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Abstract

We consider cooperative games where the coalition structure is given by the set of winning coalitions of a simple game. This type of games models some real-life situations in which some agents have economic performances while some others are endowed with a political power. On this class of cooperative games, the Myerson value has been identified as the Harsanyi power solution associated to the Equal Division power index and has been characterized in the large class of Harsanyi power solutions with respect to the associated power index. In this paper, we provide a characterization of the Myerson value for this class of games without focusing on the whole family of Harsanyi power solutions and therefore, without taking into account any power index. We identify the Myerson value as the only allocation rule that satisfies efficiency, additivity, modularity, extra-null player property, and Equal Treatment of Veto.

Key words: TU-game, Voting structure, Harsanyi dividends, Harsanyi power solution, Myerson value.

JEL classification: D71, D72.

1 Introduction

A cooperative game is with transferable utilities (TU-game) is a game in which individuals (or agents, or players) can cooperate with each other in order to increase their benefits. A *coalition* is a group of players who have decided to act together. In general, it is conceded that any coalition arise, that is, any player can cooperate with any other player, and then any coalition is feasible. However, several real-life situations highlight the plausible lack of cooperation between some agents, which implies that the set of feasible coalitions cannot be considered as the collection of all subsets of the set of players. This type of cooperative games with limited cooperation has been originally brought out by Myerson (1977), through the communication graph games. Such games are modelled by a (undirected) graph wherein each player is represented by a vertex and the edges represent the possible cooperation between players. This work paved the way for many other works such as Gilles et al. (1992), Meessen (1988), and Algaba et al. (2000) among others. The work of Algaba et al. (2000) is particularly interesting because it introduces a cooperation structure called *union stable structure*, which generalizes the communication graph structure proposed by Myerson (1977). A union stable structure is characterized by the following

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property: Given any two feasible coalitions with a non-empty intersection, the union is another feasible coalition. Recently, Algaba et al. (2019) has studied a particular type of union stable structures called *voting structures*, which capture some real-life situations where the economic power of an agent emerges if and only if it is endowed with a political power. An example of such situations is given by Algaba et al. (2019) through the Peasants and Owners problem (see Algaba et al. (2019), Example 1).

The problem of sharing the benefits resulting from the cooperation of agents remains the central problem of any cooperative game. For the class of cooperative games on union stable structures, the well-known family of allocation rules is the family of Harsanyi power solutions, which distributes the benefit of each coalition among its members, proportionally to a *power measure*. A power measure is a mapping that assigns to each player, a non-negative real number that represents his cooperation power. In the specific case of voting structures, a power measure is called a *power index*. The work of van den Brink et al. (2011) provides an axiomatic characterization of the Harsanyi power solutions for graph restricted games, and Algaba et al. (2015) provides a general characterization of Harsanyi power solutions for cooperative games on union stable structures. Next, Algaba et al. (2019) provides a characterization of these solutions for the specific class of games on voting systems. From this latter work, the Myerson value (Myerson, 1977) which is the Shapley value of the classical TU-game associated to each voting structure (generally each union stable structure), has been identified as the Harsanyi power solution associated to the equal division power index, yielding the same political power to all players. However, the characterization of the Myerson value as a Harsanyi power solution is fundamentally based on its underlying power index. Indeed, any Harsanyi power solution requires that the political power of each agent should be known in advance and the sharing of the benefits should be made with respect to the political power of agents. In other words, the Harsanyi power solutions give more credit to the political ability of agents, which is further justified by the fact that the characterization of this class of solutions uses axioms based on the (given) power index. However, since the underlying power index inducing the Myerson value yields the same (political) power to all agents, we believe that this solution can stands out from this class of rules. In this paper, we claim that for cooperative games on voting structures, the Myerson value does not need to be identified as a Harsanyi power solution and therefore, one does not need any power index to characterize the Myerson value on this class of cooperative games. To prove this claim, we provide a simple characterization of the Myerson value for games on voting structures, using five desirable axioms. The asset of this characterization is that it uses three axioms well-known in the literature, namely efficiency, additivity and modularity. We add two simple and understandable axioms to reach our result. The first additional axiom is the Extra-Null player property, which requires that for any game on a political structure (voting structure), if a player has no political power and no economic performance, his payoff should be zero. The second axiom we add is Equal Treatment of Veto, requiring that any player who is crucial for economic productivity has the same payoff than any player who is crucial for the political authority.

The rest of the paper is organized as follows: Section 2 lays out the model with some preliminary definitions, Section 3 is devoted to defining of our axioms and presenting the results, and Section 4 concludes and gives the main direction for further work on this path.

2 Preliminaries

Throughout the paper, the cardinality of a finite set S will be denoted by $|S|$ and the collection of the subsets of S will be denoted by 2^S. Moreover, for the sake of simplicity, any singleton $\{i\}$ will be written as i.

2.1 Cooperative games

Let $N = \{1, \cdots, n\}$ be a finite set of agents called *players*. Any subset S of N is called *coalition* and N is called the *grand coalition*. A cooperative game with transferable utilities (TU-game) is a pair (N, v), where N is the set of players and $v : 2^N \longrightarrow \mathbb{R}$ is the coalition function satisfying $v(\emptyset) = 0$. For each coalition $S \subseteq N$, $v(S)$ is the worth of S if its members cooperate. We denote by G^N the set of all TU-games on the player set N. Since the set of players is fixed throughout the paper[1], any TU-game (N, v) will be identified by v to simplify notations.

We say that two distinct players $i, j \in N$ are symmetric in the game v is $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N \setminus \{i, j\}$. Player i is called null player in v if $v(S \cup i) = v(S)$ for all $S \subseteq N \setminus i$.

For each coalition $S \subseteq N$, the *unanimity game* U_S is defined by $U_S(T) = 1$ if $S \subseteq T$ and $U_S(T) = 0$ otherwise. It is well-known that G^N is a vector space and that the family of unanimity games $(U_S)_{\emptyset \neq S \subseteq N}$ forms a basis of G^N. More precisely, each TU-game v can be uniquely decomposed as

$$v = \sum_{S \subseteq N, S \neq \emptyset} \Delta_v(S) U_S. \tag{1}$$

The coefficients $\Delta_v(S)$ are called the *Harsanyi dividends* (Harsanyi, 1959) of the coalitions $S \subseteq N$ in the game v, and are calculated with the following recursive formula:

$$\Delta_v(S) = v(S) - \sum_{T \subsetneq S} \Delta_v(T), \quad \forall S \subseteq N, S \neq \emptyset. \tag{2}$$

A *solution* on G^N is any mapping φ that assigns to each game $v \in G^N$ and each player $i \in N$, the numerical value $\varphi_i(v) \in \mathbb{R}$ called the *payoff* of player i in the game v. The most known solution in the literature of cooperative games is the *Shapley value* Φ (Shapley, 1953), which shares the benefit of cooperation (the Harsanyi dividend) of each coalition equally among its members. Formally, the Shapley value is defined as follows:

$$\Phi_i(v) = \sum_{S \subseteq N, i \in S} \frac{1}{|S|} \Delta_v(S), \quad \forall i \in N. \tag{3}$$

2.2 Voting games

A *voting game* is a cooperative game in which the value of each coalition is either 1 or 0. Any coalition with a value of 1 is called a *winning coalition*, and any coalition with a value of 0 is

[1] The of players is not variable.

called a *losing coalition*. A voting game is usually denoted by W, where W is the set of all winning coalitions, satisfying the following three properties:

1. $W \neq \emptyset$;
2. $\emptyset \notin W$;
3. For all $S, T \in 2^N$, if $S \subseteq T$ and $S \in W$, then $T \in W$.

Properties (1) and (2) require that there is at least one winning coalition and that the empty set is not a winning coalition. Property (3) is the monotonicity property, which requires that any superset of any winning coalition is also winning. This implies that the grand coalition is a winning coalition. We denote by $M(W) \subseteq W$ the (non-empty) subset of all minimal winning coalitions; that is, the subset of winning coalitions $T \in W$ such that there is no $S \in W$ with $S \subset T$. For any minimal winning coalition $T \in M(W)$, we denote by W_T the set of all winning coalitions containing T; i.e,

$$W_T = \{S \in W : T \subseteq S\} \tag{4}$$

It is obvious to check that

$$W = \bigcup_{T \in M(W)} W_T.$$

Moreover, it is straightforward that if W and W' are two voting games on the player set N, then $W \cup W'$ and $W \cap W'$ are also voting games. Furthermore, for any two minimal coalitions T_1 and T_2, it holds that

$$W_{T_1} \cap W_{T_2} = W_{T_1 \cup T_2}.$$

The ubiquitous query in any voting game is how to measure each player's ability to turn winning any coalition. This problem has led to the definition of several measures commonly referred to as *power indices*. Formally, a power index is a mapping σ that assigns to each voting game W and each player $i \in N$ a numerical value $\sigma_i(W) \in [0, 1]$, which is the voting power of player i in the voting game W.

The most well-known power indices in the literature are the *Shapley-Shubik power index* (Shapley and Shubik, 1954) and the *Banzhaf power index* (Banzhaf, 1964).[2] Another well-known power index is the *Equal division power index (ED)* defined by

$$ED_i(N, W) = \frac{1}{n}, \quad \forall i \in N.$$

Roughly speaking, the Equal Division power index yields the same fraction of power to each player.

[2] We do not recall the definitions of theses two power indices since this paper is not interested in.

2.3 Cooperative games on voting structure

As mentioned in the introductory section, many real-life situations highlight the restriction of the set of feasible coalitions in a cooperative game, to a specific collection of coalitions. In this paper, we are interested in situations where the set of feasible coalitions is a set W such that W is a voting game on the player set N. This type of cooperation system, which we call a *voting system*, captures situations where a coalition of players may have some economic capacity, but does not have the legal authority. A good illustration of this type of situation is given in Example 1 of Algaba et al. (2019).

Recall that the family of voting systems is included in the family of *union stable systems* introduced and studied by Algaba et al. (2000) and Algaba et al. (2001). Formally, a union stable system on the player set N is a collection $\mathcal{F}$ of coalitions such that, for any two coalitions $A, B \in \mathcal{F}$, if $A \cap B \neq \emptyset$, then $A \cup B \in \mathcal{F}$. Roughly speaking, if any two feasible coalitions have at least one common player, then that player ensures the connection between the two coalitions in such a way that the union is still feasible. Since every voting system is monotonic (property (3)), it is clear that every voting system is union stable. Therefore, denoting by U^N and V^N the families of union stable and voting systems on N respectively, it holds that $V^N \subset U^N$.

A *union stable structure* on N is a pair $(v, \mathcal{F})$ such that v is a cooperative game and $\mathcal{F}$ is a union stable system on N. Similarly, a *voting structure* on N is a pair (v, W) where v is a cooperative game and W is a voting system on N. We denote by GU^N the set of union stable structures on the player set N and by GV^N the set of voting structures on N; it then follows that $GV^N \subset GU^N$. Since not all of the coalitions are feasible, each union stable structure $(v, \mathcal{F})$ is associated with a TU-game $v^{\mathcal{F}}$, called the *$\mathcal{F}$-restricted game* associated to $(v, \mathcal{F})$. The game $v^{\mathcal{F}}$ is defined by

$$v^{\mathcal{F}}(S) = \sum_{T \in C_{\mathcal{F}}(S)} v(T) \tag{5}$$

where $C_{\mathcal{F}}(S)$ is the set of maximal feasible coalitions contained in S. It is not hard to check that the W-restricted game associated to a voting structure (v, W) is defined by

$$v^W(S) = \begin{cases} v(S) & \text{if } S \in W \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

This is due to the monotonicity property which requires that S should be feasible whenever S contains any feasible coalition.

An *allocation rule* on GU^N is any mapping φ assigning to each union stable structure $(v, \mathcal{F})$ and each player $i \in N$ the numerical value $\varphi_i(v, \mathcal{F}) \in \mathbb{R}$ which is the payoff of player i in the structure $(v, \mathcal{F})$. The best known family of allocation rules on GU^N is the family of *Harsanyi power solutions*, which share the Harsanyi dividends of the coalitions in the $\mathcal{F}$-restricted game among their members, proportionally to a power measure. A *power measure* on U^N is a mapping σ assigning to each union stable system $\mathcal{F}$ and each player $i \in N$ a non-negative real number $\sigma_i(\mathcal{F})$, which represents the cooperation power of player i in the cooperation system $\mathcal{F}$. A power measure on V^N is called power index, and some examples of power indices were given earlier in Section 2.2. Given a power measure σ, the *Harsanyi power solution* associated to σ is the allocation rule φ^σ defined by:

$$\forall i \in N,\ \varphi_i(v, \mathcal{F}) = \sum_{\substack{S \subseteq N, i \in S \\ \sum_{j \in S} \sigma_j(S, \mathcal{F}_S) > 0}} \frac{\sigma_i(S, \mathcal{F}_S)}{\sum_{j \in S} \sigma_j(S, \mathcal{F}_S)} \Delta_{v^{\mathcal{F}}}(S) \tag{7}$$

where $\mathcal{F}_S = \{F \in \mathcal{F} : F \subseteq S\}$.

For a voting structure (v, W) it is clear that for all coalitions $S \notin W$, $\Delta_{v^W}(S) = 0$.[3] The *Myerson value* μ on GV^N is the allocation rule defined as the Shapley value of the W-restricted game associated to each voting structure (v, W); that is, for each $(v, W) \in GV^N$,

$$\mu_i(v, W) = \sum_{S \subseteq N, i \in S} \frac{1}{|S|} \Delta_{v^W}(S) = \sum_{\substack{S \subseteq N, S \in W \\ i \in S}} \frac{1}{|S|} \Delta_{v^W}(S), \quad \forall i \in N. \tag{8}$$

From (7) and (8), it is clear that the Myerson value μ on GV^N is the Harsanyi power solution associated to the Equal Division power index (ED).

The works of van den Brink et al. (2011) and Algaba et al. (2015) provide a characterization of the Harsanyi power solutions on the classes of graph-restricted games and games on union stable systems and, next, Algaba et al. (2019) provide a characterization of the Harsanyi power solutions for games on voting systems. This latter work characterizes the Myerson value within the family of Harsanyi power solutions, based on the underlying power index. In this paper, we provide a simple characterization of the Myerson value in the large class of allocation rules on GV^N, without considering the family of Harsanyi solutions and without using any power index.

3 Axioms and characterization

3.1 Axioms

The axioms considered in this paper are not related to any power index and some of them are well-known in the literature. The first two axioms are *Efficiency* (E) and *Additivity* (A).

- **Efficiency (E):** For all $(v, W) \in GV^N$,

$$\sum_{i \in N} \varphi_i(v, W) = v(N);$$

- **Additivity (A):** For all $u, v \in G^N$, $W \in V^N$, and $i \in N$,

$$\varphi_i(u + v, W) = \varphi_i(u, W) + \varphi_i(v, W).$$

Remark that there is no ambiguity with Efficiency since for a voting system, the grand coalition is feasible.[4]

[3] It has been shown in Algaba et al. (2015) that this result holds for any union stable structure.

[4] For union stable system, efficiency is defined for each maximal feasible coalition and this version of efficiency is called *component efficiency*.

The third axiom considered is the *modularity* axiom (M), which is equivalent to the transfer axiom of Dubey (1975).[5]

- **Modularity (M):** For all $(v, W), (v, W') \in G^N$, and $i \in N$,

$$\varphi_i(v, W \cup W') = \varphi_i(v, W) + \varphi_i(v, W') - \varphi_i(v, W \cap W').$$

The next axiom is a strong version of the Null player axiom that merges both the economic and political abilities of a player; we call it the *Extra-Null player* axiom *(EN)*. Recall that a player i is a null player in v if for all $S \subseteq N \setminus i, v(S \cup i) = v(S)$; a player i is then a null player in W for all $S \subseteq N \setminus i$, $S \notin W \Rightarrow S \cup i \notin W$. A player i is said to be an extra-null player in (v, W) if i is a null player in both v and W.

- **Extra-Null player (EN):** For all $(v, W) \in GV^N$, if $i \in N$ is an extra-null player in (v, W), then

$$\varphi_i(v, W) = 0.$$

The last axiom aims to ensure a fairness between the economic and political abilities of players.

Definition 1 *Let $(v, W) \in GV^N$ be a voting structure and $i \in N$ be a player.*
We say that player i has a political veto if for all $S \subseteq N, S \in W \Rightarrow i \in S$.
We say that player i has an economic veto if for all $S \subseteq N, v(S) \neq 0 \Rightarrow i \in S$.

Let $PV(v, W)$ and $EV(v, W)$ the sets of players who have the political and the economic veto respectively.

- **Equal Treatment of Veto (ETV):** For all $(v, W) \in GV^N$ and $i, j \in N$, if $i \in PV(v, W)$ and $j \in EV(v, W)$, then

$$\varphi_i(v, W) = \varphi_j(v, W).$$

The next section is devoted to the characterization of the Myerson value as the only allocation rule on GV^N satisfying (E), (A), (M), (EN), and (ETV).

3.2 Characterization

We start this section with a simple but useful result.

Proposition 1 *Let $W, W' \in V^N$ be any two voting systems on N. The $W \cup W'$-restricted game associated to $(v, W \cup W')$ is defined by*

$$v^{W \cup W'}(S) = v^W(S) + v^{W'}(S) - v^{W \cap W'}(S), \ \ \forall S \subseteq N$$

Proof. Let $S \subseteq N$.

- Case 1: if $S \notin W \cup W'$, then $v^W(S) = v^{W'}(S) = v^{W \cap W'}(S) = 0 = v^{W \cup W'}(S)$.

[5]Another version of modularity can be found in Laruelle and Valenciano (2001).

- Case 2: if $S \in W \setminus W'$, then $v^{W \cup W'}(S) = v(S)$ since $S \in W \cup W'$ and $v^W(S) + v^{W'}(S) - v^{W \cap W'}(S) = v^W(S) = v(S) = v^{W \cup W'}(S)$. Similarly, if $S \in W' \setminus W$, $v^{W \cup W'}(S) = v(S) = v^{W'}(S) = v^W(S) + v^{W'}(S) - v^{W \cap W'}(S)$.

- Case 3: if $S \in W \cap W'$, then $v^{W \cup W'}(S) = v(S) = v^{W'}(S) = v^W(S) + v^{W'}(S) - v^{W \cap W'}(S)$.

■

Now let W be a voting game and $M(W)$ be the set of its minimal winning coalitions. Without loss of generality, let us write $M(W) = \{T_1, \cdots, T_m\}$, and let $\mathcal{A}(W)$ be the set of coalitions defined as follows:

$$\mathcal{A}(W) = \Big\{ A \subseteq N : A = \bigcup_{p \in H} T_p;\ H \subseteq \{1, \cdots, m\} \Big\}. \tag{9}$$

In other words, $\mathcal{A}(W)$ stands for the set of all coalitions that can be written as a union of minimal winning coalitions. Moreover, similar to (4), let us set $W_A = \{S \in W : A \subseteq S\}$ for all $A \in \mathcal{A}(W)$.

Proposition 2 *Let φ be an allocation rule on GV^N. If φ satisfies modularity, then for any voting structure (v, W), $\varphi(v, W)$ is entirely defined by the family of voting structures (v, W_A), $A \in \mathcal{A}(W)$. More precisely, for any $i \in N$, we have*

$$\varphi_i(v, W) = \sum_{p=1}^{m} \varphi_i(v, W_{T_p}) - \sum_{p<q\leq m} \varphi_i(v, W_{T_p \cup T_q}) + \cdots + (-1)^{m+1} \varphi_i\big(v, W_{\cup_{p=1}^{m} T_p}\big). \tag{10}$$

Equation (10) is equivalent to the Poincaré formula for the cardinality of any finite union of finite sets.

Proof. Let φ be an allocation rule on GV^N satisfying modularity. Let (v, W) be voting structure such that $M(W) = \{T_1, \cdots, T_m\}$.

- If $m = 1$, the result is straightforward since in this case, we have $M(W) = \{T_1\}$ and $W = W_{T_1}$.

- If $m = 2$, we have $M(W) = \{T_1, T_2\}$ and for every coalition $S \subseteq N$, it holds that

$$\begin{aligned} \varphi_i(v, W) = \varphi_i(v, W_{T_1} \cup W_{T_2}) &= \varphi_i(v, W_{T_1}) + \varphi_i(v, W_{T_2}) - \varphi_i(v, W_{T_1} \cap W_{T_2}) \\ &= \varphi_i(v, W_{T_1}) + \varphi_i(v, W_{T_2}) - \varphi_i(v, W_{T_1 \cup T_2}) \end{aligned}$$

  since $W_{T_1} \cap W_{T_2} = W_{T_1 \cup T_2}$.

  For the rest of the proof, we will denote $\varphi_i(W)$ to mean $\varphi_i(v, W)$ to simplify the notation, since the game v remains fixed.

- Assume that the result holds for $|M(W)| = m > 2$, and let us show that it remains true for $|M(W)| = m + 1$.

If $|M(W)| = m+1$, we can write $W = W' \cup W_{T_{m+1}}$ where $W' = \cup_{p=1}^{m} W_{T_p}$. It then follows that

$$\begin{aligned}
\varphi_i(W) &= \varphi_i(W') + \varphi_i(W_{T_{m+1}}) - \varphi_i(W' \cap W_{T_{m+1}}) \\
&= \varphi_i(W') + \varphi_i(W_{T_{m+1}}) - \varphi_i\Big(\cup_{p=1}^{m} (W_{T_p} \cap W_{T_{m+1}})\Big) \\
&= \sum_{p=1}^{m} \varphi_i(W_{T_p}) - \sum_{p<q} \varphi_i(W_{T_p \cup T_q}) + \cdots + (-1)^{m+1} \varphi_i\big(W_{\cup_{p=1}^{m} T_p}\big) \\
&+ \varphi_i(W_{T_{m+1}}) - \varphi_i\Big(\cup_{p=1}^{m} (W_{T_p} \cap W_{T_{m+1}})\Big) \\
&= \sum_{p=1}^{m1} \varphi_i(W_{T_p}) - \sum_{p<q} \varphi_i(W_{T_p \cup T_q}) + \cdots + (-1)^{m+1} \varphi_i\big(W_{\cup_{p=1}^{m} T_p}\big) \\
&- \varphi_i\Big(\cup_{p=1}^{m} (W_{T_p} \cap W_{T_{m+1}})\Big)
\end{aligned}$$

Now we can apply Equation (10) to $\varphi_i\Big(\cup_{p=1}^{m} (W_{T_p} \cap W_{T_{m+1}})\Big)$ and have

$$\varphi_i(W) = \sum_{p=1}^{m+1} \varphi_i(W_{T_p}) - \sum_{p<q\leq m+1} \varphi_i(W_{T_p \cup T_q}) + \cdots + (-1)^{m+2} \varphi_i\big(W_{\cup_{p=1}^{m+1} T_p}\big).$$

■

Thus, combining additivity and modularity leads to the following corollary.

Corollary 1 *Let φ be an allocation rule on GV^N. If φ satisfies Additivity (A) and Modularity (M), then for any voting structure (v, W), $\varphi(v, W)$ is entirely defined by the family of voting structures $(\alpha U_S, W_A)$, with $S \subseteq N$, $\alpha \in \mathbb{R}$, and $A \in \mathcal{A}(W)$.*

Before running to the main result of this paper, let us recall that the Shapley value on G^N satisfies the *symmetry* property. A solution φ on G^N satisfies the symmetry property if for any game v and any two symmetric players $i, j \in N$, we have $\varphi_i(v) = \varphi_j(v)$. The next theorem provides the first part of the main result of this paper.

Theorem 1 *The Myerson value μ on GV^N satisfies (E), (A), (M), (EN), and (ETV).*

Proof. It is not hard to show that μ satisfies Efficiency and Additivity since for every voting structure (v, W), $\mu(v, W)$ is simply the Shapley value of the W-restricted game v^W and since the Shapley value satisfies (E) and (A), it immediately follows that μ satisfies (E) and (A) as well.

Similarly, it is not hard to show that μ satisfies the Extra-Null player property. Indeed, if $i \in N$ is an Extra-Null player in the voting structure (v, W), we can easily check that i is a null player in the W-restricted game v^W as follows: for any subset $S \subseteq N \setminus i$,

$$v^W(S\cup i)=\begin{cases} v(S\cup i) & \text{if } S\cup i\in W\\ 0 & \text{otherwise}\end{cases}=\begin{cases} v(S) & \text{if } S\in W\\ 0 & \text{otherwise}\end{cases}=v^W(S).$$

Hence, since μ is the Shapley value of the game v^W, we have

$$\mu_i(v,W)=\Phi_i(v^W)=0$$

since the Shapley value Φ satisfies the Null player property.

Let us show that μ satisfies modularity. For any two voting structures (v,W) and (v,W') and any player $i\in N$, we have

$$\mu_i(v,W\cup W')=\Phi_i(v^{W\cup W'})=\sum_{S\subseteq N, i\in N}\frac{1}{|S|}\Delta_{v^{W\cup W'}}(S)$$

By Proposition 1, we have

$$\Delta_{v^{W\cup W'}}(S)=\Delta_{v^W}(S)+\Delta_{v^{W'}}(S)-\Delta_{v^{W\cap W'}}(S).$$

Therefore, it follows that

$$\begin{aligned}\mu_i(v,W\cup W') &= \sum_{S\subseteq N, i\in S}\frac{1}{|S|}\Delta_{v^W}(S)+\sum_{S\subseteq N, i\in S}\frac{1}{|S|}\Delta_{v^{W'}}(S)-\sum_{S\subseteq N, i\in S}\frac{1}{|S|}\Delta_{v^{W\cap W'}}(S)\\ &= \Phi_i(v^W)+\Phi_i(v^{W'})-\Phi_i(v^{W\cap W'})\\ &= \mu_i(v,W)+\mu_i(v,W')-\mu_i(v,W\cap W').\end{aligned}$$

Hence, μ satisfies modularity.

Finally, Let us show that μ satisfies Equal Treatment of Veto (ETV). Let (v,W) be a voting structure and let $i,j\in N$ be any two players such that $i\in PV(v,W)$ and $j\in EV(v,W)$. It holds that i and j are symmetric in the game v^W. Indeed, for all $S\subseteq N\setminus\{i,j\}$, we have

$$v^W(S\cup i)=0 \text{ since } j\notin S\cup i;$$

and

$$v^W(S\cup j)=0 \text{ since } i\notin S\cup j \text{ and then } S\cup j\notin W.$$

Thus it follows that $v^W(S\cup i)=v^W(S\cup j)$ which implies that i and j are symmetric in v^W and since the shapley value satisfies symmetry property, it follows that

$$\mu_i(v,W)=\Phi_i(v^W)=\Phi_j(v^W)=\mu_j(v,W).$$

Hence, μ satisfies (ETV).

■

The following theorem completes the main result of this the paper.

Theorem 2 *The Myerson value μ on GV^N is the only allocation rule on GV^N that satisfies (E), (A), (M), (EN), and (ETV).*

Proof. We showed in Theorem 1 that the Myerson value satisfies (E), (A), (M), (EN), and (ETV).

Let φ be any allocation rule on GV^N that satisfies these five axioms. Let $(v, W) \in GV^N$. To show that $\varphi(v, W) = \mu(v, W)$, we just have to show that $\varphi(\alpha U_S, W_A) = \mu(\alpha U_S, W_A)$ for each $S \subseteq N, \alpha \in \mathbb{R}$, and $A \in \mathcal{A}$, since φ satisfies (A) and (M) (Corollary (1)).

Let $(\alpha U_S, W_A)$ such a voting structure. The W_A-restricted game $(\alpha U_S)^{W_A}$ associated to $(\alpha U_S, W_A)$ is simply the game $\alpha U_{S\cup A}$. Indeed,

$$(\alpha U_S)^{W_A}(K) = \begin{cases} \alpha U_S(K) & \text{if } A \subseteq K \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \alpha & \text{if } S \cup A \subseteq K \\ 0 & \text{otherwise} \end{cases} = \alpha U_{S\cup A}(K).$$

Moreover, remark that $PV(\alpha U_S, W_A) = A$ and $EV(\alpha U_S, W_A) = S$. Since φ satisfies (ETV), we have

$$\forall i, j \in S \cup A, \varphi_i(\alpha U_S, W_A) = \varphi_j(\alpha U_S, W_A).$$

Moreover, any player i who does not belong to $S \cup A$ is an extra-null player in $(\alpha U_S, W_A)$ and since φ satisfies the (EN), it holds that

$$\varphi_i(\alpha U_S, W_A) = 0 \ \ \forall i \notin S \cup A.$$

Now, since φ satisfies (E), we have

$$\sum_{i\in S\cup A} \varphi_i(\alpha U_S, W_A) = \alpha U_S(N) = \alpha \Rightarrow |S \cup A|\varphi_i(\alpha U_S, W_A) = \alpha, \ \forall i \in S \cup A.$$

Hence it follows that for all $i \in S \cup A$,

$$\varphi_i(\alpha U_S, W_A) = \frac{\alpha}{|S \cup A|} = \Phi_i(\alpha U_{S\cup A}) = \Phi_i\big(\alpha(U_S)^{W_A}\big) = \mu_i(\alpha U_S, W_A)$$

and for all $i \notin S \cup A$,

$$\varphi_i(\alpha U_S, W_A) = 0 = \Phi_i(\alpha U_{S\cup A}) = \Phi_i\big(\alpha(U_S)^{W_A}\big) = \mu_i(\alpha U_S, W_A)$$

As result, we have $\varphi_i(\alpha U_S, W_A) = \mu_i(\alpha U_S, W_A)$ for all $i \in N$ and by Corollary (1), we deduce that $\varphi_i(v, W) = \mu_i(v, W)$ for all $i \in N$. ∎

4 Concluding remarks

The goal of this paper was to provide an axiomatic characterization of the Myerson value on the class of cooperative games on voting systems without considering the large family of Harsanyi power solutions in which this solution is included. The main worth of this work is that the characterization uses simple and understandable axioms that do not depend on any power index.

The main pending challenge of this work is to generalize this characterization on the wide class of union stable structures GU^N.

References

Algaba, E., Bilbao, J.M., and Borm, P., López, J. (2001). The position value for union stable systems. *Mathematical Methods of Operations Research*, 52:221–236.

Algaba, E., Bilbao, J.M., and Borm, P., López, J. (2001). The Myerson value for union stable structures. *Mathematical Methods of Operations Research*, 54:359–371.

Algaba, E., Bilbao, J.M., van den Brink, R. (2015). Harsanyi power solutions for games on union stable systems. *Annals of Operations Research*, 225:27–44.

Algaba, E., Béal, S., and Rémila, E., and Solal, P. (2019). Harsanyi power solutions for cooperative games on voting structures. *International Journal of General Systems*, 48(6):2575–602.

Banzhaf III, J. F. (1964). Weighted voting doesn't work: A mathematical analysis *Rutgers L. Rev.*, 19:317.

Dubey, P. (1975). On the uniqueness of the Shapley value *International Journal of Game Theory*, 4:131–139.

Gilles, R.P., Owen, G., and van den Brink, R. (1992). Games with permission structures: the conjunctive approach *International Journal of Game Theory*, 20(3):277–293.

Harsanyi, J. C. (1959). A bargaining model for the cooperative n-person game *In: Tucker, A. W., Luce, R. D. (Eds.), Contributions to the Theory of Games IV. Princeton University Press*, pp. 325–355.

Laruelle, A., and Valenciano, F. (2001). Shapley-Shubik and Banzhaf indices revisited *Mathematics of operations research*, 26(1):89–104.

Meessen, R. (1988). Communication games (in Dutch)(Ph. D. dissertation) *Department of Mathematics, University of Nijmegen, The Netherlands.*

Myerson, R.B (1977). Graphs and cooperation in games *Mathematics of operations research*, 2(3): 225–229.

Shapley, L.S. (1953). A value for n-person games *Princeton University Press Princeton.*

Shapley, L.S., and Shubik, M. (1954). A method for evaluating the distribution of power in a committee system *American political science review*, 48(3):787–792.

van den Brink, R., van der Laan, G., and Pruzhansky, V. (2011). Harsanyi power solutions for graph-restricted games *International journal of game theory*, 40:87–110.