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January 2024

Working paper No. 2024 – 04

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Weighted scoring rules for selecting a compatible committee

Clinton Gubong Gassi*

Abstract

This paper addresses the challenge of incorporating a weighting system into the committee selection process under scoring rules. Given a set of voters expressing their preferences on a set of candidates, the objective is to choose a fixed-size subset of candidates, called a *committee*, that optimally represents the voters' preferences. The main idea of the paper is that, beyond the voter's preferences, each committee of a given size may be associated with a weight vector based on a specific compatibility criterion such as communication, cooperation, connectivity, or diversity. This vector assigns weights to individual committee members, reflecting their ability to make the committee satisfactory with respect to the criterion under consideration. In order to integrate the weighting system into the committee selection process, we introduce the *continuous weighted scoring function* derived from any scoring function. This function is characterized by two crucial properties, namely *linearity* and *independence of zero-weight members*. We demonstrate how this continuous weighted scoring function facilitates a trade-off between the candidates' performance and their compatibility skills.

Keywords: Voting, committee selection, scoring functions, weight vector.
JEL classification: D71, D72.

1 Introduction

The committee selection problem is relevant in various real-life scenarios, such as the shortlisting of candidates for a competition, the election of an executive office within a classroom, or the formation of a research team for a specific project. In essence, this type of problem involves a group of individuals (or voters) expressing their preferences over a set of alternatives (or candidates), and the goal is to select a committee, i.e., a fixed-size subset of alternatives, that best represents the preferences of all voters. Numerous studies in social choice theory have been carried out in order to release this type of problem. On the one hand, some of these investigations focus on the approval ballot setting, where a voter's preference is defined as a bundle of candidates he/she approves. Examples of such investigations include Brams et al. (2007), Brill et al. (2018), Elkind (2018), Faliszewski et al. (2017), Kilgour (2010), Kilgour and

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Marshall (2012), Kilgour et al. (2006), Lackner and Skowron (2022), and Yang (2023), among others. On the other hand, some studies deal with the ordinal model, which requires voters to provide a complete ranking of candidates. Prominent contributions in this category include Barberà and Coelho (2008), Elkind et al. (2015), Elkind et al. (2017), Faliszewski et al. (2019), Faliszewski et al. (2018), and Skowron et al. (2019), among others.

Within the committee selection framework, the approaches that have been extensively examined are those based on scoring committee selection rules. These rules involve the assessment of all committees through a scoring function, ultimately selecting the committee(s) with the highest score. While scoring committee selection rules hold considerable appeal, it is crucial to emphasize that opting for the committee maximizing the scoring function might, at times, yield undesirable results. This is particularly true when other considerations beyond the excellence measured by the scoring function, such as additional parameters or requirements, come into play. Several studies, including Aziz (2019), Ianovski (2021), Diss et al. (2023), and Yang and Wang, 2018 among others, have invoked additional criteria or constraints that the committee selection may be subject to.[1] The current paper is interested in the *compatibility* criterion, emphasizing that a singular focus on committee scores may result in the selection of a committee whose members lack compatibility. To illustrate, consider a scenario where a research lab aims to recruit three researchers for collaborative work on a project, and the available candidates possess diverse language proficiencies. For instance, imagine a candidate a who speaks only English, a candidate b who exclusively speaks French, and a candidate c who solely speaks Spanish. No matter the excellence of the committee $\{a, b, c\}$, these candidates are fundamentally incompatible since, due to the inability to communicate effectively, they are not able to work efficiently together.

This paper proposes an approach to take into account a compatibility criterion to compute the scores of the committees in this context. The main idea of the paper is the introduction of a *committee weighting system* designed to gauge the level of compatibility of any committee. A committee weighting system associates each committee of any size with a *committee weight vector* which is a vector comprised of decreasing non-negative real numbers of length equal to the committee size. Each component of the weight vector represents the ability of a specific committee member to ensure group compatibility. The key contention is that, to select a committee that is not only excellent according to a scoring function but also compatible, it is imperative to integrate the committees' weight vectors into the scoring function. This integration results in a *weighted scoring function.*

Given a scoring function f, there are various ways to incorporate committee weight vectors into f in order to obtain a weighted scoring function. In this paper, we introduce the *continuous weighted scoring function* based on f. This function amalgamates the weight vector and the excellence of each committee, and it is characterized by two fundamental properties called *linearity* and *independence of zero-weight members.* The linearity property stipulates that the resulting weighted score of a committee should exhibit linearity with respect to the weight vector. The

[1] Most of these papers focus on diversity constraints within a context where candidates are labeled according to one (or more) attributes. The objective is to guarantee a specific level of diversity in the selected committee.

independence of zero-weight members necessitates that the weighted score of a committee does not depend on committee members who lack compatibility with any other member. The latter requirement holds merit since, the excellence of a candidate, no matter how outstanding, becomes ineffectual if the candidate cannot "cooperate" with any other candidate in the targeted committee. Furthermore, we establish that the continuous scoring function based on f that we introduce in the paper at hand ensures fairness between excellence and compatibility skills. Indeed, we show that analogous to the independence of zero-weight members, the continuous weighted scoring function based on f remains indifferent to null members. This implies that if a candidate fails to contribute to the (unweighted) score of a committee, it similarly does not contribute to the continuous weighted score of that committee, regardless of its compatibility skills. It is essential to note that a committee weight vector is defined as an ordered vector of non-negative real numbers, and the continuous weighted score of a committee is contingent on the order of weights in the committee weight vector. Lastly, we establish the necessary and sufficient condition so that the continuous weighted score does not depend of the order of weights: the scoring function f must be a *candidate-wise* scoring function. This type of function assesses each committee by summing up the individual scores of its members, with each member regarded as a 1-size committee.

The paper is organized as follows: Section 2 introduces the model and offers essential preliminary definitions and notations. In Section 3, we present our definition of continuous weighted scoring functions, along with the main results of the paper. Finally, Section 4 concludes the discussion and outlines the primary directions for future research.

2 Preliminary definitions

Consider a non-empty set A of m candidates and a non-empty set N of n voters with $m \geq 3$ and $n \geq 2$. Candidates are denoted by the lowercase letters a, b, c,..., or a_1, a_2, a_3, etc. We denote by $\mathbb{N}^*$ the set of all positive integers and by $\mathbb{R}_+$ the set of all non-negative real numbers. Throughout the paper, we simply write $[r]$ to denote the set $\{1, \dots, r\}$ for any positive integer $r \in \mathbb{N}^*$.

We assume that each voter $i \in N$ is endowed with a preference P_i, which can be a ranking on the set of candidates (ordinal preference), or a subset of candidates that he/she approves (approval ballot). A preference profile (or simply a profile) is a collection $P = (P_i)_{i \in N}$ specifying the preferences of all voters. Given the set of candidates and the set of voters, we consider in this paper the setting where the goal is to select a fixed-size subset of candidates, called a *committee*. For any integer $k \in [m-1]$, a committee of size k is defined as any k-element subset of A. The set containing all possible committees of size k for the set A is denoted by 2^A_k. We focus on committee sizes k such that $k \in [m-1]$, as the case $k = m$ is straightforward. In both ordinal and approval preference frameworks, a Committee Selection Rule (CSR) is defined as any mapping $\mathcal{R}$ that, for any profile P and any committee size $k \in [m-1]$, assigns the set $\mathcal{R}(P, k)$ comprising the winning committee(s). This set is referred to as the social outcome of the pair (P, k) under the CSR $\mathcal{R}$. The most studied CSRs in the literature are undoubtedly the scoring CSRs, which assign to each committee a score according to a (committee) scoring function, with respect to the profile, and select the committee(s) with the maximum score.

Definition 1 *A scoring function, denoted as f, is defined as any function assigning a score $f(P, W)$ to any profile P and any committee W of any size $k \in [m-1]$, based on the profile P. The corresponding scoring rule associated to the scoring function f is the mapping $\mathcal{R}_f$ that assigns to any profile P and any committee size $k \in [m-1]$, the set $\mathcal{R}_f(P, k)$ consisting of size-k committee(s) that maximize(s) the score $f(P, W)$.*

The above definition of scoring rules encompasses both the committee scoring rules defined for ordinal preferences by Elkind et al. (2017), as well as the approval-based committee voting examples of which can be found in works such as Elkind (2018), Kilgour (2010), Kilgour and Marshall (2012), and Lackner and Skowron (2022), among others.[2]

The selection of a set of candidates based on scores plays a pivotal role in multiple contexts extensively explored in the literature. The merits and drawbacks of scoring rules have also been thoroughly examined in the literature. Now, the committee selection process can sometimes lead to an unsatisfactory outcome when some parameters other than scores are taken into account, and Example 1 below illustrates this issue. This example uses the well-known committee scoring rules defined by Elkind et al. (2017) for the ordinal setting, where each voter's preference is a linear ranking of all candidates.[3]

Example 1 *Consider the set of five candidates $A = \{a_1, a_2, a_3, a_4, a_5\}$ from which we need to select three candidates to collaboratively work on a specific project. However, there is a language diversity among the candidates: a_1, a_2, and a_3 speak only French, a_4 exclusively speaks English, and a_5 communicates solely in Spanish. Now, let us examine the following preference profile with ordinal preferences, where each column represents the ranking of a voter:*

$$P = \begin{bmatrix} a_1 & a_1 & a_1 \\ a_4 & a_4 & a_4 \\ a_5 & a_5 & a_5 \\ a_2 & a_2 & a_2 \\ a_3 & a_3 & a_3 \end{bmatrix}$$

It can be verified that every committee scoring rule (and any other reasonable rule) would choose the committee $\{a_1, a_4, a_5\}$. However, the members of this committee cannot effectively collaborate due to the absence of communication between them, indicating complete incompatibility.

We consider in the paper at hand a setting where committee selection is subject to an additional criterion, such as communication, cooperation, or affinity among committee members,

[2]In both frameworks, each voter i gives some point $f(P_i, W)$ to any committee W according to his/her preference P_i and the score $f(P, W)$ is the sum of the points collected by W from all voters. In the approval setup, the point $f(P_i, W)$ depends on the number of candidates from W approved by voter i, and in the ordinal setup, it depends on the positions of the committee member's in voter i's ranking.

[3]This class of rules has been characterized by Skowron et al. (2019), extending the work of Young (1975) for single-winner elections.

and we talk about the selection of a *compatible* committee. In this context, we assume that each committee can be assigned with a vector of weights based on the specified compatibility criterion. The goal is to choose a committee of a fixed size, considering both the candidates' excellence measured by a scoring function and the capacities of the committee members regarding the compatibility criterion. These capacities are expressed through the weight vector associated with each committee.

Definition 2 *A committee weight vector (or simply a weight vector) of length $k \in [m-1]$ is any vector of non-negative real numbers $\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}_+^k$ such that $\theta_1 \geq \cdots \geq \theta_k$. A committee weighting system (or simply a weighting system) on A is any mapping Θ assigning to any committee W of any size $k \in [m-1]$, a committee weight vector $\Theta(W) = \theta \in \mathbb{R}_+^k$, such that each component θ_j is the weight associated to a given committee member.*

Intuitively, the weight vector assigned to a committee (by a weighting system) reflects the abilities of the members of that committee to be compatible with each other with respect to the considered compatibility criterion. More precisely, each component of the weight vector signifies the capability of an individual member within a specific committee to be compatible with other members of that same committee. For any committee size $k \in [m-1]$ and any committee weight vector $\theta = (\theta_1, \ldots, \theta_k)$ associated to a committee $W \in 2_k^A$, the component θ_j corresponding to a candidate $a \in W$ will be sometimes denoted by $\theta_W(a)$.

Example 2 *Consider the set of five candidates $A = \{a_1, a_2, a_3, a_4, a_5\}$, labeled according to spoken languages. Let $A_1 = \{a_1, a_2, a_3\}$ be the set of French speakers and $A_2 = \{a_3, a_4, a_5\}$ be the set of English speakers. Assume that the committee to be formed has a size of $k = 3$ and that the considered criterion is the communication effectiveness among committee members. The communication structure on the set of candidates can be represented by an undirected graph $G = (A, E)$, where each vertex represents a candidate and the edges are defined as $E = \{(a_1, a_2), (a_1, a_3), (a_2, a_3), (a_3, a_4), (a_3, a_5), (a_4, a_5)\}$. Assume that the weighting system Θ is the "degree system" that specifies the number of neighbors belonging to the committee for each member, i.e., the number of candidates with whom each candidate can communicate. Then, the weight vector assigned to the committee $W = \{a_1, a_3, a_4\}$ is $\theta = (2, 1, 1)$ with $\theta_W(a_1) = 1, \theta_W(a_3) = 2$, and $\theta_W(a_4) = 1$. If we consider the committee $\{a_1, a_2, a_4\}$, then the weight vector assigned to it is $\theta = (1, 1, 0)$.*

It is noteworthy to mention that there might be various approaches for defining a weighting system in this context. The degree system, as illustrated in Example 2, represents a straightforward method for assessing candidates' abilities when the communication criterion is under consideration. Now, for any two committee weighting systems Θ and Θ' on A, and any non-negative real number $\alpha \in \mathbb{R}_+$, the weighting systems $\alpha\Theta$ and $\Theta + \Theta'$ are defined by

$$(\alpha\Theta)(W) = \alpha\Theta(W) \text{ and } (\Theta + \Theta')(W) = \Theta(W) + \Theta'(W). \tag{1}$$

It is important to highlight that while it is possible to assign weights to individual candidates a priori, we assert that in the context of committee selection, the weight assigned to a candidate within a committee should depend on the other members of that committee. This is the reason why in our approach, we opt for a weighting system applied to committees, rather than individual candidates from the outset. Recall that in the single-winner setting, numerous studies explore weighting systems on candidates. We direct the reader to works such as Massó and Vorsatz (2008) and Alcalde-Unzu and Vorsatz (2009), among others, for further insights.

As mentioned in the introductory section, the current paper is concerned with the issue of how to integrate a weighting system into a scoring function in order to select a committee of fixed size. In the sequel, for any committee size $k \in [m-1]$, we use the notation I^k to represent a vector of length k where all components are set to 1. Moreover, for any vector $\theta = (\theta_1, \dots, \theta_k) \in \mathbb{R}^k_+$ and any $j \in [k]$, we denote by θ_{-j} the vector of length $k-1$ obtained by removing the component θ_j from θ.

Definition 3 *Let f be a scoring function. A weighted scoring function based on f is a function $\tilde{f}$ assigning to each profile P, each committee W of size $k \in [m-1]$, and each vector $\theta \in \mathbb{R}^k_+$, the weighted score $\tilde{f}(P, W, \theta) \in \mathbb{R}$ such that $\tilde{f}(P, W, I^k) = f(P, W)$.*

The last condition in the above definition asserts that when all committee members are assigned a weight of one, the weighted score of the committee should be identical to the score it would obtain without any weighting.

Definition 4 *Let f be a scoring function and $\tilde{f}$ be a weighted scoring function based on f. The weighted scoring rule $\mathcal{R}_{\tilde{f}}$ associated to $\tilde{f}$ selects for each profile P, each committee size $k \in [m-1]$, and each weighting system Θ, the set $\mathcal{R}_{\tilde{f}}(P, k, \Theta)$ of all size-k committees that maximize the weighted score $\tilde{f}\Big(P, W, \Theta(W)\Big)$.*

3 The Continuous weighted scoring function

Recall that there are several ways to integrate committee weight vectors into a scoring function. In this section, we propose a method to incorporate a weight vector θ into a scoring function f in such a away that the weighted score $\tilde{f}(P, W, \theta)$ is a continuous function of the weight vector θ. The resulting modified scoring function $\tilde{f}$ is called the *continuous weighted scoring function* based on f. Our proposal of the continuous weighted scoring function draws inspiration from Fagin and Wimmers (2000), where a comparable formula was proposed to adjust the values assigned to tuples of real numbers based on associated weights reflecting the significance of each component. While our framework differs entirely, we present a similar outline using the crucial notion of *θ-decomposition* of a committee.

Definition 5 *Let W be a committee of size $k \in [m-1]$ and $\theta = (\theta_1, \dots, \theta_k)$ be a weight vector assigned to W by a weighting system. A θ-decomposition of W is an increasing sequence $W^1 \subset W^2 \subset \dots \subset W^k = W$ of subsets of W such that for each $j \in [k]$, W^j contains the candidates with the weights $\theta_1, \dots, \theta_j$.*

Example 3 *In Example 2, since the committee* $W = \{a_1, a_3, a_4\}$ *is weighted by the weight vector* $\theta = (2, 1, 1)$*, it can be checked that a possible* θ*-decomposition of* W *is* $W^1 \subset W^2 \subset W^3$*, where* $W^1 = \{a_3\}, W^2 = \{a_3, a_1\}$*, and* $W^3 = W$*. A second possible* θ*-decomposition of* W *is given by* $W^1 = \{a_3\}, W^2 = \{a_3, a_4\}$*, and* $W^3 = W$*.*

Remark that the θ-decomposition of W is unique if and only if $\theta_j \neq \theta_{j+1}$ for all $j \in [k-1]$.

Henceforth, we are prepared to define the continuous weighted scoring function, derived from the scoring function f.

Definition 6 *Let* f *be a scoring function. The continuous weighted scoring function based on* f *is the weighted scoring function that assigns to any profile* P *and any committee* W *of size* $k \in [m-1]$*, weighted with a weight vector* θ*, the weighted score* $\tilde{f}(P, W, \theta)$ *defined by*

$$\tilde{f}(P, W, \theta) = \sum_{j=1}^{k-1} (\theta_j - \theta_{j+1}) f(P, W^j) + \theta_k f(P, W), \tag{2}$$

where $W^1 \subset \cdots \subset W^k = W$ *is a* θ*-decomposition of* W*. By letting* $\theta_{k+1} = 0$*, we can write*

$$\tilde{f}(P, W, \theta) = \sum_{j=1}^{k} (\theta_j - \theta_{j+1}) f(P, W^j). \tag{3}$$

The definition of the continuous weighted scoring function may naturally prompt the reader to question its well-defined nature, given that the θ-decomposition of a committee is not necessarily unique. Specifically, if there exists an integer $j \in [k]$ such that $\theta_j = \theta_{j+1}$, the θ-decomposition of the committee is not unique, potentially leading one to speculate that the continuous weighted function could assign distinct scores to the same committee. However, the ensuing claim elucidates that this concern is unfounded, since the continuous weighted score of any committee remains independent of components θ_j that are equal. This crucial insight ensures that the continuous weighted scoring function assigns a unique value to every weighted committee.

Claim 1 *Given a committee* W *of size* $k \in [m-1]$ *associated with a weight vector* $\theta \in \mathbb{R}^k_+$*, the weighted score* $\tilde{f}(P, W, \theta)$ *assigned to* W *by the continuous weighted scoring function does not depend on the chosen* θ*-decomposition of* W*.*

Proof. Let W be a committee of size k and θ be a weight vector associated to W. Furthermore, let $W^1 \subset \cdots \subset W^k = W$ be a θ-decomposition of W. For any component θ_j such that $\theta_j = \theta_{j+1}$, we have $(\theta_j - \theta_{j+1}) f(P, W^j) = 0$ and, more importantly, all the subsets W^t does not change for all $t \neq j$. Therefore, it follows that $\tilde{f}(P, W, \theta) = \sum_{\substack{t=1 \\ t \neq j}}^{k} (\theta_t - \theta_{t+1}) f(P, W^t)$. This proves that the weighted score $\tilde{f}(P, W, \theta)$ does not depend on the components that are equal and, therefore, the continuous weighted scoring function $\tilde{f}$ is well defined since it assigns a unique value to any triplet (P, W, θ). ■

We would like to argue that the basic intuition behind the definition of the continuous weighted scoring function is that the committee members are ordered according to their compatibility weights, which is modeled by the θ-decomposition. In order to take into account the compatibility skills and the excellence of all committee members, the excellence score of each subset W^j is weighted by the extra weight of its members, which is considered as the difference between the minimum weight of the members of W^j and the maximum weight of the members outside W^j. The continuous weighted score of W is the sum of these extra weighted scores.

Example 4 *Consider a set of five candidates denoted as* $A = \{a_1, a_2, a_3, a_4, a_5\}$ *and labeled according to spoken languages:* $\{a_1, a_2, a_3\}$ *as French speakers,* $\{a_3, a_4\}$ *as English speakers, and* a_5 *exclusively speaking Italian. Retain the preference profile* P *from Example 1. Consider the Borda scoring function* f*, defining the score for each committee* W *as* $f(P, W) = \sum_{a \in W} \sum_{i \in N} (m - r_i(a))$*, with* $r_i(a)$ *representing the position of candidate* a *in voter* i*'s ranking. As highlighted in Example 1, focusing solely on committee scores would lead to the selection of* $\{a_1, a_4, a_5\}$ *which is a committee with completely incompatible members. Now, introduce the weighting system assigning to each committee the vector of integers indicating the number of neighbors for each member, as detailed in Example 2. The weight vector for the committee* $\{a_1, a_3, a_4\}$ *is* $(2, 1, 1)$*, and upon verification, this committee attains the highest continuous weighted score of* 21*.*

It is worth mentioning that while all the examples throughout this paper showcase ordinal preferences, similar examples can be designed using approval preferences.

The primary merit attributed to the continuous weighted scoring function is that it is characterized by two natural and desirable properties called *linearity* and *independence of zero-weight members*. Before presenting these two properties, let us introduce some additional notations. For any committee of size $k \in [m-1]$ and any vector $\theta = (\theta_1, \ldots, \theta_k)$, we denote by $J_\theta = \{j \in [k] : \theta_j \neq 0\}$ and we use the notation θ^* to represent the restriction of θ to J_θ; that is, the vector of length less than or equal to k obtained by eliminating all null components. For any committee W weighted by a weight vector θ, we denote by $W(J_\theta)$ the set of all members of W with a non-zero weight. Obviously, if $\theta_j \neq 0$ for all $j \in [k]$, then θ has no null component and we have $\theta^* = \theta$ and $W(J_\theta) = W$. Now, let us present the two properties that allow to characterize the continuous weighted scoring function.

Let f be any scoring function.

- **Linearity:** A weighted scoring function $\tilde{f}$ based on f satisfies linearity if for any committee W of size $k \in [m-1]$, for any two vectors $\theta, \theta' \in \mathbb{R}^k_+$, and for any two non-negative real numbers α and β, we have

$$\tilde{f}(P, W, \alpha\theta + \beta\theta') = \alpha\tilde{f}(P, W, \theta) + \beta\tilde{f}(P, W, \theta'). \tag{4}$$

- **Independence of zero-weight members:** A weighted scoring function $\tilde{f}$ based on f satisfies independence of zero-weight members if for any committee W of any size $k \in$

$[m-1]$, and any vector $\theta \in \mathbb{R}_+^k$, we have

$$\tilde{f}(P, W, \theta) = \tilde{f}\Big(P, W(J_\theta), \theta^*\Big). \tag{5}$$

In essence, the linearity requirement dictates that the weighted score of any committee should be proportional to the scaling factor, denoted as α. Specifically, if the weight vector undergoes multiplication by α, the committee's weighted score should also be multiplied by α. Moreover, when considering the summation of two weight vectors, the weighted score assigned to a committee should be the sum of the weighted scores it receives concerning each of the two weight vectors. Concurrently, the independence of zero-weight members demands that the weighted score assigned to a given committee remains unaffected by the presence or absence of zero-weight members within the committee. This requirement holds merit because, regardless of a candidate's excellence across the profile, such excellence is not helpful if the candidate cannot be compatible with any other candidate within the committee.

It is important to emphasize that the paper at hand is not at all concerned with exploring how voters express their preferences or how these preferences influence the scoring rule, since these aspects have been extensively studied in existing literature. The primary focus of this paper is exclusively directed toward understanding how a committee weighting system influences the score allocated to a committee. Consequently, in instances where the preference profile is established, the paper may refer to the score and the weighted score of a committee as $f(W)$ and $\tilde{f}(W, \theta)$, respectively, without explicitly specifying the profile.

The following theorem establishes that, for any scoring function f, the continuous weighted scoring function based on f is the unique weighted scoring function based on f that satisfies linearity and independence of zero-weight members.

Theorem 1 *Let f be a scoring function. A weighted scoring function based on f satisfies linearity and independence of zero-weight members if and only if it is the continuous weighted scoring function based on f.*

Proof. Let f be a scoring function and $\tilde{f}$ be the continuous weighted scoring function based on f. It is not hard to show that $\tilde{f}$ satisfies linearity by definition of $\tilde{f}$. Let $k \in [m-1]$ and $W \in 2_k^A$ be a committee weighted by the weight vector θ. Let j_0 be the highest integer from $[k]$ such that $\theta_{j_0} \neq 0$. We have $J_\theta = \{1 \ldots, j_0\}$, $\theta^* = (\theta_1, \ldots, \theta_{j_0})$ and $W(J_\theta) = W^{j_0}$. It holds that $(\theta_j - \theta_{j+1}) f(W^j) = 0$ for all $j > j_0$. Thus,

$$\tilde{f}(W, \theta) = \sum_{j=1}^{j_0} (\theta_j - \theta_{j+1}) f(W^j) = \tilde{f}(W^{j_0}, \theta^*). \tag{6}$$

Hence, the continuous weighted function based on f satisfies independence of zero-weight members.

Conversely, assume that a weighted scoring function $\tilde{f}$ based on f obeys to linearity and independence of zero-weight members. We have to show that it is the continuous weighted scoring function based on f. Let W be a committee of size $k \in [m-1]$ weighted with a weight vector $\theta = (\theta_1, \dots, \theta_k)$ and let $W^1 \subset \dots \subset W^k = W$ be a θ-decomposition of W. Let us consider the weight vectors $E_1, E_2, \dots, E_k = I^k$ defined by $E_j = (\underbrace{1, \dots, 1}_{j\,times}, 0 \dots, 0)$ for all $j \in [k]$. We have

$$\theta = (\theta_1 - \theta_2)E_1 + (\theta_2 - \theta_3)E_2 + \dots + (\theta_{k-1} - \theta_k)E_{k-1} + \theta_k E_k. \tag{7}$$

Moreover, the $k \times k$-dimensional matrix formed by vectors $E_1, \dots, E_k$ is a triangular matrix with a determinant equal to 1. Then the vectors $(E_j)_{j=1}^k$ are linearly independent and, consequently, the decomposition of θ in equation (7) is unique. By linearity, we have

$$\tilde{f}(W, \theta) = \sum_{j=1}^{k-1} (\theta_j - \theta_{j+1}) \tilde{f}(W, E_j) + \theta_k \tilde{f}(W, E_k). \tag{8}$$

Since $\tilde{f}$ satisfies independence of zero-weight members, it holds that

$$\tilde{f}(W, E_j) = \tilde{f}(W^j, I^j) = f(W^j), \tag{9}$$

by definition of any weighted scoring function based on f. Therefore,

$$\tilde{f}(W, \theta) = \sum_{j=1}^{k-1} (\theta_j - \theta_{j+1}) f(W^j) + \theta_k f(W). \tag{10}$$

Thus, $\tilde{f}$ is the continuous weighted scoring function based on f. ■

The independence of zero-weight members necessitates that the weighted score assigned to a committee remains unaffected by members possessing a null weight. However, we believe that this characteristic should be counterbalanced by stipulating that a non-zero weight, even a maximal one, should not confer a "veto" power to a committee member. To articulate this more precisely, for any committee size $k \in [m-1]$, if a member within a committee $W \in 2^A_k$ does not contribute to the (unweighted) score $f(S)$ of any subset S of W (in accordance with the provided preference profile), then it should similarly abstain from contributing to the weighted score $\tilde{f}(W, \theta)$ for any weight vector $\theta \in \mathbb{R}^k_+$ associated with W.

Definition 7 *Let $k \in [m-1]$ be a committee size and $W \in 2^A_k$. A candidate $a \in W$ is a null member of W under f, with respect to the preference profile P, if $f(P, S \setminus \{a\}) = f(P, S)$ for all $S \subseteq W$.*

A null member of a committee W under f, with respect to the profile P, is a candidate belonging to W who does not contribute to the score of any subset of W. This definition mirrors the well-known concept of a *null player* in cooperative game theory. The following theorem asserts that the weighted score assigned to any committee W by the continuous weighted scoring function based on f remains independent of the null members within W.

Theorem 2 *Let f be a scoring function, $\tilde{f}$ be the continuous weighted scoring function based on f, and P be a preference profile. Let W be a committee of size $k \in [m-1]$, $a \in W$, and θ be* *a weight vector associated to W such that $\theta_W(a) = \theta_j$ for some $j \in [k]$. If a is a null member of W under f with respect to P, then it holds that*

$$\tilde{f}(P, W, \theta) = \tilde{f}\Big(P, (W \setminus \{a\}), \theta_{-j}\Big).$$

Proof. Let f be a scoring function, $\tilde{f}$ be the continuous weighted scoring function based on f, and P be a preference profile. Let $k \in [m-1], W \in 2^A_k$, and $a \in W$ be a null member of W under f with respect to P. Let θ be any weight vector associated to W such that the weight corresponding to a is θ_j, for some $j \in [k]$. Consider $W^1 \subset \cdots \subset W^j \subset \cdots \subset W^k$ as a θ-decomposition of W. Since a is a null member of W, it holds that $f(W^j) = f(W^j \setminus \{a\}) = f(W^{j-1})$. The continuous weighted score of W is then,

$$\begin{aligned}
\tilde{f}(P, W, \theta) &= \sum_{t=1}^{k}(\theta_t - \theta_{t+1})f(P, W^t) \text{ with } \theta_{k+1} = 0 \\
&= \sum_{t=1}^{j-1}(\theta_t - \theta_{t+1})f(W^t) + (\theta_j - \theta_{j+1})f(W^j) + \sum_{t=j+1}^{k}(\theta_t - \theta_{t+1})f(W^t) \\
&= \sum_{t=1}^{j-1}(\theta_t - \theta_{t+1})f(W^t) + \theta_j f(W^{j-1}) - \theta_{j+1} f(W^{j-1}) + \sum_{t=j+1}^{k}(\theta_t - \theta_{t+1})f(W^t \setminus \{a\}) \\
&= (\theta_1 - \theta_2)f(W^1) + \cdots + (\theta_{j-1} - \theta_{j+1})f(W^{j-1}) + \sum_{t=j+1}^{k}(\theta_t - \theta_{t+1})f(W^t \setminus \{a\}) \\
&= \tilde{f}\big(P, W \setminus \{a\}, \theta_{-j}\big).
\end{aligned}$$

Hence, the continuous weighted score $\tilde{f}(P, W, \theta)$ does not depend on a. ■

Thus, the independence of zero-weight members and the independence of null members, as articulated in Theorem 2, allow to highlight the fact that the continuous weighted scoring function based on f somehow establishes a form of fairness between the excellence of the members of a committee and their compatibility skills. Recall that by definition, the weight vector θ assigned to a committee is a vector of decreasing non-negative real numbers, inducing an order over the committee members which is modeled by the θ-decomposition, and the continuous weighted score of a committee depends on this order. However, one could naturally claim that the weighted score given to any committee W should not depend on the order of the components of the vector θ. In other words, for any permutation σ of the set $[k]$, the following assertion should hold: $\tilde{f}(W, \theta) = \tilde{f}\Big(W, (\theta_{\sigma(1)}, \ldots, \theta_{\sigma(k)})\Big)$. The response to this claim is that this requirement only holds for a particular class of (unweighted) scoring functions known as *candidate-wise* scoring functions, a class that has been introduced and thoroughly examined by Kilgour (2010) and Kilgour and Marshall (2012).[4]

Definition 8 *A scoring function f is a candidate-wise scoring function if for any committee W*

[4] The candidate-wise scoring functions include the (weakly) separable scoring functions defined by Elkind et al. (2017) and studied by Faliszewski et al. (2018) and Faliszewski et al. (2019), among others.

of size $k \in [m-1]$, it holds that $f(W) = \sum_{a \in W} f(\{a\})$.

Alternatively stated, a scoring function is said to be candidate-wise if the score it assigns to any committee is equivalent to the sum of the individual scores assigned to each member of that committee, with each member treated as a 1-size committee.

Theorem 3 *Let f be a scoring function and $\tilde{f}$ be the continuous weighted scoring function based on f. The weighted score $\tilde{f}(W,\theta)$ of any couple (W,θ) does not depend on the order within θ if and only if the scoring function f is a candidate-wise function.*

Proof. Let f be a scoring function and assume that f is a candidate-wise scoring function. Let $W = \{a_1, \dots, a_k\}$ be any committee of size $k \in [m-1]$ weighted by any weight vector θ such that a θ-decomposition of W is $W^1 \subset \cdots \subset W^k$, with $W^j = \{a_1, \dots, a_j\}$. It holds that the continuous weighted score of W is

$$\begin{aligned}
\tilde{f}(W,\theta) &= \sum_{j=1}^{k-1}(\theta_j - \theta_{j+1}) f(W^j) + \theta_k f(W) \\
&= \sum_{j=1}^{k-1}\Big[(\theta_j - \theta_{j+1}) \sum_{t=1}^{j} f(\{a_t\})\Big] + \theta_k \sum_{j=1}^{k} f(\{a_t\}) \\
&= \theta_1 f(\{a_1\}) + \cdots + \theta_k f(\{a_k\}) \\
&= \sum_{a \in W} \theta_W(a) f(\{a\}).
\end{aligned}$$

This proves that the continuous weighted score $\tilde{f}(W,\theta)$ does not depend on the ranking within the weight vector θ. Conversely, assume that for any committee W of size $k \in [m-1]$, weighted by any weight vector θ, the continuous weighted score $\tilde{f}(W,\theta)$ does not depend on the order of the components of θ. Then, for any permutation σ of the set $[k]$, we have $\tilde{f}(W,\theta) = \tilde{f}\Big(W, (\theta_{\sigma(1)}, \dots, \theta_{\sigma(k)})\Big)$. However, any unordered vector $(\theta_{\sigma(1)}, \dots, \theta_{\sigma(k)})$ can be uniquely decomposed as

$$(\theta_{\sigma(1)}, \dots, \theta_{\sigma(k)}) = \theta_{\sigma(1)} e_1 + \cdots + \theta_{\sigma(k)} e_k, \tag{11}$$

where $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0 \dots, 0), \dots, e_k = (0, \dots, 1)$. Without loss of generality, let us write $W = \{a_1, \dots, a_k\}$. Thus, we have

$$\begin{aligned}
f(W) &= \tilde{f}(W, I^k) \text{ by definition} \\
&= \sum_{j=1}^{k} \tilde{f}(W, e_j) \text{ by linearity, since } I^k = \sum_{j=1}^{k} e_k \\
&= \sum_{j=1}^{k} \tilde{f}(\{a_j\}, I^1) \text{ by independence of zero-weight members} \\
&= \sum_{a \in W} f(\{a\}).
\end{aligned}$$

This proves that f is a candidate-wise scoring function. ■

4 Conclusion

The main objective of this paper was to introduce a framework for integrating a weighting system into the committee selection process under a scoring rule. This becomes particularly pertinent when committee selection involves a compatibility criterion. Here, we proposed a method to amalgamate the individual excellence of committee members with the degree of compatibility, preserving several essential properties. However, we focused in this paper on the main properties that the derived weighted scoring function should guarantee when a weighting system is embedded. We contend that the next crucial step is to study the properties of the weighted scoring rules defined through a weighted scoring function in general and, in particular, the properties that could be used to characterize the weighted scoring rule defined through the continuous weighted scoring function.

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