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Multiwinner elections with diversity constraints on individual preferences

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Abstract

We address the problem of selecting a committee of a specified size from a given set of candidates, where individuals are requested to provide their preferences in the form of linear rankings of the candidates. In this framework, the selection of a committee depends on the multiwinner voting rule, also known as the committee selection rule. In this paper, we assume that the candidates possess an official attribute, namely the gender identity. Additionally, the linear ordering of voters should meet some diversity requirements (such as alternating males and females positions, ranking a certain number/percentage of males and females in the top half of the linear ranking, etc.) in order to be considered as admissible for the voting process. The objective of this paper is to assess the cost incurred by implementing diversity restrictions on the preferences of voters. We present a measure for assessing the cost of diversity and calculate the maximum cost, or upper bound, for a commonly used family of multiwinner voting methods known as (weakly) separable committee scoring rules.

Key-words: Voting, multiwinner elections, (weakly) separable committee scoring rules, diversity constraints, price of diversity.

JEL Codes: D71, D72.

1. Introduction

Many countries have been faced with the scarcity of females in the political sphere, certain positions of responsibility, and various areas of activity, including sport, engineering, army, and hard sciences. To address this issue, some countries have implemented public

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policies aimed at reducing gender inequality and discrimination. For instance, in the political sphere (including local, regional, senatorial, and parliamentary elections), several countries have implemented gender quotas. The UN WOMEN [2023] review presents a comprehensive list of countries detailing their electoral systems, gender quota regulations, and sanctions for non-compliance. The majority of countries mentioned in this review employ either proportional representation or a combination of majority/plurality systems for their electoral processes. The gender quota imposed to each candidate (or party) list include the candidate quota, the candidate quota with a ranking rule, and the candidate quota with a partial ranking rule. Concerning the countries that use the candidate quota, the legislator impose a minimal (or a maximal) percentage of males and females on a candidate list. For instance, Croatia, Greece, and Norway impose that each candidate list must be composed of at least 40% of the candidates of each gender.[1] For the countries that use the candidate quota with a ranking rule, the legislator first imposes a quota for each gender on the candidate list, and then a ranking rule for the candidates of the list. There are many ranking rules and the well-known ranking rule is the alternation by gender (female-male or male-female) in such a way that two people of the same gender cannot be consecutively on the list. Many countries such as Belgium, Bolivia, Costa Rica, Ecuador, France (for elections concerning regions, departments, and municipalities of over than 1000 inhabitants), Senegal and Tunisia impose the principle of parity between males and females, and the application of the alternation mechanism. There are some countries that apply the candidate quota with a partial ranking rule. In Spain for instance, each political party must propose at least 40% of males and 40% of females in the list for the municipalities of more than 3000 inhabitants; and moreover this quota is also applied to every top-five-positions (at least 2 males and 2 females among the top-five-positions). In Slovenia, each candidates list must be made up of at least 40% of each gender, and candidates in the first half of the list must alternate between men and women.[2]

While the gender quota has been welcomed and implemented in numerous countries, it remains a topic of political contention. For example, certain political parties can not participate in an election due to a lack of male or female candidates meeting the required minimum quota. In addition, political parties may need to replace highly influential members with less influential ones to meet diversity requirements. In other words, gender quotas have a price/cost. For instance, Bagues and Campa (2021) studied the effects of gender quotas in candidate lists using evidence from local elections in Spain. They found that gender quotas in candidate lists do not remove the barriers that prevent females from playing an influential role in politics, as gender quotas do not seem to increase the likelihood of females reaching powerful positions. In this article, we examine the issue

[1]Many other countries such as Andorra, Argentina, Brazil, Columbia, Mongolia, Panama, and Poland also apply the candidate quota.

[2]For more details regarding other gender quotas used by other countries, we refer the reader to the UN WOMEN [2023] review.

of the price of gender quotas, that we call the *price of diversity*, within the framework of multi-winner elections when we impose some diversity constraints on the ranking of voters. To address this question one can retain some key elements from the literature on multiwinner elections.

A multiwinner election, also called committee voting, is an electoral system in which a collection of voters aims to elect multiple candidates, called a committee, from a larger set of available candidates. The number of candidates elected is usually fixed in advance and it is denoted by the symbol k. Voter preferences can take different forms but we are interested here in the case where each voter proposes a linear ranking over the set of candidates, i.e., without the possibility of ties between candidates. Many rules are defined in order to aggregate these rankings. The most commonly rules used in the literature are those that are based on scoring functions and called the committee scoring rules. This class of rules, introduced by Elkind et al. (2017), is adapted from the well-known class of single-winner scoring rules. Roughly speaking, and similarly to the single-winner setting, under a committee scoring rule each voter assigns a predefined score to each committee based on the positions of the committee members in the considered voter's ranking and, at the end, the winning committee is the one with the maximum total score computed as the sum of the scores received from all voters to every candidate.[3] The well-known examples of committee scoring rules are the k-plurality rule, the k-antiplurality rule, the Bloc rule, the k-Borda rule, the Chamberlin-Courant rule (Chamberlin and Courant, 1983) and the proportional approval voting rule (Kilgour, 2010). Among these rules, Elkind et al. (2017) defined a family of committee scoring rules that are (weakly) separable. These rules select the committee(s) containing the k-best candidates in terms of aggregated scores received from the voters. In other words, under those rules we compute a separate aggregated score for each candidate (using a single-winner scoring rule) taking into account all of the voters and then we pick k candidates with the top aggregated scores, possibly using a tie-breaking rule. The aforementioned first four committee scoring rules are (weakly) separable. Many other families of committee scoring rules have been introduced in the literature and to further explore these we refer the reader to Faliszewski et al. (2016, 2018) and Skowron et al. (2016, 2019), among others.

Choosing a committee can sometimes be limited by various standards and objectives that need to be accomplished. Our paper is concerned with the question of diversity. The literature devoted to the framework of diverse committee selection is rich and recent. It mainly concerns the computational complexity of managing such a problem based on different ideas and goals and furthermore provides some properties that should be satisfied in this framework. To the best of our knowledge, this literature can be categorized into two groups. The first group regards the multi-attribute structure, which supposes that each candidate is portrayed by a single label for each attribute. In this framework, Bredereck

[3]Note that Skowron et al. (2019) provided an axiomatic characterization of committee scoring rules.

et al. (2017), Celis et al. (2018), and Lang and Skowron (2018), among others, consider that the selected committee must contain at least a predefined quota of candidates from each attribute. The second group assumes that candidates are sorted into multiple classes based on a single specific attribute. We can cite the work of Aziz (2019) who considered that the set of candidates is structured into several non-disjoint classes according to a specific attribute, and defined a diversity constraint as a vector of integers specifying the lowest number of candidates to be selected from each class. We also refer the reader to Ianovski (2022), Kagita et al. (2021), and Relia (2021), among others.[4]

In this paper, we consider a multiwinner election in which the candidates are subdivided into two groups according to only two gender identities: males and females. Each voter is required to propose a ranking over the set of all candidates which satisfies some diversity constraints. More precisely, our paper introduces a novel approach that takes into account diversity constraints on the input, i.e., on individual preferences. We propose three natural diversity constraints that can be imposed on the individual preferences. The first constraint imposes to each voter to rank the candidates by alternating males and females up to the last candidate if possible. The second constraint imposes to each voter a quota of each gender group among the top-k-positions of each linear ranking. The last type of constraint imposes that the first half of the ranking of each voter must contain 50% of candidates of each group. We believe that this approach makes it possible to account for the changes that diversity constraints can implicitly induce in the mental process that voters use to adjust the representation of their preferences. Using the method of Bredereck et al. (2017), we introduce a measure of the price of diversity and we determine the highest price of diversity to pay for the family of (weakly) separable committee scoring rules. We classify the well-known (weakly) separable committee scoring rules in terms of the maximal price of diversity to pay. Finally, we propose a second boundary of the price of diversity specific to each diversity constraint and then we deduce a classification of these diversity constraints considering again the family of (weakly) separable committee scoring rules.

Our paper is structured as follows. Section 2 presents some basic definitions. Section 3 presents the diversity constraints that we consider. Section 4 studies the price of diversity and Section 5 concludes.

[4]Diversity constraints ha ve also been introduced in the context of matching problems (Benabbou et al., 2020; Echenique and Yenmez, 2015; Ehlers et al., 2014). For instance, Benabbou et al. (2020) consider the matching problems between a finite set of individuals and a finite set of houses. They suppose that the houses are partitioned into different categories; and the individuals are partitioned into different groups or types (e.g., ethnicity groups). They also assume that there are a utility function which measures the satisfaction of each individual over the set of houses. They impose a quota (minimal number) of each category of houses to each group of individuals. They measure the price of diversity (measure of the loss in welfare incurred by imposing diversity constraints) and proposed two upper bounds of this price.

2. Preliminaries

For any integer $r \in \mathbb{N}^*$, for simplicity we denote by $[r]$ the set $\{1, \dots, r\}$. An election is a pair (N, A) where $N = [n]$ is a finite set of voters with $|N| = n \geq 2$, and $A = \{a_1, \dots, a_m\}$ is a finite set of candidates (or alternatives) with $|A| = m \geq 3$. Candidates can also be denoted by a, b, c, etc. We assume that A is partitioned into two groups according to the gender identity, i.e., $A = A_1 \bigcup A_2$ where A_1 is the set of males and A_2 is the set of females. Any subset $W \subseteq A$ of candidates of size $k < m$ is called a committee. We denote by 2^A_k the set of possible committees with cardinality k.

We assume that individual preferences are strict (or linear), which means that ties are not possible. For any voter $i \in N$, the notation $\succ_i$ denotes the (linear) preference or the ranking of voter i over the set A so that $a \succ_i b$ means that voter i strictly prefers a to b. We will sometimes use the notation ab instead of $a \succ_i b$. The n-tuple $\succ := (\succ_1, \succ_2, \dots, \succ_n)$ is called a preference profile or simply a profile. We denote by $\mathcal{P}^n$ the set of all possible profiles with n voters. In this paper, we consider voters' preferences to be sincere, meaning that each voter votes according to his/her true preference. The rank of a candidate $a \in A$ in preference $\succ_i$ is denoted by $\text{rank}(a, \succ_i)$ and it is given by

$$\text{rank}(a, \succ_i) := |\{b \in A : b \succ_i a\}| + 1 = m - |\{b \in A : a \succ_i b\}|. \tag{1}$$

Similarly, the rank of a committee $W \in 2^A_k$ in $\succ_i$ is denoted by $\text{rank}(W, \succ_i)$ and it is given by

$$\text{rank}(W, \succ_i) := (i_1, i_2, \dots, i_k) \tag{2}$$

where $(i_1, i_2, \dots, i_k)$ is an increasing sequence of ranks of W members according to the ranking given by $\succ_i$. In order words, position i_1 represents the rank of W's highest-ranked candidate in $\succ_i$, i_2 is the rank of W's second-highest-ranked candidate in $\succ_i$, and so on up to i_k, which is the rank of W's worse candidate in $\succ_i$. We denote by $[m]_k$ the set of all length-k increasing sequences of numbers from $[m]$.

A scoring vector is a vector $s = (s_1, s_2, \dots, s_m) \in \mathbb{R}^m_+$ such that $s_1 \geq s_2 \geq \dots \geq s_m$ and $s_1 > s_m$. A single-winner scoring function for m candidates is a non-increasing function $\gamma : [m] \longrightarrow \mathbb{R}_+$ that assigns a given score value to each rank in each preference order. The most commonly-used scoring function is the Borda scoring function denoted by β_m and given by $\beta_m(r) = m - r$, where $r = \text{rank}(a, \succ_i)$. The second most commonly-used scoring function is the t-approval scoring function α_t (with $t \in [m]$) defined by $\alpha_t(r) = 1$ if $r \leq t$, and 0 otherwise. In particular, α_1 is the plurality scoring function, and α_{m-1} is the antiplurality scoring function. The score of a candidate $a \in A$ in a profile $\succ$ following the scoring vector s is denoted by $Sc(a, s, \succ)$ and it is given by

$$Sc(a, s, \succ) = \sum_{i \in N} s_{\text{rank}(a, \succ_i)}. \tag{3}$$

Let $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ be two sequences from $[m]_k$. We will say that the sequence I (weakly) dominates J, if $i_\sigma \leq j_\sigma$ holds for all $\sigma \in [k]$. Following Elkind et al. (2017), and Faliszewski et al. (2016), a committee scoring function for m candidates and a fixed committee size k is a function $f_{m,k} : [m]_k \longrightarrow \mathbb{R}_+$ such that for any two sequences $I, J \in [m]_k$, $f_{m,k}(I) \geq f_{m,k}(J)$ holds if I (weakly) dominates J. Given that m and k are fixed in advance, we will simply write f instead of $f_{m,k}$.

The score of a committee $W \in 2^A_k$ in a profile $\succ$ is denoted by $Sc(W, f, \succ)$ and it is given by

$$Sc(W, f, \succ) = \sum_{i \in N} f(\text{rank}(W, \succ_i)). \tag{4}$$

A committee scoring rule associated to a committee scoring function f is a function that assigns to each profile $\succ \in \mathcal{P}^n$ the winning committee(s) of size k (or the set of committees of size k that tied as winners) with the highest score. In other words, the committee scoring rule outputs the committee(s) that belong(s) to $\text{argmax}_{W \in 2^A_k} Sc(W, f, \succ)$. Recently, Elkind et al. (2017) introduced a particular family of committee scoring rules called **(weakly) separable** committee scoring rules. This family is commonly used in the literature of multiwinner elections and has been adapted from the well-known family of single-winner scoring rules.

A committee scoring function f is said to be (weakly) separable if there exists a scoring vector s such that for any sequence $(i_1, i_2, \ldots, i_k) \in [m]_k$, we have $f(i_1, i_2, \ldots, i_k) = \sum_{r \in [k]} s_{i_r}$. In other words, f is (weakly) separable if the score a voter assigns to any committee is simply the sum of the scores he/she assigns to each candidate belonging to that committee. If the scoring vector does not depend on k then we say that f is separable. A committee scoring rule is (weakly) separable if the associated committee scoring function is (weakly) separable.

Among the well-known **(weakly) separable** committee scoring rules, we can mention the following:

k-plurality rule:[5] Each voter gives 1 point to the top-ranked candidate and 0 to the others, i.e., $s = (1, 0, \ldots, 0)$. The k candidates with the highest aggregated scores are selected.

k-antiplurality rule: Each voter gives 1 point to each candidate in his/her ranking except the last candidate who obtains 0, i.e., $s = (1, 1, \ldots, 1, 0)$. The k candidates with the highest aggregated scores are selected.

k-Borda rule: Each voter gives to each candidate $(m - r)$ points where $r \in [m]$ is the rank of the candidate, i.e., $s = (m - 1, m - 2, \ldots, 1, 0)$. The k candidates with the highest aggregated scores are selected.

[5] Also known as Single Non-Transferable Vote (SNTV).

Bloc rule: Each voter gives 1 point to the candidates ranked at the top-k-positions and 0 point to the others, i.e., $s = (\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$. The k candidates with the highest aggregated scores are selected.

Clearly, the k-antiplurality rule, the k-plurality rule, and the k-Borda rule are separable. In addition, the Bloc rule is weakly separable.

Note that Bredereck et al. (2017) define the price that can be paid in terms of scores in the case of imposing predefined constraints on the composition of the selected committee (i.e., they impose the diversity constraints on the **output**). The price of diversity, denoted by $pod(f, \succ)$, is defined for a given profile $\succ$ and a given committee scoring function f as follows:

$$pod(f, \succ) = \frac{\max_{W \in 2_k^A} Sc(W, f, \succ)}{\max_{W \in \mathcal{D}} Sc(W, f, \succ)} \tag{5}$$

where $\mathcal{D} = \{W \in 2_k^A : W \text{ satisfies the predefined diversity constraints on the output}\}$, $\max_{W \in 2_k^A} Sc(W, f, \succ)$ is the score of the selected committee without considering diversity constraints, and $\max_{W \in \mathcal{D}} Sc(W, f, \succ)$ is the score of the selected committee when the diversity constraints are taken into account.

Bredereck et al. (2017) show that if the family of committee scoring functions $(f_{m,k'})_{k' \leq m}$ is submodular[6] and monotone[7] then the price of diversity is at most 2 when we enforce the constraint of balanced committees.[8] In other words, the constraint consisting at balanced committees may reduce the score of the selected committee by up to 50% when $(f_{m,k'})_{k' \leq m}$ is submodular and monotone.

We enforce the diversity constraints on the **input**, i.e., on voters preferences. In other words, we consider a framework in which each voter holds a sincere preference, but he/she may revise it if it fails to satisfy the requirements enforced for promoting diversity. For instance, we can enforce each voter to rank the candidates by alternating the positions of males and females. In this case, we obtain a new profile when the original one does not meet the alternation principle. This means that for a given voting rule and a given preference profile, due to the diversity constraints on the preferences, we may select a new committee different from the one obtained through the original profile. We will then be interested in the price of diversity in that framework. Before answering this question, we start by presenting some diversity constraints that can be imposed on individual preferences.

[6]A family of committee scoring functions $(f_{m,k'})_{k' \leq m}$ is submodular if $\forall S, S' \subseteq A$ with $S \subseteq S'$ and $\forall a \in A \backslash S'$, we have $Sc(S \cup a, f_{m,|S \cup a|}, \succ) - Sc(S, f_{m,|S|}, \succ) \geq Sc(S' \cup a, f_{m,|S' \cup a|}, \succ) - Sc(S', f_{m,|S'|}, \succ)$ for any $\succ \in \mathcal{P}^n$.

[7]A family of committee scoring functions $(f_{m,k'})_{k' \leq m}$ is monotone if $\forall S, S' \subseteq A$ with $S \subseteq S'$, we have $Sc(S, f_{m,|S|}, \succ) \leq Sc(S', f_{m,|S'|}, \succ)$ for any $\succ \in \mathcal{P}^n$.

[8]A balanced committee is a committee (with an even number of candidates) containing the same number of males and females.

3. Some diversity constraints on admissible preferences

We introduce three natural diversity constraints that can be imposed on individual preferences.

Alternation of males and females

In this case, voters are required to rank the candidates by placing them in alternate positions (i.e., male, female, male, female,...; or female, male, female, male,...). From the original preference profile $\succ$, we can construct a new profile denoted by $\succsim^{\text{alt}} = (\succsim_1^{\text{alt}}, \succsim_2^{\text{alt}}, \ldots, \succsim_n^{\text{alt}})$ where every $\succsim_i^{\text{alt}}$ satisfies the alternation principle as follows:

- The original preference $\succ_i$ of any voter i induces a ranking of candidates within each group.
- If voter i's preferred candidate is male (respectively, female), he/she should propose a new ranking consisting of the top male and female candidates (respectively, female and male) according to $\succ_i$, followed by the second highest male and female candidates (respectively, female and male) according to $\succ_i$, and so on.
- During the process, when the candidates in one group are all ranked, the remaining candidates in the other group are added to the current ranking in accordance with their order given by $\succ_i$.

Constraints on the top-k-positions on individual preferences

In this case, we consider a vector of natural numbers $q := (q_1, q_2) \neq (0, 0)$, where q_1 and q_2 are respectively the minimal numbers of males and females occupying the top-k-positions in each voter's ranking and satisfying the following conditions: $q_1 \leq |A_1|$, $q_2 \leq |A_2|$, and $q_1 + q_2 \leq k$. In other words, we require each voter to propose a ranking such that the top-k-positions contain at least q_1 males and q_2 females.

Let $\succ$ be the original preference profile. If we impose the constraints on the top-k-positions then we can construct a new profile denoted by $\succsim^{\text{top-}k}$. The original preference of a voter i induces a ranking of candidates within each group. The top-k-candidates who will occupy the top-k-positions of the preference $\succsim_i^{\text{top-}k}$ are presented as follows: we first consider the increasing sequence of candidates according to their order given by $\succ_i$, and containing the top-q_1-males and the top-q_2-females. From the remaining candidates, we select those ranked higher by $\succ_i$ and add them to the end of this sequence in order to complete the size of this sequence at k (since $q_1 + q_2 \leq k$). Candidates occupying top-k-positions of $\succsim_i^{\text{top-}k}$ correspond to this sequence. From position $k+1$ to m in $\succsim_i^{\text{top-}k}$, we rank the remaining candidates according to the order induced by $\succ_i$, and we obtain the new preference $\succsim_i^{\text{top-}k}$.

Constraints on the first half of the ranking

We impose here to each voter to rank the candidates in such a way that the first half of his/her ranking (i.e., the $\lceil\frac{|A_1|}{2}\rceil + \lceil\frac{|A_2|}{2}\rceil$ first positions within the ranking) must contain at least 50 % of candidates of each group (i.e., $\lceil\frac{|A_1|}{2}\rceil$ males and $\lceil\frac{|A_2|}{2}\rceil$ females). From a preference profile $\succ$, we can construct a new preference profile denoted by $\tilde{\succ}^{\text{half}}$ satisfying the constraints on the top half of the ranking. The original preference $\succ_i$ of each voter i induces the ranking of the candidates within each group, and we can then select the top-$\lceil\frac{|A_j|}{2}\rceil$-candidates for each group. The new preference $\tilde{\succ}_i^{\text{half}}$ of voter i is described as follows:

- On the top-$\lceil\frac{|A_1|}{2}\rceil + \lceil\frac{|A_2|}{2}\rceil$-positions of voter i's new ranking, we rank all his/her top-$\lceil\frac{|A_j|}{2}\rceil$-candidates (with $j \in [2]$) from the order induced by $\succ_i$.
- From position $\lceil\frac{|A_1|}{2}\rceil + \lceil\frac{|A_2|}{2}\rceil + 1$ to m, we rank the rest of the candidates using the order induced by $\succ_i$.

Let us consider an example in order to illustrate the different constraints that we consider.

Example 3.1. *We consider an election where we have 8 candidates such that the set of males is $A_1 = \{a_1, a_2, a_3, a_4\}$ and the set of females is $A_2 = \{b_1, b_2, b_3, b_4\}$. We assume that we have 6 voters with the following profile: $\succ = (a_3a_2b_2b_1a_1b_3a_4b_4,\ b_2a_1a_2b_1a_3b_3b_4a_4,\ b_1a_3a_1a_2b_2b_3b_4a_4,\ a_3a_1b_2a_2b_1b_3a_4b_4,\ a_1a_3a_2b_2b_1b_3a_4b_4,\ a_1b_2b_1a_3a_2b_3a_4b_4)$. We obtain a new preference profile for each diversity constraint.*

- *The alternation of males and females*

  The new profile is $\tilde{\succ}^{alt} = (a_3b_2a_2b_1a_1b_3a_4b_4, b_2a_1b_1a_2b_3a_3b_4a_4, b_1a_3b_2a_1b_3a_2b_4a_4, a_3b_2a_1b_1a_2b_3a_4b_4,\ a_1b_2a_3b_1a_2b_3a_4b_4, a_1b_2a_3b_1a_2b_3a_4b_4)$.

- *The constraints on the top-2-positions with $q_1 = q_2 = 1$*

  The new profile is: $\tilde{\succ}^{top\text{-}k} = (a_3b_2a_2b_1a_1b_3a_4b_4,\ b_2a_1a_2b_1a_3b_3b_4a_4,\ b_1a_3a_1a_2b_2b_3b_4a_4,\ a_3b_2a_1a_2b_1b_3a_4b_4,\ a_1b_2a_3a_2b_1b_3a_4b_4,\ a_1b_2b_1a_3a_2b_3a_4b_4)$.

- *The constraints on the first half of the ranking*

  The new profile is: $\tilde{\succ}^{half} = (a_3a_2b_2b_1a_1b_3a_4b_4,\ b_2a_1a_2b_1a_3b_3b_4a_4,\ b_1a_3a_1b_2a_2b_3b_4a_4,\ a_3a_1b_2b_1a_2b_3a_4b_4,\ a_1a_3b_2b_1a_2b_3a_4b_4,\ a_1b_2b_1a_3a_2b_3a_4b_4)$.

In the next section, we will study the price of diversity constraints imposed on individual preferences. For each diversity constraint introduced in this paper, we will also exhibit a profile that reaches the boundary of the price of diversity.

4. The price of diversity constraints on admissible preferences

Let $\succ$ be a given preference profile. Using a committee scoring rule with the associated committee scoring function f, we can determine an optimal committee $W \in 2^A_k$ such that $W \in \arg\max_{C \in 2^A_k} Sc(C, f, \succ)$. Now, if we impose some diversity constraints on the preferences, we can construct from the profile $\succ$ a new profile $\tilde{\succ}$ satisfying the chosen diversity constraints. From this new profile, we can determine an optimal committee $\widetilde{W} \in 2^A_k$, i.e., $\widetilde{W} \in \arg\max_{C \in 2^A_k} Sc(C, f, \tilde{\succ})$. We are interested in the price that could be paid if we select the committee $\widetilde{W}$ instead of the committee W. Using the approach proposed by Bredereck et al. (2017), we define a new measure that quantifies the cost of diversity constraints enforced on individual preferences. This measure is denoted by pod^1 and it is given by the following definition.

Definition 1. *Let $\succ \in \mathcal{P}^n$ be a profile and f be a committee scoring function associated to a given committee scoring rule. If we impose some diversity constraints on individual preferences and select the committee of size k with profile $\tilde{\succ}$ instead of $\succ$, then the price of diversity is given by:*

$$pod^1(f, \succ) = \frac{\max_{W \in 2^A_k} Sc(W, f, \succ)}{\min_{\widetilde{W} \in \, arg \max_{C \in 2^A_k} Sc(C, f, \tilde{\succ})} Sc(\widetilde{W}, f, \succ)}. \tag{6}$$

In Equation (6),

$$\max_{W \in 2^A_k} Sc(W, f, \succ)$$

is the score of the committee selected without considering diversity constraints on preferences,

$$\arg\max_{C \in 2^A_k} Sc(C, f, \tilde{\succ})$$

is the set of all winning committees in the new profile $\tilde{\succ}$, and

$$\min_{\widetilde{W} \in \, \arg\max_{C \in 2^A_k} Sc(C, f, \tilde{\succ})} Sc(\widetilde{W}, f, \succ)$$

is the lowest score of the selected committee when diversity constraints are taken into account. Note that Equation (6) may be normalized as follows:

$$pod^1(f, \succ) = 1 - \frac{\min\limits_{\widetilde{W} \in \arg\max\limits_{C \in 2^A_k} Sc(C, f, \tilde{\succ})} Sc(\widetilde{W}, f, \succ)}{\max\limits_{W \in 2^A_k} Sc(W, f, \succ)}.$$

Let us now consider the following example.

Example 4.1 (Example 3.1 continued)**.** *Let us find the price of diversity when we enforce the diversity constraints introduced in Section 3. We assume that the objective is to select a committee of size $k = 2$ using the k-Borda rule. The candidates have the following scores: $Sc(a_1, s, \succ) = 34$, $Sc(a_2, s, \succ) = 27$, $Sc(a_3, s, \succ) = 33$, $Sc(a_4, s, \succ) = 4$, $Sc(b_1, s, \succ) = 26$, $Sc(b_2, s, \succ) = 30$, $Sc(b_3, s, \succ) = 12$, and $Sc(b_4, s, \succ) = 2$. The elected committee for the profile $\succ$ is $\{a_1, a_3\}$ with a score of 67.*

- *If we impose alternation on the gender position, we obtain the new profile $\tilde{\succ}^{alt}$ described in Example 3.1 and the selected committees are $\{a_1, b_2\}$ and $\{a_3, b_2\}$. Thus,*

$$pod^1(f, \succ) = \frac{Sc(\{a_1, a_3\}, s, \succ)}{Sc(\{a_3, b_2\}, s, \succ)} = \frac{67}{63}.$$

  The diversity constraints on the preferences reduce the score of the selected committee by 1-63/67=5.97%.

- *If we impose the constraints on the top-2-positions with $q_1 = q_2 = 1$ then we obtain the new profile $\tilde{\succ}^{top\text{-}k}$ and the selected committee is $\{a_1, b_2\}$. So,*

$$pod^1(f, \succ) = \frac{Sc(\{a_1, a_3\}, s, \succ)}{Sc(\{a_1, b_2\}, s, \succ)} = \frac{67}{64}.$$

  The diversity constraints on the preferences reduce the score of the selected committee by 1-64/67=4.48%.

- *If we impose the constraints on the first half of the ranking then the new profile is $\tilde{\succ}^{half}$ and the elected committee is $\{a_1, a_3\}$. Thus,*

$$pod^1(f, \succ) = \frac{Sc(\{a_1, a_3\}, s, \succ)}{Sc(\{a_1, a_3\}, s, \succ)} = 1.$$

  In this particular case, diversity constraints on preferences do not affect the composition of the selected committee.

4.1. Boundary of the price of diversity and classification of the well-known committee scoring rules

For any preference profile, any diversity constraint on preferences and any committee scoring rule, we determine the boundary of the price of diversity for the family of (weakly) separable committee scoring rules. Through some examples and using the diversity constraints given in the previous section, we show that the boundary is reached.

Proposition 1. *Let f be a (weakly) separable committee scoring function and s a scoring vector of f. For any profile $\succ \in \mathcal{P}^n$, when we enforce the diversity constraints on preferences, then $1 \leq pod^1(f, \succ) \leq \frac{\sum_{\sigma=1}^{k} s_\sigma}{\sum_{\sigma=1}^{k} s_{m-k+\sigma}}$ if $s_{m-k+1} > 0$ and $pod^1(f, \succ) \leq +\infty$ otherwise.*

Proof of Proposition 1. If the preference profile $\succ$ satisfies the diversity constraints then $pod^1(f, \succ) = 1$. Let W be the selected committee obtained without considering diversity constraints on preferences and $\widetilde{W}$ the selected committee obtained when the diversity constraints on preferences are taken into account. The following two polar cases are possible:

- In the profile $\succ$, each voter ranks the members of W in the top-k-positions (from position 1 to position k), i.e., each voter assigns the highest score $\sum_{\sigma=1}^{k} s_\sigma$ to the committee W.
- In the profile $\succ$, each voter ranks the members of $\widetilde{W}$ in the last-k-positions (from position $m-k+1$ to position m), i.e., each voter assigns the smallest score $\sum_{\sigma=1}^{k} s_{m-k+\sigma}$ to the committee $\widetilde{W}$.

When diversity constraints are taken into account (alternating males and females positions, constraints on the top k-positions, etc.), the committee $\widetilde{W}$ can be selected in the new profile $\widetilde{\succ}$. So,

$$pod^1(f, \succ) \leq \frac{n. \sum_{\sigma=1}^{k} s_\sigma}{n. \sum_{\sigma=1}^{k} s_{m-k+\sigma}}$$

if $\sum_{\sigma=1}^{k} s_{m-k+\sigma} \neq 0$ and $pod^1(f, \succ) \leq +\infty$ otherwise. If $s_{m-k+1} > 0$ then $\sum_{\sigma=1}^{k} s_{m-k+\sigma} \neq 0$. □

For any diversity constraint on preferences, Proposition 1 gives the highest price of diversity to pay. It is important to exhibit a profile that reaches the upper bound. Examples 4.2 and 4.3 given below are useful to show that the upper bound defined in Proposition 1 is reached when we impose the diversity constraints described in Section 3.

Example 4.2. *We consider an election where we have 4 voters and 6 candidates such that the set of males is $A_1 = \{a_1, a_2, a_3, a_4\}$ and the set of females $A_2 = \{b_1, b_2\}$. We assume that the original preference profile is $\succ = \big(a_1a_2a_3a_4b_1b_2, a_1a_2a_3a_4b_1b_2, a_2a_1a_3a_4b_2b_1, a_2a_1a_3a_4b_2b_1\big)$ and the size of the selected committee is $k = 2$.*

- *If the committee scoring rule is the k-antiplurality rule then the committee elected for the preference profile $\succ$ is $\{a_1, a_2\}$ (i.e., the committee occupying the top-2-positions for each voter) with a score $Sc(\{a_1, a_2\}, f, \succ) = 8$. Note that, the committee occupying the last-2-positions for each voter in the profile $\succ$ is $\{b_1, b_2\}$ and* *its score is* $Sc(\{b_1, b_2\}, f, \succ) = 4$. *Now, if we impose alternating position for males and females on preferences then the new profile will be as follows:* $\tilde{\succ}^{alt} = \big(a_1b_1a_2b_2a_3a_4, a_1b_1a_2b_2a_3a_4,\ a_2b_2a_1b_1a_3a_4, a_2b_2a_1b_1a_3a_4\big)$. *The committee* $\{b_1, b_2\}$ *is one of the committees elected in the profile* $\tilde{\succ}^{alt}$. *Finally,* $pod^1(f, \succ) = \dfrac{Sc(\{a_1, a_2\}, f, \succ)}{Sc(\{b_1, b_2\}, f, \succ)} = \dfrac{8}{4} = 2$. *These constraints reduce the score of the selected committee by 1-1/2=50%.*

- *If we consider a separable committee scoring rule where the scoring vector is* $s = (1, 1, 0.5, 0.5, 0.5, 0.5)$ *then the selected committee in the preference profile* $\succ$ *remains* $\{a_1, a_2\}$ *with a score of 8. If we impose the constraints on the top-2-positions (with* $q_1 = q_2 = 1$*) then the new profile will be* $\tilde{\succ}^{top\text{-}k} = \big(a_1b_1a_2a_3a_4b_2, a_1b_1a_2a_3a_4b_2, a_2b_2a_1a_3a_4b_1, a_2b_2a_1a_3a_4b_1\big)$. *Moreover, the committee* $\{b_1, b_2\}$ *is one of the committees elected in the profile* $\tilde{\succ}^{top\text{-}k}$. *Finally,* $pod^1(f, \succ) = \dfrac{Sc(\{a_1, a_2\}, f, \succ)}{Sc(\{b_1, b_2\}, f, \succ)} = \dfrac{8}{4} = 2$.

Example 4.3. *We consider an election where we have 4 candidates such that* $A_1 = \{a_1, a_2\}$ *and* $A_2 = \{b_1, b_2\}$. *We assume that we have 2 voters with the following profile:* $\succ = \big(a_1a_2b_1b_2, a_2a_1b_2b_1\big)$. *We also assume that* $k = 2$ *and that the committee scoring rule is a separable rule with the scoring vector* $s = (1, 1, 0.5, 0.5)$. *The committee selected in profile* $\succ$ *is* $\{a_1, a_2\}$. *If we impose the constraints on the first half of the ranking then we obtain the following profile:* $\tilde{\succ}^{half} = \big(a_1b_1a_2b_2, a_2b_2a_1b_1\big)$. *The committee* $\{b_1, b_2\}$ *is one of the committees elected in the profile* $\tilde{\succ}^{half}$. *Thus,* $pod^1(f, \succ) = \dfrac{Sc(\{a_1, a_2\}, f, \succ)}{Sc(\{b_1, b_2\}, f, \succ)} = \dfrac{4}{2} = 2$.

The boundary of the price of diversity given by the Proposition 1 is too large for some well-known (weakly) separable committee scoring rules. The diversity constraints on preferences defined in this paper have no impact on the first ranking position of each voter. Thus, $pod^1(f, \succ) = 1$ for the k-plurality rule. For any $\succ \in \mathcal{P}^n$, we respectively denote by $pod^1_{\text{alt}}(f, \succ)$, $pod^1_{\text{top-}k}(f, \succ)$ and $pod^1_{\text{half}}(f, \succ)$ the price to pay when diversity constraints on preferences are alternating males and females positions (alt), constraints on the top-k positions (top-k) and constraints on the first half of the ranking (half). The next result suggests the right boundaries of the price of diversity for the well-known committee scoring rules defined above when the constraints on preferences are alternating males and females positions.

Corollary 1. *If we enforce alternating males and females positions on preferences then the boundaries of the price of diversity for the well-known (weakly) separable committee scoring rules can be summarized in Table 1.*

Table 1: Boundary of pod^1_{alt} for the well-known (weakly) separable committee scoring rules

Committee scoring rule	Upper bound of $pod^1_{\text{alt}}(f, \succ)$
k-plurality	1
k-antiplurality	$\dfrac{k}{k-1}$
k-Borda	$\dfrac{2m-k-1-(4/(nk))}{k-1}$
Bloc	$\begin{cases} k & \text{if } k \geq \frac{m+1}{2} \\ +\infty & \text{otherwise} \end{cases}$

Proof of Corollary 1.

i) As mentioned above in the case of k-plurality rule, $pod^1_{\text{alt}}(f, \succ) = 1$ since the diversity constraints do not affect the position of the top ranked candidate in the original preferences.

ii) The upper boundary of the k-antiplurality rule and the Bloc rule follows directly from Proposition 1.

iii) Let us consider the k-Borda rule. Note that, the highest score attributed by voters to a committee is $n.\sum_{\sigma=1}^{k} s_\sigma$ (this is the score of the committee ranked at the top-k-positions by all voters), the second highest score is $(n.\sum_{\sigma=1}^{k} s_\sigma)-1$ and the third highest score is $(n.\sum_{\sigma=1}^{k} s_\sigma)-2$. Moreover, the smallest score attributed by voters to a committee is $n.\sum_{\sigma=1}^{k} s_{m-k+\sigma}$ (this is the score of the committee ranked at the last-k-positions by all voters) and the previous lowest score is $(n.\sum_{\sigma=1}^{k} s_{m-k+\sigma})+1$. The highest boundary of the price of diversity is therefore

$$\frac{n.\sum_{\sigma=1}^{k} s_\sigma}{n.\sum_{\sigma=1}^{k} s_{m-k+\sigma}} = \frac{2m-k-1}{k-1}.$$

The second boundary of the price of diversity could be $\dfrac{(n.\sum_{\sigma=1}^{k} s_\sigma)-1}{n.\sum_{\sigma=1}^{k} s_{m-k+\sigma}}$ if the first one is too large. If the first and the second boundaries of the price of diversity are too large then the boundary of the price of diversity could be

$$\begin{cases} \dfrac{(n.\sum_{\sigma=1}^{k} s_\sigma)-2}{n.\sum_{\sigma=1}^{k} s_{m-k+\sigma}} & \text{if } 2m \geq 3k, \\[2ex] \dfrac{n.\sum_{\sigma=1}^{k} s_\sigma}{(n.\sum_{\sigma=1}^{k} s_{m-k+\sigma})+1} & \text{otherwise} \end{cases}$$

because the inequality

$$\frac{(n.\sum_{\sigma=1}^{k} s_\sigma) - 2}{n.\sum_{\sigma=1}^{k} s_{m-k+\sigma}} \geq \frac{n.\sum_{\sigma=1}^{k} s_\sigma}{(n.\sum_{\sigma=1}^{k} s_{m-k+\sigma}) + 1}$$

holds when $2m \geq 3k$ $\Big($respectively, $\frac{(n.\sum_{\sigma=1}^{k} s_\sigma) - 2}{n.\sum_{\sigma=1}^{k} s_{m-k+\sigma}} < \frac{n.\sum_{\sigma=1}^{k} s_\sigma}{(n.\sum_{\sigma=1}^{k} s_{m-k+\sigma}) + 1}$ when $2m < 3k\Big)$.

Without loss of generality, we assume that $k \leq |A_2| \leq |A_1|$. We obtain the highest price of diversity when all voters rank all males before females in the profile $\succ$. Let us assume that in the profile $\succ$, all voters rank the members of $W \subseteq A_1$ at the top-k-positions, and also rank the members of $\widetilde{W} \subseteq A_2$ at the last-k-positions. For any voter $i \in N$, the increasing sequence $I = (i_1, i_2, \ldots, i_k)$ of the ranks of W members in $\tilde{\succ}_i^{\text{alt}}$ dominates the increasing sequence $J = (j_1, j_2, \ldots, j_k)$ of the ranks of $\widetilde{W}$ members in $\tilde{\succ}_i^{\text{alt}}$. Moreover, the committee W beats the committee $\widetilde{W}$ with at least 4 points in the new profile $\tilde{\succ}^{\text{alt}}$ since, in the new profile, W members occupy the odd positions 1, 3, etc. and $\widetilde{W}$ members occupy the even positions 2, 4, etc. Thus, the first and second boundaries are too large since there is no profile $\succ \in \mathcal{P}^n$ such that the committee W with a score $Sc(W, f, \succ) \geq (n.\sum_{\sigma=1}^{k} s_\sigma) - 1$ is selected and the committee $\widetilde{W}$ with a score $Sc(\widetilde{W}, f, \succ) = n.\sum_{\sigma=1}^{k} s_{m-k+\sigma}$ is also selected in the new profile $\tilde{\succ}^{\text{alt}}$. If one voter moves a member of $\widetilde{W}$ up (That is, when the voter improves his rank) in the profile $\succ$ such that the score of $\widetilde{W}$ in $\succ$ adds by 1 point then the committee W will still beat the committee $\widetilde{W}$ in the profile $\tilde{\succ}^{\text{alt}}$. That is, there is no profile $\succ \in \mathcal{P}^n$ such that W has $n.\sum_{\sigma=1}^{k} s_\sigma$ points in $\succ$ and $\widetilde{W}$ has $(n.\sum_{\sigma=1}^{k} s_{m-k+\sigma})+1$ points in $\succ$. If voters rank W members so that their score decreases by at least 2 then after constraints the score of W will also decrease by at least 4 in the new profile $\tilde{\succ}^{\text{alt}}$ (W and $\widetilde{W}$ could have the same score in $\tilde{\succ}^{\text{alt}}$, and $\widetilde{W}$ could be selected in $\tilde{\succ}^{\text{alt}}$). In order to reach the boundary, some voters must move down some members of W in the profile $\succ$ such that the score of W decreases by at least 2 points. Finally, $pod^1_{\text{alt}}(f, \succ) \leq \frac{(n.\sum_{\sigma=1}^{k} s_\sigma) - 2}{n.\sum_{\sigma=1}^{k} s_{m-k+\sigma}} = \frac{2m - k - 1 - (4/(nk))}{k-1}$.

□

We will now propose an example in order to show that we reach the boundaries given in Table 1.

Example 4.4. i) *By using the k-antiplurality rule, we show in Example 4.2 that we reach the boundary of the price of diversity.*

ii) • *Let us assume that the set of males is $A_1 = \{a_1, a_2, a_3\}$, the set of females is $A_2 = \{b_1, b_2\}$, $k = 3$, and the committee scoring rule is Bloc rule. If $\succ = (a_1a_2a_3b_1b_2,$*

$a_2a_3a_1b_2b_1$), *then* $\tilde{\succ}^{alt} = (a_1b_1a_2b_2a_3, a_2b_2a_3b_1a_1)$. *Moreover,* $W = \{a_1, a_2, a_3\}$ *with a score* $Sc(W, f, \succ) = 6$, *and* $\widetilde{W} = \{a_2, b_1, b_2\}$ *with a score* $Sc(\widetilde{W}, f, \succ) = 2 = Sc(\{a_3, b_1, b_2\}, f, \succ) = Sc(\{a_1, b_1, b_2\}, f, \succ)$. *Thus,* $pod^1_{alt}(f, \succ) \leq \frac{6}{2} = 3$.

- *Let us assume that* $A_1 = \{a_1, a_2\}$, $A_2 = \{b_1, b_2\}$, $k = 2$, *and the committee scoring rule is Bloc rule. If* $\succ = (a_1a_2b_1b_2,\ a_2a_1b_2b_1)$, *then* $\tilde{\succ}^{alt} = (a_1b_1a_2b_2, a_2b_2a_1b_1)$. *Moreover,* $W = \{a_1, a_2\}$ *with a score* $Sc(W, f, \succ) = 4$, *and* $\widetilde{W} = \{b_1, b_2\}$ *with a score* $Sc(\widetilde{W}, f, \succ) = 0$. *Thus,* $pod^1_{alt}(f, \succ) = +\infty$.

iii) *Let us assume that* $A_1 = \{a_1, a_2, a_3, a_4\}$, $A_2 = \{b_1, b_2\}$, $k = 2$, *and the committee scoring rule is the* k*-Borda rule. If* $\succ = (a_1a_2a_3a_4b_1b_2,\ a_4a_2a_1a_3b_2b_1)$, *then* $\tilde{\succ}^{alt} = (a_1b_1a_2b_2a_3a_4,\ a_4b_2a_2b_1a_1a_3)$. *Moreover,* $W = \{a_1, a_2\}$ *is selected in* $\succ$ *with a score* $Sc(W, f, \succ) = 16$, *and* $\widetilde{W} = \{b_1, b_2\}$ *is selected in* $\tilde{\succ}^{alt}$ *and its score in the original profile* $\succ$ *is* $Sc(\widetilde{W}, f, \succ) = 2$. *Thus,* $pod^1_{alt}(f, \succ) \leq \frac{16}{2} = 8 = \frac{2*6-2-1-4/4}{2-1}$.

We respectively denote by f^{Plu}, f^{Ant}, f^{Bloc} and f^{Borda} the committee scoring functions associated to the k-plurality rule, the k-antiplurality rule, the Bloc rule and the k-Borda rule. Using Table 1, we distinguish the well-known committee scoring rules in terms of the highest price of diversity to pay as shown in the remark below.

Remark 1.

i) If $m = 3$, *then*

$\max_{\succ \in \mathcal{P}^n} pod^1_{alt}(f^{Plu}, \succ) \leq \max_{\succ \in \mathcal{P}^n} pod^1_{alt}(f^{Ant}, \succ) \leq \max_{\succ \in \mathcal{P}^n} pod^1_{alt}(f^{Bloc}, \succ) \leq \max_{\succ \in \mathcal{P}^n} pod^1_{alt}(f^{Borda}, \succ)$.

ii) If $m \geq 4$, *then*

$\max_{\succ \in \mathcal{P}^n} pod^1_{alt}(f^{Plu}, \succ) \leq \max_{\succ \in \mathcal{P}^n} pod^1_{alt}(f^{Ant}, \succ) \leq \max_{\succ \in \mathcal{P}^n} pod^1_{alt}(f^{Borda}, \succ) \leq \max_{\succ \in \mathcal{P}^n} pod^1_{alt}(f^{Bloc}, \succ)$.

For a given election, Remark 1 classifies the various well-known (weakly) separable committee scoring rules according to the highest price of diversity to pay when the diversity constraints are alternating males and females positions on individual preferences. Now, using (weakly) separable committee scoring rules and diversity constraints on the top-k-positions of each preference, we also propose the right boundaries of the price of diversity as shown below.

Corollary 2. *If we enforce the diversity constraints on the top-k-positions of individual preferences then the boundaries of the price of diversity for the well-known (weakly) separable committee scoring rules can be summarized in Table 2:*

Table 2: Boundary of $pod^1_{\text{top-}k}$ for well-known (weakly) separable committee scoring rules

Committee scoring rule	Upper bound of $pod^1_{\text{top-}k}(f,\succ)$
k-plurality	1
k-antiplurality	$\frac{k}{k-1}$
Bloc	$\begin{cases} k & \text{if } k \geq \frac{m+1}{2} \\ +\infty & \text{otherwise} \end{cases}$
k-Borda	$\frac{2m-k-1}{k-1}$

Proof of Corollary 2. The upper boundary of the k-antiplurality rule, the k-Borda rule and the Bloc rule are respectively given by Proposition 1. □

Through the following example, we show that we reach the boundaries given in Table 2.

Example 4.5. i) *Let us assume that* $A_1 = \{a_1, a_2\}$*,* $A_2 = \{b_1\}$*,* $k = 2$*,* $q_1 = q_2 = 1$*, and the committee scoring rule is the* k*-antiplurality rule. If* $\succ= (a_1a_2b_1,\ a_2a_1b_1)$*, then* $\tilde{\succsim}^{\textit{top-k}} = (a_1b_1a_2, a_2b_1a_1)$*. Moreover,* $W = \{a_1, a_2\}$ *with a score* $Sc(W, f, \succ) = 4$*, and* $\widetilde{W} = \{a_1, b_1\}$ *with a score* $Sc(\widetilde{W}, f, \succ) = 2$*. Thus,* $pod^1_{\textit{top-k}}(f, \succ) \leq \frac{4}{2} = 2 = \frac{2}{2-1}$*.*

ii)

- *Let us assume that* $A_1 = \{a_1, a_2, a_3\}$*,* $A_2 = \{b_1, b_2\}$*,* $k = 3$*,* $q = 1$*,* $q_2 = 2$*, and the committee scoring rule is the Bloc rule. If* $\succ= (a_1a_2a_3b_1b_2,\ a_2a_3a_1b_2b_1,\ a_3a_1a_2b_1b_2)$*, then* $\tilde{\succsim}^{\textit{top-k}} = (a_1b_1b_2a_2a_3,\ a_2b_2b_1a_3a_1,\ a_3b_1b_2a_1a_2)$*. Moreover,* $W = \{a_1, a_2, a_3\}$ *with a score* $Sc(W, f, \succ) = 9$*, and* $\widetilde{W} = \{a_2, b_1, b_2\}$ *with a score* $Sc(\widetilde{W}, f, \succ) = 3 = Sc(\{a_3, b_1, b_2\}, f, \succ) = Sc(\{a_1, b_1, b_2\}, f, \succ)$*. Thus,* $pod^1_{\textit{top-k}}(f, \succ) \leq \frac{9}{3} = 3$*.*
- *Let us assume that* $A_1 = \{a_1, a_2\}$*,* $A_2 = \{b_1, b_2\}$*,* $k = 2$*,* $q_1 = q_2 = 1$*, and the committee scoring rule is the Bloc rule. If* $\succ= (a_1a_2b_1b_2,\ a_2a_1b_2b_1)$*, then* $\tilde{\succsim}^{\textit{top-k}} = (a_1b_1a_2b_2, a_2b_2a_1b_1)$*. Moreover,* $W = \{a_1, a_2\}$ *with a score* $Sc(W, f, \succ) = 4$*, and* $\widetilde{W} = \{b_1, b_2\}$ *with a score* $Sc(\widetilde{W}, f, \succ) = 0$*. Thus,* $pod^1_{\textit{top-k}}(f, \succ) = +\infty$*.*

iii) *Let us assume that* $A_1 = \{a_1, a_2, a_3, a_4\}$*,* $A_2 = \{b_1, b_2, b_3, b_4\}$*,* $k = 4$*,* $q_1 = 1$*,* $q_2 = 3$*, and the committee scoring rule is the* k*-Borda rule. If* $\succ= (a_1a_2a_3a_4b_1b_2b_3b_4,\ a_2a_3a_4a_1b_2b_3b_4b_1,\ a_3a_4a_1a_2b_3b_4b_1b_2,\ a_4a_1a_2a_3b_4b_1b_2b_3)$*, then* $\tilde{\succsim}^{\textit{top-k}} = (a_1b_1b_2b_3a_2a_3a_4b_4,\ a_2b_2b_3b_4a_3a_4a_1b_1,\ a_3b_3b_4b_1a_4a_1a_2b_2,\ a_4b_4b_1b_2a_1a_2a_3b_3)$*. Moreover,* $W = \{a_1, a_2, a_3, a_4\}$ *with a score* $Sc(W, f, \succ) = 88$*, and* $\widetilde{W} = \{b_1, b_2, b_3, b_4\}$ *with a score* $Sc(\widetilde{W}, f, \succ) = 24$*. Thus,* $pod^1_{\textit{top-k}}(f, \succ) \leq \frac{88}{24} = \frac{11}{3} = \frac{2*8-4-1}{4-1}$*.*

Note that, Remark 1 also classifies the different well-known (weakly) separable committee scoring rules in terms of the highest price of diversity to pay when the diversity constraints are the constraints on the top-k-positions of individual preferences.

In the next corollary, we consider the diversity constraints on the first half of the ranking.

Corollary 3. *If we enforce the diversity constraints on the first half of the ranking then the boundaries of the price of diversity for the well-known (weakly) separable committee scoring rules can be summarized in Table 3:*

Table 3: Boundary of pod^1_{half} for well-known (weakly) separable committee scoring rules

Committee scoring rule	Upper bound of $pod^1_{\text{half}}(f, \succ)$
k-plurality	1
k-antiplurality	1
Bloc	$\begin{cases} k & \text{if } k \geq \frac{m+1}{2} \\ +\infty & \text{otherwise} \end{cases}$
k-Borda	$\dfrac{2m-k-1-4/(nk)}{k-1}$

Proof of Corollary 3.

i) When we use the k-antiplurality rule, the diversity constraints on the first half of the ranking do not affect the candidates occupying the last positions. So, the elected committee in both profiles (the original profile and the new profile) remains the same.

ii) The proof of Proposition 1 for the Bloc rule and the k-Borda rule remains valid here.

□

Through the following example, we show that the boundaries given in Table 3 are reached.

Example 4.6. i)

- *Let us assume that $A_1 = \{a_1, a_2, a_3\}$, $A_2 = \{b_1, b_2\}$, $k = 3$, and the committee scoring rule is the Bloc rule. If $\succ = (a_1a_2a_3b_1b_2,\ a_3a_2a_1b_2b_1)$, then $\tilde{\succ}^{half} = (a_1b_1a_2a_3b_2,\ a_3b_2a_2a_1b_1)$. Moreover, $W = \{a_1, a_2, a_3\}$ with a score $Sc(W, f, \succ) = 6$ and $\widetilde{W} = \{a_2, b_1, b_2\}$ with a score $Sc(\widetilde{W}, f, \succ) = 2 = Sc(\{a_3, b_1, b_2\}, f, \succ) = Sc(\{a_1, b_1, b_2\}, f, \succ)$. Thus, $pod^1_{half}(f, \succ) \leq \frac{6}{2} = 3$.*
- *Let us assume that $A_1 = \{a_1, a_2\}$, $A_2 = \{b_1, b_2\}$, and the committee scoring rule is the Bloc rule. If $\succ = (a_1a_2b_1b_2,\ a_2a_1b_2b_1)$, then $\tilde{\succ}^{half} = (a_1b_1a_2b_2, a_2b_2a_1b_1)$. Moreover, $W = \{a_1, a_2\}$ with a score $Sc(W, f, \succ) = 4$ and $\widetilde{W} = \{b_1, b_2\}$ with a score $Sc(\widetilde{W}, f, \succ) = 0$. Thus, $pod^1_{half}(f, \succ) = +\infty$.*

ii) *Let us assume that* $A_1 = \{a\}$*,* $A_2 = \{b_1, b_2\}$*,* $k = 2$*, and the committee scoring rule is the k-Borda rule. If* $\succ = (b_1b_2a, b_2b_1a)$*, then* $\widetilde{\succ}^{top\text{-}k} = (b_1ab_2,\ b_2ab_1)$*. Moreover,* $W = \{b_1, b_2\}$ *with a score* $Sc(W, f, \succ) = 6$*, and* $\widetilde{W} \in \big\{\{b_1, a\}, \{b_2, a\}\big\}$ *with a score* $Sc(\widetilde{W}, f, \succ) = 3$*. Thus,* $pod^1_{top\text{-}k}(f, \succ) \leq \frac{6}{3} = 2 = \frac{2*3-2-1-4/(2*2)}{2-1}$.

Remark 1 also classifies the different well-known (weakly) separable committee scoring rules in terms of the highest price of diversity to pay when we require the diversity constraints on the first half of the ranking.

4.2. Boundary of the price of diversity and classification of diversity constraints

Proposition 1 does not depend on the constraints imposed on individual preferences. In other words, the three diversity constraints introduced in this paper have the same upper bound in terms of the price of diversity when we apply Proposition 1. From Corollaries 1, 2 and 3, we can remark that some well-known (weakly) separable committee scoring rules do not have the same upper bound of the price of diversity when the diversity constraints on preferences change. It is therefore necessary to propose an alternative boundary of the price of diversity depending on the diversity constraints so that we can exhibit a domain for distinguishing these constraints.

4.2.1. The alternation of males and females positions

In the following result, we propose a second boundary of the price of diversity for the family of (weakly) separable committee scoring rules when the diversity constraints on preferences are alternating males and females positions.

Proposition 2. *Let* f *be a (weakly) separable committee scoring function and* s *a scoring vector of* f*. For any profile* $\succ \in \mathcal{P}^n$*,*

$$1 \leq pod^1_{alt}(f, \succ) \leq \frac{\sum_{\sigma=1}^{k} s_\sigma}{\frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=1}^{k}\big(s_{\sigma+1} - s_{m-k+\sigma}\big)}$$

if

$$\frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=1}^{k}\big(s_{\sigma+1} - s_{m-k+\sigma}\big) > 0,$$

and $pod^1(f, \succ) \leq +\infty$*, otherwise.*

Proof of Proposition 2. Let f be a (weakly) separable committee scoring function and s a scoring vector of f. Let W be the selected committee in profile $\succ \in \mathcal{P}^n$ without considering diversity constraints on preferences, and $\widetilde{W}$ the committee selected in profile $\widetilde{\succ}^{\text{alt}}$ when alternating males and females positions on individual preferences are taken into account. Let $\xi = \min_{\sigma\in[m-1]} \big(s_\sigma - s_{\sigma+1}\big)$ be the smallest score lost by a candidate when a voter move that candidate down on his/her ranking. Also, let μ_i (respectively, ν_i) be the number of members of $\widetilde{W}$ who move down[9] (respectively, up[10]) in the new preference $\widetilde{\succ}_i^{\text{alt}}$ of voter i. Let ϱ_i be the number of $\widetilde{W}$ members whose their positions do not change when voter i votes according to $\widetilde{\succ}_i^{\text{alt}}$ instead of $\succ_i$.

The variation of the score of $\widetilde{W}$ in the two profiles is given by

$$\Delta_f(\widetilde{W},\widetilde{\succ}^{\text{alt}},\succ) := Sc(\widetilde{W},f,\widetilde{\succ}^{\text{alt}}) - Sc(\widetilde{W},f,\succ) \leq \sum_{i\in N}\sum_{\sigma=1}^{\nu_i}\big(s_{\sigma+1} - s_{m-\nu_i+\sigma}\big) - \xi\sum_{i\in N}\mu_i.$$

That is,

$$Sc(\widetilde{W},f,\succ) \geq Sc(\widetilde{W},f,\widetilde{\succ}^{\text{alt}}) + \xi\sum_{i\in N}\mu_i - \sum_{i\in N}\sum_{\sigma=1}^{\nu_i}\big(s_{\sigma+1} - s_{m-\nu_i+\sigma}\big).$$

Note that, the variation of the score of the "ϱ_i candidates"[11] for each voter i is 0 since their positions do not change. We obtain the boundary of $\Delta_f(\widetilde{W},\widetilde{\succ}^{\text{alt}},\succ)$ by assuming that for each voter i, the "ν_i candidates"[12] occupy the last ν_i-positions in $\succ_i$ and then occupy at most position 2, 4, ..., and $2\nu_i$ in $\widetilde{\succ}_i^{\text{alt}}$ (with $2\nu_i < m$). The score of the candidates occupying positions 2, 4, ..., and $2\nu_i$ in $\widetilde{\succ}_i^{\text{alt}}$ can be equal to the score of the candidates occupying position 2 to position ν_i+1 in $\widetilde{\succ}_i^{\text{alt}}$.

Since $\widetilde{W}$ is the winning committee in $\widetilde{\succ}^{\text{alt}}$, we have $Sc(\widetilde{W},f,\widetilde{\succ}^{\text{alt}}) \geq \frac{nk\sum_{\sigma=1}^m s_\sigma}{m}$. In fact, let us assume by contradiction that $Sc(\widetilde{W},f,\widetilde{\succ}^{\text{alt}}) < \frac{nk\sum_{\sigma=1}^m s_\sigma}{m}$. There exists a candidate $a_0\in\widetilde{W}$ such that $Sc(a_0,s,\widetilde{\succ}^{\text{alt}}) < \frac{n\sum_{\sigma=1}^m s_\sigma}{m}$. Moreover, $Sc(A\backslash\widetilde{W},f,\widetilde{\succ}^{\text{alt}}) > \frac{nk\sum_{\sigma=1}^m s_\sigma}{m}$. This implies that there exists a candidate $b\in A\backslash\widetilde{W}$ such that $Sc(b,s,\widetilde{\succ}^{\text{alt}}) > \frac{n\sum_{\sigma=1}^m s_\sigma}{m}$. This is absurd since $Sc(b,s,\widetilde{\succ}^{\text{alt}}) > Sc(a_0,s,\widetilde{\succ}^{\text{alt}})$, $a_0\in\widetilde{W}$, and $b\in A\backslash\widetilde{W}$.

So, we have

$$\begin{aligned} Sc(\widetilde{W},f,\succ) &\geq \frac{nk\sum_{\sigma=1}^m s_\sigma}{m} + \xi\sum_{i\in N}\mu_i - \sum_{i\in N}\sum_{\sigma=1}^{\nu_i}\big(s_{\sigma+1} - s_{m-\nu_i+\sigma}\big)\\ &\geq \frac{nk\sum_{\sigma=1}^m s_\sigma}{m} - \sum_{i\in N}\sum_{\sigma=1}^{\nu_i}\big(s_{\sigma+1} - s_{m-\nu_i+\sigma}\big). \end{aligned}$$

[9]That is, when their ranks decrease or when they are moved to the right in the voter's ranking.
[10]That is, when their ranks improve or when they are moved to the left in the voter's ranking.
[11]That is, those whose their positions do not change in $\widetilde{\succ}_i^{\text{alt}}$ and $\succ_i$.
[12]In other words, candidates whose ranking improves when voter i moves from $\succ_i$ to $\widetilde{\succ}_i^{\text{alt}}$.

We obtain the lowest boundary of $Sc(\widetilde{W}, f, \succ)$ when $\nu_i = k$ for each voter i. Then,

$$Sc(\widetilde{W}, f, \succ) \geq \frac{nk \sum_{\sigma=1}^{m} s_\sigma}{m} - n \sum_{\sigma=1}^{k} \big(s_{\sigma+1} - s_{m-k+\sigma}\big).$$

The members of $\widetilde{W}$ occupy at most positions 2, 4, ..., and $2k$ in the preference $\widetilde{\succ}_i^{\text{alt}}$ (with $2k \leq m$). There are some (weakly) separable committee scoring rules such that, the score of $\widetilde{W}$ in $\widetilde{\succ}_i^{\text{alt}}$ is equal to the score of the committee occupying position 2 to position $k+1$ in $\widetilde{\succ}_i^{\text{alt}}$. Figure 1 presents the worst position of committee $\widetilde{W}$ in the preference $\succ_i$ of a voter i without considering diversity constraints and the best position of the committee having the same score with $\widetilde{W}$ in the preference $\widetilde{\succ}_i^{\text{alt}}$ of voter i when alternating males and females positions are taken into account.

Figure 1: Position of $\widetilde{W}$ before and after the alternation constraints

Remember that $Sc(W, f, \succ) \leq n \sum_{\sigma=1}^{k} s_\sigma$. We conclude that

$$pod^1_{\text{alt}}(f, \succ) \leq \frac{\sum_{\sigma=1}^{k} s_\sigma}{\frac{k \sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=1}^{k} \big(s_{\sigma+1} - s_{m-k+\sigma}\big)}$$

if

$$\frac{k \sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=1}^{k} \big(s_{\sigma+1} - s_{m-k+\sigma}\big) > 0,$$

and $pod^1(f, \succ) \leq +\infty$, otherwise. □

The following example helps us to show that the boundary given by Proposition 2 is reached when the committee scoring rule is (weakly) separable.

Example 4.7. *We consider an election where we have 2 voters, 4 candidates such that* $A_1 = \{a_1, a_2\}$ *and* $A_2 = \{b_1, b_2\}$. *We assume that* $k = 2$ *and the selection rule is a separable committee scoring rule with a scoring vector* $s = (1, 1, 0.5, 0.5)$. *By using the profile* $\succ = (a_1a_2b_1b_2, a_2a_1b_2b_1)$, *the winning committee is* $W = \{a_1, a_2\}$. *In the profile* $\widetilde{\succ}^{alt} = (a_1b_1a_2b_2, a_2b_2a_1b_1)$, *the committee* $\widetilde{W} = \{b_1, b_2\}$ *is among the winning committees. Thus,* $pod^1_{alt}(f, \succ) = \frac{Sc(W,f,\succ)}{Sc(\widetilde{W},f,\succ)} = \frac{2*2}{2*1} = 2$ *and* $\frac{\sum_{\sigma=1}^{k} s_\sigma}{\frac{k \sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=1}^{k} \big(s_{\sigma+1} - s_{m-k+\sigma}\big)} = \frac{2}{\frac{2*3}{4} - (1.5 - 1)} = 2.$

4.2.2. The top-k-positions constraint

The next proposition gives the highest price of diversity for the family of (weakly) separable committee scoring rules when we enforce the constraints on the top-k-positions on preferences.

Proposition 3. *Let f be a (weakly) separable committee scoring function and s a scoring vector of f. For any profile $\succ \in \mathcal{P}^n$,*

$$1 \leq pod^1_{top\text{-}k}(f, \succ) \leq \frac{\sum_{\sigma=1}^{k} s_\sigma}{\frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=1+\min_{j\in[2]} q_j}^{k} \left(s_\sigma - s_{m-k+\sigma}\right)}$$

if

$$\frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=1+\min_{j\in[2]} q_j}^{k} \left(s_\sigma - s_{m-k+\sigma}\right) > 0,$$

and $pod^1(f, \succ) \leq +\infty$, otherwise.

Proof of Proposition 3. The approach remains the same as the one used in the proof of Proposition 2. We obtain the boundary of $\Delta_f(\widetilde{W}, \widetilde{\succ}^{\text{top-}k}, \succ)$ by assuming that for each voter i, the "ν_i candidates" occupy the last ν_i-positions in $\succ_i$ and then occupy at most position $1 + \min_{j\in[2]} q_j \leq k - \nu_i + 1$ to position k in $\widetilde{\succ}_i^{\text{top-}k}$. The lowest boundary of $Sc(\widetilde{W}, f, \succ)$ is obtained when $\nu_i = k - \min_{j\in[2]} q_j$. So, we have

$$Sc(\widetilde{W}, f, \succ) \geq \frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=1+\min_{j\in[2]} q_j}^{k} \left(s_\sigma - s_{m-k+\sigma}\right).$$

Figure 2 presents the worst positions of $\widetilde{W}$ members in $\succ_i$ and the best positions of some $\widetilde{W}$ members in $\widetilde{\succ}_i^{\text{top-}k}$ (or the position of the committee/candidates equivalent to $\widetilde{W}$ in terms of scores given by voter i in the preference $\succ_i$ without considering diversity constraints, and in the preference $\widetilde{\succ}_i^{\text{top-}k}$ when the constraints on the top-k-positions are taken into account).

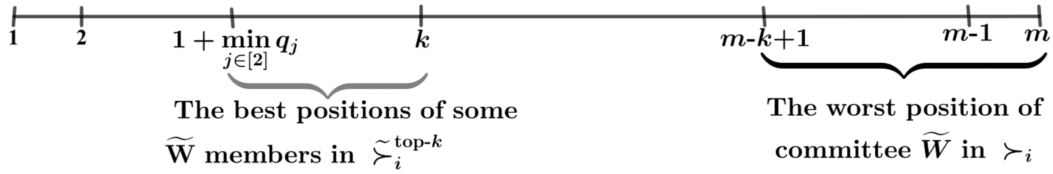

Figure 2: Positions of $\widetilde{W}$ members before and after the top-k-position constraint

□

In the next example, we show that the boundary given by Proposition 3 is reached for the family of (weakly) separable committee scoring rules.

Example 4.8. *We consider an election with 5 voters and 10 candidates such that* $A_1 = \{a_1, a_2, a_3, a_4, a_5\}$ *and* $A_2 = \{b_1, b_2, b_3, b_4, b_5\}$*. We assume that* $k = 5$*,* $q_1 = 2$*,* $q_2 = 3$*, and the selection rule is a separable committee scoring rule with a scoring vector* $s = (2, 1.5, 1.5, 1.5, 1, 0.5, 0.5, 0.5, 0.5, 0.5)$*. By using the profile* $\succ = (a_1a_2a_3a_4a_5b_1b_2b_3b_4b_5,$ $a_2a_3a_4a_5a_1b_2b_3b_4b_5b_1,$ $a_3a_4a_5a_1a_2b_3b_4b_5b_1b_2,$ $a_4a_5a_1a_2a_3b_4b_5b_1b_2b_3,$ $a_5a_1a_2a_3a_4b_5b_1b_2b_3b_4)$*, the winning committee is* $W = \{a_1, a_2, a_3, a_4, a_5\}$*. In the profile* $\tilde{\succ}^{top\text{-}k} = (a_1a_2b_1b_2b_3a_3a_4a_5b_4b_5,$ $a_2a_3b_2b_3b_4a_4a_5a_1b_5b_1,$ $a_3a_4b_3b_4b_5a_5a_1a_2b_1b_2,$ $a_4a_5b_4b_5b_1a_1a_2a_3b_2b_3,$ $a_5a_1b_5b_1b_2a_2a_3a_1b_3b_1)$*, the winning committee is* $\widetilde{W} = \{b_1, b_2, b_3, b_4, b_5\}$*. Thus,* $pod^1_{alt}(f, \succ) = \frac{Sc(W,f,\succ)}{Sc(\widetilde{W},f,\succ)} = \frac{5*7.5}{5*2.5} =$ $\frac{\sum_{\sigma=1}^{k} s_\sigma}{\frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=1+\min_{j\in[2]} q_j}^{k}\left(s_\sigma - s_{m-k+\sigma}\right)} = \frac{7.5}{\frac{5*10}{10} - (4-1.5)} = 3.$

4.2.3. The constraints on the first half of the ranking

Our next result gives the highest price of diversity for the family of (weakly) separable committee scoring rules when we enforce the constraints on the top half of the ranking. For a given size k, the maximal number of selected candidates in $\widetilde{W}$ who can move up when voter i changes his/her ranking $\succ_i$ to $\tilde{\succ}_i^{\text{half}}$ is $\min_{j\in[2]}\left\lceil\frac{|A_j|}{2}\right\rceil$. We denote by $\theta := \sum_{j\in[2]}\left\lceil\frac{|A_j|}{2}\right\rceil$ the position separating the ranking into two "equal" parts. This position is either the median position or the median position plus one, depending on the number of males and females.

Proposition 4. *Let f be a (weakly) separable committee scoring function and s a scoring vector of f. For any profile $\succ \in \mathcal{P}^n$,*

i) When $k \le \min_{j\in[2]}\left\lceil\frac{|A_j|}{2}\right\rceil$, then we have

$$1 \le pod^1_{half}(f, \succ) \le \frac{\sum_{\sigma=1}^{k} s_\sigma}{\frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=1}^{k}\left(s_{\theta-k+\sigma} - s_{m-k+\sigma}\right)},$$

if

$$\frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=1}^{k}\left(s_{\theta-k+\sigma} - s_{m-k+\sigma}\right) > 0,$$

and $pod^1(f, \succ) \le +\infty$ otherwise.

ii) When $k > \min_{j\in[2]}\left\lceil\frac{|A_j|}{2}\right\rceil$, then we have

$$1 \le pod^1_{half}(f, \succ) \le \frac{\sum_{\sigma=1}^{k} s_\sigma}{\frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=\max_{j\in[2]}\left\lceil\frac{|A_j|}{2}\right\rceil+1}^{\theta}\left(s_\sigma - s_{m-\theta+\sigma}\right)},$$

if

$$\frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=\max_{j\in[2]}\left\lceil\frac{|A_j|}{2}\right\rceil+1}^{\theta}\left(s_\sigma - s_{m-\theta+\sigma}\right) > 0,$$

and $pod^1(f, \succ) \leq +\infty$ *otherwise.*

Proof of Proposition 4. The approach remains the same as the one used in the proof of Proposition 2.

i) If $k \leq \min_{j\in[2]} \lceil \frac{|A_j|}{2} \rceil$ then Figure 3 describes the position of $\widetilde{W}$ in the preference $\succ_i$. The worst situation is the case where the committee $\widetilde{W}$ is (or equivalent to) the committee occupying the last-k-positions in terms of scores given by voter i in $\succ_i$.

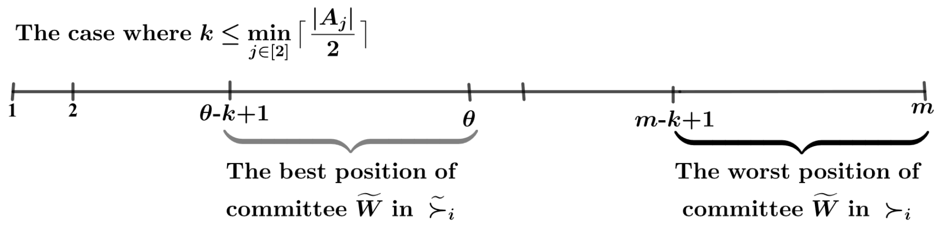

Figure 3: Position of $\widetilde{W}$ before and after the constraints on the first half of the ranking

So, the lowest boundary of $Sc(\widetilde{W}, f, \succ)$ is given by

$$Sc(\widetilde{W}, f, \succ) \geq \frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=1}^{k} \big(s_{\theta-k+\sigma} - s_{m-k+\sigma}\big).$$

ii) If $k > \min_{j\in[2]} \lceil \frac{|A_j|}{2} \rceil$, then Figure 4 illustrates the worst positions of $\widetilde{W}$ in the preference $\succ_i$ of voter i. At most $\min_{j\in[2]} \lceil \frac{|A_j|}{2} \rceil$ members of $\widetilde{W}$ can move up by voter i in his/her new preference $\tilde{\succ}_i^{\text{half}}$. The score of $\widetilde{W}$ in $\succ_i$ can be equal to the score of the last-k-candidates in $\succ_i$.

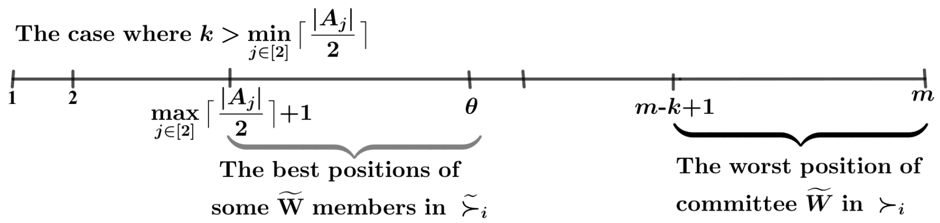

Figure 4: Position of $\widetilde{W}$ before and after the constraints on the first half of the ranking

Thus, the lowest boundary of $Sc(\widetilde{W}, f, \succ)$ is given by

$$Sc(\widetilde{W}, f, \succ) \geq \frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=\max_{j\in[2]} \lceil \frac{|A_j|}{2} \rceil + 1}^{\theta} \big(s_\sigma - s_{m-\theta+\sigma}\big).$$

□

In the example below, we show that the boundary given by Proposition 4 is reached for the family of (weakly) separable committee scoring rules.

Example 4.9. *We consider an election where we have 4 voters and 8 candidates such that* $A_1 = \{a_1, a_2, a_3, a_4\}$ *and* $A_2 = \{b_1, b_2, b_3, b_4\}$*. We assume that* $k = 4$ *and the selection rule is a separable committee scoring rule with a scoring vector* $s = (1.5, 1.5, 1.5, 1.5, 0.5, 0.5, 0.5, 0.5)$*. By using the following profile* $\succ = (a_1a_2a_3a_4b_1b_2b_3b_4,\ a_2a_3a_4a_1b_2b_3b_4b_1,\ a_3a_4a_1a_2b_3b_4b_1b_2,\ a_4a_1a_2a_3b_4b_1b_2b_3,\ a_1a_2a_3a_4b_1b_2b_3b_4)$*, the selected committee is* $W = \{a_1, a_2, a_3, a_4\}$*. In the profile* $\widetilde{\succ}^{half} = (a_1a_2b_1b_2a_3a_4b_3b_4,\ a_2a_3b_2b_3a_4a_1b_4b_1,\ a_3a_4b_3b_4a_1a_2b_1b_2,\ a_4a_1b_4b_1a_2a_3b_2b_3,\ a_1a_2b_1b_2a_3a_1b_3b_1)$*, the selected committee is* $\widetilde{W} = \{b_1, b_2, b_3, b_4\}$*. Thus,* $pod^1_{alt}(f, \succ) = \frac{Sc(W,f,\succ)}{Sc(\widetilde{W},f,\succ)} = \frac{4*6}{4*2} = \frac{\sum_{\sigma=1}^{k} s_\sigma}{\frac{k\sum_{\sigma=1}^{m} s_\sigma}{m} - \sum_{\sigma=\max_{j\in[2]}\lceil\frac{|A_j|}{2}\rceil+1}^{\theta}\left(s_\sigma - s_{m-\theta+\sigma}\right)} = \frac{6}{\frac{4*8}{8} - (3-1)} = 3.$

By using Propositions 2, 3 and 4, we can distinguish the three diversity constraints as follows:

Corollary 4. *Let f be a (weakly) separable committee scoring function. The following statements hold:*

i) $\max_{\succ\in\mathcal{P}^n} pod^1_{alt}(f, \succ) \geq \max_{\succ\in\mathcal{P}^n} pod^1_{top\text{-}k}(f, \succ)$.

ii) $\max_{\succ\in\mathcal{P}^n} pod^1_{alt}(f, \succ) \geq \max_{\succ\in\mathcal{P}^n} pod^1_{half}(f, \succ)$.

iii) If $k \geq \theta$ *and* $\max_{j\in[2]}\lceil\frac{|A_j|}{2}\rceil \geq \min_{j\in[2]} q_j$*, then*

$$\max_{\succ\in\mathcal{P}^n} pod^1_{top\text{-}k}(f, \succ) \geq \max_{\succ\in\mathcal{P}^n} pod^1_{half}(f, \succ).$$

iv) If $k \leq \theta$ *and* $\max_{j\in[2]}\lceil\frac{|A_j|}{2}\rceil \leq \min_{j\in[2]} q_j$*, then*

$$\max_{\succ\in\mathcal{P}^n} pod^1_{half}(f, \succ) \geq \max_{\succ\in\mathcal{P}^n} pod^1_{top\text{-}k}(f, \succ).$$

Proof of Corollary 4. Let f be a (weakly) separable committee scoring function and s a scoring vector of f.

i) By using Propositions 2 and 3, the relation

$$\max_{\succ\in\mathcal{P}^n} pod^1_{\text{alt}}(f, \succ) \geq \max_{\succ\in\mathcal{P}^n} pod^1_{\text{top-}k}(f, \succ)$$

implies that

$$\sum_{\sigma=1}^{k}\left(s_{\sigma+1} - s_{m-k+\sigma}\right) \geq \sum_{\sigma=1+\min_{j\in[2]} q_j}^{k}\left(s_\sigma - s_{m-k+\sigma}\right). \tag{7}$$

Equation (7) is equivalent to

$$\sum_{\sigma=1}^{k} s_{\sigma+1} - \sum_{\sigma=1+\min_{j\in[2]} q_j}^{k} s_\sigma \geq \sum_{\sigma=1}^{k} s_{m-k+\sigma} - \sum_{\sigma=1+\min_{j\in[2]} q_j}^{k} s_{m-k+\sigma}. \tag{8}$$

The expression to the left of the symbol " $\geq$" in Equation (8) is illustrated in Figure 5 and corresponds to the variation of $\widetilde{W}$ colored in gray (i.e., the score of the candidates occupying the large gray brace minus the score of those occupying the small gray brace). While, the expression to the right of the symbol " $\geq$" in Equation (8) corresponds to the variation of $\widetilde{W}$ colored in black (i.e., the score of the candidates occupying the large black brace minus the score of those occupying the small black brace). It is clear that the first variation (in gray) is greater than the second one (in black).

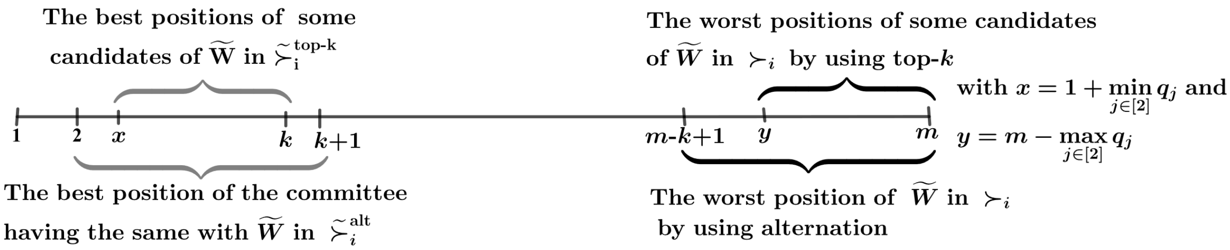

Figure 5: Alternation constraints and constraints on the top-k-positions

ii) By using the same reasoning given in i), we show that when $k \leq \min_{j\in[2]} \left\lceil \frac{|A_j|}{2} \right\rceil$, the inequality $\sum_{\sigma=1}^{k} \left(s_{\sigma+1} - s_{m-k+\sigma}\right) \geq \sum_{\sigma=1}^{k} \left(s_{\theta-k+\sigma} - s_{m-k+\sigma}\right)$ holds. We can illustrate this in Figure 6.

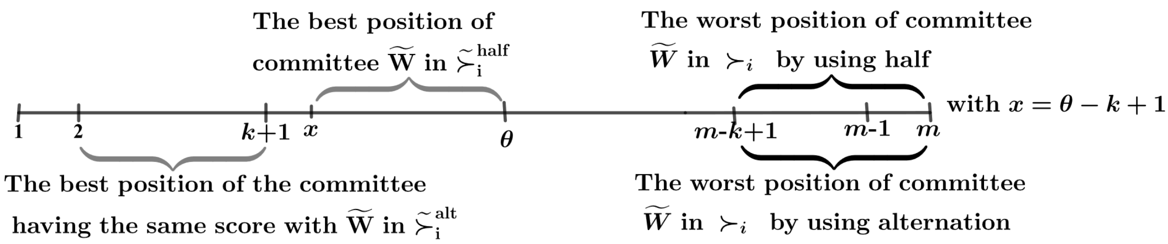

Figure 6: Alternation constraints and top half diversity constraints

Moreover, the following inequality also holds when $k > \min_{j\in[2]} \left\lceil \frac{|A_j|}{2} \right\rceil$,

$$\sum_{\sigma=1}^{k} \left(s_{\sigma+1} - s_{m-k+\sigma}\right) \geq \sum_{\sigma=\max_{j\in[2]} \left\lceil \frac{|A_j|}{2} \right\rceil + 1}^{\theta} \left(s_{\sigma} - s_{m-\theta+\sigma}\right).$$

This is illustrated in Figure 7.

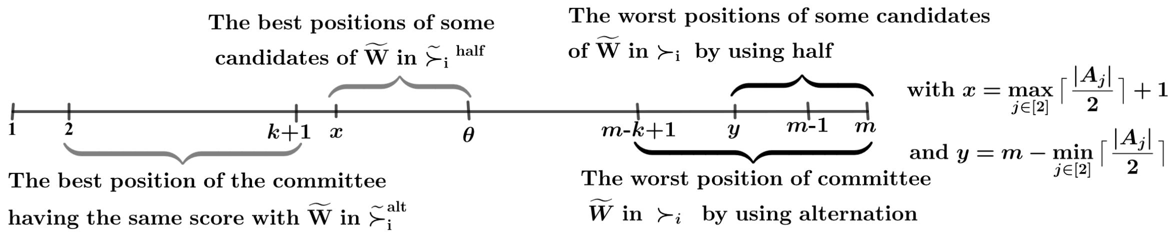

Figure 7: Alternation constraints and top half diversity constraints

iii) By using Figure 8 and the approach given above, we have the result for $k \geq \theta$ and $\max_{j\in[2]} \lceil \frac{|A_j|}{2} \rceil \geq \min_{j\in[2]} q_j$.

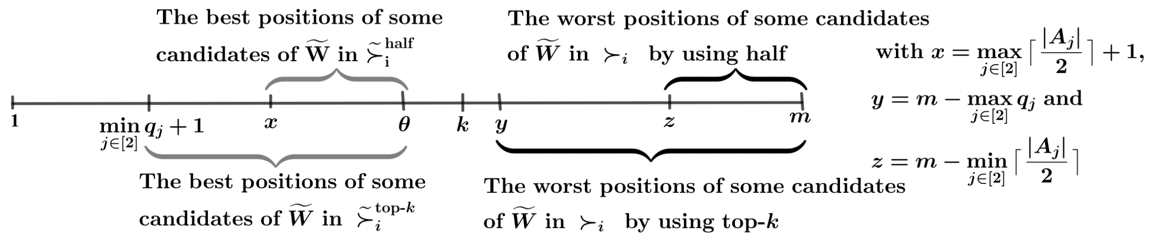

Figure 8: top-k and top half diversity constraints

iv) By analyzing Figure 9 for the case $k \leq \theta$ and $\max_{j\in[2]} \lceil \frac{|A_j|}{2} \rceil \leq \min_{j\in[2]} q_j$, we have the result.

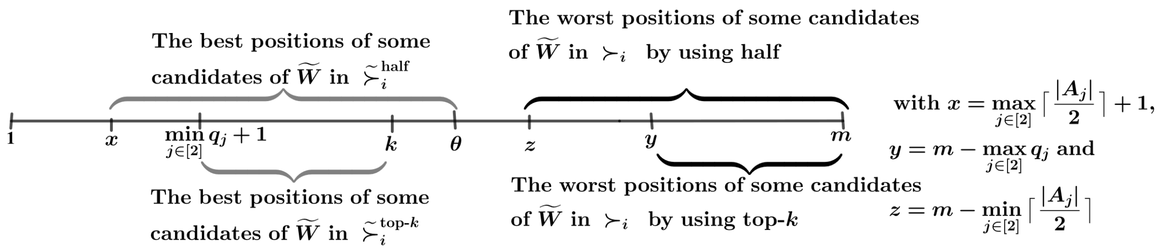

Figure 9: top-k and top half diversity constraints

□

For a given election, Corollary 4 compares the diversity constraints (introduced in this paper) in terms of the highest price to be paid for (weakly) separable committee scoring rules. Note that, there are no conclusion to be made when we have $k \geq \theta$ and $\max_{j\in[2]} \lceil \frac{|A_j|}{2} \rceil < \min_{j\in[2]} q_j$. Indeed, there are quotas such that we have $\max_{\succ\in\mathcal{P}^n} pod^1_{\text{top-}k}(f,\succ) \geq \max_{\succ\in\mathcal{P}^n} pod^1_{\text{half}}(f,\succ)$, and there are also other quotas such that we have $\max_{\succ\in\mathcal{P}^n} pod^1_{\text{top-}k}(f,\succ) < \max_{\succ\in\mathcal{P}^n} pod^1_{\text{half}}(f,\succ)$. We make the same observation when we have $k \leq \theta$ and $\max_{j\in[2]} \lceil \frac{|A_j|}{2} \rceil > \min_{j\in[2]} q_j$.

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5. Conclusion

The aim of this paper is to measure the impact of imposing diversity constraints on individual preferences. We propose a new measure of the price of diversity and we determine the boundary of the price of diversity constraints for the family of (weakly) separable committee scoring rules. Moreover, we present three diversity constraints on preferences and for each diversity constraint, we propose the boundaries of the price of diversity for the well-known (weakly) separable committee scoring rules. Through some examples, we show that the boundaries proposed in this paper are reached. In our model, the set of candidates A is partitioned according to one attribute which is the gender identity.

We conclude by suggesting a few research projects opened up by this article. Firstly, it would be useful to extend our results to the class of all committee scoring rules. Secondly, it is also possible to take into consideration the intersectionality (that is, the partition of A according to some official attributes such as race, gender identity, qualification, etc., such that any candidate belongs to many different groups). Thirdly, extra measures of the price of diversity can be introduced by regarding within each group (or within the set A), the highest price to pay when we impose the diversity constraints on the preferences. Fourthly, another important exercise to do will be to measure the price of diversity when the rule is a party-list proportional representation.

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