediate contributors (group 2) and the latest contributors (group 2). Our reward rule is such that rewards are decreasing with time: the members of group 1 get more than the members of group 2 who get more than the members of group 3. Furthermore, the rewards within each group are equal except within group 2 in which they increase with financial contributions. Finally, if the reward function is additive and if there are no contributors contributing at the same time, then we show that the cooperative game associated with a crowdfunding problem is the same as the cooperative game arising from the so-called pure sequential liability situations in Dehez and Ferey (2013).

The rest of the article is organized as follows. Section 2 and 3 introduce cooperative games and crowdfunding problems, respectively. Section 4 motivates our assumption regarding the contributors' willingness to contribute and presents the axiomatic characterization of our reward rule. Section 5 provides the other aforementioned results for specific reward functions. Alternative assumptions about the willingness to contribute are discussed as concluding remarks in section 6. All proofs are relegated to the appendix.

2. Preliminaries on cooperative games

If A is a finite set, then we use the lower case a to denote its cardinality $|A|$. A **cooperative game** (with transferable utility) is a pair (N, v) such that $N = \{1, \dots, n\}$ is a finite set of players and v is a characteristic function which assigns to each coalition of players $S \in\subseteq N$ a worth $v(S)$ and such that $v(\varnothing) = 0$. For a game (N, v) and a nonempty coalition $S \subseteq N$, the **subgame** of (N, v) induced by S is the game $(S, v_{|S})$ such that, fore each $T \subseteq S$, $v_{|S}(T) = v(S)$.

A player $i \in N$ is a **null player** in a game (N, v) if, for each $S \subseteq N\backslash\{i\}$, it holds that $v(S \cup \{i\}) = v(S)$. A player $i \in N$ is a **necessary player** in a game (N, v) if, for each $S \subseteq N\backslash\{i\}$, it holds that $v(S) = 0$. A player $i \in N$ is **at least as desirable** as a player $j \in N$ is a game (N, v) if, for each $S \subseteq N\backslash\{i, j\}$, it holds that $v(S \cup \{i\}) \geq v(S \cup \{j\})$. Two players $i, j \in N$ are **equal players** in a game (N, v) if each is at least as desirable as the other. Two necessary players are equal but the converse is not always true.

A game (N, v) is **monotonic** if, for each pair of coalitions $S \subseteq T \subseteq N$, it holds that $v(S) \leq v(T)$. A game (N, v) is **convex** if, for each pair of coalitions $S, T \subseteq N$, it holds that $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$.

An **allocation rule** is a function f which assigns to each game (N, v) an allocation $f(N, v) \in \mathbb{R}^N$ specifying the payoff $f_i(N, v)$ obtained by each player $i \in N$ for her participation to game (N, v). The **Shapley value** (Shapley, 1953) is the allocation rule Sh which assigns to each game (N, v) and to each player $i \in N$ the payoff

$$Sh_i(N, v) = \sum_{S \subseteq N\backslash\{i\}} \frac{s!(n-s-1)!}{n!} \big(v(S \cup \{i\} - v(S)\big). \tag{1}$$

It is well-known that the Shapley value satisfies the following axioms, stated for an arbitrary allocation rule f.

Null player axiom. For each game (N, v) and each null player $i \in N$ in (N, v), $f_i(N, v) = 0$.

Desirability. (Maschler and Peleg, 1966) For each game (N, v) and each pair of players $i, j \in N$ such that i is at least as desirable as j in (N, v), $f_i(N, v) \geq f_j(N, v)$.

Equal treatment of equal players. For each game (N, v) and each pair of equal players $i, j \in N$ in (N, v), $f_i(N, v) = f_j(N, v)$.

The **core** of a game (N, v) is the (possibly empty) set of allocations $C(N, v)$ distributing the worth of the grand coalition in such a way that each coalition gets at least as much as its worth, i.e.

$$C(N, v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in S} x_i \geq v(S) \text{ for each } S \subseteq N \text{ and } \sum_{i \in N} x_i = v(N) \right\}.$$

Core allocations are often considered as stable whereas the Shapley value is often considered as a fair allocation rule.

3. Crowdfunding environments

In this section, we introduce crowdfunding environments and we discuss some special cases that will be taken up in the following sections. A **crowdfunding environment** models a completed crowdfunding project and is given by a four-tuple $C = (N, B, R, (c_i)_{i \in N})$ such that

- $N = \{1, \ldots, n\}$ is a finite set of **contributors**. Typical contributors are denoted by i and j;
- $B = (B_1, \ldots, B_k)$ is an ordered partition of N, $1 \leq k \leq n$, i.e. $\bigcup_{q \in \{1,\ldots,k\}} B_q = N$ and for each $q, q' \in \{1, \ldots, k\}$, $B_q \cap B_{q'} = \varnothing$, which models the **contribution timing**. If $i \in B_q$ and $j \in B_{q'}$, $q < q'$, this means that i is an earlier contributor than j. Each set B_q can be considered as time window containing contributors who cannot be distinguished with respect to the timing of their contribution. The case in which $B_q = \varnothing$ is allowed. Furthermore, let $q(i)$ be the index of the sole element of B containing contributor i;
- $R : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a **reward function** which assigns to each total of contributions $c \in \mathbb{R}_+$ a total reward $R(c) \in \mathbb{R}_+$ that must be used to reward contributors. We simply assume that $R(0) = 0$ and that R is non-decreasing. Function R can be considered as a proxy for the value of the project or not;
- for each contributor $i \in N$, $c_i \in \mathbb{R}_+$ is her **contribution**.

The general form of the reward function R includes the following special cases. A **threshold** reward function R is such that there are two real number c^* and r^* such that, for each $c \in \mathbb{R}$

$$R(c) = \begin{cases} r^* & \text{if } c \geq c^* \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

A **convex** reward function is such that R is a convex function. The **additive** reward function R is such that for each $c \in \mathbb{R}$, $R(c) = c$. The additive function is special instance of convex functions. Figure 3 illustrates these three types of reward functions.

For each crowdfunding environment $C = (N, B, R, (c_i)_{i \in N})$, the objective is to determine, for each contributor $i \in N$, a fair reward for her participation in C. This reward aims to assess the importance of the contributor's role in the success of the campaign, which may depend on the value of her contribution, but also on the timing of this contribution. In this article, we would like to highlight the fact that early contributors can be particularly important because of the signal their contribution sends to future contributors about the value of the project.

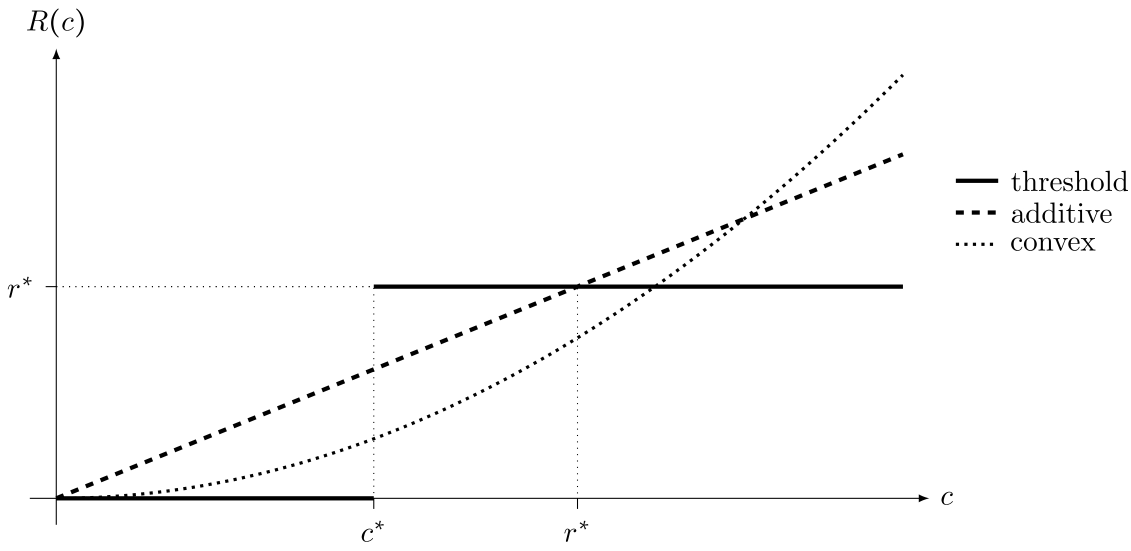

Figure 3: An illustration of three types of reward functions

4. The willingness to contribute and an axiomatic study

In order to deal with the aforementioned problem, we adopt counter-factual reasoning, which considers what would have happened in the absence of some contributors. Starting from a crowdfunding environment $C = (N, B, R, (c_i)_{i \in N})$ and any nonempty subgroup of contributors $S \subseteq N$, this means that we have to define the crowdfunding situation arising from the crowdfunding environment C if the other contributors in $N \backslash S$ leave or are absent. There are several ways to proceed, depending on the factors that are considered necessary for a contributor to be willing to contribute. Put differently, the crowdfunding situation on S is shaped by the impact that the leaving contributors may have on the remaining contributors, and in particular on the (late) contributors that do not receive the signal of the (early) missing contributions anymore. Should we assume that these contributors will maintain their contributions or, on the contrary, that they will renounce them?

We answer this question by retaining the following principle, which captures the fact that contributors are influenced by the situation they observe when deciding whether to contribute: to contribute when S is the set of contributors, a contributor needs all contributors prior to her in C to be made also in the crowdfunding situation on S. This principle can be considered as a pessimistic/prudent view in that a contributor of S gives up contributing as soon as she does not observe the same contributions prior to hers. While this principle is adapted to bring out significant effects, we also discuss in section 6 alternative principles to model the contributors' willingness to contribute.

To formalize these concepts, we need the following definitions. For each $S \subseteq N$ and each $i \in S$, let P_i^S denote the set of contributors in S who contribute before i, that is

$$P_i^S = \{j \in S : q(j) < q(i)\}.$$

Furthermore, define $c(P_i^S)$ as the total contribution of these contributors in P_i^S, that is

$$c(P_i^S) = \sum_{j \in P_i^S} c_j.$$

In the crowdfunding environment $C = (N, B, R, (c_i)_{i \in N})$, note that P_i^N corresponds to the set of contributors whose contributions contributor i observes when making her own contribution, and so $c(P_i^S)$ indicates the total funding of the project when i contributes.

From the crowdfunding environment $C = (N, B, R, (c_i)_{i \in N})$ and any coalition of contributors S, we define the crowdfunding situation $C^S = (S, B^S, R, (c_i^S)_{i \in S})$ induced by S in which

- $B^S = (B_1^S, \ldots, B_k^S)$ with $B_q^S = B_q \cap S$ for each $q \in \{1, \ldots, k\}$. In words, B^S is just the restriction of B to S;
- for each $i \in S$, $c_i^S = c_i$ if $P_i^N \subseteq S$ and $c_i^S = 0$ otherwise.

The second item means that contributor i makes her original contribution c_i in C^S only if all contributors in P_i^N are in S as well. We denote by $\mathcal{C}$ the set of all crowdfunding situations arising from the original crowdfunding environment $C = (N, B, R, (c_i)_{i \in N})$. Hence $\mathcal{C}$ also contains C since $C = C^N$. Next, we introduce two axioms for a reward rule f on $\mathcal{C}$. The first one imposes that the sum of the contributors' rewards is equal to the total reward generated by the sum of all their contributions.

Full distribution. For each $C^S = (S, B^S, R, (c_i^S)_{i \in S}) \in \mathcal{C}$,

$$\sum_{i \in S} f_i(C^S) = R\left(\sum_{i \in S} c_i^S\right). \tag{3}$$

The second axiom aims at translating fairness considerations into a property for a reward rule. More specifically, we impose that the withdrawal of a first contributor must have the same impact on the reward of a second contributor as if we measure the impact of the withdrawal of the second contributor on the reward of the first contributor.

Fair rewarding. For each $C^S = (N, B^S, R, (c_i^S)_{i \in S}) \in \mathcal{C}$ and each $i, j \in S$,

$$f_i(C^S) - f_i(C^{S \setminus \{j\}}) = f_j(C^S) - f_j(C^{S \setminus \{i\}}). \tag{4}$$

An alternative interpretation of the axiom in terms of threats is possible. Let us imagine that a contributor considers threatening to give up his contribution in order to obtain a larger share of the total rewards. Each other contributor contemplates the consequences that this threat would have on their own share of the rewards. The axiom simply states that each other contributor can neutralise such a threat by making the same threat of withdrawal to the first contributor, who would suffer the same impact as that created by his threat.

It should be clear that the axiom of Fair rewarding does not imply that two contributors are always equally important to the success of a campaign. The withdrawal of a large contributor will result in a sharp decline in both the value of the project and the total amount of reward to be distributed to the remaining contributors. The impact on a small contributor who can legitimately claim a small share of the total rewars, however, will be moderate. Conversely, the withdrawal of a small contributor will result in a small decrease in the total amount of reward to be distributed to the remaining contributors. Hence, although a large contributor will absorb a significant portion of this change, the overall effect on that contributor will be moderate as well. The axiom of Fair rewarding requires precisely that the two effects be identical. It is part of a long tradition of axioms in the literature on cooperative games which require similar balanced effects (see Myerson, 1977, 1980; Herings et al., 2008; Kamijo and Kongo, 2010; Béal et al., 2016; Yokote and Kongo, 2017; Yokote et al., 2018).

It turns out the there is a unique reward rule satisfying these two axioms, and that this reward rule coincides with the Shapley value of a specific cooperative game. This result is inspired by the characterization of the Shapley value provided by Myerson (1980).

Proposition 1. *There is a unique reward rule f^* on $\mathcal{C}$ that satisfies Full distribution and Fair rewarding. For each $C^S = (S, B^S, R, (c_i^S)_{i \in S}) \in \mathcal{C}$, it is given by $f^*(C^S) = Sh(S, v_{C^S})$ where (S, v_{C^S}) is the cooperative game defined, for each coalition of contributors $T \subseteq S$, as*

$$v_{C^S}(T) = R\left(\sum_{i \in T : P_i^S \subseteq T} c_i^S\right). \tag{5}$$

In order to understand the cooperative game given by (5), assume that the set of participating contributors is $S \subseteq N$ and that $T \subseteq S$ is the coalition under consideration. Then, $v_{C^S}(T)$ indicates the total rewards that should go to the members of T in this counterfactual situation, which corresponds to the total rewards calculated when the only contributions are those of T members whose predecessors in the original crowdfunding campaign (with N as the set of contributors) are also present in T.

It is easy to see that two axioms invoked in Proposition 1 are logically independent. The **null** reward rule which assigns unconditionally a null reward to each contributor satisfies

Fair rewarding but not Full distribution. The **equal split** reward rule ES such that for each $C^S = (S, B^S, R, (c_i^S)_{i \in S}) \in \mathcal{C}$ and each $i \in S$,

$$ES_i(C^S) = \frac{1}{s} R\left(\sum_{i \in S} c_i^S\right)$$

satisfies Full distribution but not Fair rewarding.

Remark 1. *From definition 5 of function v_{C^S}, it is obvious that (S, v_{C^S}) is a monotonic game for each nonempty $S \subseteq N$. Reformulated in the context of crowdfunding, this property has the intuitive meaning that the ability of a coalition to claim a significant share of the total available reward cannot decrease if the coalition expands.* □

5. Properties and special reward functions

In this section, we point out several interesting properties of our allocation rule f^* that are valid either for all crowdfunding environments and/or for special crowdfunding environments.

5.1. The signaling effect

The **signaling effect** reflects the phenomenon that early contributors are more important than late contributors, all other things being equal. We measure this effect by comparing the result attributed by a reward rule to two contributors with the same contribution level but at different time. Formally, a reward rule rule f conveys the signaling effect if for each crowdfunding environment $C = (N, B, R, (c_i)_{i \in N})$ and each pair of contributors $i, j \in N$ such that $c_i = c_j$ but $q(i) < q(j)$, it holds that $f_i(C) > f_j(C)$.

Proposition 2. *If R is (strictly) increasing, then the reward rule f^* conveys the signaling effect.*

In the same vein, it is easy to figure out that the rewards of two contributors with the same contribution timing are ranked according to their contributions by our reward rule: if $q(i) = q(j)$ and $c_i > c_j$, then $f_i^*(C) > f_j^*(C)$.

5.2. Convex reward functions

Whenever the reward function is convex, the associated cooperative game possesses extra properties.

Proposition 3. *For each crowdfunding environment $C = (N, B, R, (c_i)_{i \in N})$ such that the reward function R is convex, the associated cooperative game (N, v_C) is convex.*

Proposition 3 implies that the Shapley value of game (N, v_C) is a core allocation: if a group of contributors $N\backslash S$ eventually decide not to contribute to the project, then the group of remaining contributors S cannot end up is a better situation. More specifically, the members of S get a total reward according to the Shapley value which is not less than the total reward that they could obtain by their own, that is

$$\sum_{i \in S} Sh_i(N, v_C) \geq v_C(S) = v_{C^S}(S).$$

Another consequence of Proposition 3 is that another relevant allocation belongs to the core. In fact, from Shapley (1971), it is core stable to reward the contributors according to the extra value they add to the project at the time of their contribution (the contributors belonging to a given time unit B_q can be sorted in any way).

Proposition 3 obviously holds for any crowdfunding situation $C^S = (S, B^S, R, (c_i^S)_{i \in S})$, $S \subseteq N$. As a consequence, in this context we get from Sprumont (1990, Corollary 2) that the Shapley value is a population monotonic allocation scheme. In our framework, this means that for each $i \in N$ and each pair of coalitions $S \subseteq T$ such that $i \in S$, $f_i^*(C^S) \leq f_i^*(C^T)$, that is the rewards obtained by any contributor cannot increase after the leave of some other contributors or, equivalently, that the reward received by a contributor is weakly increasing in the population of participating contributors.

5.3. The threshold function

Now, we examine the specific shape of the allocation $f^*(C)$ for crowdfunding environment $C = (N, B, R, (c_i)_{i \in N})$ in which R is a threshold reward function. In order to state this result, let q^* be the minimal $q \in \{1, \ldots, k\}$ such that

$$\sum_{i \in \bigcup_{q' \leq q} B_{q'}} c_i \geq c^*,$$

that is, B_{q^*} is the earliest time unit during which total of contributions reaches the threshold c^*. We partition the contributors into three groups:

$$G_1 = \bigcup_{q < q^*} B_q \cup \left\{ i \in B_{q^*} : \sum_{j \in \bigcup_{q \leq q^*} B_q} c_j - c_i < c^* \right\}.$$

$$G_2 = \left\{ i \in B_{q^*} : \sum_{j \in \bigcup_{q \leq q^*} B_q} c_j - c_i \geq c^* \right\}.$$

$$G_3 = \bigcup_{q > q^*} B_q.$$

Group G_1 contains the contributors that are needed to reach the threshold c^*. Group G_2 contains some contributors in B_{q^*} without whom the earliest contributors up to B_{q^*} can still reach the threshold. Group G_3 contains the remaining (later) contributors. Figure 4 provides a schematic representation of these groups in which the nodes within group B_{q^*} represent the contributors belonging to this group, who are assumed to be positioned from left to right in ascending order of individual contributions.

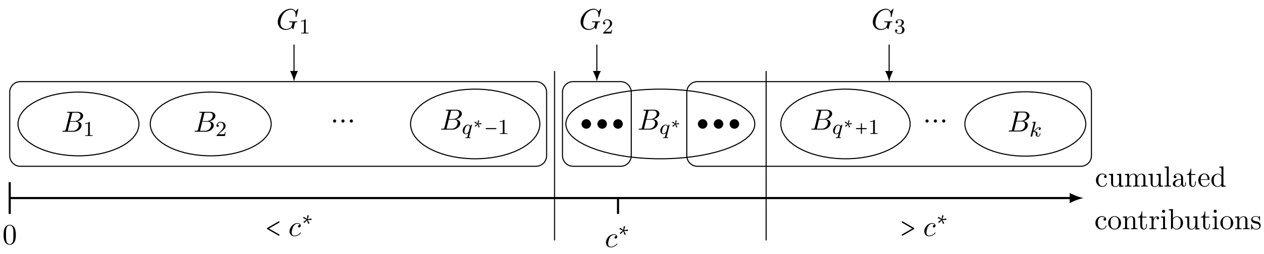

Figure 4: Illustration of groups G_1, G_2 and G_3

The next result shows that contributors in G_1 get the same reward, which is greater than the reward allocated to contributors in G_2, that the rewards of contributors in G_2 are non-decreasing with respect to their contributions, and that contributors in G_3 are not rewarded at all.

Proposition 4. *For each crowdfunding environment $C = (N, B, R, (c_i)_{i \in N})$ such that the reward function R is the threshold function given by* (2), *the allocation $f^*(C)$ satisfies the following properties:*

(i) if $i, j \in G_1$, then $f_i^*(C) = f_j^*(C)$;

(ii) if $i, j \in G_2$ and $c_i \geq c_j$, then $f_i^*(C) \geq f_j^*(C)$;

(iii) if $i \in G_1$, $j \in G_2$ and $l \in G_3$, then $f_i^*(C) > f_j^*(C) > f_l^*(C) = 0$.

5.4. Additive reward functions

In this paragraph, we establish links between our model and two other models in the literature: games arising from liability situations (Dehez and Ferey, 2013) and from aircraft landing fee problems (Littlechild and Owen, 1973).

A **liability situation** L on a player set N (the set of tortfeasors) is given by an ordered partition $B^L = (B_1^L, \ldots, B_k^L)$ of N, as for a crowdfunding situation, which reflects the sequence of wrongful acts to a victim, and a vector of damages $d = (d_1, \ldots, d_k)$ where d_q, $q \in \{1, \ldots, k\}$ is the extra damage caused by the group of tortfeasors B_q to the victim. Let D_q be the cumulative damage induced by the first q groups of tortfeasors in the sequence, i.e.

$$D_q = \sum_{h=1}^{q} d_h.$$

Then, the corresponding liability game (N, v_L) is such that, for each $S \subseteq N$,

$$v_L(S) = D_q,$$

where q is the largest integer such that $\cup_{h=1}^{q} B_h^L \subseteq S$.

An **aircraft landing fee problem** A on a player set N (the set of aircrafts which are to land on a runway) is given by an ordered partition $B^A = (B_1^A, \ldots, B_k^A)$ of N, which groups aircraft by type, from the smallest to the largest. The cost of building the runway depends upon the largest plane for which the runway is designed. The cost vector $a = (a_1, \ldots, a_k)$ specifies, for each $q \in \{1, \ldots, k\}$, the cost a_q necessary to make the runway suitable for landings by planes of type q. Then, the corresponding airport game (N, v_A) is such that, for each $S \subseteq N$,

$$v_A(S) = \max_{q \in \{1,\ldots,q\} : S \cap B_q^A \neq \emptyset} a_q.$$

In both situations, the pure sequential case is the one in which $|B_q^L| = 1$ and $|B_q^A| = 1$ for each $q \in \{1, \ldots, k\}$, respectively. Similarly, a crowdfunding environment $C = (N, B, R, (c_i)_{i \in N})$ is called purely sequential if $|B_q| = 1$ for each $B_q \in B$.

Proposition 5. *The cooperative games arising from purely sequential crowdfunding environments in which the reward function is additive are purely sequential liability games.*

Furthermore, Dehez and Ferey (2013) show that liability games and airport games are dual to each other, where the dual of a game (N, v) is the game (N, v^*) such that, for each $S \subseteq N$, $v^*(S) = v(N) - v(N \backslash S)$, and it is well-known that the Shapley value prescribes the same allocation in a game and in its dual. Hence, in the specific case described in Proposition 5 and for purely sequential aircraft landing fee problems, our reward rule f^* prescribes the same allocation as the Shapley value of the corresponding liability and airport games.

Proposition 5 does not extend to the non-sequential case as illustrated by the next example.

Example 1. Consider the crowdfunding environment $C = (N, B, R, (c_i)_{i \in N})$ such that $N = \{1, 2, 3\}$, $B_1 = \{1\}$ and $B_2 = \{2, 3\}$. Assume further that $c_i > 0$ for each $i \in N$ and that R is a (strictly) increasing function. In the corresponding game (N, v_C), it holds that $v_C(\{1\}) = R(c_1)$, $v_C(\{1, 2\}) = R(c_1 + c_2)$, $v_C(\{1, 3\}) = R(c_1 + c_3)$, $v_C(\{1, 2, 3\}) = R(c_1 + c_2 + c_3)$ and $v_C(S) = 0$ for each other coalition S. Next, assume that there is a liability situation $L = (N, B^L, d)$ from which the corresponding liability game (N, v_D) coincides with (N, v_C). From $v_C(\{1\}) = R(c_1) > 0$, it must be that $R(c_1) = d_1$ and $B_1^L = \{1\}$. From $v_C(\{1, 2\}) = R(c_1 + c_2) > R(c_1) = v_C(\{1\})$, it must be that $d_2 = R(c_1 + c_2) - R(c_1)$ and $B_2 = \{2\}$. But from $v_C(\{1, 3\}) = R(c_1 + c_3) > R(c_1) = v_C(\{1\})$, it must be that $d_3 = R(c_1 + c_3) - R(c_1)$ and $B_2 = \{3\} \neq \{2\}$, a contradiction which shows that there is no liability situation L such that $(N, v_L) = (N, v_C)$. □

6. Conclusion

As a conclusion, we discuss alternative principles which can be used to model the willingness to contribute and which generalize the principle considered in section 4. The first two principles discussed here have in common that they personalize the conditions that must be met for a contributor to actually decide to contribute to the project in the counterfactual scenario in which some other contributors are absent. The first is possibly based on the identity of contributors who preceded a given contributor. The second depends only on an observed amount of contribution. So, for the rest of this section, fix some crowdfunding environment $C = (N, B, R, (c_i)_{i \in N})$ and consider any crowdfunding (counterfactual) situation $C^S = (S, B^S, R, (c_i^S)_{i \in S})$, $S \subseteq N$.

In order to describe the first alernative principle, for each $i \in S$, denote by $E_i^S \subseteq P_i^S$ the set of contributors preceding i whose contribution must be effective for i to make its own contribution when the set of contributors is S. This approach is flexible: the principle considered in section 4 can be obtained by setting $E_i^S = P_i^S$ for each $i \in S$, but E_i^S and E_j^S can be different for two contributors i and j belonging to the same time window. In particular, this principle allows to highlight the identity of contributors in E_i^S in order to reflect, for example, the leading role of an influencer.

In order to describe the second alernative principle, for each $i \in S$, denote by e_i^S the accumulated contribution that the project must display in order for i to make its own contribution when the set of contributors is S. This principle implies that the identity of the contributors preceding i does not matter; only the level of contributions is relevant from the perspective of i. Once again, it generalizes the principle considered in section 4 which is the special case obtained by setting $e_i^S = \sum_{j \in P_i^S} c_j^S$.

The two approaches allow the benchmark case in which each contributor makes exactly the same contribution in the original crowdfunding environment and when the set of contributors is restricted to S by setting $E_i^S = \varnothing$ and $e_i^S = 0$ for each $i \in S$, respectively. Finally, note that under these two general principles it is necessary to adapt the cooperative game defined in equation 5.

These two principles take up the idea developed in section 4 that if a contributor does not observe certain signals, then she does not contribute at all. It is also possible to relax this assumption so that a contributor contributes partially if it partially observes the original signals. A natural way to take this principle into account is to assume that when the set of contributors is S, a contributor contributes in proportion to her original contribution, where the proportion is measured by the ratio between the contributions of her predecessors in S

and the contributions of its original predecessors in N.

Finally, remark that in all these principles, the contributions when the set of contributors is S are at most equal to (and many times strictly less) the original contributions for the participating contributors. An interesting consequence is that the signaling effect highlighted in Proposition 2 is robust to the adoption of any of these alternative principles.

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Appendix

Proof. (**Proposition 1**) We split the proof into two parts. Firstly, we begin by proving that the reward rule f^* which assigns to each crowdfunding situation $C^S = (S, B^S, R, (c_i^S)_{i\in S})$ the Shapley value of the game (S, v_{C^S}) satisfies the axioms of Full distribution and Fair rewarding. Regarding Full distribution, we have

$$\sum_{i\in S} f_i^*(C^S) = \sum_{i\in S} Sh_i(S, v_{C^S}) = v_{C^S}(S) = R\Bigg(\sum_{i\in S: P_i^S \subseteq S} c_i^S\Bigg) = R\Bigg(\sum_{i\in S} c_i^S\Bigg),$$

as desired, where the second equality comes from the fact that the Shapley value is always an efficient allocation and the last equality follows from the fact that $P_i^S \subseteq S\setminus\{i\}$ for each $i \in S$.

Regarding the axiom of Fair rewarding, we start with the following observation. For the original crowdfunding environment $C = (N, B, R, (c_i)_{i\in N})$, the cooperative game defined by (5) is the game (N, v_C) such that, for each $S \subseteq N$,

$$v_C(S) = R\Bigg(\sum_{i\in S: P_i^N \subseteq S} c_i\Bigg).$$

By definition of the subgame $(S, v_{C|S})$ of (N, v_C) induced by S, we have, for each $T \subseteq S$, $v_{C|S}(T) = v_C(T)$. Now, the cooperative game (S, v_{C^S}) defined by (5) for the crowdfunding situation $C^S = (S, B^S, R, (c_i^S)_{i\in S}) \in \mathcal{C}$ is such that, for each $T \subseteq S$,

$$v_{C^S}(T) = R\Bigg(\sum_{i\in T: P_i^S \subseteq T} c_i^S\Bigg).$$

By definition of c_i^S, the previous expression can be rewritten as

$$v_{C^S}(T) = R\Bigg(\sum_{i\in T:P_i^N\subseteq T, P_i^S\subseteq T} c_i\Bigg) = R\Bigg(\sum_{i\in T:P_i^N\subseteq T} c_i\Bigg) = v_C(T),$$

where the second equality comes from the fact that $P_i^S \subseteq P_i^N$. We have shown that $v_{C^S} = v_{C|S}$ for each $S \subseteq N$. Next, this equality can be used to write that, for each $S \subseteq N$ and each $i, j \in S$,

$$f_i^*(C^S) - f_i^*(C^{S\backslash\{j\}}) = Sh_i(S, v_{C^S}) - Sh_i(S\backslash\{j\}, v_{C^{S\backslash\{j\}}}) = Sh_i(S, v_{C|S}) - Sh_i(S\backslash\{j\}, v_{C|S\backslash\{j\}}).$$

From Myerson (1980), it is known that the Shapley value satisfies the axiom of balanced contributions, which imposes, for each pair $\{i, j\} \subseteq S$, that $Sh_i(S, v_{C|S}) - Sh_i(S\backslash\{j\}, v_{C|S\backslash\{j\}}) = Sh_j(S, v_{C|S}) - Sh_j(S\backslash\{i\}, v_{C|S\backslash\{i\}})$. As a consequence, we can write that

$$\begin{array}{rl} & f_i^*(C^S) - f_i^*(C^{S\backslash\{j\}}) \\ = & Sh_i(S, v_{C|S}) - Sh_i(S\backslash\{j\}, v_{C|S\backslash\{j\}}) \\ = & Sh_j(S, v_{C|S}) - Sh_j(S\backslash\{i\}, v_{C|S\backslash\{i\}}) \\ = & f_j^*(C^S) - f_j^*(C^{S\backslash\{i\}}), \end{array}$$

from which we conclude that f^* satisfies the axiom of Fair rewarding.

Secondly, we prove that if an arbitrary reward rule f on $\mathcal{C}$ satisfies the two axioms of Full distribution and Fair rewarding, then it must be that $f = f^*$. We proceed by induction on the number of contributors.

Initial step. Consider any $i \in N$ and the crowdfunding situation $C^{\{i\}} = (\{i\}, B^{\{i\}}, R, c_i^{\{i\}})$. By Full distribution, it is clear that $f_i(C^{\{i\}}) = R(c_i^{\{i\}}) = f_i^*(C^{\{i\}})$.

Induction hypothesis. Assume that $f(C^S) = f^*(C^S)$ for all crowdfunding situations $C^S = (S, B^S, R, (c_i^S)_{i\in S}) \in \mathcal{C}$ such that $s \leq m$, $1 \leq m < n$.

Induction step. Consider any crowdfunding situations $C^S = (S, B^S, R, (c_i^S)_{i\in S}) \in \mathcal{C}$ such that $s = m + 1$. For any pair of players $i, j \in S$, we have that

$$\begin{array}{rl} & f_i^*(C^S) - f_j^*(C^S) \\ = & f_i^*(C^{S\backslash\{j\}}) - f_j^*(C^{S\backslash\{i\}}) \\ = & f_i(C^{S\backslash\{j\}}) - f_j(C^{S\backslash\{i\}}) \\ = & f_i(C^S) - f_j(C^S). \end{array}$$

where the first and third equalities comes from the fact that both f^* and f satisfy Fair rewarding and the second equality follows from our induction hypothesis. Hence, we have $f_i^*(C^S) - f_i(C^S) = f_j^*(C^S) - f_j(C^S)$ for each $i, j \in S$. Summing on $j \in S$ and using the fact that both f and f^* satisfy Full distribution, we get

$$\begin{aligned} & \sum_{j\in S}\left(f_i^*(C^S)-f_i(C^S)\right)=\sum_{j\in S}\left(f_j^*(C^S)-f_j(C^S)\right) \\ \iff\ & s\left(f_i^*(C^S)-f_i(C^S)\right)=R\left(\sum_{i\in S}c_i^S\right)-R\left(\sum_{i\in S}c_i^S\right) \\ \iff\ & f_i^*(C^S)-f_i(C^S)=0, \end{aligned}$$

for each $i\in S$, which proves that $f(C^S)=f^*(C^S)$ and completes the proof. ■

Proof. (**Proposition 2**) Consider any crowdfunding environment $C=(N,B,R,(c_i)_{i\in N})$ and any pair of contributors $i,j\in N$ such that $c_i=c_j$ but $q(i)<q(j)$. Assume that R is an increasing function. We have to show that $f_i^*(C)>f_j^*(C)$. By Proposition 1, this inequality is equivalent to $Sh_i(N,v_C)>Sh_j(N,v_C)$. By definition (1) of the Shapley value, it is enough to show that i is at least as desirable as j in (N,v_C), with a strict inequality $v_C(S\cup\{i\})>v_C(S\cup\{j\})$ for at least one coalition $S\subseteq N\backslash\{i,j\}$. As a start, note that $q(i)<q(j)$ implies that $i\in P_j^N$,

$$P_i^N\subsetneq P_j^N, \tag{6}$$

and

$$\left[P_l^N\subseteq S\cup\{j\}\right]\Longrightarrow\left[P_l^N\subseteq S\cup\{i\}\right]\quad\forall S\subseteq N\backslash\{i,j\},\forall l\in S. \tag{7}$$

Using (6) and (7), we can consider two cases.

Firstly, if $P_j^N\subseteq S\cup\{j\}$, then $P_i^N\subseteq S\cup\{i\}$ and so

$$v_C(S\cup\{j\})=R\left(c_j+\sum_{l\in S:P_l^N\subseteq S\cup\{j\}}c_l\right)\leq R\left(c_i+\sum_{l\in S:P_l^N\subseteq S\cup\{i\}}c_l\right)=v_C(S\cup\{i\}).$$

Secondly, if $P_j^N\nsubseteq S\cup\{j\}$, then it holds that

$$v_C(S\cup\{j\})=R\left(\sum_{l\in S:P_l^N\subseteq S\cup\{j\}}c_l\right)\leq R\left(\sum_{l\in S:P_l^N\subseteq S\cup\{i\}}c_l\right)\leq v_C(S\cup\{i\}).$$

We conclude that $v_C(S\cup\{i\})\geq v_C(S\cup\{j\})$ holds for any $S\subseteq N\backslash\{i,j\}$. Finally, since $i\in P_j^N$, it follows that

$$v_C(P_i^N\cup\{i\})=R\left(c_i+\sum_{l\in P_i^N}c_l\right)>R\left(\sum_{l\in P_i^N}c_l\right)=v_C(P_i^N\cup\{j\}),$$

which is the desired strict inequality. ■

Proof. (**Proposition 3**) Consider any crowdfunding environment $C = (N, B, R, (c_i)_{i\in N})$ and assume that R is a convex function. From Shapley (1971), we have to show that for each $i \in N$ and each $S \subseteq T \subseteq N\backslash\{i\}$,

$$v_C(S \cup \{i\}) - v_C(S) \leq v_C(T \cup \{i\}) - v_C(T). \tag{8}$$

Recall that for each $S \subseteq N$, from (5), we have

$$v_C(S) = R\Bigg(\sum_{j\in S: P_i^N \subseteq S} c_j \Bigg).$$

In order to save on notations, let c_S stand for $\sum_{j\in S:P_i^N\subseteq S} c_j$. It is easy to figure out that for two coalitions $S \subseteq T$, $\{j \in S : P_i^N \subseteq S\} \subseteq \{j \in T : P_i^N \subseteq T\}$, so that we get the following inequalities:

$$c_S \leq c_{S\cup\{i\}} \leq c_{T\cup\{i\}}, \tag{9}$$

and

$$c_S \leq c_T \leq c_{T\cup\{i\}}. \tag{10}$$

Both cases $c_{S\cup\{i\}} \leq c_T$ and $c_T \leq c_{S\cup\{i\}}$ are possible but it does not matter for the rest of the proof. Furthermore, remark that

$$\Bigg(\Big\{j \in S\cup\{i\} : P_j^N \subseteq S\cup\{j\}\Big\}\backslash\Big\{j \in S : P_j^N \subseteq S\Big\}\Bigg) \subseteq \Bigg(\Big\{j \in T\cup\{i\} : P_j^N \subseteq T\cup\{j\}\Big\}\backslash\Big\{j \in T : P_j^N \subseteq T\Big\}\Bigg),$$

which implies that

$$c_{S\cup\{i\}} - c_S \leq c_{T\cup\{i\}} - c_T. \tag{11}$$

From (9), (10) and the convexity of R, we get

$$\frac{R\big(c_{S\cup\{j\}}\big) - R\big(c_S\big)}{c_{S\cup\{i\}} - c_S} \leq \frac{R\big(c_{T\cup\{j\}}\big) - R\big(c_T\big)}{c_{T\cup\{i\}} - c_T} \leq \frac{R\big(c_{T\cup\{j\}}\big) - R\big(c_T\big)}{c_{S\cup\{i\}} - c_S},$$

where the last inequality comes from (11). Therefore, it follows that

$$R\big(c_{S\cup\{j\}}\big) - R\big(c_S\big) \leq R\big(c_{T\cup\{j\}}\big) - R\big(c_T\big),$$

which is equivalent to (8) as desired. ∎

Proof. (**Proposition 4**) Part *(i)*. We start by showing that any player in group G_1 is a necessary player in the associated game (N, v_C). So pick any $i \in G_1$ and any $S \subseteq N\backslash\{i\}$. Then,

$$\sum_{j\in S:P_j^N\subseteq S} c_j \leq \sum_{j\in S\cap G_1} c_j < c^*,$$

where the strict inequality comes from the definition of G_1 and the fact that $i \in G_1 \backslash S$. Hence, since R is the threshold function, we obtain that $v_C(S) = 0$, proving that i is a necessary player in (N, v_C). Because i was chosen arbitrarily in G_1, conclude that all players in G_1 are necessary in (N, v_C), which in turn implies that these players are all equal in (N, v_C). By the axiom of equal treatment of equals, we get that $Sh_i(N, v_C) = Sh_j(N, v_C)$ for each $i, j \in G_1$, which is equivalent to $f_i^*(C) = f_j^*(C)$ for each $i, j \in G_1$ as desired.

Part *(ii)*. We prove that if $i, j \in G_2$ and $c_i \geq c_j$, then i is at least as desirable as j in (N, v_C). So, pick any $S \subseteq N \backslash \{i, j\}$. Because i and j are in G_2, they belong to the same time unit B_{q^*}. This implies that $P_i^N = P_j^N$ and so that

$$\sum_{l \in S \cup \{i\} : P_l^N \subseteq S \cup \{i\}} c_l \geq \sum_{l \in S \cup \{j\} : P_l^N \subseteq S \cup \{j\}} c_l.$$

The latter inequality is equivalent to $v_C(S \cup \{i\}) \geq v_C(S \cup \{j\})$, proving that i is at least as desirable as j in (N, v_C). From the axiom of desirability, we get that $Sh_i(N, v_C) \geq Sh_j(N, v_C)$ for each $i, j \in G_2$ such that $c(i) \geq c(j)$, or equivalently that $f_i^*(C) \geq f_j^*(C)$ for each $i, j \in G_2$ such that $c(i) \geq c(j)$.

Part *(iii)*. We proceed in two steps. Firstly, we show that any contributor in G_3 is a null player in game (N, v_C). So let $i \in G_3$ and choose any $S \subseteq N \backslash \{i\}$. Two cases can be distinguished. If $\sum_{j \in S : P_j^N \subseteq S} c_j \geq c^*$, then obviously $v_C(S) = r^* = v_C(S \cup \{i\})$. To the contrary, if $\sum_{j \in S : P_j^N \subseteq S} c_j < c^*$, then it must be that $(G_1 \cup G_2) \nsubseteq S$, which implies that $P_i^N \nsubseteq S \cup \{i\}$ since $(G_1 \cup G_2) \subseteq P_i^N$. Similarly, if $j \in S$ is such that $P_j^N \nsubseteq S$, then $P_j^N \nsubseteq S \cup \{i\}$ as well. Therefore, $\sum_{j \in S : P_j^N \subseteq S} c_j = \sum_{j \in S \cup \{i\} : P_j^N \subseteq S \cup \{i\}} c_j$, leading once again to the equality $v_C(S) = v_C(S \cup \{i\})$. Conclude that i is a null player in (N, v_C).

Secondly, we show that if $i \in G_1$, $j \in G_2$ and $l \in G_3$, then i is at least as desirable as j and j is at least as desirable as l in (N, v_C). Consider any $S \subseteq N \backslash \{i, j\}$. From part *(i)*, we have that $v_C(S \cup \{j\}) = 0$ since contributor i is necessary in (N, v_C). By monotonicity of (N, v_C) (see remark 1) and $v_C(\varnothing) = 0$, we obtain that

$$v_C(S \cup \{j\}) = 0 \leq v_C(S \cup \{i\}), \tag{12}$$

proving that i is at least as desirable as j in (N, v_C). Next, pick any $S \subseteq N \backslash \{j, l\}$. We already proved that l is a null player in (N, v_C) at the beginning of part *(iii)*. Together with the monotonicity of (N, v_C), this implies that

$$v(S \cup \{l\}) = v(S) \leq v(S \cup \{j\}), \tag{13}$$

proving that j is at least as desirable as l in (N, v_C). It is easy to show that the previous two inequalities can be strict by choosing $S = P_i^N$ in (12) and $S = P_j^N$ in (13). It follows that $Sh_i(N, v_C) > Sh_j(N, v_C) > Sh_l(N, v_C) = 0$ or, equivalently, $f_i^*(C) > f_j^*(C) > f_l^*(C)$. ■

Proof. (**Proposition 5**) Suppose that the reward function of a crowdfunding environment is additive and assumme further that each time window contains one and only one contibutor. For simplicity, assume that the contribution timing is consistent with the natural order on natural numbers, i.e. $B = (B_1, \ldots, B_n)$ with $B_i = \{i\}$ for each $i \in N$. In such a case, let us show that the corresponding cooperative game (N, v_C) coincides with the liability game obtained from the pure sequential liability situation in which $d_i = c_i$ and $B_i^L = \{i\}$ for each $i \in N$. In fact, letting i_S be the greatest player in S for which $P_i^N \subseteq S$, for each $S \subseteq N$, we have

$$v_C(S) = R\left(\sum_{i \in S : P i^N \subseteq N} c_i\right) = R\left(\sum_{i=1}^{i_S} c_i\right) = \sum_{i=1}^{i_S} c_i,$$

where the second equality follows from the fact that $B = B^L$ contains n singletons and the third equality comes from the additivity of R. This completes the proof. ■