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CRESE
30, avenue de l'Observatoire
25009 Besançon
France
http://crese.univ-fcomte.fr/

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Cooperative games with diversity constraints*

Sylvain Béal[a,*], Marc Deschamps[a], Mostapha Diss[a,c], Rodrigue Tido Takeng[b]

[a] *Université de Franche-Comté, CRESE, F-25000 Besançon, France*
[b] *Université de Caen Normandie, CREM, UMR 6211, F-14000 Caen, France*
[c] *Africa Institute for Research in Economics and Social Sciences (AIRESS), University Mohamed VI Polytechnic, Rabat, Morocco.*

Abstract

A cooperative game with diversity constraints is given by a cooperative game, a coalition structure which partitions the set of players into communities, and a vector of integers specifying, for each community, the minimal number of its members that a coalition must possess to be considered as diverse. We provide axioms for a value on the class of such cooperative games with diversity constraints. Some combinations of axioms characterize two values inspired by the Shapley value (Shapley, 1953) and the Owen value (Owen, 1977) for games with a coalition structure. More specifically, the Diversity Owen value is characterized as the Owen value of the diversity-restricted game with a coalition structure, where the diversity-restricted game assigns a null worth to a coalition if it does not meet the diversity requirements or its original worth otherwise. Similarly, the Diversity Shapley value is characterized as the Shapley value of the diversity-restricted game (without coalition structure). Some of our axiomatic characterizations can be adapted to the class of simple games by replacing the Additivity axiom by the Transfer axiom (Dubey, 1975).

Keywords: Cooperative game, diversity constraints, axiomatic characterization, Owen Value, Shapley value.

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*Corresponding author

Email addresses: sylvain.beal@univ-fcomte.fr (Sylvain Béal), marc.deschamps@univ-fcomte.fr (Marc Deschamps), mostapha.diss@univ-fcomte.fr (Mostapha Diss), rodriguetido@yahoo.fr (Rodrigue Tido Takeng)

URL: https://sites.google.com/site/bealpage/ (Sylvain Béal), https://sites.google.com/site/marcdeschampspage/home (Marc Deschamps), https://sites.google.com/site/mostaphadiss/ (Mostapha Diss), https://sites.google.com/view/rodriguetido/home (Rodrigue Tido Takeng)

1. Introduction

In recent years, inclusion and diversity has been underline as important criteria in numerous aspects of political and economic life (e.g. Page, 2008). It takes many forms: increased participation of women in company management committees, gender parity on certain electoral lists of candidates (this is the case for instance in some elections in Belgium, Ecuador, and France to name few countries) or requirement for a minimum percentage of each gender to form a committee (40% for recruiting committees at the university level in France), representation of minorities (according to Haughton et al., 2023, at least 44 states or state-like entities impose minority representation within their parliaments), representation of political parties (standing parliamentary committees in France must include at least one member from each political group), different districts of a city in the context of a municipal participatory budgeting campaign, diversified selection of movies from airlines and digital platforms, etc.

In the academic world, diversity has already been taken into account in theoretical studies. In social choice theory, it is analyzed in the context of multiwinner elections by Bredereck et al. (2018), Celis et al. (2018), Aziz (2019), Kagita et al. (2021), Relia (2021), and Ianovski (2022). Diversity is also examined in the context of matching problems (see Ehlers et al., 2014; Echenique and Yenmez, 2015; Benabbou et al., 2020).

The objective of this article is to introduce diversity considerations into the classical framework of cooperative games with transferable utility (simply games henceforth). To the best of our knowledge, we provide the first study accomplishing this. Cooperative games model situations in which a finite set of players can obtain certain payoffs by cooperation and have been used extensively to analyze many applications, including the a priori measure of the power of each player in a collective decision-making process modelled as a simple game (see Barr and Passarelli, 2009, among others). A game with diversity constraints is described by a classical game, which reflects the economic possibilities that the various coalitions of players can achieve by cooperating, a partition of the player set into groups that we interpret as the different communities in the participating population and a vector specifying, for each community, an integer or quota representing the number of targeted members for this community that a coalition must contain in order to be considered diverse. Our model extends both classical games and games with coalition structures. Classically, a value for this class of games specifies, for each game and each player, a payoff that captures the importance of that player in the game.

A reader familiar with the literature on games with a coalition structure will find some similarities with our model in the sense that both approaches rely on a partition of the set of players. At the level of the models, the similarities end there as this partition is used in a very different way to ours, whether in the approach of Aumann and Dreze (1974), where the cooperation between players is limited within communities, or that of Owen (1977), where communities are considered as bargaining units without preventing cooperation possibilities. At the level of the results, as we will explain later, however, the values that we characterize show links with this literature.

We adopt the axiomatic approach and invoke several axioms in order to design a value for games with diversity constraints. Some axioms are new in order to highlight the influence of the diversity constraints on the allocation process. Some others are inspired by the literature on the Shapley value (Shapley, 1953) and the Owen value (Owen, 1977), which are probably the most well-studied values for classical games and for games with coalition structures, respectively. A key axiom is the axiom of *Independence from non-diverse coalitions*, which imposes that the payoff allocation only depends on the susbset of coalitions that meet the diversity requirements. This axiom clearly makes sense in the applications that we have in mind. For instance, in the case of certain participatory budgeting campaign, subset of projects that did not include at least one project in each district would not be eligible. Another axiom which emphasize the importance of the diversity requirement is the axiom of *Equality through diversity*, which imposes equal total payoff for each community in the population. Again, this is relevant in the context of participatory budgeting, where many municipalities (for instance in Wroclaw, Poland) allocates the same budget to each district. We also invoke four other axioms that adapt classical axioms to the class of games with diversity constraints. *Additivity* requires that the characterized value is an additive function. *Efficiency* imposes that the worth of the grand coalition is fully distributed to the players. *Null player out for preserving-diversity games* adapts the null player out axiom (Derks and Haller, 1999) and states that if a player is unproductive, all productive coalitions meet the diversity criteria and some coalition can achieve the diversity requirements without this player, then the payoff of the other players should remain the same when this player leaves the game. Finally, *Intra-coalitional equal treatment of necessary players* impose that two necessary players (a player is necessary if a coalition not containing this player is unproductive) belonging to the same community obtain the same payoff. We prove that the combination of these six axioms singles out a unique value for games with diversity constraints. It turns out that this value coincides with the Owen value of the so-called diversity-restricted game with coalition structure in which the characteristic function assigns a null worth to coalitions that do not meet the diversity standards and its original worth to the other coalitions. For that reason, we call this value the Diversity Owen value.

Dropping Equality through diversity in the previous result and strengthening Intra-coalitional equal treatment of necessary players by the axiom of *Equal treatment of necessary players*, which imposes equal payoff for two necessary players even if they belong to distinct communities, yields a characterization of the Diversity Shapley value, which assigns to each game with diversity constraints the Shapley value of its diversity-restricted game (without coalition structure). Furthermore, two types of variation are proposed for these characterisations. The first variation is inspired by the characterizations of the Shapley value by Young (1985) and Chun (1989) and that of the Owen value by Khmelnitskaya and Yanovskaya (2007).[1] It replaces the axioms of *Additivity* and *Null player out for preserving-diversity games* with *Coalitional strategic equivalence for diverse game*, *Marginality for diverse games*, or *Strong monotonicity for diverse games*. All three axioms involve diverse games, i.e., games with diversity constraints in which each productive coalition meets the diversity criteria. *Coalitional strategic equivalence for diverse game* requires invariance to a player's payoff if a diverse game in which she is null is added to any game. *Marginality for diverse games* imposes equal payoffs for a player in two diverse games in which she has identical marginal contributions. *Strong monotonicity for diverse games* states that if the marginal contributions of a player are weakly greater in a first diverse game than in a second diverse game, then she does not obtain less payoff in the first game than in the second game. The resulting characterization of the Diversity Owen value via Coalitional strategic equivalence for diverse game can be seen as an adaptation to our framework of the characterization of the Owen value by Hu (2021, Theorem 3.2). The second variation demonstrates that our two characterizations can be applied to the class of simple games with diversity constraints by substituting the Additivity axiom with the Transfer axiom suggested by Dubey (1975). This latter result is relevant since many natural applications of games with diversity constraints are linked to voting procedures modelled by simple games.

We would like to distinguish our approach in terms of diversity from games featuring complementarities. In multi-sided assignment games (Quint, 1991) or multi-glove games (Moretti and Norde, 2021), the players are grouped into categories according to the type of resource they own, and a coalition is productive if and only if it contains at least one player of each category. Hence, complementarities have a direct effect on the characteristic function while diversity has not, a priori, since coalitions that do not meet the diversity requirements can still be productive. It is true that our diversity-restricted game acts like the above constraints of complementarities, but it is also possible to follow another road and to design values that take non-diverse productive coalitions into account.

The rest of this article is organized as follows. In Section 2, we recall basic notions on games and games with coalition structures. In Section 3, we present the class of games with diversity constraints. In Section 4, we introduce axioms for games with diversity constraints. In Section 5, we characterize two allocation rules for games with diversity constraints. In Section 6, we adapt our characterizations to the class of simple games with diversity constraints. Section 7 concludes the paper.

[1] More characterizations of the Shapley value can be found in Hart and Mas-Colell (1989) and Casajus (2011), among others, whereas alternative characterizations of the Owen value are due to Hart and Kurz (1983) and Casajus (2010), among others.

2. Cooperative games

2.1. Cooperative games with transferable utility

A cooperative game with transferable utility (simply a game) is a pair (N, v) such that, for each coalition $S \subseteq N$, $v(S) \in \mathbb{R}$ is the worth of coalition S, i.e., the best result that the players in S can achieve by cooperating without the help of the other players, and $v(\emptyset) = 0$ by convention. Denote by G the set of all games in which the player set is finite. Players $i, j \in N$ are symmetric in the game (N, v) if for each $S \subseteq N \backslash \{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$. Player $i \in N$ is null in the game (N, v) if for each $S \subseteq N \backslash \{i\}$, $v(S \cup \{i\}) = v(S)$. Player $i \in N$ is necessary in the game (N, v) if for each $S \subseteq N \backslash \{i\}$, $v(S) = 0$. The sum of two games (N, v) and (N, w) is the game $(N, v + w)$ such that, for each $S \subseteq N$, $(v + w)(S) = v(S) + w(S)$. For any $S \subseteq N$, the subgame of (N, v) induced by S is the game $(S, v_{|S})$, where, for each $T \subseteq S$, $v_{|S}(T) = v(T)$. Since Shapley (1953), it is known that any function v can be uniquely expressed as:

$$v = \sum_{S \subseteq N, S \neq \emptyset} \Delta_v(S) u_S,$$

where (N, u_S) is the unanimity game on N induced by coalition S given by $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise, and $\Delta_v(S)$ is called the Harsanyi dividend (Harsanyi, 1959) of S and defined recursively by setting $\Delta_v(\emptyset) = 0$ and

$$\Delta_v(S) = v(S) - \sum_{T \subsetneq S} \Delta_v(T).$$

A value on G is a function f which assigns to each game $(N, v) \in G$ a payoff vector $f(N, v)$ such that $f_i(N, v) \in \mathbb{R}$ is the payoff obtained by player $i \in N$ for her participation in game (N, v). The Shapley value Sh (Shapley, 1953) is the value on G which shares the dividend of each coalition equally among its members:

$$Sh_i(N, v) = \sum_{S \subseteq N, S \ni i} \frac{\Delta_v(S)}{|S|}.$$

2.2. Games with a coalition structure

A game with a coalition structure is a triple $(N, v, \mathcal{B})$, where (N, v) is a game and $(N, \mathcal{B})$ is a coalition structure, i.e., $\mathcal{B} = \{B_1, \ldots, B_m\}$ is a partition of N. Denote by CSG the set of all games with a coalition structure in which the player set is finite. Let $M = \{1, \ldots, m\}$ and denote by $\mathcal{B}(i)$ is coalition containing player i in $\mathcal{B}$. The quotient game $\big(M, (v)_{\mathcal{B}}\big)$ is a game where $(v)_{\mathcal{B}}(Q) = v\big(\bigcup_{k \in Q} B_k\big)$ for all $Q \subseteq M$. In order to be more specific, the coalitions in $\mathcal{B}$ are called components. For any $S \subseteq N$, the coalition sub-structure of $(N, \mathcal{B})$ induced by S is the coalition structure $(S, \mathcal{B}_{|S})$ where $\mathcal{B}_{|S} = \{B_1 \cap S, \ldots, B_m \cap S\}$. For any $S \subseteq N$, the subgame with a coalition structure of $(N, v, \mathcal{B})$ induced by S is the game with a coalition structure $(S, v_{|S}, \mathcal{B}_{|S})$.

As for a value on G, a value on CSG is a function f which assigns to each game $(N,v,\mathcal{B}) \in CSG$ and each player $i \in N$ a payoff $f_i(N,v,\mathcal{B})$ reflecting the participation of i to $(N,v,\mathcal{B})$. The most well-known value for games with a coalition structure is the Owen value Ow (Owen, 1977), which can be defined as follows:

$$Ow_i(N,v,\mathcal{B}) = \sum_{S\subseteq N, S\ni i} \frac{\Delta_v(S)}{|\mathcal{B}(i)\cap S| \cdot |\{B_k \in \mathcal{B} : S\cap B_k \neq \varnothing\}|}. \tag{1}$$

Hence, the dividend of a coalition S is shared according to a two-step procedure. First, the dividend is split in equal shares among the components in $\mathcal{B}$ intersecting S. Second, the share obtained by a component of the coalition structure is split in equal shares among the component members who are in S. The Owen value generalizes the Shapley value on CSG since $Ow(N,v,\mathcal{B}) = Sh(N,v)$ if either $\mathcal{B} = \{N\}$ or $\mathcal{B} = \{\{i\} : i \in N\}$.

3. Games with diversity constraints

In this section, the diversity of coalition members is taken into account. As for a game with a coalition structure, we rely on a partition $\mathcal{B}$. Each component $B_k \in \mathcal{B}$ is considered as a **community** and we require a coalition to include at least $d_k \in \{1,\dots,|B_k|\}$ members of community B_k to be considered diverse. Hence, a coalition structure with diversity constraint is a triple $(N,\mathcal{B},d)$ where $d = (d_1,\dots,d_m)$. Formally, a coalition $S \subseteq N$ is a **diverse coalition** in $(N,\mathcal{B},d)$ if $|S\cap B_k| \geq d_k$ for all $k \in M$. We denote by $D(N,\mathcal{B},d)$ the set of diverse coalitions in $(N,\mathcal{B},d)$. Remark that if $S \in D(N,\mathcal{B},d)$, then $S\cap B_k \neq \varnothing$ for each $k \in M$ and if $T \supseteq S$, then $T \in D(N,\mathcal{B},d)$.

A game with diversity constraints $(N,v,\mathcal{B},d)$ is called **diverse** if $v(S) \neq 0 \Longrightarrow S \in D(N,\mathcal{B},d)$, i.e., all coalitions enjoying a non-null worth are diverse. A game $(N,v,\mathcal{B},d)$ is called ***i*-out diverse**, $i \in B_k$, if $(N,v,\mathcal{B},d)$ is diverse and $|B_k| - d_k \geq 1$. This last condition means that the presence of player i is not necessary for a coalition to be diverse or, equivalently, diverse coalitions exist in the coalition structure with diversity constraints resulting from the removal of i. A **game with diversity constraints** is a four-tuple $(N,v,\mathcal{B},d)$ where $(N,v,\mathcal{B})$ is a game with a coalition structure and $d = (d_k)_{k\in M}$. We denote by GD the set of all games with diversity constraints with a finite player set. For any $S \subseteq N$, the subgame with diversity constraints of $(N,v,\mathcal{B},d)$ induced by S is the game with diversity constraints $(S,v_{|S},\mathcal{B}_{|S},d)$. Remark that d is identical in both $(N,v,\mathcal{B},d)$ and $(S,v_{|S},\mathcal{B}_{|S},d)$. Hence, if diverse coalitions exist in $(N,v,\mathcal{B},d)$, there may exist none in $(S,v_{|S},\mathcal{B}_{|S},d)$.

We illustrate games with diversity constraints in the context of three examples.

Example 1. ***(Constitutional reforms in France)*** The French Parliament of the Fifth Republic is bicameral, comprising the National Assembly and the Senate. The National Assembly is composed of 577 deputies and the Senate is composed of 381 senators. The constitutional reforms initiated by the President of the Republic can be done without using a referendum if:

- Firstly, on the National Assembly at least 50% of the deputies vote for the reform and also on the Senate at least 50% of the senators vote for the reform.
- Secondly, on the Parliament meeting in Congress 60% of the parliamentarians (deputies and senators) vote for the reforms.

This voting system can be apprehended by a game with diversity constraint $(N, v, \mathcal{B}, d)$ where (N, v) is the simple game such that N is the set of all 958 members of the Parliament, $v(S) = 1$ if $|S| \geq 575$ and $v(S) = 0$ otherwise, $\mathcal{B} = (B_1, B_2)$, where B_1 is the set of 577 deputies and B_2 the set of 381 senators and $d = (d_1, d_2)$ with $d_1 = 289$ and $d_2 = 191$. In words, the aforementioned first condition is reflected in the 50% quota required for each of the two communities, one corresponding to the National Assembly, the other corresponding to the Senate. □

Example 2. ***(Participatory budgeting with districts)*** Participatory budgeting refers to a form of public consultation in which residents of a city decide how to spend a part of the municipal budget. It can be described as a tuple $(N, K, \mathcal{B}, r, c, sc)$, where $N = \{1, \ldots, n\}$ is a set of projects, $K = \{1, \ldots, k\}$ is a set of voters/residents, $\mathcal{B}$ is a partition of the set of projects into m groups, one for each of the m municipality districts, $r \in \mathbb{R}_+$ is the available budget, $c : N \longrightarrow \mathbb{R}_+$ specifies the cost $c(i) \in \mathbb{R}_+$ of each project $i \in N$ and sc is a profile of score functions such that, for each voter $j \in K$ and each project $i \in N$, j assigns the score $sc_j(i) \in \mathbb{R}_+$ to i (a point system is often used to reflect the preferences of the voters). In this context, the objective is to select of subset of feasible projects, i.e., a set $S \subseteq N$ such that $\sum_{i \in S} c(i) \leq r$. For each subset of projects $S \subseteq N$, the score of S is $sc(S) = \sum_{j \in K} \sum_{i \in S} sc_j(i)$, i.e., the total score given by the voters to the projects in S. Inspired by Faliszewski et al. (2018), we associate this problem to a cooperative game (N, v) where N is the set of projects and, for each coalition of projects $S \subseteq N$,

$$v(S) = \max_{T \subseteq S : \sum_{i \in T} c(i) \leq r} sc(T).$$

In words, $v(S)$ is the maximal score achieved by a feasible subset of projects among S. In the close context of a multi-winner election, Faliszewski et al. (2018) rely on a value (the Banzhaf value) in order to assess the importance of each project in the game (N, v). However, the approach presented so far does not fully capture the system used by some municipalities, where the highest-ranked project from each district must be included in the subset of selected projects. This is the case, for instance in Brest, France.[2] A game with

[2] https://jeparticipe.brest.fr/pages/le-reglement-du-budget-participatif

diversity constraints $(N, v, \mathcal{B}, d)$ in which $d_q = 1$ for each district $q \in \{1, \ldots, m\}$ better reflects such a situation. □

Example 3. ***(United Nations Security Council)*** The United Nations Security Council is composed of five permanent members — China, France, Russia, United Kingdom and United States of America — and ten non-permanent members elected by the General Assembly for a two-year term — currently Algeria, Ecuador, Guyana, Japan, Malta, Mozambique, Republic of Korea, Sierra Leone, Slovenia, Switzerland. Under Article 27 of the UN Charter, Security Council decisions on all substantive matters require the affirmative votes of three-fifths (i.e., nine) of the members. A negative vote or a "veto" by a permanent member prevents adoption of a proposal, even if it has received the required votes. As an alternative of the usual simple game used to model this voting system (see Shapley and Shubik, 1954, for instance), we can define the game with diversity constraint $(N, v, \mathcal{B}, d)$ where

- N is the set of fifteen members of the United Nations Security Council;
- $v(S) = 1$ if $|S| \geq 9$ and $v(S) = 0$ otherwise;
- $\mathcal{B} = (B_1, B_2)$, where B_1 contains the five permanent members and B_2 the ten non-permanent members;
- $d = (d_1, d_2)$ with $d_1 = 5$ and $d_2 = 4$.

In words, the underlying cooperative game only includes the requirement that 9 countries are needed to adopt a proposal but not the veto power of the permanent members. This veto power is rather reflected by the organization of the countries into communities: while only four of the ten members of the community of non-permanent members is required, all five members of the community of permanent members are needed. □

4. Axioms

We list below some axioms for an arbitrary value f on GD. The first five axioms are classical and do not rely on the diversity constraints. The axiom of Efficiency is stated on GD and imposes that the sum of all distributed payoffs coincides with the worth of the grand coalition in each TU-game with diversity constraints.

Efficiency (E). For each $(N, v, \mathcal{B}, d) \in GD$, $\sum_{i \in N} f_i(N, v, \mathcal{B}, d) = v(N)$.

Additivity (w.r.t the characteristic function) requires that the payoffs distributed in the sum of two games are equal to the sum of the payoffs distributed in the two games.

Additivity (A). For each $(N, v, \mathcal{B}, d), (N, w, \mathcal{B}, d) \in GD$, $f(N, v + w, \mathcal{B}, d) = f(N, v, \mathcal{B}, d) + f(N, w, \mathcal{B}, d)$.

The third axiom, invoked by Owen (1977) for games with a coalition structure states that if two players are symmetric in the associated TU-game and belong to the same component in the coalition structure, then they get the same payoff.

Intra-coalitional symmetry (ICS). For any $(N, v, \mathcal{B}, d) \in GD$, if $i, j \in B_k$ and i and j are symmetric in (N, v), then $f_i(N, v, \mathcal{B}, d) = f_j(N, v, \mathcal{B}, d)$.

Intra-coalitional symmetry weakens the classical axiom of Symmetry, which requires that any two symmetric players get the same payoff.

Symmetry (S). For any $(N, v, \mathcal{B}, d) \in GD$, if i and j are symmetric in (N, v), then $f_i(N, v, \mathcal{B}, d) = f_j(N, v, \mathcal{B}, d)$.

Owen (1977) also invokes the following coalitional variant of symmetry for games with a coalition structure, which coincides with the requirement of the Symmetry axiom in the quotient game.

Coalitional symmetry (CS). For any $(N, v, \mathcal{B}, d) \in GD$, if two players $k, q \in M$ are symmetric in $(M, (v)_{\mathcal{B}})$, then

$$\sum_{i \in B_k} f_i(N, v, \mathcal{B}, d) = \sum_{i \in B_q} f_i(N, v, \mathcal{B}, d).$$

The next axiom is an adaptation of the Equal treatment of necessary players recently introduced by Béal and Navarro (2020) for the class of classical games where they show that this axiom weakens the classical Symmetry.

Equal treatment of necessary players (ETNP). For each $(N, v, \mathcal{B}, d) \in GD$ and each $i, j \in N$, if i and j are necessary in (N, v) then, $f_i(N, v, \mathcal{B}, d) = f_j(N, v, \mathcal{B}, d)$.

This axiom states that all players that are necessary to produce worth should have the same payoff. By using the spirit of Béal and Navarro (2020), **ETNP** and **ICS** can be weakened into the following axiom.

Intra-coalitional equal treatment of necessary players (ICETNP). For each $(N, v, \mathcal{B}, d) \in GD$ and each $i, j \in B_h \in \mathcal{B}$, if i and j are necessary in (N, v) then, $f_i(N, v, \mathcal{B}, d) = f_j(N, v, \mathcal{B}, d)$.

This axiom states that all players within a component necessary to produce worth should have the same payoff.

The next six axioms are new and describe how the diversity constraint can affect the allocation process. If diversity is viewed as a requirement for cooperation, then only the diverse coalitions should matter as imposed by the next axiom.

Independence from non-diverse coalitions (INDC). For each $(N, v, \mathcal{B}, d), (N, w, \mathcal{B}, d)$ such that $v(S) = w(S)$ for all $S \in D(N, \mathcal{B}, d)$, $f(N, v, \mathcal{B}, d) = f(N, w, \mathcal{B}, d)$.

Independence from non-diverse coalitions is similar to the axiom of Independence of irrelevant coalitions invoked by van den Brink et al. (2011) for cooperative games on union closed systems. As above, any community can be considered as equality important in order to meet the diversity constraints. As a consequence, it makes sense to distribute an equal total payoff to each such component, as required by the next axiom.

Equality through diversity (ED). For each $(N, v, \mathcal{B}, d) \in GD$ and each pair $k, q \in M$, $\sum_{i \in B_k} f_i(N, v, \mathcal{B}, d) = \sum_{i \in B_q} f_i(N, v, \mathcal{B}, d)$.

As mentioned in the introduction, this echoes the equal budget allocated by some municipalities allocate to each of their districts in the context of participatory budgeting.

The next axiom takes place in a i-out diverse game in which i is further a null player in the associated game. We can argue that the presence of player i is inconsequential for two reasons. First, her presence is not necessary to achieve diversity as her community contains more members than the required quota. Second, her presence is not necessary to achieve any productivity since she is a null player. In this case, the next axiom imposes that removing such a useless player does not alter the payoffs received by the remaining players.

Null player out for preserving-diversity games (NPOPD). If $(N, v, \mathcal{B}, d)$ is i-out diverse and player i is null in (N, v), then $f_j(N, v, \mathcal{B}, d) = f_j(N \backslash \{i\}, v_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}, d)$ for each $j \in N \backslash \{i\}$.

This axiom is a variant of the Null player out axiom (Derks and Haller, 1999) for classical games, which requires that the removal of any null player from a game does not alter the payoffs of the other players.

Next, **A** and **NPOPD** can be combined into the following new axiom.

Coalitional strategic equivalence for diverse game (CSEDG). If $(N, v, \mathcal{B}, d) \in GD$ is diverse and player i is null in (N, v), then for any game $(N, w, \mathcal{B}, d) \in GD$, $f_i(N, v+w, \mathcal{B}, d) = f_i(N, w, \mathcal{B}, d)$.

This axiom states that the payoff of a player in a game with diversity constraints remains the same in another game with diversity constraints if the latter game is obtained from the former by adding a diverse game in which the considered player is null. This axiom is an adaptation of the Coalitional strategic equivalence proposed by Chun (1989) for classical games and by Hu (2021) for games with coalition structures.

The last two axioms are the adaptations of the axioms introduced by Young (1985) to characterize the Shapley value.

Marginality for diverse games (MDG). For any diverse games $(N, v, \mathcal{B}, d), (N, w, \mathcal{B}, d) \in GD$, and any $i \in N$ such that $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \backslash \{i\}$, we have $f_i(N, v, \mathcal{B}, d) = f_i(N, w, \mathcal{B}, d)$.

This axiom states that if a player has the same marginal contributions in two diverse games then she receives the same payoff in these two games.

Strong monotonicity for diverse games (MoDG). For any pair of diverse games $(N, v, \mathcal{B}, d), (N, w, \mathcal{B}, d) \in GD$, and any $i \in N$ such that $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \backslash \{i\}$, we have $f_i(N, v, \mathcal{B}, d) \geq f_i(N, w, \mathcal{B}, d)$.

This last axiom states that if the marginal contributions of a player weakly increase from one diverse game to another diverse game, then her payoff weakly increases as well.

5. Characterizations of the Diversity Owen and Shapley values

In this section, we present four axiomatic characterizations that single out two values for games with diversity constraints inspired by the Shapley value for classical games and the Owen value for games with coalition structures. The first two results invoke the axiom of Additivity.

Proposition 1. *There is a unique value on GD satisfying Efficiency* **(E)**, *Additivity* **(A)**, *Intra-coalitional equal treatment of necessary players* **(ICETNP)**, *Null player out for preserving-diversity games* **(NPOPD)**, *Equality through diversity* **(ED)** *and Independence from non-diverse coalitions* **(INDC)**. *It is the* **Diversity Owen value** *DOw, such that, for each $(N, v, \mathcal{B}, d) \in GD$, $DOw(N, v, \mathcal{B}, d) = Ow(N, v^d, \mathcal{B})$, where $(N, v^d, \mathcal{B}) \in CSG$*

is the **diversity-restricted game** *with a coalition structure constructed from* $(N, v, \mathcal{B}, d)$ *such that, for each* $S \subseteq N$,

$$v^d(S) = \begin{cases} v(S) & \text{if } S \in D(N, \mathcal{B}, d), \\ 0 & \text{otherwise.} \end{cases}$$

The diversity-restricted game points out that only diverse coalitions are likely to influence the allocation process. This approach is similar to Myerson (1977), who constructs a graph-restricted classical game in which the role of the graph is incorporated into the graph-restricted characteristic function. More specifically, we construct a diversity-restricted game with a coalition structure in which the quotas are not an input anymore but are indirectly taken into account through the diversity-restricted characteristic function.

Before proving Proposition 1, we highlight several properties regarding the diversity constraints.

Lemma 1. *For any* $(N, v, \mathcal{B}, d) \in GD$, *it holds that:*

(i) For each $S \subseteq N$, $D(S, \mathcal{B}_{|S}, d) = \{T \subseteq S : T \in D(N, \mathcal{B}, d)\}$.

(ii) For each $S \subseteq N$, $(S, (v_{|S})^d, \mathcal{B}_{|S}, d) = (S, (v^d)_{|S}, \mathcal{B}_{|S}, d)$.

(iii) For each $k, q \in M$, k *and* q *are symmetric in* $(M, (v^d)_{\mathcal{B}})$.

(iv) For each $k \in M$ *and* $i, j \in B_k$, *if* i *and* j *are symmetric in* (N, v), *then* i *and* j *are symmetric in* (N, v^d).

(v) For each $S \notin D(N, \mathcal{B}, d)$, $\Delta_{v^d}(S) = 0$.

Proof. Consider any $(N, v, \mathcal{B}, d) \in GD$.

(i) For each $S \subseteq N$,

$$D(S, \mathcal{B}_{|S}, d) = \{T \subseteq S : |T \cap (B_k \cap S)| \geq d_k\} = \{T \subseteq S : |T \cap B_k| \geq d_k\} = \{T \subseteq S : T \in D(N, \mathcal{B}, d)\}.$$

(ii) Fix any $S \subseteq N$. For each $T \subseteq S$, from point *(i)*, $T \in D(S, \mathcal{B}_{|S}, d)$ if and only if $T \in D(S, \mathcal{B}_{|S}, d)$. If $T \in D(S, \mathcal{B}_{|S}, d)$, then

$$(v_{|S})^d(T) = v^d(T) = v(T) = v_{|S}(T) = (v^d)_{|S}(T).$$

If $T \notin D(S, \mathcal{B}_{|S}, d)$, the previous equation still holds except that the middle term is 0 instead of $v(T)$.

(iii) In the quotient game $(M, (v^d)_{\mathcal{B}})$ associated with $(N, v^d, \mathcal{B})$, if a coalition $Q \subseteq M$ is such that $Q \neq M$, then there is $q \in M \backslash Q$, which implies that $B_q \cap (\cup_{k \in Q} B_k) = \varnothing$. Hence, $\cup_{k \in Q} B_k \notin D(N, \mathcal{B}, d)$, so that

$$(v^d)_{\mathcal{B}}(Q) = (v^d)\big(\cup_{k \in Q} v(B_k)\big) = 0.$$

Therefore, all players in M are symmetric in $(M, (v^d)_{\mathcal{B}})$, or equivalently, all components in $\mathcal{B}$ are symmetric in $(N, v, \mathcal{B})$.

(iv) Consider any $B_k \in \mathcal{B}$ and any two players $i, j \in B_k$ who are symmetric in (N, v). Pick any $S \subseteq N \backslash \{i, j\}$. We distinguish two cases. If $S \cup \{i\} \notin D(N, \mathcal{B}, d)$, this means that there is $q \in M$ such that $|S \cap B_q| < d_q$. But then $S \cup \{j\} \notin D(N, \mathcal{B}, d)$ since $\{i, j\} \in B_k$. Thus, $v^d(S \cup \{i\}) = 0 = v^d(S \cup \{j\})$. If $S \cup \{i\} \in D(N, \mathcal{B}, d)$, obviously $S \cup \{j\} \in D(N, \mathcal{B}, d)$, which implies that

$$v^d(S \cup \{i\}) = v(S \cup \{i\}) = v(S \cup \{j\}) = v^d(S \cup \{j\}),$$

where the second equality comes from the fact that i and j are symmetric in (N, v).

(v) Consider any $S \notin D(N, \mathcal{B}, d)$. Clearly, $T \notin D(N, \mathcal{B}, d)$ for each $T \subseteq S$. This yields that $v^d(T) = 0$ for each $T \subseteq S$, from which we immediately get $\Delta_{v^d}(T) = 0$ for each $T \subseteq S$. ■

Proof. (Proposition 1) EXISTENCE. We show that DOw satisfying the six axioms.

E. Since Ow satisfies **E** on CSG and $v^d(N) = v(N)$ for each $(N, v, \mathcal{B}, d) \in DG$, DOw satisfies **E** as well.

A. For any two games $(N, v, \mathcal{B}, d), (N, w, \mathcal{B}, d) \in DG$, it holds that $(v+w)^d = v^d + w^d$ since the set of diverse coalitions does not depend on the characteristic function. Hence DOw inherits **A** on DG from the fact that Ow satisfies **A** on CSG.

ICETNP. It is known that Ow satisfies **ICS** on CSG. Combined with point *(iv)* in Lemma 1, we obtain that DOw satisfies **ICS** on GD as well. DOw satisfies **ICETNP** on GD since **ICETNP** weakens **ICS**.

NPOPD. Let $(N, v, \mathcal{B}, d)$ be an i-out diverse game with diversity constraints and suppose that i is a null player in (N, v), $i \in B_k$. Since $(N, v, \mathcal{B}, d)$ is diverse, $v^d = v$, and so i is null in (N, v^d). Recall that $|B_k| - d_k \geq 1$. Hence, if $S \subseteq N \backslash \{i\}$ is in $D(N, \mathcal{B}, d)$, then $S \in D(N \backslash \{i\}, \mathcal{B}_{|N \backslash \{i\}}, d)$. This implies that $(N \backslash \{i\}, v_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}, d)$ is a diverse game as well. Therefore, $DOw(N, v, \mathcal{B}, d) = Ow(N, v, \mathcal{B})$ and $DOw(N \backslash \{i\}, v_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}, d) = Ow(N \backslash \{i\}, v_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}})$. Now, consider any $j \in N \backslash \{i\}$. From definition (1), we have

$$\begin{aligned}
DOw_j(N, v, \mathcal{B}, d) &= Ow_j(N, v, \mathcal{B}) \\
&= \sum_{S \subseteq N, S \ni j} \frac{\Delta_v(S)}{|\mathcal{B}(j) \cap S| \cdot |\{B_k \in \mathcal{B} : S \cap B_k \neq \emptyset\}|} \\
&= \sum_{S \subseteq N \backslash \{i\}, S \ni j} \frac{\Delta_v(S)}{|\mathcal{B}(j) \cap S| \cdot |\{B_k \in \mathcal{B} : S \cap B_k \neq \emptyset\}|} \\
&= \sum_{S \subseteq N \backslash \{i\}, S \ni j} \frac{\Delta_v(S)}{|(\mathcal{B}(j) \backslash \{i\}) \cap S| \cdot |\{B_k \in \mathcal{B}_{|N \backslash \{i\}} : S \cap (B_k \backslash \{i\}) \neq \emptyset\}|} \\
&= Ow_j(N \backslash \{i\}, v_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}), \\
&= DOw_j(N \backslash \{i\}, v_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}, d),
\end{aligned}$$

where the third equality follows from the fact that $\Delta_S(v) = 0$ for each $S \subseteq N$ with $S \ni i$ since i is null in (N, v).

ED. By point *(iii)* of Lemma 1, we know that for each $k, q \in M$, B_k and B_q are symmetric components in $(N, v^d, \mathcal{B})$ (i.e., k and q are symmetric in the quotient game $(M, (v^d)_{\mathcal{B}})$). Since Ow satisfies the Coalitional symmetry axiom on CSG and $DOw(N, v, \mathcal{B}, d) = Ow(N, v^d, \mathcal{B})$, DOw satisfies **ED** on GD.

INDC. If two games $(N, v, \mathcal{B}, d)$ and $(N, w, \mathcal{B}, d)$ are such that $v(S) = w(S)$ for each $S \in D(N, \mathcal{B}, d)$, then we have $v^d = w^d$. Therefore, $DOw(N, v, \mathcal{B}, d) = Ow(N, v^d, \mathcal{B}) = Ow(N, w^d, \mathcal{B}) = DOw(N, w, \mathcal{B}, d)$.

Uniqueness. Let f be any value on GD meeting the six axioms. Consider any game $(N, v, \mathcal{B}, d) \in GD$. By **INDC**, we have $f(N, v, \mathcal{B}, d) = f(N, v^d, \mathcal{B}, d)$. By **A** and point *(v)* in Lemma 1, we also have

$$f(N, v^d, \mathcal{B}, d) = \sum_{S \subseteq N, S \neq \varnothing} f(N, \Delta_{v^d}(S) u_S, \mathcal{B}, d) = \sum_{S \in D(N, \mathcal{B}, d)} f(N, \Delta_{v^d}(S) u_S, \mathcal{B}, d).$$

As a consequence, it is enough to prove that $f(N, cu_S, \mathcal{B}, d)$ is uniquely determined for each $S \in D(N, \mathcal{B}, d)$ and each $c \in \mathbb{R}^*$. Remark that $(N, cu_S, \mathcal{B}, d)$ is diverse since $S \in D(N, \mathcal{B}, d)$. Furthermore, each player $i \in N \backslash S$ is null in (N, cu_S). The game $(N, cu_S, \mathcal{B}, d)$ is i-out diverse since $i \in N \backslash S$. For any $j \in N \backslash i$, we have

$$f_j(N, cu_S, \mathcal{B}, d) \overset{(NPOPD)}{=} f_j(N \backslash \{i\}, (cu_S)_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}, d).$$

Hence, for each $i \in N \backslash S$, from **E**, note also that

$$\sum_{j \in N} f_j(N, cu_S, \mathcal{B}, d) = cu_S(N) = c = cu_S(N \backslash \{i\}) = \sum_{j \in N \backslash \{i\}} f_j(N \backslash \{i\}, (cu_S)_{|N \backslash \{i\}}, \mathcal{B}_{|N \backslash \{i\}}, d).$$

This forces, for each $i \in N \backslash S$, that

$$f_i(N, cu_S, \mathcal{B}) = 0. \tag{2}$$

For each $i \in N \backslash S$, $i \in B_k$, and each $T \subseteq N \backslash (S \cup \{i\})$, consider the subgame $(N \backslash T, (cu_S)_{|N \backslash T}, \mathcal{B}_{|N \backslash T}, d)$. Since $S \in D(N, \mathcal{B}, d)$ and $S \subseteq N \backslash T$, we have $S \in D(N \backslash T, \mathcal{B}_{|N \backslash T}, d)$ by point *(i)* of Lemma 1. Hence, $|B_k \cap S| \geq d_k$, $S \subseteq N \backslash T$, and $i \in N \backslash T$ imply that

$$|B_k \cap (N \backslash T)| - d_k \geq |B_k \cap (N \backslash T)| - |S \cap B_k| \geq 1.$$

Thus, all conditions are met to apply **NPOPD** $(n - s)$ times successively from $(N, cu_S, \mathcal{B}, d)$ to $(S, (cu_S)_{|S}, \mathcal{B}_{|S}, d)$ to get that, for each $j \in S$,

$$f_j(N, cu_S, \mathcal{B}, d) = f_j(S, (cu_S)_{|S}, \mathcal{B}_{|S}, d). \tag{3}$$

Since $(S, (cu_S)_{|S}, \mathcal{B}_{|S}, d)$ is diverse, $(cu_S)^d = cu_S$ on S. As a consequence, combining point *(iii)* of Lemma 1 with **ED** and **E**, we obtain, for each $k \in M$, that

$$\sum_{j \in B_k \cap S} f_j(S, (cu_S)_{|S}, \mathcal{B}_{|S}, d) = c/m. \tag{4}$$

Moreover, since any two players $j, l \in B_k \cap S$ are necessary in $(S, (cu_S)_{|S})$, **ICETNP** and (4) yield that

$$f_j(S, (cu_S)_{|S}, \mathcal{B}_{|S}, d) = \frac{c}{m|B_k \cap S|}. \tag{5}$$

Combining (2), (3), and (5), we have proved that $f(N, cu_S, \mathcal{B}, d)$ is uniquely determined for each $S \in D(N, \mathcal{B}, d)$ and each $c \in \mathbb{R}^*$, which completes the proof. ■

Dropping Equality through diversity **(ED)** and strengthening Intra-coalitional equal treatment of necessary players **(ICETNP)** by Equal treatment of necessary players **(ETNP)** yields a characterization of the Diversity Shapley value, which assigns to each game with diversity constraints its Shapley value in the associated diversity game.

Proposition 2. *There is a unique value on GD satisfying Efficiency* **(E)**, *Additivity* **(A)**, *Equal treatment of necessary players* **(ETNP)**, *Null player out for preserving-diversity games* **(NPOPD)** *and Independence from non-diverse coalitions* **(INDC)**. *It is the Diversity Shapley value DSh, such that, for each* $(N, v, \mathcal{B}, d) \in GD$, $DSh(N, v, \mathcal{B}, d) = Sh(N, v^d)$.

Proof. EXISTENCE. By definition, DSh satisfies **INDC**. As in the proof of Proposition 1, DSh inherits the four other axioms from the fact that Sh satisfies their counterpart on G.

UNIQUENESS. Consider any value f on GD satisfying the five axioms. The first steps of the proof of Proposition 1 can be replicated until reaching the subgame with diversity constraints $f(S, (cu_S)_{|S}, \mathcal{B}_{|S}, d)$, $S \in D(N, \mathcal{B}, d)$, and $c \in \mathbb{R}^*$. Here, **ETNP** and **E** yield that $f_i(S, (cu_S)_{|S}, \mathcal{B}_{|S}, d) = c/|S|$ for each $i \in S$, completing the proof. ■

Note that, in the presence of the Efficiency, Additivity, and Null player out for preserving-diversity games can be weakened into Coalitional strategic equivalence for diverse game. Moreover, Coalitional strategic equivalence for diverse game and Marginality for diverse games do not imply each other. We obtain other axiomatizations of the Diversity Owen value which do not rely of the Additivity axiom.

Proposition 3. *The Diversity Owen value is the unique value on GD that satisfies Efficiency* **(E)**, *Intra-coalitional equal treatment of necessary players* **(ICETNP)**, *Equality through diversity* **(ED)**, *Independence from non-diverse coalitions* **(INDC)** *and Coalitional strategic equivalence for diverse game* **(CSEDG)** *or Marginality for diverse games* **(MDG)** *or Strong monotonicity for diverse games* **(MoDG)**.

Proof. (Proposition 3) Existence. We already proved that DOw satisfies **E**, **ICETNP**, **ED**, and **INDC**.
Regarding **MDG** or **MoDG**, if $(N, v, \mathcal{B}, d) \in GD$ is diverse then $v^d = v$. Thus, DOw also satisfies **MDG** or **MoDG** since the Owen value satisfies Marginality and Strong monotonicity for games with coalition structures.

Regarding **CSEDG**, let $(N, v, \mathcal{B}, d) \in GD$ be a diverse game and i a null player in (N, v). For any game $(N, w, \mathcal{B}, d) \in GD$, since DOw satisfies **A**, we have $DOw_i(N, v + w, \mathcal{B}, d) = DOw_i(N, w, \mathcal{B}, d) + DOw_i(N, v, \mathcal{B}, d)$. Since $(N, v, \mathcal{B}, d)$ is a diverse game and i is null in (N, v), i is also null in (N, v^d) and we have $DOw_i(N, v, \mathcal{B}, d) = 0$. Thus, $DOw_i(N, v + w, \mathcal{B}, d) = DOw_i(N, w, \mathcal{B}, d)$.

Uniqueness. Let f be a value on GD that satisfies **E**, **ICETNP**, **ED**, **INDC**, and **CSEDG** or **MDG** or **MoDG**. We show that $f(N, v, \mathcal{B}, d) = DOw(N, v, \mathcal{B}, d)$ for each $(N, v, \mathcal{B}, d) \in GD$. For any game $(N, v, \mathcal{B}, d) \in GD$, by **INDC**, we have $f(N, v, \mathcal{B}, d) = f(N, v^d, \mathcal{B}, d)$. Hence, it is enough to prove that $f(N, v^d, \mathcal{B}, d) = DOw(N, v^d, \mathcal{B}, d)$. We prove this by induction on the cardinality of the set

$$\mathcal{T}(v^d) := \{T \in D(N, \mathcal{B}, d) : \Delta_{v^d}(T) \neq 0\}$$

of coalitions with non-zero dividends in v^d.

By induction on the cardinality of $\mathcal{T}(v^d)$, let us show that $DOw = f$.
Induction basis (IB): For any game $(N, v, \mathcal{B}, d) \in GD$ such that $|\mathcal{T}(v^d)| = 0$, (N, v^d) is a null game. All players are symmetric in (N, v^d). By using **E**, **ICETNP** and **ED**, for any $i \in N$, we have

$$f_i(N, v^d, \mathcal{B}, d) = \frac{0}{m.|\mathcal{B}(i)|} = 0 = Ow_i(N, v^d, \mathcal{B}) = DOw(N, v^d, \mathcal{B}, d).$$

Induction hypothesis (IH): Suppose that $f(N, v^d, \mathcal{B}, d) = DOw(N, v^d, \mathcal{B}, d)$ for all $(N, v, \mathcal{B}, d) \in GD$ such that such that $|\mathcal{T}(v^d)| \leq \bar{t}$ with $\bar{t} \in \mathbb{N}$.
Induction step: Let $(N, v, \mathcal{B}, d) \in GD$ such that $|\mathcal{T}(v^d)| = \bar{t} + 1$. By setting

$$\mathbb{T}(v^d) := \{i \in N : i \in T, \text{ for all } T \in \mathcal{T}(v^d)\},$$

we distinguish the following two cases:

Case 1. We assume that $\mathbb{T}(v^d) = \varnothing$. Note that this rules out the case where $|B_h| = d_h$ for some component $B_h \in \mathcal{B}$ since we would have $B_h \subseteq S$ for each diverse coalition. Thus, assume that there are no $B_h \in \mathcal{B}$ such that $|B_h| = d_h$. Then, for any $i \in N$, there exists some $T_o \in \mathcal{T}(v^d)$ such that $i \notin T_0$. Note that, T_0 is diverse. We set $w = \sum_{S \in D(N,\mathcal{B},d) \setminus \{T_0\}} \Delta_{v^d}(S) u_S$. Obviously, $(N, w, \mathcal{B}, d)$ is a diverse game (as the sum of diverse games) and then $w^d = w$.

Hence, $v^d = w + \Delta_{v^d}(T_0)u_{T_0}$ and $|\mathcal{T}(w)| = \bar{t}$. Thus, $f_i(N, w, \mathcal{B}, d) \overset{\textit{(IH)}}{=} Ow_i(N, w, \mathcal{B})$. Moreover, the game $(N, \Delta_{v^d}(T_0)u_{T_0}, \mathcal{B}, d)$ is diverse (since T_0 is a diverse coalition) and player i is null in $(N, \Delta_{v^d}(T_0)u_{T_0})$. Each of the following two situations hold.

- By **CSEDG**, we have $f_i(N, v^d, \mathcal{B}, d) = f_i(N, w, \mathcal{B}, d) = Ow_i(N, w, \mathcal{B}) = Ow_i(N, v^d, \mathcal{B})$.
- Since $\Delta_{v^d}(K \cup \{i\}) = \Delta_w(K \cup \{i\})$ for all $K \subseteq N \backslash \{i\}$. By **MDG** (or **MoDG**), we have $f_i(N, v^d, \mathcal{B}, d) = f_i(N, w, \mathcal{B}, d) = Ow_i(N, w, \mathcal{B}) = Ow_i(N, v^d, \mathcal{B})$.

In the two situations, the equality $f(N, v^d, \mathcal{B}, d) = DOw(N, v^d, \mathcal{B}, d)$ holds, as desired.

Case 2. Now, we assume that $\mathbb{T}(v^d) \neq \varnothing$. For any $i \in N$, if $i \in N \backslash \mathbb{T}(v^d)$ then from *Case 1* we have the claim. Let $i \in N$ be a player such that $i \in \mathbb{T}(v^d)$. If $\mathbb{T}(v^d) = \{i\}$ then from *Case 1* and **E**, we have $f_i(N, v^d, \mathcal{B}, d) = Ow_i(N, v^d, \mathcal{B})$ since DOw and f satisfy **E**. Let us assume that $|\mathbb{T}(v^d)| > 1$. Every pair of players in $\mathbb{T}(v^d)$ are necessary in (N, v^d). If $\mathbb{T}(v^d) \subseteq \mathcal{B}(i)$ then by applying **E** and **ICETNP**, we have

$$
\begin{aligned}
f_i(N, v^d, \mathcal{B}, d) &= \frac{v^d(N) - \sum_{j \in N \backslash \mathbb{T}(v^d)} f_j(N, v^d, \mathcal{B}, d)}{|\mathbb{T}(v^d)|} \\
&= \frac{v^d(N) - \sum_{j \in N \backslash \mathbb{T}(v^d)} Ow_j(N, v^d, \mathcal{B})}{|\mathbb{T}(v^d)|} \\
&= Ow_i(N, v^d, \mathcal{B}),
\end{aligned}
$$

for each $i \in \mathbb{T}(v^d)$. If $\mathbb{T}(v^d) \nsubseteq \mathcal{B}(i)$ then we set $R_{\mathbb{T}(v^d)} = \{B_\ell \in \mathcal{B} : B_\ell \cap \mathbb{T}(v^d) \neq \varnothing\}$. By applying **E** and **ED**, we have

$$
\begin{aligned}
\sum_{k \in \mathcal{B}(i)} f_k(N, v^d, \mathcal{B}, d) &= \frac{\sum_{B_\ell \in R_{\mathbb{T}(v^d)}} \sum_{k \in B_\ell} f_k(N, v^d, \mathcal{B}, d)}{|R_{\mathbb{T}(v^d)}|} \\
&= \frac{v^d(N) - \sum_{B_\ell \in \mathcal{B} \backslash R_{\mathbb{T}(v^d)}} \sum_{k \in B_\ell} f_k(N, v^d, \mathcal{B}, d)}{|R_{\mathbb{T}(v^d)}|} \\
&= \frac{v^d(N) - \sum_{B_\ell \in \mathcal{B} \backslash R_{\mathbb{T}(v^d)}} \sum_{k \in B_\ell} Ow_k(N, v^d, \mathcal{B})}{|R_{\mathbb{T}(v^d)}|} \\
&= \sum_{k \in \mathcal{B}(i)} Ow_k(N, v^d, \mathcal{B}).
\end{aligned}
$$

Moreover, i and every $j \in \mathbb{T}(v^d) \cap \mathcal{B}(i)$ are necessary in (N, v^d). Applying **ICETNP** gives

$$\begin{aligned} f_i(N, v^d, \mathcal{B}, d) &= \frac{\sum_{k \in \mathbb{T}(v^d) \cap \mathcal{B}(i)} f_k(N, v^d, \mathcal{B}, d)}{|\mathbb{T}(v^d) \cap \mathcal{B}(i)|} \\ &= \frac{\sum_{k \in \mathcal{B}(i)} f_k(N, v^d, \mathcal{B}, d) - \sum_{k \in \mathcal{B}(i) \setminus \mathbb{T}(v^d)} f_k(N, v^d, \mathcal{B}, d)}{|\mathbb{T}(v^d) \cap \mathcal{B}(i)|} \\ &= \frac{\sum_{k \in \mathcal{B}(i)} Ow_k(N, v^d, \mathcal{B}) - \sum_{k \in \mathcal{B}(i) \setminus \mathbb{T}(v^d)} Ow_k(N, v^d, \mathcal{B})}{|\mathbb{T}(v^d) \cap \mathcal{B}(i)|} \\ &= Ow_i(N, v^d, \mathcal{B}), \end{aligned}$$

for each $i \in \mathbb{T}(v^d)$, which completes the proof. ■

The counterpart of Proposition 2 can be stated as well to obtain additional characterizations of the Diversity Shapley value.

Proposition 4. *The Diversity Shapley value is the unique value on GD that satisfies Efficiency* **(E)***, Equal treatment of necessary players* **(ETNP)***, Independence from non-diverse coalitions* **(INDC)** *and Coalitional strategic equivalence for diverse game* **(CSEDG)** *or Marginality for diverse games* **(MDG)** *or Strong monotonicity for diverse games* **(MoDG)**.

Since the proof follows the same steps as in the proof of Proposition 3, we do not detail it here. Through the following remark, we show that the characterizations in Propositions 1, 2, 3, and 4 are non-redundant.

Remark 1. *i) For the case of Propositions 1 and 3, we consider the following examples:*

- *The value f on GD defined by $f_i(N, v, \mathcal{B}, d) = \sum_{S \subseteq N : S \ni i} \frac{\Delta_v(S)}{m|\mathcal{B}(i) \cap S|}$ for each $(N, v, \mathcal{B}, d) \in GD$ and $i \in N$ satisfies* **E**, **A**, **ICETNP**, **MDG**, **MoDG**, **ED**, **CSEDG** *and* **NPOPD***; but does not satisfy* **INDC**.
- *The value f on GD defined by $f_i(N, v, \mathcal{B}, d) = Sh(N, v^d)$ for each $(N, v, \mathcal{B}, d) \in GD$ and $i \in N$ satisfies* **E**, **A**, **ICETNP**, **MDG**, **MoDG**, **CSEDG**, **NPOPD** *and* **INDC***; but violates* **ED**.
- *The value f on GD defined by $f_i(N, v, \mathcal{B}, d) = \frac{v(N)}{m|\mathcal{B}(i)|}$ for each $(N, v, \mathcal{B}, d) \in GD$ and $i \in N$ satisfies* **E**, **A**, **ICETNP**, **ED** *and* **INDC***; except* **NPOPD**,**MDG**, **MoDG** *and* **CSEDG**. *If fact, let consider $S \in D(N, \mathcal{B}, d)$, for any $i \notin S$, $(N, u_S, \mathcal{B}, d)$ is i-out diverse and i is null in (N, u_S). So, $f_i(N, v + u_S, \mathcal{B}, d) = \frac{v(N)+u_S(N)}{m|\mathcal{B}(i)|} = \frac{v(N)+1}{m|\mathcal{B}(i)|} \neq f_i(N, v, \mathcal{B}, d)$.*
- *The value f on GD defined by*

$$f_i(N, v, \mathcal{B}, d) = \sum_{S \subseteq N : S \ni i} \frac{\sum_{j \in \mathcal{B}(i) \cap S} j}{i} \cdot \frac{\Delta_v(S)}{|\{B_k \in \mathcal{B} : S \cap B_k \neq \varnothing\}|}$$

for each $(N, v, \mathcal{B}, d) \in GD$ *and* $i \in N$ *satisfies* **E**, **A**, **ED**, **MDG**, **MoDG**, **CSEDG**, **NPOPD** *and* **INDC**; *but does not satisfy* **ICETNP**.

– *The value* f *on* GD *defined by* $f_i(N, v, \mathcal{B}, d) = Ow_i(N, v^d, \mathcal{B})$ *if* $v(N) \neq 0$ *and* $f_i(N, v, \mathcal{B}, d) = 0$ *if* $v(N) = 0$ *for each* $(N, v, \mathcal{B}, d) \in GD$ *and* $i \in N$ *satisfies* **E**, **ICETNP**, **ED**, **NPOPD** *and* **INDC**. *It does not satisfy* **A**.

– *The null value on* GD *defined by* $f_i(N, v, \mathcal{B}, d) = 0$ *for each* $(N, v, \mathcal{B}, d) \in GD$ *and* $i \in N$ *satisfies* **A**, **ICETNP**, **ED**, **MDG**, **MoDG**, **CSEDG**, **NPOPD** *and* **INDC**; *but does not satisfy* **E**.

ii) For the case of Propositions 2 and 4, let us consider the following examples:

– *The value* f *on* GD *defined by* $f_i(N, v, \mathcal{B}, d) = Sh_i(N, v)$ *for each* $(N, v, \mathcal{B}, d) \in GD$ *and* $i \in N$ *satisfies* **E**, **A**, **MDG**, **MoDG**, **CSEDG**, **ETNP** *and* **NPOPD**; *but does not satisfy* **INDC**.

– *The value* f *on* GD *defined by* $f_i(N, v, \mathcal{B}, d) = Ow_i(N, v^d, \mathcal{B})$ *for each* $(N, v, \mathcal{B}, d) \in GD$ *and* $i \in N$ *satisfies* **E**, **A**, **NPOPD**, **MDG**, **MoDG**, **CSEDG** *and* **INDC**; *violates* **ETNP**.

– *The equal division value* f *on* GD *defined by* $f_i(N, v, \mathcal{B}, d) = \frac{v(N)}{|N|}$ *for each* $(N, v, \mathcal{B}, d) \in GD$ *and* $i \in N$ *satisfies* **E**, **A**, **ETNP** *and* **INDC**; *except* **NPOPD**,**MDG**, **MoDG** *and* **CSEDG**.

– *The value* f *on* GD *defined by* $f_i(N, v, \mathcal{B}, d) = Sh_i(N, v^d)$ *if* $v(N) \neq 0$ *and* $f_i(N, v, \mathcal{B}, d) = 0$ *if* $v(N) = 0$ *for each* $(N, v, \mathcal{B}, d) \in GD$ *and* $i \in N$ *satisfies* **E**, **ETNP**, **NPOPD** *and* **INDC**. *It does not satisfy* **A**.

– *The null value on* GD *defined by* $f_i(N, v, \mathcal{B}, d) = 0$ *for each* $(N, v, \mathcal{B}, d) \in GD$ *and* $i \in N$ *satisfies* **A**, **ETNP**, **NPOPD**, **MDG**, **MoDG**, **CSEDG** *and* **INDC**; *but does not satisfy* **E**.

6. Characterizations on simple games with diversity constraints

In this section, we show that our two previous characterizations invoking the Additivity axiom can be adapted to the class of simple TU-games with diversity constraints in which the Diversity Owen and Shapley values can be used as relevant power indices. A game $(N, v) \in G$ is a simple game if for any $S \subseteq N$, $v(S) \in \{0, 1\}$; $v(N) = 1$ and v is monotonic, i.e., for any coalitions $S, T \subseteq N$, $S \subset T$ implies $v(S) \leq v(T)$. We denote by SG the set of simple games on a finite set of players. A coalition S is said to be winning in a game $(N, v) \in SG$ if $v(S) = 1$, and losing otherwise. A coalition S is said to be minimal winning in a game $(N, v) \in SG$ if it is winning and for any $T \subset S$, we have $v(T) = 0$. For two simple games $(N, v), (N, w) \in SG$, we define the simple games $(N, v \vee w)$ and $(N, v \wedge w)$ as follows: $v \vee w(S) = \max\{v(S), w(S)\}$ and $v \wedge w(S) = \min\{v(S), w(S)\}$ for all $S \subseteq N$.

The four-tuple $(N, v, \mathcal{B}, d)$ is called a simple game with diversity constraints if $(N, v) \in SG$. We denote by SGD the set of simple games with diversity constraints with a finite set of players and by $CSSG$ the set of all simple games with a coalition structure in which player set is finite. Any value f restricted to SGD is called an index and assigns to each game $(N, v, \mathcal{B}, d) \in SGD$ and each player $i \in N$ a positive real number $f_i(N, v, \mathcal{B}, d)$ which can be seen as the power of i or her influence in $(N, v, \mathcal{B}, d)$. Following the literature, the Owen value restricted to SGD can be called the Owen index and the Shapley value on SGD can be called the Shapley-Shubik index (Shapley and Shubik, 1954).

It is obvious that the sum of two simple games is not a simple game, which prevents to use of axiom of Additivity on SGD. Dubey (1975) suggests to replace this axiom by the axiom of Transfer, which can be stated as follows on SGD.

Transfer (Tr). For any $(N, v, \mathcal{B}, d), (N, w, \mathcal{B}, d) \in SGD$, we have $f(N, v \vee w, \mathcal{B}, d) + f(N, v \wedge w, \mathcal{B}, d) = f(N, v, \mathcal{B}, d) + f(N, w, \mathcal{B}, d)$.

The transfer axiom is invoked to characterize the Shapley-Shubik index on SG (Dubey, 1975; Einy and Haimanko, 2011) and other values in various context (see Feltkamp, 1995; Dubey et al., 2005, among others). The transposition of our two first results to the class SGD is not totally immediate since this requires to ensure that the diversity-restricted game induced by a simple game remains a simple game. This is done in the next lemma.

Lemma 2. *Let $(N, v, \mathcal{B}, d) \in SGD$ be any simple game with diversity constraints. Then it holds that:*

(i) the game (N, v^d) is a simple game.

(ii) if an index f on SGD satisfies Transfer **(Tr)** *and Independence from non-diverse coalitions* **(INDC)** *then*

$$f(N, v, \mathcal{B}, d) = \sum_{I \subset \{1,2,\ldots,\ell\}, I \neq \varnothing} (-1)^{|I|+1} f(N, u_{\cup_{j \in I} L_j}, \mathcal{B}, d),$$

where $\{L_j : j \in \{1, 2, ..., \ell\}\}$ is the set of minimal winning coalitions in the game (N, v^d).

Proof. Consider an arbitrary simple game with diversity constraints $(N, v, \mathcal{B}, d) \in SGD$.

Part (i) Firstly, we have $v^d(N) = v(N) = 1$ since $N \in D(N, \mathcal{B}, d)$. Secondly, for any $S \subseteq N$, we have either $v^d(S) = v(S) \in \{0, 1\}$ if $S \in D(N, \mathcal{B}, d)$ or $v^d(S) = 0$ if $S \notin D(N, \mathcal{B}, d)$. Thus, $v^d(S) \in \{0, 1\}$. Thirdly, for any $S, T \subseteq N$ such that $S \subseteq T$, if $T \in D(N, \mathcal{B}, d)$ then $v^d(T) = v(T) \geq v(S) \geq v^d(S)$, where the first inequality comes from the fact that v is monotonic and the second inequality comes from the definition of v^d. If $T \notin D(N, \mathcal{B}, d)$ then $S \notin D(N, \mathcal{B}, d)$ and $v^d(T) = v^d(S) = 0$. Therefore, v^d is monotonic as well.

Part (ii) From **INDC**, we have $f(N,v,\mathcal{B},d) = f(N,v^d,\mathcal{B},d)$. Furthermore, from Dubey (1975) and the fact that $(N,v^d) \in SG$, it is known that v^d can be written as a maximum of a finite number of unanimity games: $v^d = u_{L_1} \vee u_{L_2} \vee ... \vee u_{L_\ell}$ where $\{L_j : j \in \{1,2,...,\ell\}\}$ is the set of minimal winning coalitions in (N,v^d). By definition, all the winning coalitions in the game (N,v^d) are diverse, which obviously implies that all minimal winning coalitions in (N,v^d) are diverse too. Since f satisfies **Tr** and (N,v^d) is a simple voting game, then Einy (1987, Lemma 2.3) yields that

$$f(N,v^d,\mathcal{B},d) = \sum_{I\subset\{1,2,...,\ell\}, I\neq\emptyset} (-1)^{|I|+1} f(N, u_{\cup_{j\in I} L_j}, \mathcal{B}, d),$$

as desired. ■

Building on the previous Lemma and Propositions 1 and 2, we obtain the following two corollaries.

Corollary 1. *There is a unique index on SGD satisfying Efficiency* **(E)***, Transfer* **(Tr)***, Intra-coalitional equal treatment of necessary players* **(ICETNP)***, Null player out for preserving-diversity games* **(NPOPD)***, Equality through diversity* **(ED)** *and Independence from non-diverse coalitions* **(INDC)***. It is the* **Diversity Owen index** *$DOwI$, such that, for each $(N,v,\mathcal{B},d) \in SGD$, $DOwI(N,v,\mathcal{B},d) = Ow(N,v^d,\mathcal{B})$.*

Corollary 2. *There is a unique index on SGD satisfying Efficiency* **(E)***, Transfer* **(Tr)***, Equal treatment of necessary players* **(ETNP)***, Null player out for preserving-diversity games* **(NPOPD)** *and Independence from non-diverse coalitions* **(INDC)***. It is the Diversity Shapley-Shubik index DSS, such that, for each $(N,v,\mathcal{B},d) \in SGD$, $DSS(N,v,\mathcal{B},d) = Sh(N,v^d)$.*

We conclude this section by noting that Propositions 3 and 4 do not hold anymore on the class of simple games with diversity constraints. While the axiom sets in these proposition are still valid on this class, there are other values than DOw and DSh satisfying them as pointed out by the following two examples:

- The index f on SGD defined by $f_i(N,v,\mathcal{B},d) = \frac{1}{m|\mathcal{B}(i)|}$ for each $(N,v,\mathcal{B},d) \in SGD$ and $i \in N$ satisfies **E**, **ICETNP**, **ED**, **INDC**, **MDG**, **MoDG** and **CSEDG**.
- The index f on SGD defined by $f_i(N,v,\mathcal{B},d) = \frac{1}{|N|}$ for each $(N,v,\mathcal{B},d) \in SGD$ and $i \in N$ satisfies **E**, **ETNP**, **INDC**, **MDG**, **MoDG** and **CSEDG**.

7. Conclusion

As a conclusion, we would like to allude to some extensions of our work. There are some situations in which diversity matters but that our model cannot capture. We detail briefly two such situations.

Firstly, it may be the case that some but not all communities must be represented within a diverse coalition. An example is the so-called Victoria charter in Canada that required in 1971 a Constitutional amendment to be approved by Quebec, Ontario, two of the four Atlantic Provinces and British Columbia and one central province OR all three central provinces (see Straffin, 1977). If one naturally considers British Columbia as a single community and the three central provinces as another community, then this last requirement implies that all but one communities are needed for a Constitutional amendment, i.e., there are diverse coalitions not containing members of all communities.

Secondly, in some cases, it makes sense that a player belongs to more than one communities. As an example, go back to the Participatory budgeting with districts developed in Example 2. While most projects are implemented within a single districts, there are also projects that have an impact on multiple districts or even the entire city. In many cases but not always, such projects are handled through a separate procedure.

These extensions are left for future works.

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