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Léa Munich

December 2023

Working paper   No. 2023−08

ᴜFR **SJEPG**

Sciences juridiques économiques
politiques et de gestion

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Abstract

In this paper, we optimize and allocate the costs of a non-rival common-pool resource among several users. In such a so-called schedule situation the players have different demands given by distinct subsets of periods satisfying their needs. The total costs resulting from shared use of the resource are allocated by natural allocations called Equal Pooling allocations, in which the cost of each needed period is shared equally among the users of this period. The associated schedule game gives, for each coalition of players, the minimal cost of a period configuration satisfying the needs of all its members. We have three main contributions. First, we provide several sufficient conditions for the non-emptiness of the core of a schedule game. Second, we prove that under some of these conditions the Shapley value is in the core and coincides with some Equal pooling allocation. Third, we establish connections with other classes of operational research games. Furthermore, we present an application to the allocation of the common costs of the mail carrier route of La Poste, the french postal operator.

Keywords: (B) Game theory, Schedule, OR-game, Cost allocation, Equal pooling allocations.
JEL codes: C71, L87.

1. Introduction

In this article we introduce a new scheduling cost allocation problem called a schedule situation. Several players share a non-rival common-pool infrastructure whose consumption is possible during several periods but is costly. The per-period cost can vary from one period to the next. The needs of each player are expressed in the form of consumption schedules, i.e. each schedule specifies a minimum set of periods that meets the player's needs.

The objective is to find the combination of consumption schedules for all the players that minimizes the overall cost while all the needs are satisfied, in particular through potential mutualisation since the resource is non-rival. Once this total cost has been determined, the natural

next step is to allocate it among the participating players, taking into account how they managed to jointly use the infrastructure. We investigate this cost allocation problem by means of the theory of cooperative games with transferable utility. The resulting game specifies, for each coalition of players, the cheapest cost of a set of periods that meets the needs of all members of the coalition.

Our approach originates from the concrete problem of allocating the common cost of the mail carrier route in France, which is an ongoing challenge for La Poste, the postal operator in charge of the postal universal service in France, and Arcep[2], the French national regulatory authority. The European directive 97/67/CE in article 14-3 states that the universal service providers shall keep separate accounts within their internal accounting systems between the postal products belonging the universal service scope and the other. For that reason, the common cost of the mail carrier is allocated between the different postal products that are delivered. In addition, this article states that " whenever possible, common costs shall be allocated on the basis of direct analysis of the origin of the costs themselves; [...]". Therefore, to allocate the common cost of the mail carrier route two cost drivers are taken into account, the delivery speed and the format/volume of the postal products. Currently, the common cost of the mail carrier route is allocated in two steps. In the first step, postal products are grouped into three categories according to their delivery speed: D^7, D^3 and D^1 with a delivery target on the 7st, 3rd and 1th business day after posting, respectively. Given that La Poste must organize the delivery network in order to be in capacity to visit all recipients' addresses six days a week and given the logistical constraints, a theoretical delivery frequency of one, three, and six days per week would be required to respectively deliver D^7, D^3 and D^1. Arcep's decision 2008-0165 states that the common cost of the six weekly mail carrier routes is allocated to the three categories in proportion to their aforementioned delivery frequency: 60% of the delivery costs to D^1, 30% to D^3 and 10% to D^7. In the second step, the share of the cost previously calculated for each category is then allocated to the postal products belonging to this category according to their format/volume.

The schedule situations provide a good insight into the first step of this process (a detailed description of the second step can be found in Bohorquez Suarez and Munich, 2023). The infrastructure is the mail carrier route, which can be used six days/periods per week with an identical daily cost, and the players are the three postal product categories. The cheapest/minimal consumption schedules for the three categories are as follows. For D^1, the unique consumption schedule is the set of all six days of the week (or equivalently a mail carrier route every business day) since the postal products in this category must be delivered on the next business day. On the contrary, for D^7, there are six singleton possible alternative consumption schedules, one for each day of the week (one mail carrier route is enough, no matter which day), since a postal product belonging to this category must be delivered not later than 7 days after being posted. For D^3, due to the logistic constraints, the set of minimal consumption schedules contains all the triple of days which are not consecutive two by two such as, for example, $\{day\ 1, day\ 4, day\ 6\}$. In the last part of the article, we explain how our model of schedule situations can lead to a relevant alternative allocation.

We make two types of contributions to the literature, on the structure of schedule games and on cost allocations. These contributions provided sufficient conditions of the non-emptiness of the core of a schedule game and highlight natural allocations lying in the core.

Regarding the first type of contributions, Proposition 1 is a characterization of the class of schedule games: a cooperative game is a schedule game if and only if it is monotonic and sub-additive. One of the particularities of this new class of games is that specific schedule situations can be linked to other classes of operational research games. First, a schedule situation is called uniform if, for each player, only the number of consumption time periods matters but not their timing. Proposition 2 shows that a schedule game is uniform if and only if it is an airport game (Littlechild and Owen, 1973). This result follows Bohorquez Suarez and Munich (2023) in which the aforementioned postal allocation problem is addressed by an airport game. The latter article draws an analogy with airport games, but does not deepen or generalize the analysis as in the present article. Second, a schedule situation is called singleton if, for each player, there is a unique minimal schedule satisfying its needs. Proposition 4 proves that a schedule game is a singleton schedule game in which each period costs one unit if and only if it is a carpool game (Naor, 2005).

Regarding the second type of contributions, we provide natural allocations for schedule situations based on an equal pooling principle. According to given subsets of periods, one for each player, the cost of each period is shared equally among the users of the period. On the subclass of uniform schedule situations, Proposition 3 shows that there is always at least one optimal consumption time schedule for the grand coalition such that the corresponding Equal pooling allocation is a core allocation and coincides with the Shapley value of the associated schedule game. On the subclass of singleton schedule situations, there is a unique Equal pooling allocation and, similarly, Proposition 5 demonstrates that it is a core allocation and coincides with the Shapley value of the associated schedule game. Since airport games are concave, uniform schedule games are concave too. Moreover, we also show in Proposition 6 that singleton schedule games are concave as well. In these types of schedule games the core is nonempty. Proposition 1 reveals that some schedule games have an empty core. Nevertheless, we provide two more general sufficient conditions for the non-emptiness of the core by proving that the core contains specific Equal Pooling allocations. In the first condition, an optimal configuration for the grand coalition is called "coherent" if its restriction to each sub-coalition remains optimal. Corollary 1 establishes that the Equal Pooling allocation constructed from any coherent configuration is a core allocation and coincides with the Shapley value of the associated schedule game. Coherent schedule situations include the aforementioned classes of uniform and singleton schedule situations. The second condition is weaker and relies on specific subsets of periods, one for each player, constituting what we call a coherent covering. Proposition 8 shows that the Equal Pooling allocation constructed from any coherent covering is a core allocation even if it can be different from the Shapley value of the associated schedule game.

Our model is formally equivalent to one model presented in Moulin (2013), even if the formulation and the interpretation are quite different. The interpretation is different in the sense that periods are called items and that these items are often associated with the sharing of network connectivity costs (for instance, the items are the costly edges that help to connect a network). The formulation is also different: while we express the needs of a players by the minimal suitable sets of items/periods, Moulin (2013) lists all suitable such sets and not just the minimal ones. The formulation of the associated cooperative game is different as well, although equivalent

to ours. Moulin (2013) only highlights some basic properties and then suggests directions for further research that have been followed by Moulin and Laigret (2011) and Hougaard and Moulin (2014). These articles differ from our approach in two important aspects. Firstly, the cost of all items/periods is shared among the players even if some items/periods are not necessary (or even not demanded) to meet the needs of the players. To the contrary, we only share the minimal cost of a set of items/periods satisfying the players' needs. In many cost problems such as the one concerning La Poste, there is a real stake in selecting an optimal schedule in order to minimize the use of a costly infrastructure. Minimum cost spanning trees (see Norde et al., 2004) and their variants (see, for instance, Hougaard and Tvede, 2022) are classical examples of such cost problems. Secondly, a solution in Moulin and Laigret (2011) and Hougaard and Moulin (2014) only specifies an allocation of the costs (of all items/periods) whereas a solution in our setting is a pair consisting of an optimal schedule and an associated allocation of the total cost induced by this schedule. In other words, we consider that an allocation is inextricably linked to the resource's usage schedule and to the precise demands of the players within this schedule.

Moulin and Laigret (2011) impose a non-redundancy condition: dropping any single item implies that at least one user is no longer being satisfied. They share the cost of all items by the so-called Equal Need solution: the cost of each item is divided equally among the group of players whose needs cannot be met in the absence of this item. Unless in trivial cases, the Equal need solution is different from our Equal Pooling allocations. Moulin and Laigret (2011) provide an elegant axiomatic characterization of the Equal Need solution and show that it is a core allocation. Hence, the non-redundancy condition is another sufficient condition to ensure the non-emptiness of the core of schedules games. Nevertheless, the class of non-redundant schedule situations is a bit narrow since it cannot model the fact that different disjoint subsets of periods can alternatively satisfy a player's needs, which is often realistic. Furthermore, our Proposition 9 shows that the non-redundancy condition is stronger than our condition based on coherent coverings. Hence, our analysis of the core of schedule games substantially extends the existing literature.

As in our article, Hougaard and Moulin (2014) do not impose the non-redundancy condition. Again, they consider the problem of sharing the costs of all items, which prevent them from relying on cooperative games and therefore limits comparisons with our approach in terms of core. They propose a family of cost ratios and axiomatically characterize it. These allocation rules divide the total cost of each item in proportion to some indice which, roughly speaking, represents the importance of the item for each user. Except in very specific cases, this family does not include our equal pooling allocations. We present these allocation rules and the Equal Need solution in detail in Section 3.2. To be complete, let us mention that Fopa et al. (2022) provide an axiomatic characterization of one of the allocation rules studied in Hougaard and Moulin (2014), and that Hougaard and Moulin (2018) extend the analysis by assuming that items may fail because they have limited reliability.

More generally, this article is in line with the growing literature on operations research (OR) games in which the players wish to minimize total joint costs and then must distribute these joint costs among them. Borm et al. (2001) and Fiestras-Janeiro et al. (2011) provide a general view of the literature of OR problems and applications of cooperative games to cost allocation in transportation, connection, sequencing/queuing, production and inventory issues, among others (see Csóka et al., 2022; Slikker, 2023, for more recent references). Our model can also be considered as a generalization of airport games and thus is in line with the other

generalizations of the class of airport games proposed in Fragnelli et al. (1999), Kuipers et al. (2013), Rosenthal (2017) and Sudhölter and Zarzuelo (2017), among others.

The rest of the article is organized as follows. After giving the preliminaries on cooperative games in Section 2, we introduce the schedule situations and the associated games in Section 3. The equal pooling allocations are also presented in this section. In Section 4, we link schedule games to airport games and carpool games. In Section 5, we provide the two most general sufficient conditions for the non-emptiness of the core. Section 6 comes back to the application of allocating the cost of the mail carrier route in France. Section 7 concludes.

2. Preliminaries on cooperative games

Let N be a nonempty and finite set of players. Each subset $E \in 2^N$ is referred to as a coalition of cooperating players. The grand coalition N represents a situation in which all players cooperate. Coalition $\emptyset$ represents a situation in which no player cooperates, it is called the empty coalition. For each $E \in 2^N$, the integer $|E| \in \mathbb{N}$ denotes the cardinality of coalition E.

A transferable utility game, or simply a TU-game, is a couple (N, v) consisting of a finite players set N and a characteristic function $v : 2^N \to \mathbb{R}$, with the convention that $v(\emptyset) = 0$. The real number $v(E)$ can be interpreted as the worth the players in E generate when they cooperate. This worth can be perceived by the players as desirable (like profits) or, on the contrary, undesirable (like costs). We will focus on the second case: the players share cost. Thus the game (N, v) is a cost game. For ease of writing the game (N, v) will be designated by its characteristic function v where N is fixed. A game v may satisfy some interesting properties:

Monotone For each $E \subseteq S \subseteq N$, $v(E) \leq v(S)$.

Adding a player to a coalition does not reduce its cost.

Sub-additive For each couple of coalitions $E, S \subseteq N$ such that $E \cap S = \emptyset$, $v(E \cup S) \leq v(E) + v(S)$.

When two disjoint coalitions come together, the resulting joint cost is at most equal to the sum of their initial costs. Merging two coalitions is not detrimental to their members.

Concave For each $i \in N$ and each $E \subseteq S \subseteq N \setminus \{i\}, v(E \cup \{i\}) - v(E) \geqslant v(S \cup \{i\}) - v(S)$.

This property indicates that the incremental cost due to the arrival of a new player in a coalition does not increase if this coalition grows.

The basic issue in the theory of cooperative games is to divide fairly the cost of the grand coalition among its members. This issue may be addressed using allocations for TU-games. An allocation $x \in \mathbb{R}^{|N|}$ is a $|N|$-dimensional vector that assigns a share of the cost $x_i \in \mathbb{R}$ to each player $i \in N$.

An efficient allocation shares exactly $v(N)$ among the players and it is called coalitionally rational if no coalition would be better off by splitting from the grand coalition and paying its cost. The core of a game v, is the set $Core(v)$ of efficient and coalitionally rational allocations:

$$Core(v) = \Big\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \text{ and for each } E \subseteq N, \sum_{i \in E} x_i \leqslant v(E) \Big\}.$$

The core of a game can be empty. However, Shapley (1971) demonstrates that the core of a concave game is nonempty. The core can contain often several allocations from which it can be difficult to choose one and only one. Alternatively, the Shapley value assigns to each game v a unique allocation $Sh(v)$ such that for each $i \in N$:

$$Sh_i(v) = \sum_{E \subseteq N \setminus \{i\}} \frac{|E|!(|N| - |E| - 1)!}{|N|!} \Big(v(E \cup \{i\}) - v(E) \Big).$$

Shapley (1971) proves that the Shapley value of a concave game lies in its core.

3. Schedule situations and schedule games

A group of players share a common-pool resource whose consumption is possible during several periods. The use of this resource induces a cost that can vary from one period to the next. The players have different demands represented by the subsets of periods allowing to satisfy their needs. Let us formalize this framework and illustrate its features.

3.1. Schedule situations and schedule games

Let N be a fixed finite set of n players. A **schedule situation** on N is a tuple $M = (T, (T_i)_{i \in N}, (c_t)_{t \in T})$ where

- $T = \{1, \ldots, |T|\}$ is a finite set of **time periods**;

- for each $i \in N$, $T_i \subset 2^T$ is the nonempty set of minimal (w.r.t. inclusion) **time configurations** satisfying the needs of player i;

- for each $t \in T$, $c_t \in \mathbb{R}_+$ is the cost of using the resource in period t.

The minimality condition in the definition of set T_i implies that if $S, E \in T_i$, then neither $S \subset E$ nor $E \subset S$. In words, each player needs a certain schedule for the consumption of a common-pool resource. Such a schedule specifies the needed subset of consumption time periods. The set T_i collects all minimal (with respect to set inclusion) schedules or time configurations satisfying the consumption needs of player i. Although T_i is always nonempty, it can contain the emptyset. In this case, the emptyset is the unique element of T_i, i.e. $T_i = \{\emptyset\}$, and this should be interpreted as the fact that player i has no need (over the time span T studied). The common-pool resource is non-rival but costly, so that the objective is to minimize the overall cost of a set of consumption time periods while satisfying the needs of all players. In order to do so, we introduce an associated cooperative game called the **schedule game**. For each $E \subseteq N$, if $T_E := \prod_{i \in N} T_i$ denote the time configurations satisfying E, then the schedule game v_M associated with M is defined, for each $E \subseteq N$, by

$$v_M(E) = \min_{R \in T_E} \sum_{t \in \cup_{Q \in R} Q} c_t.$$

The real number $v_M(E)$ is the minimal cost of a subset of periods that satisfies the need of all the members of E. For each nonempty coalition E, we also denote by $O(E)$ the set of all

optimal time configurations, i.e. those which minimize the overall cost of a set of consumption time periods satisfying the needs of E:

$$O(E) = \left\{ R \in T_E : \sum_{t \in \cup_{Q \in R} Q} c_t = v_M(E) \right\}.$$

The periods belonging to this set of optimal time configuration of the grand coalition $O(N)$ are called **active**.

Example 1. *Set* $N = \{A, B, C\}$, $T = \{1, \ldots, 8\}$, $T_A = \{\{1, 2\}, \{3, 4, 5\}\}$, $T_B = \{\{1, 2\}, \{7, 8\}\}$, $T_C = \{\{3, 4, 5\}, \{6, 7, 8\}\}$ *and the costs:*

t	1	2	3	4	5	6	7	8
c_t	1.1	0.8	2	0.6	0.4	0.5	1.4	0.9

Then, the resulting schedule game is given by:

E	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{A, C\}$	$\{B, C\}$	$\{A, B, C\}$
$v_M(E)$	1.9	1.9	2.8	1.9	3	2.8	4.7

As an example, consider coalition $\{B, C\}$, which can be satisfied by the following four minimal schedules

$$T_{\{B,C\}} = T_B \times T_C = \big\{ (\{1, 2\}, \{3, 4, 5\}), (\{1, 2\}, \{6, 7, 8\}), (\{7, 8\}, \{3, 4, 5\}), (\{7, 8\}, \{6, 7, 8\}) \big\},$$

with corresponding costs 4.9 for $(\{1, 2\}, \{3, 4, 5\})$, 4.7 for $(\{1, 2\}, \{6, 7, 8\})$, 5.3 for $(\{7, 8\}, \{3, 4, 5\})$ and 2.8 for $(\{7, 8\}, \{6, 7, 8\})$. Hence,

$$v_M(\{B, C\}) = \min \big\{ 4.9, 4.7, 5.3, 2.8 \big\} = 2.8,$$

which means that player B can completely pool its two-period demand $\{7, 8\}$ with the demand $\{6, 7, 8\}$ of player C in order to save on costs within this two-player coalition. Here, we have $O(\{B, C\}) = \{(\{7, 8\}, \{6, 7, 8\})\}$. There may be multiple optimal time configurations for a given coalition. For instance $O(N)$ contains two elements: $(\{1, 2\}, \{1, 2\}, \{6, 7, 8\})$ and $(\{1, 2\}, \{7, 8\}, \{6, 7, 8\})$. □

The first result below characterizes the class of schedule games.

Proposition 1. *The class of all schedule games on N coincides with the class of monotone sub-additive TU-games on N.*

Proof. It is obvious that v_M is monotone for each schedule situation M. Furthermore, for a schedule situation M on N, consider any pair of coalitions $E, S \subseteq N$ such that $E \cap S = \emptyset$. Pick any time configurations $R^1 \in O(E)$ and $R^2 \in O(S)$. Since $(R^1, R^2) \in T_{E \cup S}$, i.e., the time configurations R^1 and R^2 for E and S are still available, when combined, as a time configuration for $E \cup S$, we immediately get $v_M(E) + v_M(S) \geq v_M(E \cup S)$, proving that v_M is sub-additive.

Conversely, let v be any monotone subadditive game on N. To show: there is a schedule situation M on N such that $v_M = v$. Consider any ordering π of the $2^n - 1$ nonempty coalitions on N, where, for each nonempty coalition E, $\pi(E)$ stands for the position of E according to π. For each nonempty $E \subseteq N$, let $A_E = \{\pi(E)\}$. These sets being singletons, for each $E, S \subseteq N$ with $E \neq S$, it holds that

$$A_E \cap A_S = \emptyset. \tag{1}$$

We construct the schedule situation $M = (T, (T_i)_{i \in N}, (c_t)_{t \in T})$ such that $|T| = 2^n - 1$, i.e. there are as many periods as the number of nonempty coalitions. Moreover, for each player $i \in N$, define $T_i = \{A_E : E \ni i\}$. In words, T_i contains 2^{n-1} singletons, one for each coalition containing player i. Hence, any time configuration R for any nonempty coalition E is of the form $R = (A_{S_i})_{i \in E}$ where, for each $i \in E$, S_i is a coalition containing player i. In addition, for each nonempty coalition $E \subseteq N$, set $c_{\pi(E)} = v(E)$. From now on, we focus on an arbitrary nonempty coalition E in order to prove that $v_M(E) = v(E)$. From E and any time configuration $R \in T_E$, $R = (A_{S_i})_{i \in E}$, define $x^E(R) = \{S \subseteq N : A_{S_i} = A_S \text{ for some } i \in E\}$ and $y^E(R) = \sum_{t \in \cup_{i \in E} A_{S_i}} c_t$. It holds that $v_M(E) = \min_{R \in T_E} y^E(R)$ or equivalently, from (1) and the fact that $c_{\pi(S)} = v(S)$ for each nonempty $S \subseteq N$, $v_M(E) = \min_{R \in T_E} \sum_{S \in x^E(R)} v(S)$. Consider the configuration $R^E := (A_i^E)_{i \in N}$ such that $A_i^E = A_E$ for each $i \in E$. Remark that R^E belongs to T_E and that $x^E(R^E) = \{E\}$, which yields $y^E(R^E) = c_{\pi(E)} = v(E)$. It remains to show that if $R \in T_E$, $R = (A_{S_i})_{i \in E}$, then $y^E(R) \geqslant v(E)$. Given R and $S \in x^E(R)$, define $E_R(S) = \{i \in E : A_{S_i} = S\}$ and note that

$$\emptyset \subsetneq E_R(S) \subseteq S. \tag{2}$$

From R, construct the collection $\bar{R} = (A_{E_R(S_i)})_{i \in E}$, which implies that $\bar{R} \in T_E$. By definition, $x^E(\bar{R})$ is a partition of E. We can write that

$$y^E(R) = \sum_{S \in x^E(R)} v(S) \geqslant \sum_{S \in x^E(R)} v(E_R(S)) = \sum_{S' \in x^E(\bar{R})} v(S') = y^E(\bar{R}) \geqslant v(E),$$

where the first inequality comes from the monotonicity of v and equation (2), and the second inequality comes from the subadditivity of v and the fact that $x^E(\bar{R})$ is a partition of E. We conclude that $v_M(E) = v(E)$, as desired. ∎

Proposition 1 implies that not all schedule games have a nonempty core as pointed out in the introduction. This is illustrated by 1. Assume that an allocation x is candidate to belong to the core. Note that $x_A + x_B \leq 1.9$ and efficiency of any core allocation leads to $x_C \geq 2.8$. Similarly the use of efficiency together with $x_A + x_C \leq 3$ and $x_B + x_C \leq 2.8$ yields that $x_B \geq 1.7$ and $x_A \geq 1.9$. Summing these three inequalities, we get $x_A + x_B + x_C \geq 6.4$, which is incompatible with the efficiency constraint. Thus, $Core(v) = \emptyset$. Two conditions ensuring the non-emptiness of the core are introduced in section 5.

In order to illustrate the proof, we consider the following four-player game in which brackets and commas are omitted in order to save space and in which π orders coalitions by size and lexicographically within a given size.

E	a	b	c	d	ab	ac	ad	bc	bd	cd	abc	abd	acd	bcd	$abcd$
$\pi(E)$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$v(E) = c_{\pi(E)}$	1.2	0.7	3.5	5	1.6	4.7	6	3.9	5.2	7.9	5	6.5	8.5	8.1	9.1

It is easy to check that v satisfies the conditions imposed in Proposition 1. As a start, let us build the schedule situation M as in the proof. Focusing on player a, we have

$$T_a = \{A_{\{a\}}, A_{\{a,b\}}, A_{\{a,c\}}, A_{\{a,d\}}, A_{\{a,b,c\}}, A_{\{a,b,d\}}, A_{\{a,c,d\}}, A_{\{a,b,c,d\}}\},$$

where, for instance, $A_{\{a,b,c\}} = \{\pi(\{a,b,c\})\} = \{11\}$ since coalition $\{a,b,c\}$ is in position 11 according to π. We now focus on the previous coalition $E = \{a,b,c\}$ in order to sketch why that $v_M(\{a,b,c\}) = v(\{a,b,c\})$. We proceed in three steps.

Step 1. We show that a specific time configuration for E costs exactly $v(E)$. In fact, since $A_{\{a,b,c\}}$ belongs to T_a, T_b and T_c, $R^{\{a,b,c\}} := (A_{\{a,b,c\}}, A_{\{a,b,c\}}, A_{\{a,b,c\}}) \in T_{\{a,b,c\}}$. Obviously, $y^E(R^{\{a,b,c\}}) = c_{\pi(\{a,b,c\})} = c_{11} = 5 = v(\{a,b,c\})$, so that $v_M(E) \leq v(\{a,b,c\})$. In the final two steps, we show that no other time configuration $R \in T_{\{a,b,c\}}$ can do better. We only illustrate these steps with $R = (A_{\{a,b,c,d\}}, A_{\{a,b,c,d\}}, A_{\{c,d\}})$.

Step 2. This step is a "reduction" step in which the individual time configurations in R are reduced by eliminating unnecessary needs, in some sense. From R and E, we have $x^E(R) = \{\{a,b,c,d\}, \{c,d\}\}$ so that $y^E(R) = v(\{a,b,c,d\}) + v(\{c,d\})$. We drop from coalition $\{a,b,c,d\}$ the two players c and d that do not choose $A_{\{a,b,c,d\}}$ in R and similarly, we drop d from $\{c,d\}$. The resulting coalitions, called $E_R(\{a,b,c,d\}) = \{a,b\}$ and $E_R(\{c,d\}) = \{c\}$ in the proof, are subsets of the original coalitions and $\bar{R} := (A_{\{a,b\}}, A_{\{a,b\}}, A_{\{c\}})$ is also in $T_{\{a,b,c\}}$. The monotonicity of v yields that $v(\{a,b\}) \leq v(\{a,b,c,d\})$ and $v(\{c\}) \leq v(\{c,d\})$, which is equivalent to $c_{\pi(\{a,b\})} \leq c_{\pi(\{a,b,c,d\})}$ and $c_{\pi(\{c\})} \leq c_{\pi(\{c,d\})}$, respectively. Hence, $y^E(\bar{R}) \leq y^E(R)$. Thus, $\bar{R}$ is already not worse than R for coalition E.

Step 3. This step is a "partition" step in which non-pooled consumption in $\bar{R}$ are compared to the fully pooled consumption in $R^{\{a,b,c\}}$ from step 1. To see this, note that $\{a,b\}$ and $\{c\}$ form a partition of E so that the sub-additivity of v yields that $v(\{a,b,c\}) < v(\{a,b\}) + v(\{c\})$. Equivalently, $c_{\pi(\{a,b,c\})} \leq c_{\pi(\{a,b\})} + c_{\pi(\{c\})}$. Thus, we conclude that $y^E(R^{\{a,b,c\}}) < y^E(\bar{R})$, proving that $R^{\{a,b,c\}}$ from step 1 cannot be worse than $\bar{R}$ for coalition E. Thus $v_M(\{a,b,c\}) = v(\{a,b,c\})$.

3.2. The equal pooling principle

Several types of allocations have been studied in the literature, all of which are based on a general **equal pooling principle**. This section presents the new relevant way in which we use this equal pooling principle and highlights the differences with other related articles in the literature.

For schedule situations, the equal pooling principle can be formulated in two steps. Firstly, for each player, a subset of time periods is selected. Secondly, according to the chosen subsets, the cost of each period is shared equally among the users of the period. The resulting allocation is called the **Equal Pooling allocation**.

Definition 1. *Fix any schedule situation M. Let $R = (A_1, ..., A_n)$ be a time configuration such that, for each player $i \in N$, $A_i \subseteq T$. The* **Equal Pooling allocation** *$EP^R(M)$ on M associated with R is such that, for each $i \in N$,*

$$EP_i^R(M) = \sum_{t \in T : t \in A_i} \frac{c_t}{|\{j \in N : t \in A_j\}|}.$$

In this article, we almost always use the Equal Pooling allocation in which the chosen time configuration is optimal for the grand coalition, i.e. when $R \in O(N)$. We advocate that this

choice is the most natural one in our context. Clearly, there can be numerous optimal time configurations in $O(N)$, which implies that many Equal pooling allocations can be computed for a given schedule situations. For instance, in Example 1, there are two optimal time configurations for N: $O(N) = \{R^1, R^2\}$ where $R^1 = \{\{1,2\}, \{1,2\}, \{6,7,8\}\}$ and $R^2 = \{\{1,2\}, \{7,8\}, \{6,7,8\}\}$, leading to two equal pooling allocations:

$$EP^{R^1}(M) = (0.95, 0.95, 2.8) \quad \text{and} \quad EP^{R^2}(M) = (1.9, 1.15, 1.65).$$

In the next sections, we identify conditions guaranteeing that our Equal Pooling allocations belong to the core of the associated schedule game. Furthermore, we weaken theses conditions in section 5.2, but at the cost of a change in the application of the equal pooling principle. More specifically, instead of computing the Equal Pooling allocation from an optimal time configuration $R = (A_1, ..., A_n)$ in $O(N)$, we restrict ourselves to specific subsets of sets A_i.

We can now compare our Equal Pooling allocations based on optimal time configurations and the Equal Need solution introduced in Moulin and Laigret (2011). For any schedule situation $M = (T, (T_i)_{i \in N}, (c_t)_{t \in T})$, define $G^i(M) = \cap_{A_i \in T_i} A_i$ the set, possibly empty, of critical periods for player i, i.e. the set of periods that are necessary to satisfy the needs of player i.[3] Similarly, define $H^i(M) = \cup_{A_i \in T_i} A_i$ as the set of periods that are relevant to player i. Moulin and Laigret (2011) only consider the subclass of so-called non-redundant schedule situations in which each period is critical for at least one player, i.e. M such that $\cup_{i \in N} G^i(M) = T$. The **Equal Need solution** applies the equal pooling principle to the configuration $G := (G^1(M), \ldots, G^n(M))$: for each $i \in N$,

$$EP_i^G(M) = \sum_{t \in T : t \in G^i(M)} \frac{c_t}{|\{j \in N : t \in G^j(M)\}|}.$$

There are at least two major differences between the Equal Need solution and our Equal Pooling allocations. The first one is that any of our Equal Pooling allocations is inseparable from the associated optimal schedule. Hence, not only the Equal Pooling allocation provides a relevant cost allocation but it also highlights an actual schedule showing which players use the resource at which periods. This is not the case of the Equal Need solution which can be seen as a global approach that does not select a specific schedule. The second difference is that the Equal Need solution ignores the periods that are relevant but not necessary for a player, which leads sometimes to puzzling allocations. For instance, in the context of Example 2, The Equal Need solution leads player C to pay entirely the cost of all periods even if players A and B need to consume the resource one period and four periods, respectively. This is explained by the fact all periods are critical for player C, who needs to consume the resource every period while players A and B have more flexibility in their consumption needs.

Let us add that Moulin and Laigret (2011) provide an axiomatic characterization of the Equal Need solution on the class of non-redundant schedule situations by means of three axioms (their definition of a solution already include the requirement that the cost of all periods is entirely distributed). Additivity in cost imposes that the sum of the two solutions in two schedule situations that only differ with respect to their cost vectors is equal to the solution of the schedule situation obtained by summing the two cost vectors. The symmetry axiom imposes to assign the same cost share to two players i and j for which the the sets T_i and T_j are identical

---

[3]In this section, we use the formulation of our model to define the concept introduced in the related literature.

when restricted to the set of periods with a strictly positive cost. The third axiom requires that the solution always selects a core allocation of the associated schedule game, which implies that the non-redundancy is an alternative condition to ours to ensure the non-emptiness of the core (see sections 4 and 5 and the picture in the conclusion).

Hougaard and Moulin (2014) apply variants of the Equal Pooling principle. More specifically, for each schedule situation, they share the cost of all periods, without considering the possible existence of a strict subset of less expensive periods that would be sufficient to satisfy the needs of all players. Even on the class of non-redundant schedule situations, the family of solutions they study does not contain the Equal Need solution and differs from ours Equal Pooling allocations except in very special cases. This family is constructed as follows. Firstly, a solution is supposed to satisfy the axiom of additivity in cost. As a consequence, Hougaard and Moulin (2014) focus on schedule situations in which there is a unique costly period, the cost of which is one:

$$M^{t^*} = (T, (T_i)_{i \in N}, (c_t)_{t \in T}),$$

with $c_{t^*} = 1$ and $c_t = 0$ if $t \in T \backslash \{t^*\}$. A solution on the class of such schedule situations is called a cost-ratio index if it satisfies four properties (in addition to the fact that the sum of the players' ratio is equal to one):

1. If t^* is critical for some player i, then i's share is not less than in any other schedule situation obtained by replacing T_i by any other set of time configurations $T_{i'}$;

2. If t^* is relevant for player i, then i's cost share is strictly positive;

3. If t^* is not relevant for i but relevant for at least one other player, then i's share is null;

4. If t^* is relevant for no player, then each of the n players' share is $1/n$.

Moulin (2013) characterize the following family of cost ratio indices with a proportional flavor:

$$f_i(M^{t^*}) = \frac{\theta(T_i, t^*)}{\sum_{j \in N} \theta(T_j, t^*)},$$

where

$$\theta(T_i, t^*) = \left(\frac{|\{A_i \in T_i : A_i \ni t^*\}|}{|T_i|} \right)^{\alpha}$$

for any number $\alpha \geq 0$. The case $\alpha = 1$ yields the so-called counting cost ratio index, which is studied in Fopa et al. (2022).

Rather than presenting the axioms that characterize this family in detail, we would like to come back to the major differences that distinguishes this family from our Equal Pooling allocations. Sharing the cost of all periods/items as in Moulin (2013) can make sense in network connection problems. Note that this prevent the authors from re-exploiting the approach in terms of cooperative games. In our context of selecting an optimal time schedule, sharing the cost of all periods is rather questionable since there is no obvious reason to charge players when the resource is not being used. Finally, as for the Equal Need solution, and even if the non-redundancy condition is further imposed, this family of cost ratio indices does not allow to exhibit an optimal schedule.

11

4. Particular schedule situations

In this section we introduce two specific schedule situations and show their close links with other classes of operational research games.

4.1. Uniform schedule situations and airport situations

In the first particular category of schedule situations, the number of consumption time periods matters for a player but not their timing.

Definition 2. *A schedule situation is called* **uniform** *if for each player all time configurations of a certain size satisfy the needs of this player. Formally, for each $i \in N$, there is $p_i \in \{1, \ldots, t\}$ such that $T_i = \{Q \subseteq T : |Q| = p_i\}$.*

This definition can be illustrated by the following example:

Example 2. *Let $N = \{A, B, C\}$, $T = \{1, \ldots, 6\}$,*

$$T_A = \{\{t\}, t \in T\},$$
$$T_B = \{E \subseteq T : |E| = 4\},$$
$$T_C = \{T\},$$

and

t	1	2	3	4	5	6
c_t	2.5	1.2	4.1	3.1	1.9	2

Then the resulting game is:

E	$\{A\}$	$\{B\}$	$\{C\}$	$\{A,B\}$	$\{A,C\}$	$\{B,C\}$	$\{A,B,C\}$
$v_M(E)$	1.2	7.6	14.8	7.6	14.8	14.8	14.8

$\square$

Uniform schedule situations are closely related to airport situations. An **airport situation** (Littlechild and Owen, 1973) on N is a tuple $A = ((N_j, co_j)_{j \in \{1, \ldots, m\}})$ where N_j denotes the set of n_j aircrafts of type j, for $j = 1, \ldots, m$, and $N = \cup_{j=1}^{m} N_j$ and $n = \sum_{j=1}^{m} n_j$. The cost associated with an aircraft of type j is given by co_j. These types of aircraft are ordered so that $co_0 < co_1 < \ldots < co_m$, where $co_0 = 0$ by convention. Any airport situation A gives rise to an **airport game** v_A such that for each $E \subseteq N$,

$$v_A(E) = \max_{j \in \{1, \ldots, m\}: E \cap N_j \neq \emptyset} co_j.$$

Note that the schedule game in Example 2 coincides with the airport game induced by the airport situation in which $N_1 = \{A\}$, $N_2 = \{B\}$, $N_3 = \{C\}$ and $co_1 = 1.2$, $co_2 = 7.6$, $co_3 = 14.8$. We show below that this property holds for any uniform schedule situation. To see this, The next result shows the converse implication as well: any airport game can be obtained from some uniform schedule situation.

Proposition 2. *A game v is an airport game if and only if $v = v_M$ for some uniform schedule situation M.*

Proof. Firstly, we prove the "only if" part. From any uniform schedule situation M, construct the airport situation $A^M = ((N_j^M, co_j^M)_{j \in \{1,\dots,m\}})$ as follows. First, let $p(0) = 0$. Then, for each $j \in \{1, \dots, m\}$, define

$$p(j) = \min_{i \in N : p_i > p(j-1)} p_i,$$

$$co_j^M = \min_{Q \subseteq T : |Q| = p(j)} \sum_{t \in Q} c_t, \tag{3}$$

and

$$N_j^M = \{i \in N : p_i = p(j)\}.$$

Now, let E be any nonempty coalition in N. Define further $p^* = \max_{i \in E} p_i$ and A^* as a subset of p^* periods such that $\sum_{t \in A^*} c_t = \min_{Q \subseteq N : |Q| = p^*} \sum_{t \in Q} c_t$. We will prove that,

$$v_M(E) \overset{\text{Step 1}}{=} \sum_{t \in A^*} c_t \overset{\text{Step 2}}{=} v_{A^M}(E).$$

Step 1. Denote by i one of the players in E such that $p_i = p^*$. Then since the schedule situation M is uniform, any $A_i \in T_i$ is such that $|A_i| = p^*$. We immediately get that $v_M(E) \geqslant \sum_{t \in A^*} c_t$. Next, for each $k \in E$, there is $A_k^* \in T_k$ such that $|A_k^*| = p_k$ and $A_k^* \subseteq A^*$. Hence, $\cup_{k \in E} A_k^* = A^*$ and $(A_k^*)_{k \in E}$ belongs to T_E, which implies that $v_M(E) \leqslant \sum_{t \in A^*} c_t$.

Step 2. Consider the airport situation A^M. By definition of an airport game, for each nonempty $E \subseteq N$, we have:

$$v_{A^M}(E) = \max_{j \in \{1,\dots,m\} : E \cap N_j^M \neq \emptyset} co_j^M.$$

It is clear that player $i \in E$ such that $p_i = p^*$ is the player belonging to the group $N_{j^*}^M$ with the greatest index j^* among E, from which one gets,

$$v_{A^M}(E) = co_{j^*} = \min_{Q \subseteq T : |Q| = p^*} \sum_{t \in Q} c_t = \sum_{t \in A^*} c_t.$$

This completes "only if" part.

Secondly, we deal with the "if" part. From any airport situation A on N, construct the uniform schedule situation $M^A = (T^A, (T_i^A)_{i \in N}, (c_t^A)_{t \in T^A})$ on N as follows. Letting $co_0 = 0$ and $p(0) = 0$, consider a sequence of integers $(p(j))_{j \in \{1,\dots,m\}}$ such that $p(0) < p(1) < \cdots < p(m)$. Since the two sequences $(co_j)_{j \in \{1,\dots,m\}}$ and $(p(j))_{j \in \{1,\dots,m\}}$ are non negative and strictly increasing, the sequences $(co_j - co_{j-1})_{j \in \{0,\dots,m\}}$ and $(p(j) - p(j-1))_{j \in \{0,\dots,m\}}$ are strictly positive. Therefore, it is always possible to specify the sequence $(p(j))_{j \in \{1,\dots,m\}}$ so that, for each $j \in \{1, \dots, m-1\}$,

$$\frac{co_j - co_{j-1}}{p(j) - p(j-1)} < \frac{co_{j+1} - co_j}{p(j+1) - p(j)}. \tag{4}$$

Now, define $T^A = \{1, \dots, p_m\}$, for each $j \in \{1, \dots, m\}$ and each $i \in N_j$, define $p_i^A = p(j)$ and for each $j \in \{1, \dots, m\}$ and each $t \in T^A$ with $t \in \{p(j-1) + 1, \dots, p(j)\}$,

$$c_t^A = \frac{co_j - co_{j-1}}{p(j) - p(j-1)}. \tag{5}$$

As in the "only if" part, For an arbitrary nonempty coalition E, let $p^* = \max_{i \in E} p_i^A$ and $A^* = \min_{Q \subseteq N: |Q| = p^*} \sum_{t \in Q} c_t^A$. Moreover, denote by j^* the index of the group N_j^* in A to which belong the player(s) i such that $p_i^A = p^*$ in M^A. Let us show that

$$v^A(E) \overset{\text{Step 1}}{=} co_{j^*} \overset{\text{Step 2}}{=} v_{M^A}(E).$$

Step 1. It is clear that player $i \in E$ such that $p_i = p^*$ is the player from to the group N_j with the greatest index j among E, i.e. N_{j^*}, so that we immediately get $v^A(E) = co_{j^*}$.

Step 2. Because M^A is uniform, we know that $v_{M^A}(E) = \sum_{t \in A^*} c_t^A$. From (4), this equality can be rewritten as $v_{M^A}(A) = \sum_{t=1}^{p^*} c_t$ or equivalently, by definitions of j^* and of c_t^A in (5), as:

$$v_{M^A}(E) = \sum_{j=1}^{j^*} \big(p(j) - p(j-1)\big) \times \frac{co_j - co_{j-1}}{p(j) - p(j-1)} = co_{j^*},$$

as desired. ∎

It is well-known that airport games are concave. According to the Proposition 2, uniform schedule games are concave too. The next result below reveals that for uniform schedule situations, there is always an optimal time configuration for the grand coalition such that the associated equal pooling allocation is the Shapley value of the uniform schedule game.

Proposition 3. *If M is an uniform schedule situation on N, then there is $R^* \in O(N)$ such that the Equal pooling allocation $EP^{R^*}(M)$ coincides with the Shapley value of game v_M.*

Proof. Consider any uniform schedule situation M on N. For the sake of simplicity and without any loss of generality, we can assume that

$$c_1 \leqslant c_2 \leqslant \cdots \leqslant c_{|T|} \tag{6}$$

and that

$$p_1 \leqslant p_2 \leqslant \cdots \leqslant p_n. \tag{7}$$

In this demonstration, we make use of the specific airport situation A^M constructed in the proof of Proposition 2. Littlechild and Owen (1973) give the following expression for the Shapley value of an airport game: for each $j \in \{1, \ldots, m\}$ and each $i \in N_j^M$,

$$Sh_i(v_{A^M}) = \sum_{q=1}^{j} \frac{co_q^M - co_{q-1}^M}{\sum_{g=q}^m |N_g^M|}. \tag{8}$$

The Equal pooling allocation associated to some $R^* = (\{A_i^*\})_{i \in N} \in O(N)$, according to definition 1, is given, for each $i \in N$, by

$$EP_i^{R^*}(M) = \sum_{t \in T: t \in A_i^*} \frac{c_t}{|\{k \in N : t \in A_k^*\}|}. \tag{9}$$

To prove Proposition 3, firstly, we select a specific optimal time allocation R^* for N and, secondly, we rewrite (9) in the form of (8). So, choose $R^* = (A_i^*)_{i \in N}$ such that $A_i^* = \{1, \ldots, p_i\}$ for each

$i \in N$. From (6), $\min_{Q \subseteq T : |Q| = p_i} \sum_{t \in Q} c_t = \sum_{t=1}^{p_i} c_t$, which means that $A_i^* \in T_i$ and in turn that $R^* \in O(N)$. Now, we can rewrite $EP_i^{R^*}(M)$:

$$
\begin{aligned}
EP_i^{R^*}(M) &\overset{(9)}{=} \sum_{t \in T : t \in A_i^*} \frac{c_t}{|\{k \in N : t \in A_k^*\}|} \\
&\overset{(7)}{=} \sum_{t=1}^{p_i} \frac{c_t}{|\{k \in N : p_k \geq t\}|} \\
&= \sum_{t=1}^{p(j)} \frac{c_t}{|\{k \in N : p_k \geq t\}|} \\
&= \sum_{q=1}^{j} \sum_{t=p(q-1)+1}^{p(q)} \frac{c_t}{\sum_{g=q}^{m} |N_g^M|} \\
&= \sum_{q=1}^{j} \frac{\sum_{t=1}^{p(q)} c_t - \sum_{t=1}^{p(q-1)} c_t}{\sum_{g=q}^{m} |N_g^M|} \\
&\overset{(3)}{=} \sum_{q=1}^{j} \frac{co_q^M - co_{q-1}^M}{\sum_{g=q}^{m} |N_g^M|} \\
&\overset{(8)}{=} Sh_i(v_{A^M}),
\end{aligned}
$$

which completes the proof. ∎

4.2. Singleton schedule situations and carpool situations

In the second particular category of schedule situations, the players have no flexibility: each seeks a unique particular minimal time configuration.

Definition 3. *A schedule situation is called **singleton** if for each player there is a unique minimal time configuration satisfying the needs of this player. Formally, for each $i \in N$ we have $|T_i| = 1$. In this case let us denote by A_i the unique element of T_i, for each $i \in N$. Furthermore, a singleton schedule situation is called **unit-cost** if $c_t = 1$ for each $t \in T$.*

Example 3. *Consider the singleton unit-cost schedule situation M such that $N = \{A, B, C\}$, $T = \{1, ..., 6\}$, $T_A = \{\{1\}\}$, $T_B = \{\{5\}\}$ and $T_C = \{\{1, 3, 5\}\}$ and $c_t = 1$ for each $t \in T$. Then,*

E	$\{A\}$	$\{B\}$	$\{C\}$	$\{A, B\}$	$\{A, C\}$	$\{B, C\}$	$\{A, B, C\}$
$v_M(E)$	1	1	3	2	3	3	3

□

The singleton schedule situations are closely related to the class of carpool situations. A **carpool situation** (Naor, 2005) is a situation in which players form a carpool and decide to use it on different days. Formally, a carpool situation on N is a tuple $D = (D_j)_{j=1,...,l}$ where each $D_j \subseteq N$ corresponds to the nonempty set of players who showed on day $j \in \{1, ..., l\}$. The use of the carpool system is costly: any carpool situation D gives rise to a **carpool game** v_D

such that, for any subset $E \subseteq N$, $v_D(E)$ associates for each coalition E a cost measured by the number of days on which at least one player of the coalition E shows up, i.e.,

$$v_D(E) = \left| \{1 \leqslant j \leqslant l : D_j \cap E \neq \emptyset\} \right|.$$

From any singleton cost-unit schedule situation M it is possible to construct a specific carpool situation $D^M = (D_j^M)_{j=1,\ldots,l}$, such that $l = |T|$ for each $j \in \{1,\ldots,l\}$, $D_j^M = \{i \in N : j \in A_i\}$, where A_i is the unique element in T_i. It is easy to get the correspondence between a carpool situation and a singleton cost-unit schedule situation. In Example 3, the set of periods T can represent the set of days where the players A, B and C "showed up" or must be distributed. The following table gives the relationship between the carpool and the schedule situations:

$j \backslash i$	A	B	C	D_j^M
1	X		X	$\{A, C\}$
2				$\emptyset$
3			X	$\{C\}$
4				$\emptyset$
5		X	X	$\{B, C\}$
6				$\emptyset$
A_i	$\{1\}$	$\{5\}$	$\{1, 3, 5\}$	

Table 1: Relationship between carpool and singleton unit-cost schedule situations

Proposition 4. *A game v is a carpool game if and only if $v = v_M$ for some singleton unit-cost schedule situation M.*

Proof. Regarding the "only if" part, let $M = (T, (T_i)_{i \in N}, (c_t)_{t \in T}$ with $T_i = \{A_i\}, i \in N$, and $c_t = 1$ for each $t \in T$, be any singleton unit-cost schedule situation. For any E nonempty subset of N, $v_M(E)$ can be rewritten as:

$$v_M(E) = \left| \bigcup_{i \in E} A_i \right|. \tag{10}$$

Let us show that $v_{D^M}(E) = v_M(E)$. Since $l = |T|$, we have

$$
\begin{aligned}
v_{D^M}(E) &= |\{1 \leq j \leq |T| : D_j^M \cap E \neq \emptyset\}| \\
&= |\{1 \leq j \leq |T| : \{i \in N : j \in A_i\} \cap E \neq \emptyset\}| \\
&= |\{1 \leq j \leq |T| : j \in A_i \text{ for some } i \in E\}| \\
&= |\bigcup_{i \in E} A_i| \\
&= v_M(E),
\end{aligned}
$$

as desired.

The "if" part follows an inverted but similar straightforward path compared to the "only if" part and is not detailed here. ∎

In a carpool game Fagin and Williams (1983) and Ajtai et al. (1998) propose and study an equal share of the cost of each days between the players who used it. The resulting allocation rule, denoted by α, assigns to each D and each $i \in N$ the share

$$\alpha_i(D) = \sum_{j \in \{1,\ldots,l\}: i \in D_j} \frac{1}{|D_j|}.$$

Naor (2005) demonstrates that $\alpha(D)$ is the Shapley value of the game v_D, i.e. $\alpha(D) = Sh(v_D)$. Note that there is trivially a unique optimal time configuration $R^* \in O(N)$ such that $R^* = (\{A_i\})_{i \in N}$ for each singleton schedule situation M.

Proposition 5. *If M is a singleton unit-cost schedule situation on N, then the unique Equal pooling allocation $EP^{R^*}(M)$ coincides with the allocation $\alpha(D^M)$ of the carpool situation D^M.*

Proof. The claim follows from viewing the allocation rule $EP^{R^*}(M)$ as the sum of the inverse of the number of players who consume the active time period t simultaneously. In the carpool situation the active time period t is expressed by a set of days $D_j^M \subseteq N$ corresponding to the players who showed on day j and $|D_j^M|$ is the number of these players. Hence, $|D_j^M|$ and $|\{t \in T : t \in A_i\}|$ express the same thing and $EP_i(M) = \alpha_i(D^M)$. See table 1. Therefore, $EP_i^{R^*}(M)$ is the Shapley value of the game v_M when M is a singleton schedule situation on N. ∎

Singleton (not necessarily unit-cost) schedule games are concave as demonstrated below.

Proposition 6. *If M is a singleton schedule situation on N, then the associated schedule game v_M is concave.*

Proof. Let M be a singleton schedule situation on N with $T_i = \{A_i\}$ for each $i \in N$. We will prove that for each $E \subseteq S \subseteq N \setminus \{i\}$, $v_M(E \cup \{i\}) - v_M(E) \geqslant v_M(S \cup \{i\}) - v_M(S)$. From (10), we can rewrite both sides of the inequality as follows:

$$v_M(E \cup \{i\}) - v_M(E) = \sum_{t \in A_i \setminus (\cup_{j \in E} A_j)} c_t,$$

$$v_M(S \cup \{i\}) - v_M(E) = \sum_{t \in A_i \setminus (\cup_{j \in S} A_j)} c_t.$$

Next, since $E \subseteq S$, we have that:

$$\cup_{j \in E} A_j \subseteq \cup_{j \in S} A_j \iff A_i \setminus \left(\cup_{j \in E} A_j \right) \supseteq A_i \setminus \left(\cup_{j \in S} A_j \right),$$

which implies that

$$\sum_{t \in A_i \setminus (\cup_{j \in E} A_j)} c_t \geqslant \sum_{t \in A_i \setminus (\cup_{j \in S} A_j)} c_t$$

and thus

$$v_M(E \cup \{i\}) - v_M(E) \geqslant v_M(S \cup \{i\}) - v_M(S),$$

for each $i \in N$ and $E \subseteq S \subseteq N \setminus \{i\}$. ∎

From Propositions 4 and 6, we conclude that carpool games are concave.

5. Non-emptiness of the core of schedule games

Proposition 1 reveals that the core of a schedule game can be empty (see Example 1). Nevertheless, uniform and singleton schedule games have nonempty cores as concave games as underlined in section 4. In this section, we provide two sufficient conditions for the non-emptiness of the core of a schedule game. The resulting classes of schedule situations encompass uniform and singleton schedule situations. The two conditions are based on the specification of the subset of time periods used by the players and thus explicitly provide core allocations.

5.1. Coherent schedule situations

An optimal time configuration $R^* \in O(N)$ with $R^* = (\{A_i^*\})_{i \in N}$ is called **coherent** for M if for each nonempty $E \subseteq N$, it holds that $R_E^* \in O(E)$, where R_E^* is the restriction of R^* to E. In words, a time configuration for N is coherent if no player has an incentive to change its consumption schedule in smaller coalitions.

Definition 4. *A schedule situation is called* **coherent** *if it admits a coherent optimal time configuration.*

This type of schedule situation is illustrated in the example below.

Example 4. *Let M be given by $N = \{A, B, C\}$, $T = \{1, \ldots, 5\}$, $T_A = \{\{1, 2\}, \{2, 3, 4\}, \{2, 5\}\}$, $T_B = \{\{1, 3, 4\}, \{1, 5\}\}$, $T_C = \{\{2, 4, 5\}, \{3, 4, 5\}\}$ and the costs:*

t	1	2	3	4	5
c_t	2.3	0.8	4	1.1	2.3

There are two optimal time configurations for N:

$$O(N) = \{\{R^1\}, \{R^2\}\} \quad \text{where} \quad R^1 = (\{1, 2\}, \{1, 5\}, \{2, 4, 5\}) \quad \text{and} \quad R^2 = (\{2, 5\}, \{1, 5\}, \{2, 4, 5\}).$$

These time configurations and those of smaller coalitions are listed in the table below in which the game v_M is reported as well.

E	$v_M(E)$	$O(E)$
$\{A\}$	3.1	$(\{1, 2\})$
		$(\{\mathbf{2, 5}\})$
$\{B\}$	4.6	$(\{\mathbf{1, 5}\})$
$\{C\}$	4.2	$(\{\mathbf{2, 4, 5}\})$
$\{A, B\}$	5.4	$(\{1, 2\}, \{1, 5\})$
		$(\{\mathbf{2, 5}\}, \{\mathbf{1, 5}\})$
$\{A, C\}$	4.2	$(\{\mathbf{2, 5}\}, \{\mathbf{2, 4, 5}\})$
$\{B, C\}$	6.5	$(\{\mathbf{1, 5}\}, \{\mathbf{2, 4, 5}\})$
$\{A, B, C\}$	6.5	$R^1 = (\{1, 2\}, \{1, 5\}, \{2, 4, 5\})$
		$\mathbf{R^2} = (\{\mathbf{2, 5}\}, \{\mathbf{1, 5}\}, \{\mathbf{2, 4, 5}\})$

Table 2: The set of all optimal time configurations

The time configuration R^2 is the unique coherent time configuration (this is highlighted in bold characters in Table 2). $\square$

The presence of a coherent time configuration is sufficient to guarantee that the core is nonempty as a corollary of the next result.

Proposition 7. *If M is coherent, then $v_M = v_{M'}$ for some singleton schedule situation M'.*

Proof. Let $R^* = (A_i^*)_{i \in N}$ be any coherent time configuration on M. From M and R^*, construct the schedule situation M^{R^*} such that $M^{R^*} = (T^{R^*}, (T^{R^*})_{i \in N})$ with $T^{R^*} = T$ and $T_i^{R^*} = \{A_i^*\}$. Consequently, M^{R^*} is a singleton schedule situation. In addition, R^* is coherent for M^{R^*}. Therefore, $v_{M^{R^*}} = v_M$ follows from the fact that R^* is coherent for both M^{R^*} and M. $\blacksquare$

From Propositions 5, 6 and 7, we get the following corollary.

Corollary 1. *If R^* on M is coherent, then the Equal pooling allocation $EP^{R^*}(M)$ is in the core of v_M and coincides with the Shapley value of v_M.*

The condition in Corollary 1 is sufficient but not necessary. In the following example an allocation $EP^{R^*}(M)$ is in the core of v_M even if $R^* \in O(N)$ is not coherent for a schedule situation M.

Example 5. *Let $N = \{A, B, C\}, T = \{1, \dots, 5\}, T_A = \{\{1,2,3\}, \{1,4\}, \{2,4\}\}, T_B = \{\{1,5\}, \{2,5\}\}, T_C = \{\{1,2,3,5\}\}$ and $c_t = 1$ for each $t \in T$. Then we obtain the following table,*

E	$v_M(E)$	$O(E)$	$\sum_{i \in E} x_i$
$\{A\}$	2	$(\{1,4\})$	1
		$(\{2,4\})$	
$\{B\}$	2	$(\{1,5\})$	1
		$(\{2,5\})$	
$\{C\}$	4	$(\{1,2,3,5\})$	2
$\{A,B\}$	3	$(\{1,4\},\{1,5\})$	2
		$(\{2,4\},\{2,5\})$	
$\{A,C\}$	4	$(\{1,4\},\{1,2,3,5\})$	3
		$(\{2,4\},\{1,2,3,5\})$	
$\{B,C\}$	4	$(\{1,5\},\{1,2,3,5\})$	3
		$(\{2,5\},\{1,2,3,5\})$	
$\{A,B,C\}$	4	$(\{1,2,3\},\{1,5\},\{1,2,3,5\})$	4
		$(\{1,2,3\},\{2,5\},\{1,2,3,5\})$	

Table 3: The set of all optimal time configurations

Here, only the number of time periods matters. The time configuration of player A used to compute $v_M(N)$ is not its smaller time configuration. More specifically, the selected time configuration for A is $(\{1, 2, 3\})$ whereas on its own its smaller time configurations are $(\{1, 4\})$ or $(\{2, 4\})$. Hence, none of the two optimal time configurations on N is coherent. However, the core of this example is nonempty since it contains allocation $x = (1, 1, 2)$ as shown by the above table. Remark that the two Equal pooling allocations constructed from $O(N)$ are also in the core. □

5.2. Coherent covering

This section introduces a new condition for the nonemptiness of the core of a schedule game. It is weaker than the coherence condition: the condition is automatically met when the corresponding schedule situation is coherent but is also satisfied on a larger class of schedule situations.

Fix any schedule situation M and let $R^* = (A_i^*)_{i \in N}$ be any optimal configuration in $O(N)$. A time configuration $\bar{R}^* = (\bar{A}_i^*)_{i \in N}$ is called a **coherent covering** of R^* if the following three conditions are satisfied:

(a) For each player $i \in N$, $\bar{A}_i^* \subseteq A_i^*$;

(b) $\cup_{i \in N} \bar{A}_i^* = \cup_{i \in N} A_i^*$;

(c) For each $E \subsetneq N$, there is $R^E \in O(E)$, $R^E = (A_j^E)_{j \in E}$, such that $\bar{A}_i^* \subseteq A_i^E$ for each $i \in E$.

Before going any further, let us comment on these conditions. Condition (a) means that we only select a subset of time periods for each player. This condition even allows an empty subset (see Example 6 below for an illustration). Condition (b) means that grouping the aforementioned subsets is optimal for the grand coalition. Put differently, subsets $\bar{A}_i^*$, $i \in N$, cover all active time periods in R^*. Condition (c) means that $\bar{A}_i^*$ is a subset of A_i^* that player i is always able to use optimaly in each coalition she belongs to. In other words, sets $\bar{A}_i^*$, $i \in N$, can be seen as coherent subsets.

Example 6. *Here, we slightly modify Example 1. Set $N = \{A, B, C\}$, $T = \{1, \ldots, 8\}$, $T_A = \{\{1, 2\}, \{3, 4, 5\}\}$, $T_B = \{\{1, 2\}, \{7, 8\}\}$, $T_C = \{\{3, 4, 5\}\}$ and the costs given by:*

t	1	2	3	4	5	6	7	8
c_t	1.1	0.8	2	0.6	0.4	0.5	1.4	0.9

Then, the resulting game and the coalitions' optimal time configurations are summarized in the following table.

E	$v_M(E)$	$O(E)$
$\{A\}$	1.9	$(\{1,2\})$
$\{B\}$	1.9	$(\{1,2\})$
		$(\{7,8\})$
$\{C\}$	3	$(\{3,4,5\})$
$\{A,B\}$	1.9	$(\{1,2\},\{1,2\})$
$\{A,C\}$	3	$(\{3,4,5\},\{3,4,5\})$
$\{B,C\}$	4.9	$(\{1,2\},\{3,4,5\})$
		$(\{7,8\},\{3,4,5\})$
$\{A,B,C\}$	4.9	$R^1 = (\{1,2\},\{1,2\},\{3,4,5\})$
		$R^2 = (\{3,4,5\},\{1,2\},\{3,4,5\})$
		$R^3 = (\{3,4,5\},\{7,8\},\{3,4,5\})$

Table 4: A schedule situation that is not coherent but admits a coherent covering

The only change compared to Example 1 is that T_C is now a singleton. This schedule situation is not coherent (and obviously neither uniform nor singleton). In fact, R^1 is not coherent since player A does not use $\{1,2\}$ within the optimal configuration of coalition $\{A,C\}$, R^2 is not coherent since player A does not use $\{3,4,5\}$ within the optimal configuration of coalitions $\{A\}$ and $\{A,B\}$, and R^3 is not coherent since player B does not use $\{7,8\}$ within the optimal configuration of coalition $\{A,B\}$. To the contrary, it is easy to check that $\bar{R} = (\emptyset, \{1,2\}, \{3,4,5\})$ is a coherent covering of both R^1 and R^2. $\qquad\square$

Remark that if R is a coherent time configuration on M, then R is trivially a coherent covering of itself. This means that any coherent schedule situation admits a coherent covering while the converse implication is not always true as illustrated by Example 6.

It can be checked that the core of the schedule game v_M in Example 6 is the singleton $Core(v_M) = \{x\}$ where $x = (0, 1.9, 3)$. Furthermore x is different from the two distinct equal pooling allocations that can be obtained from the optimal time configurations R^1, R^2 and R^3 and different from the Shapley value of v_M. Still, we provide below a variant of the equal pooling allocation which coincides with x in Example 6.

More precisely, instead of applying the Equal Pooling principle directly to an optimal time configuration R^*, we apply it to any of its coherent covering $\bar{R}^*$. Before stating the main result of this section, consider the unique coherent covering $\bar{R} = (\emptyset, \{1,2\}, \{3,4,5\})$ of an optimal time configuration (either of R^1 or R^2) in Example 6. We easily obtain $EP^{\bar{R}}(M) = (0, 1.9, 3) = x$. In words, in Example 6, the equal pooling allocation associated with the unique coherent covering singles out the unique core allocation.

Proposition 8. *If a schedule situation M admits a coherent covering $\bar{R}^*$ of some optimal time configuration $R^* \in O(N)$, then $EP^{\bar{R}^*}(M) \in Core(v_M)$.*

Proof. Let M be any schedule situation which admits a coherent covering $\bar{R}^*$ of some optimal time configuration $R^* \in O(N)$ and let $R^* = (A_i^*)_{i \in N}$ and $\bar{R}^* = (\bar{A}_i^*)_{i \in N}$. As a start, construct the singleton schedule situation $M^{\bar{R}^*}$ induced by $\bar{R}^*$, i.e., $M^{\bar{R}^*} = (T, (T_i^{\bar{R}^*})_{i \in N}, (c_t)_{t \in T})$ with $T_i^{\bar{R}^*} = \{\bar{A}_i^*\}$ for each $i \in N$. By definition of $M^{\bar{R}^*}$, it holds that

$$EP^{\bar{R}^*}(M) = EP^{\bar{R}^*}(M^{\bar{R}^*}). \tag{11}$$

Furthermore, from Propositions 5 and 6, we know that $EP^{\bar{R}^*}(M^{\bar{R}^*}) \in Core(v_{M^{\bar{R}^*}})$. Combined with (11), we get

$$EP^{R^*}(M) \in Core(v_{M^{\bar{R}^*}}). \tag{12}$$

Now, from condition (b) in the definition of a coherent covering, we obtain $v_{M^{\bar{R}^*}}(N) = v_M(N)$. Moreover, for each nonempty $E \subsetneq N$, condition (a) in the definition of a coherent covering implies that

$$v_{M^{\bar{R}^*}}(E) = \sum_{t \in \cup_{i \in E} \bar{A}_i^*} c_t \leq v_M(E),$$

from which (12) yields that

$$\sum_{j \in E} EP_j^{\bar{R}^*}(M) \leq v_{M^{\bar{R}^*}}(E) \leq v_M(E).$$

Hence, we conclude that $EP^{\bar{R}^*}(M) \in Core(v_M)$. ∎

Example 6 is a rather extreme case in which the Equal Pooling allocation obtained from the unique coherent covering results in player A paying no costs. Therefore, even if such an allocation is a core allocation in this example, it can be considered more unfair than equal pooling allocations calculated from $O(N)$. To sum up, in a schedule situation that is not coherent but admits a coherent covering, there may be a trade-off to be made between the stability of an allocation (in the sense of the core) and the fairness of an allocation (in the sense of the equal pooling of all active time periods of a given optimal time configuration for the grand coalition).

Example 4 is an example of a schedule situation which admits multiple coherent coverings of a non-coherent optimal time configuration. In order to see this, consider the optimal time configuration $R^1 = (\{1,2\},\{1,5\},\{2,4,5\})$ which is not coherent as shown is table 2. The time configuration $\bar{R}^1 = (\{2\},\{1,5\},\{2,4,5\})$ is the unique maximal coherent covering of R^* with respect to set inclusion. However $\bar{R}' = (\{2\},\{1\},\{2,4,5\})$ and $\bar{R}'' = (\emptyset,\{1,5\},\{2,4\})$ are two other coherent coverings of R^*, among others. From Proposition 8, we obtain that the corresponding Equal Pooling allocations $EP^{\bar{R}^1}(M) = (0.4, 3.45, 2.65)$, $EP^{\bar{R}'}(M) = (0.4, 2.3, 3.8)$ and $EP^{\bar{R}''}(M) = (0, 4.6, 1.9)$ are in the core of v_M.

The condition in Proposition 8 is sufficient but not necessary. In the following example, the core is nonempty even if the underlying schedule situation admits no coherent covering.

Example 7. Assume that $N = \{A, B, C\}$, $T = \{1, \ldots, 9\}$, $T_A = \{\{1,2\},\{3,4\},\{7,8,9\}\}$, $T_B = \{\{1,2\},\{5,6\},\{7,8,9\}\}$, $T_C = \{\{3,4\},\{5,6\},\{7,8,9\}\}$ and $c_t = 1$ for each $t \in T$. Then we obtain the following table.

E	$v_M(E)$	$O(E)$	$\sum_{i \in E} x_i$
$\{A\}$	2	$(\{1,2\})$	1
		$(\{3,4\})$	
$\{B\}$	2	$(\{1,2\})$	1
		$(\{5,6\})$	
$\{C\}$	2	$(\{3,4\})$	1
		$(\{5,6\})$	
$\{A,B\}$	2	$(\{1,2\},\{1,2\})$	2
$\{A,C\}$	2	$(\{3,4\},\{3,4\})$	2
$\{B,C\}$	2	$(\{5,6\},\{5,6\})$	2
$\{A,B,C\}$	3	$(\{7,8,9\},\{7,8,9\},\{7,8,9\})$	3

Table 5: The set of all optimal time configurations

The time configuration used to compute $v_M(N)$ contains, for each player, time periods that are never used to compute the worth of smaller coalitions. Therefore, the unique optimal time configuration on N neither is coherent nor admits a coherent covering. However, the core is a singleton containing allocation $x = (1,1,1)$. $\qquad\square$

The previous example does not ensure that the game v_M cannot be obtained as the schedule game of an alternative schedule situation which admits a coherent covering. In other words, the following question remains open: Can every schedule game with a nonempty core be obtained from a schedule situation admitting a coherent covering?

We conclude this section by pointing out the relationships between schedule situations that admit coherent configurations and/or coherent coverings and non-redundant schedule situations studied in Moulin and Laigret (2011). It is easy to figure out that the non-redundancy condition implies that the corresponding game admits a coherent covering consisting, for each player, of her set of critical periods (see section 3.2 for the definition).

Proposition 9. Let M be a non-redundant schedule situation, that is $\cup_{i \in N} G^i(M) = T$, where G^i is the set of critical periods for player $i \in N$. Then, for each $R \in O(N)$, $G = (G^i)_{i \in N}$ is a coherent covering of R.

Proof. Let M be a non-redundant schedule situation and $R \in O(N)$, $R = (A_i^*)_{i \in N}$. By definition, since $A_i^* \in T_i$,

$$G^i(M) = \bigcap_{A_i \in T_i} A_i \subseteq A_i^*$$

for each $i \in N$, which means that condition (a) in the definition of coherent covering is satisfied. Next, the non-redundancy condition implies that

$$\bigcup_{i \in N} G^i = T = \bigcup_{i \in N} A_i^*,$$

so that condition (b) in the definition of coherent covering is satisfied as well. Finally, $G^i(M) \subseteq A_i$ for each $A_i \in T_i$ yields that for each $E \subseteq N$ and each $R^E \in O(E)$, $R^E = (A_i^E)_{i \in E}$, we have $G^i(M) \subseteq A_i^E$. Hence condition (c) in the definition of coherent covering is satisfied for all optimal configurations for E (and not just at least one as required in the definition of a coherent covering). ∎

Example 7 illustrates that the converse implication does not hold: some schedule situations admit some coherent covering but fail non-redundancy. Finally, non-redundancy and coherence conditions are not related to each other. Non-redundancy does not implies coherence. To see this, it is to enough revisit Example 7 by adding player D with $T_D = \{\{1, 2, 6, 7, 8\}\}$, ceteris paribus. The new augmented schedule situation becomes non-redundant but is not coherent because its sub-schedule situation induced by coalition $\{A, B, C\}$ is not as underlined in Example 7. Similarly, Coherence does not imply non-redundancy as highlighted by Example 4.[4]

6. An application to the French postal case

Below, we present once again the problem of allocating the cost of the mail carrier route in France which was already mentioned in the introduction. To meet its universal service obligations, La Poste must organize the delivery network in order to be in capacity to visit all recipients' addresses six days a week and meet the **delivery speed** of the postal products. The delivery speed refers to the time period within which a particular postal product must be delivered, from the moment between it is posted until its actual delivery at the customers' location choice. In its decision 2008-0165 the French national regulatory authority Arcep, in charge of defining the allocation rules of universal products' costs, distinguished three delivery speed categories: D^7 for a delivery target on the 7th business day after posting, D^3 for a delivery target on the 3rd business day after posting and D^1 for a delivery target on the 1st business day after posting. Considering logistical constraints, a **delivery frequency** of one day per week would be enough to satisfy D^7, delivery frequency of three days per week would be enough to satisfy D^3 and delivery frequency of six days per week would be required to satisfy D^1. Arcep's decision states that the common cost of the six weekly mail carrier routes, first, is allocated to the three categories in proportion to their aforementioned delivery frequency, i.e. 10% of the delivery costs to D^7, 30% to D^3 and 60% to D^1. Secondly, the share of the cost previously calculated for each category is then allocated to the postal products belonging to this category according to their format/volume. We will only focus on the first part of this process which can be apprehended by an uniform unit-cost schedule game.

We can use the rich possibilities offered by the schedule situations in order to model the cost sharing of the mail carrier route as the following unit-cost schedule situation M^1. The infrastructure is the mail carrier route which can be used once during six days per week. The costs are quantified in number of routes, i.e. one tour used equal to one unit of cost. So that $T = \{1, 2, 3, 4, 5, 6\}$ and $c_t = 1$ for each $t \in T$. Period 1 represents the delivery day Monday and so on. The players are the three postal product categories, i.e. $N = \{D^7, D^3, D^1\}$. For category D^7, there are six singleton possible alternative consumption schedules, one for each day of the week, since a postal product in this category must be delivered not later than 7 days after being

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[4]In Example 4, period 3 is never selected, for any nonempty coalition, in any optimal time configuration. This property no longer holds if one adds period 6 which costs $c_6 = 0.6$ and $\{3, 6\}$ in T_B.

posted. On the contrary, for category D^1, the unique consumption schedule is the set of all six days of the week since the postal products in this category must be delivered on the next business day. For category D^3, the set of minimal consumption schedules contains all the triple of days which are not consecutive two by two[5]. Therefore:

$$T_{D^7} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\},$$
$$T_{D^3} = \{\{1,3,5\}, \{1,3,6\}, \{1,4,6\}, \{2,4,6\}\},$$
$$T_{D^1} = \{\{1,2,3,4,5,6\}\}.$$

We obtain the associated schedule game v_{M^1} below, where superscript 1 is here added to distinguish the two schedule situations presented in this section.

E	$\{D^7\}$	$\{D^3\}$	$\{D^1\}$	$\{D^7, D^3\}$	$\{D^7, D^1\}$	$\{D^3, D^1\}$	$\{D^7, D^3, D^1\}$
$v_{M^1}(E)$	1	3	6	3	6	6	6

The content of decision 2008-0165 only considers the number of delivery days. This process could naively be formulated as the following uniform unit-cost schedule situation M^2 where $T = \{1,2,3,4,5,6\}$, $N = \{D^7, D^3, D^1\}$ and:

$$T_{D^7} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\},$$
$$T_{D^3} = \{E \subseteq T : |E| = 3\},$$
$$T_{D^1} = \{\{1,2,3,4,5,6\}\}.$$

However, the set T_{D^3} does not reflect correctly the constraints imposed on category D^3. As an example, $\{1,2,3\} \in T_{D^3}$ which means that postal products in category D^3 can, a priori, be distributed in the three consecutive days Monday, Tuesday and Wednesday. However, this would prevent postal products posted on Wednesday to be delivered on time. This is legally not possible because La Poste has to meet the delivery speed of each postal product. This may seem inconsequential from the point of view of cost sharing since the resulting uniform unit-cost schedule game v_{M^2} is identical to v_{M^1}. This will no longer be the case if further changes are incorporated to this problem. As an illustration, imagine that we add a new category of postal products that must be delivered on some specific days, such as newspapers or advertisements, the mutualization of the time configurations can be different from v_{M^1} to v_{M^2}. More specifically, suppose that we add a fourth category of postal products D^2 corresponding to direct marketing mail that must be distributed in two specific consecutive days $\{2,3\}$. Consider for instance a first advertisement sent Tuesday that proposes discounts on Wednesday and, to encourage consumers, the store send back coupons on Wednesday, as a reminder. We get the following set of minimal time configurations for category D^2:

$$T_{D^2} = \{\{2,3\}\}.$$

Denote by $v_{M^{1'}}$ and $v_{M^{2'}}$ the two four-player schedule games obtained by adding player D^2 to the schedule situations M^1 and M^2, respectively. These two games are distinct. To see this consider coalition $\{D^2, D^3\}$. In the schedule game $v_{M^{1'}}$, the delivery category D^2 pools only one of its two days with D^3, which yields that $v_{M^{1'}}(\{D^2, D^3\}) = 4$, i.e. four routes per week are

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[5]Time periods 1 and 6 belonging to T_{D^3} are not consecutive due to Sunday which is not a delivery day.

needed. On the contrary, in the uniform unit-cost schedule game $v_{M^{2'}}$ the label of the delivery days does not matter, hence the delivery category D^2 can completely (but inaccurately) pool its time periods with category D^3, which implies that $v_{M^{2'}}(\{D^2, D^3\}) = 3$.

To conclude this application, let us back to the original three-player problem. To determine allocations of the schedule game v_{M^1} we will apply the Equal pooling allocation to a coherent optimal time configuration R^* and to a non-coherent optimal time configuration R. Let $R^* = \big\{\{1\}, \{1, 3, 5\}, \{1, 2, 3, 4, 5, 6\}\big\}$ and $R = \big\{\{1\}, \{2, 4, 6\}, \{1, 2, 3, 4, 5, 6\}\big\}$. According to proposition 3 the Equal pooling allocation of a coherent optimal time configuration is the Shapley value. It gives the following percentages: D^7 incurs 5.56% of the costs, D^3 incurs 22.22% and D^1 incurs 72.22% of the costs. This corresponds to the efficient allocation $(\frac{1}{3}, \frac{4}{3}, \frac{13}{3})$ in the game v_{M^1}, as calculated in Bohorquez Suarez and Munich (2023). The Equal pooling allocation on R gives the following percentages: D^7 incurs 8.33% of the costs, D^3 incurs 25% and D^1 incurs 66.67% of the costs. This corresponds to the efficient allocation $(\frac{1}{2}, \frac{3}{2}, 4)$ in the game v_{M^1}. Compared to the two previous allocations the current allocation incurs less costs to D^1. Although, the three allocations are close to each other, they rely on distinct principles. The Shapley value is based on the incremental costs of each category to coalitions, the Equal pooling allocation takes into account the routes needed by each category and the current allocation shares the costs according to a proportional principle. Hence, the Equal pooling allocations can be considered as an alternative to the current allocation.

7. Concluding remarks

We conclude briefly with a summary of the relation between schedule games and schedule situations and propose possible extensions for future works. Figure 1a illustrates the relation between schedule games and Figure 1b the relation between schedule situations. Proposition 1 states that the class of all schedule games on N coincides with the class of monotone and sub-additive TU-games on N thus, some schedule games may have an empty-core. However, the schedule games arising from schedule situation with a coherent covering (CCSG) have a non-empty core. Coherent schedule situations are a subset of the class of schedule situations that admit a coherent covering: any coherent M admits some coherent covering while the converse is not always true. Therefore, schedule games resulting from a coherent schedule situation (CSG) are included in the CCSG set. In addition, Proposition 7 states that CSG coincide with singleton schedule games (SSG). However, as pictured in Figure 1a, coherent schedule situations include singleton schedule situations while the converse is not always true, unlike the corresponding classes of schedule games. Uniform schedule games (USG), by definition, are coherent schedule situations so that they are included in the set of singleton schedule games (SSG). However, sets of uniform schedule situations (USS) and singleton schedule situations (SSS) obviously are disjoint. Finally, Proposition 9 states that the set of schedule games with non-redundant periods (NRSG) is contained by the set of CCSG and it is easy to figure out that it intersects CSG, SSG and USG. The same connection applies to the sets of corresponding schedule situations.

We know that some situations do not admit a coherent covering while they induce a schedule game with a non-empty core, see Example 7. Nevertheless, it remains an open question whether any schedule games with a non-empty core can be obtained from a schedule situation with a coherent covering. Moreover, it could be interesting to find a condition which would be both necessary and sufficient for the non-emptiness of the core of a schedule game.
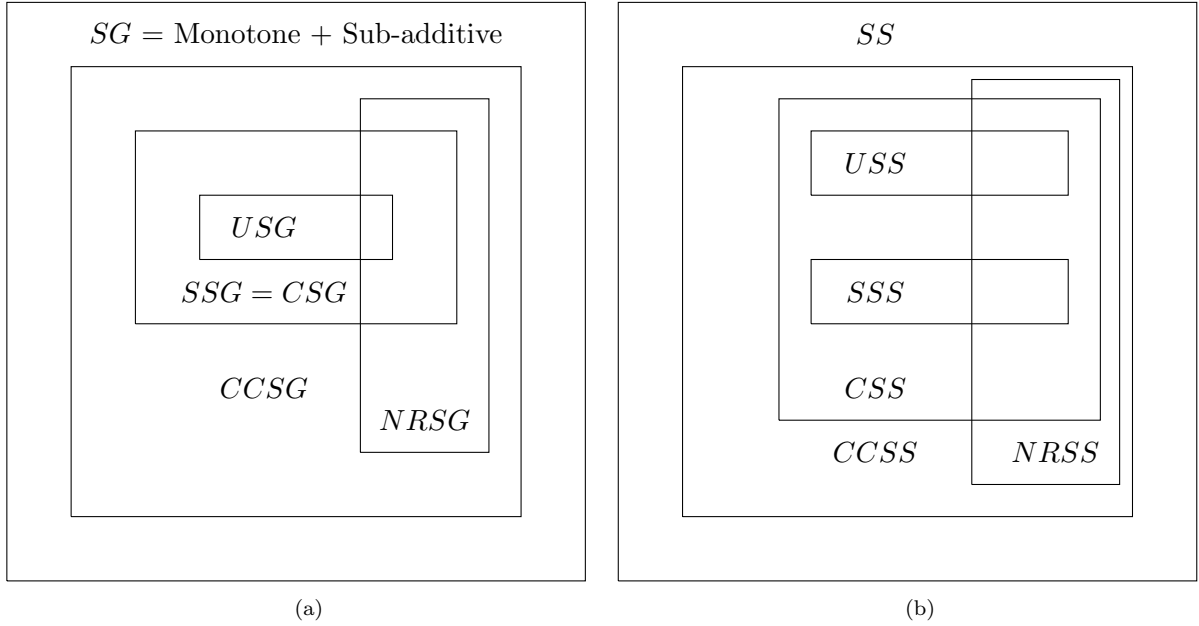
Figure 1: Relations between schedule games and schedule situations

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