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Cooperative games with unpaid players*

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Abstract

We consider cooperative TU-games with unpaid players, which are described by a TU-game and two categories of players, paid and unpaid. Unpaid players participate in the cooperative game but are not rewarded for their participation, for instance for legal reasons. The objective is then to determine how the contributions of unpaid players are redistributed among the paid players. To meet this goal, we introduce and characterize axiomatically three values that are inspired by the Shapley value but differ in the way they redistribute the contributions of unpaid players. These values are unified as instances of a more general two-step allocation procedure.

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1. Introduction

Cooperative games with transferable utility (TU-games for short) describe situations where players can generate certain worths by cooperating. A payoff vector for a TU-game is a vector that assigns a real payoff to each player in order to reflect its participation to the game. A value on a class of TU-games assigns to each game in this class a payoff vector. In the classical model, it is assumed that all players are paid. In this article, we relax this assumption, which makes sense in several situations where some players cannot be paid. Examples are the following.

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Firstly, according to the French law, voluntary work in a company may be authorized in certain cases (occasional family assistance, help from a spouse, "a helping hand" from a non-executive partner),[1] but is still common in not-for-profit organisations. In all cases, a voluntary worker cannot be paid. Secondly, in a worker cooperative company, employees and machines all participate in the production process (and therefore in the creation of value), but only employees are paid. So the contribution of machines can be seen as voluntary work. Thirdly, in France, for certain minor offenses, criminal courts may sentence the offender to perform community service. This is unpaid work that facilitates reintegration by engaging in a socially useful professional activity. Fourthly, in certain committees, members may hold either deliberative or advisory voting rights. In the latter case, their vote carries no weight in the final decision. This is for instance the case of the supervisory board at Curie institute, a leading French cancer treatment and research center, which is composed of 24 members; 18 of them with deliberative votes and 6 with advisory votes.[2]

Formally, we consider TU-games with unpaid players. Such a game is given by a TU-game in which the player set is partitioned into paid and unpaid players. A value in this context assigns to any TU-game with unpaid players a payoff vector on the set of paid players. In this sense, one of the questions we are trying to answer is how the collective contributions of unpaid players are distributed among the paid players. In the classical case where the set of unpaid players is empty, the Shapley value (Shapley, 1953) is widely recognized as the most prominent value. In our setting, we believe there are several relevant ways to extend the Shapley value. We have chosen three, which we call the P-priority value, the U-priority value and the Redistribution value.

The P-priority value is inspired by the formulation of the Shapley in terms of Harsanyi dividends. The Shapley values splits the Harsanyi dividend of each coalition equally among the coalition members. The P-priority value splits the dividend of each coalition equally among the paid players in the coalition and equally among all paid players when the coalition only contains unpaid players.

The U-priority value assigns to each paid player its Shapley value in the subgame among paid players and then split what remains of the worth of the grand coalition (the incremental contribution of the set of unpaid players to the set of paid players) equally among all paid players.

The Redistribution value assigns to each paid player its Shapley value in the original TU-game (and not in the subgame among paid players as for the U-priority value) and then redistributes the payoffs that the unpaid players would have obtained according to the Shapley value equally among all paid players.

[1] See the Circular of the 18^{th} January 2010 on relations between associations and public authorities, which recalls that volunteering is unpaid.

[2] See `https://institut-curie.org/supervisory-board`

It is interesting to remark that these three values each have an asymmetrical component that differentiates between paid players according to their productivity, and an egalitarian component where each paid player receives the same additional payoff. Our three values essentially differ in the perimeter used to calculate the productivity of paid players. We highlight these similarities and differences through a two-step allocation procedure that unifies our three values within a single framework. In a first step, a value is used to determine payoffs of all players in the original TU-game with unpaid players. This value may depend on the status of paid players relative to unpaid players or the type of interaction they may have. In the second step, a quasi-additive TU-game is constructed on the set of paid players from the resulting allocation, i.e. the worth of the coalition containing all paid players is the sum of all payoffs distributed in the first step and for any other subcoalition, the worth is the sum of payoffs obtained by its members in the first step. This unified approach is useful to link axioms for the class of TU-games with unpaid players and axioms in the classical setting but also to make connections with the literature on TU-games with a priority structure. Our two-step procedure is reminiscent, to some extent, of the literature on operators for claim problems initiated by Thomson and Yeh (2008) and continued by Hougaard et al. (2012), among others.

We characterize axiomatically these three values in the tradition initiated by Shapley. Some axioms are natural adaptations of classical axioms in the literature to the framework of TU-games with unpaid players. Some other axioms are new and have no counterpart in the classical model of TU-games. In this introduction, we illustrate our results by focusing on a single characterization. An instance of a new axiom is the axiom of Equal impact of promoting an unpaid player, which requires that any two paid players should enjoy the same payoff variation if an arbitrary initially unpaid player becomes a paid player. This changes clearly makes sense in applications where, for example, an unpaid trainee becomes a fully-fledged employee following recruitment. This axiom is satisfied by the Redistribution value but not by the two other values. In order to characterize the Redistribution value, we also invoke two extra axioms. The first one is Efficiency for paid players, which requires that the worth of the grand coalition is fully shared among paid players. The second one is Balanced contributions for paid players with null unpaid players, which adapts Myerson (1980)'s celebrated axiom of Balanced contributions by comparing only pairs of paid players in a situation where, in addition, unpaid players are null players. Efficiency for paid players and Balanced contributions for paid players with null unpaid players are also satisfied by the P-priority and U-priority values, which means that the axiom of Equal impact of promoting an unpaid player is really characteristic of the Redistribution value. Some of our results are comparable in the sense that they invoke common axioms and differ with respect to axioms with similar flavor.

The rest of this article is organized as follows. Section 2 presents TU-games and TU-games with unpaid players. Section 3 presents the two-step procedure. Section 4 defines the axioms that we invoke in Section 5 for the characterizations of three values emerging from this two-step procedure. Section 6 presents alternative approaches and proves the logical independence of the set of axioms used in each characterization result. Section 7 concludes.

2. Cooperative games with unpaid players

Notation. The cardinality of a finite set S will be denoted by s.

TU-games. A cooperative game with transferable utility (TU-game) is a pair (N,v) where $N \subseteq \mathbb{N}$ is finite player set and $v : 2^N \longrightarrow \mathbb{R}$ is a characteristic function such that $v(\varnothing) = 0$. The **null game** on N is denoted by $(N, \mathbf{0}_N)$ and is such that $\mathbf{0}_N(S) = 0$ for each $S \subseteq N$. The sum of two TU-games (N,v) and (N,w) is the TU-game $(N, v+w)$ such that $(v+w)(S) = v(S) + w(S)$ for each $S \subseteq N$. The TU-game (N,v) multiplied by a scalar $\alpha \in \mathbb{R}$ is the TU-game $(N, \alpha v)$ such that $(\alpha v)(S) = \alpha v(S)$ for each $S \subseteq N$. This way, the set of all TU-games on a fixed player set N of size n can be viewed as a linear subspace of $\mathbb{R}^{2^n}$. The Dirac game induced on N by a nonempty coalition $S \subseteq N$ is the TU-game $(N, 1_S)$ such that

$$1_S(T) = 1 \text{ if } T = S \quad \text{and} \quad 1_S(T) = 0 \text{ if } T \neq S.$$

Obviously, for each TU-game (N,v),

$$v = \sum_{S \subseteq N, S \neq \varnothing} v(S) 1_S.$$

The **unanimity TU-game** on N induced by a nonempty coalition $S \subseteq N$ is the TU-game (N, u_S) such that

$$u_S(T) = 1 \text{ if } T \supseteq S \quad \text{and} \quad u_S(T) = 0 \text{ if } T \nsupseteq S.$$

It is known that each TU-game (N,v) can be linearly decomposed in a unique way into a weighted sum of the unanimity TU-games (N, u_S), $S \subseteq N, S \neq \varnothing$:

$$v = \sum_{S \subseteq N, S \neq \varnothing} \Delta_S(v) u_S, \tag{1}$$

where the coordinate $\Delta_S(v)$ is called the Harsanyi dividend (Harsanyi, 1959) of coalition S. A TU-game (N,v) is **quasi-additive** if its linear decomposition according to the unanimity games is given by:

$$v = \sum_{i \in N} \Delta_{\{i\}}(v) u_{\{i\}} + \Delta_N(v) u_N. \tag{2}$$

The TU-game (N,v) is **additive** if $\Delta_N(v) = 0$ in the above linear decomposition. Quasi-additive games are considered in Moulin (1987) (under the name joint venture games) and van den Brink et al. (2020), among others. For each game (N,v), the **subgame** induced by a nonempty coalition $S \subseteq N$ is denoted by $(S, v_{|S})$, where $v_{|S}$ stands for the restriction of the original characteristic function to 2^S. If no confusion arise, the subgame $(S, v_{|S})$ of (N, v) will be denoted by (S, v). The **(marginal) contributions** of player $i \in N$ in (N, v) are defined as:

$$v(S \cup \{i\}) - v(S), \quad \forall S \subseteq N \setminus \{i\}. \tag{3}$$

Given a TU-game (N, v), two players $i, j \in N$ are **equal** in (N, v) if

$$v(S \cup \{i\}) = v(S \cup \{j\}), \quad \forall S \subseteq N \setminus \{i, j\}.$$

A player $i \in N$ is **null** in (N, v) if

$$v(S \cup \{i\}) = v(S), \quad \forall S \subseteq N \setminus \{i\}.$$

Remark 1. *If $i \in N$ is a null player in a game (N, v), then $\Delta_S(v) = 0$ for each $S \subseteq N$ such that $S \ni i$.*

Values for TU-games. Let $\mathbf{G}$ be the set of all TU-games. A value on $\mathbf{G}$ is a function f on $\mathbf{G}$ that associates to each TU-game $(N, v) \in \mathbf{G}$, a unique payoff vector $f(N, v)$ in $\mathbb{R}^N$, specifying the payoff of each player $i \in N$ for its participation in the TU-game (N, v). For $S \subseteq N$, $f_S(N, v)$ stands for the sum of the payoffs $f_i(N, v)$, $i \in S$.

Here are some usual properties or **axioms** for a value on a domain $\mathbf{G}$, which we will invoke in this article.

Efficiency. For each $(N, v) \in \mathbf{G}$, it holds that:

$$\sum_{i \in N} f_i(N, v) = v(N).$$

Equal treatment of equal players. For each $(N, v) \in \mathbf{G}$, and each pair $\{i, j\} \subseteq N$ of equal players in (N, v), it holds that:

$$f_i(N, v) = f_j(N, v).$$

Null player axiom. For each $(N, v) \in \mathbf{G}$, and each null player $i \in N$ in (N, v), it holds that:

$$f_i(N, v) = 0.$$

Marginality. For each pair of TU-games (N, v) and (N, w) of $\mathbf{G}$ and each $i \in N$ such that its contributions as defined in (3) are identical in (N, v) and (N, w), it holds that:

$$f_i(N, v) = f_i(N, w).$$

Additive. The value f on $\mathbf{G}$ is additive, that is,

$$f(N, v + w) = f(N, v) + f(N, w), \quad \forall (N, v), (N, w) \in \mathbf{G}.$$

Covariance. For each pair of TU-games (N,v) and (N,w) in $\mathbf{G}$ such that (N,w) is additive, and each $\alpha \in \mathbb{R}$, it holds that:

$$f(N,\alpha v + w) = \alpha f(N,v) + (w(\{i\}))_{i\in N}.$$

Coalitional strategic equivalence. For each TU-game $(N,v) \in \mathbf{G}$, each unanimity game $(N,u_S) \in \mathbf{G}$ and each $\alpha \in \mathbb{R}$, it holds that:

$$f_i(N,v+\alpha u_S) = f_i(N,v), \quad \forall i \in N \setminus S.$$

Coalitional strategic equivalence, introduced by Chun (1989), is equivalent to Marginality (see footnotes 3 in van den Brink (2007) and Casajus (2011)).

Balanced contributions. For each TU-game $(N,v) \in \mathbf{G}$ and each pair of players $\{i,j\} \subseteq N$, it holds that:

$$f_i(N,v) - f_i(N \setminus \{j\},v) = f_j(N,v) - f_j(N \setminus \{i\},v).$$

One of the most well-known values for TU-games is the Shapley value Sh (Shapley, 1953). For each TU-game in $\mathbf{G}$, the Shapley value distributes to the players a weighted average of their contributions as defined in (3). Equivalently, the Shapley value distributes the Harsanyi dividend of each coalition equally among its members:

$$\begin{aligned} Sh_i(N,v) &= \sum_{S\subseteq N\setminus\{i\}} \frac{s!(n-s-1)!}{n!}(v(S\cup\{i\})-v(S) \\ &= \sum_{S\subseteq N:S\ni i} \frac{\Delta_S(v)}{s}, \quad \forall (N,v)\in G, \forall i \in N. \end{aligned} \tag{4}$$

Proposition 1 below states three popular axiomatic characterizations of the Shapley value Sh on $\mathbf{G}$. The first one is due to Shapley (1953), the second one is due Myerson (1980) and the last one can be found in Young (1985).

Proposition 1. *The following equivalent characterizations results hold:*

1. *The Shapley value Sh on $\mathbf{G}$ is the only value that satisfies Efficiency, Linearity (or Additivity), Equal treatment of equals players, and the Null player axiom;*
2. *The Shapley value Sh on $\mathbf{G}$ is the only value that satisfies Efficiency and Balanced contributions;*
3. *The Shapley value Sh on $\mathbf{G}$ is the only value that satisfies Efficiency, Marginality (or Coalitional strategic equivalence) and Equal treatment of equal players.*

TU-games with unpaid players. A **TU-game with unpaid players** is a triple (N, v, B) where (N, v) is a TU-game in $\mathbf{G}$ and B is a partition of N. We consider only two types of partitions: either B is a bipartition $\{P, U\}$, that is $U \cup P = N$ and $U \cap P = \varnothing$, or B is the coarsest partition $\{N\}$. In the first case, the elements of P represent the set of **paid players** and the elements of U represent the set of **unpaid players**. In the second case, all players are paid players and the TU-game with unpaid players boils down to a TU-game. For convenience, we sometimes continue to write $(N, v, \{P, U\})$ in case $U = \varnothing$. Denote by $\mathbf{B}$ the set of such partitions that one can construct from finite player sets, and let $\mathbf{GB}$ be the set of all such TU-games with unpaid players. By construction $\mathbf{G} \subseteq \mathbf{GB}$.

A TU-game with unpaid players (N, v, B) of $\mathbf{GB}$ is formally equivalent to a game with a coalition structure.[3] A **coalitional value** for TU-games with a coalition structure is a value f on $\mathbf{GB}$ that associates to each $(N, v, B) \in \mathbf{GB}$, a unique payoff vector $f(N, v, B)$ in $\mathbb{R}^N$, specifying the payoff of each player $i \in N$ for its participation in the TU-game (N, v, B). When triplets (N, v, B) are viewed as TU-games with unpaid players, a **value for TU-games with unpaid players** on $\mathbf{GB}$ is a function f on $\mathbf{GB}$ which associates to each TU-game with unpaid players $(N, v, \{P, U\})$, the payoff vector $f(N, v, \{P, U\})$ in $\mathbb{R}^P$, specifying the payoff of each paid player in P for its participation in $(N, v, \{P, U\})$. Equivalently, f could be viewed as a specific coalitional value where the unpaid players receive a null payoff. In the following, the term coalitional value is used when all participants in the game can receive a payoff, and the term value is used when the players of U do not receive any payoff. The objective is therefore to determine fair payoffs for paid players in P, which requires determining how the contributions of unpaid players in U are distributed among paid players.

3. A two-step procedure

For TU-games with a coalition structure, Owen (1977) defines a two-step procedure. In the first step, the coalitions in the partition B bargain for the value $v(N)$ through a TU-game on B constructed from (N, v). As a result, each coalition of B receives a payoff; in the second step, each coalition of B allocates payoffs to its members using a TU-game on that coalition, which takes into account the payoffs obtained in the first step when each subcoalition of that coalition represents it.

Here we also propose a two-step procedure, but the objective is different. In the first step, the procedure determines the payoffs that members of N can obtain in $(N, v, \{P, U\})$ viewed as a TU-game with a coalition structure. The payoff allocation may depend on the status of members of P relative to members of U or the type of interaction they may have.

[3] In a TU-game with a coalition structure, the latter may contain more than two elements. Here, the coalition structure contains at most two elements, P and U. A more general framework could allow the members of P and U to be partitioned to obtain the coalition structure $\{P_1, \ldots, P_k, U_{k+1}, \ldots U_m\}$.

For example, if the members of P represent workers and U represents machines, the status of workers can be considered different from that of machines. If the elements of P represent skilled employees and U represents unskilled volunteers, a hierarchical relationship may exist between the members of these two groups, with the former having priority over the appropriation of dividends obtained by cooperating. If the members of P represent the official languages of a jurisdiction and U represents the non-official languages of that jurisdiction, it is possible that a government whose objective is to rank the official languages of that jurisdiction may choose to ignore or minimize the possible effects of non-official languages on communication between citizens of that jurisdiction. The second step determines a transfer of payoffs from the members of U to the members of P, taking into account the payoffs obtained in the first step.

Formally, a value f on **GB** is obtained from a two-step procedure, if there exist a coalitional value $f^{(1)}$ on **GB** and a value $f^{(2)}$ on **G** such that:

$$f(N,v,\{P,U\}) = f^{(2)}(P, v_{f^{(1)}}), \tag{5}$$

where $(P, v_{f^{(1)}}) \in G$ is a quasi-additive TU-game constructed from the payoffs $f^{(1)}(N,v,\{P,U\})$ obtained in the first step of the procedure and defined as:

$$v_{f^{(1)}} = \sum_{i\in P} f_i^{(1)}(N,v,\{P,U\})u_{\{i\}} + f_U^{(1)}(N,v,\{P,U\})u_P. \tag{6}$$

Assume that $f^{(2)}$ satisfies Covariance, then (5) rewrites as:

$$f_i(N,v,\{P,U\}) = f_i^{(1)}(N,v,\{P,U\}) + f^{(2)}(P, f_U^{(1)}(N,v,\{P,U\})u_P), \quad \forall i \in P,$$

meaning that each player i in P receives the payoff $f_i^{(1)}(N,v,\{P,U\})$ in the first step plus the payoff $f_i^{(2)}(P, f_U^{(1)}(N,v,\{P,U\})u_P)$ in the second step that represents a transfer of money from the members of U to the members of P. If, in addition, $f^{(2)}$ is Efficient and satisfies Equal treatment of equal players, then the transfer is an equal share of the total payoffs $f_U^{(1)}(N,v,\{P,U\})$ obtained by the members of U at the first step:

$$f_i(N,v,\{P,U\}) = f_i^{(1)}(N,v,\{P,U\}) + \frac{f_U^{(1)}(N,v,\{P,U\})}{p}, \quad \forall i \in P.$$

4. Axioms

In this section, we introduce axioms for values on the domain **GB** of TU-games with unpaid players. We begin by restating the axioms of Efficiency, Additivity, and Equal treatment of equals and Balanced contributions adapted to the values of the domain **GB**. Next, we introduce variants of the axioms of Equal treatment of equals, Null player, Marginality, Coalitional strategic equivalence and Balanced contributions. Finally, we introduce two new axioms for values on **GB**.

The first axiom states that the worth generated by the cooperation of all players is shared among the paid players.

Efficiency for paid players. For each $(N, v, \{P, U\}) \in \mathbf{GB}$, it holds that

$$f_P(P, U, v) = v(N).$$

The second axiom extends the axiom of Additivity from **G** to **GB**.

Additivity. The value of f is additive on **GB**:

$$f(N, v{+}w, \{P, U\}) = f(N, v, \{U, P\}){+}f(N, w, \{U, P\}), \quad \forall (N, v, \{P, U\}), (N, w, \{P, U\}) \in \mathbf{GB}.$$

The third axiom is similar to the axiom of Equal treatment of equals for values on **G** but restricted to pairs of paid players.

Equal treatment of equal paid players. For each $(N, v, \{P, U\}) \in \mathbf{GB}$ and each pair $\{i, j\} \subseteq P$ of equal paid players in $(N, v, \{P, U\})$, it holds that:

$$f_i(N, v, \{P, U\}) = f_j(N, v, \{P, U\}).$$

Balanced contributions for paid players. For each $(N, v, \{P, U\}) \in \mathbf{GB}$ and each pair $\{i, j\} \subseteq P$, it holds that:

$$f_i(N, v, \{P, U\}) - f_i(N \setminus \{j\}, v, \{P \setminus \{j\}, U\}) = f_j(N, v, \{P, U\}) - f_j(N \setminus \{i\}, v, \{P \setminus \{j\}, U\}).$$

Next, we introduce axioms that are variant of well-known axioms.

Weak equal treatment of equal paid players. For each $(N, v, \{P, U\}) \in \mathbf{GB}$ and each pair $\{i, j\} \subseteq P$ of equal paid players in the subgame (P, v), it holds that:

$$f_i(N, v, \{P, U\}) = f_j(N, v, \{P, U\}).$$

The next axiom distributes a null payoff to null paid player, provided that all members of U are also null players.

Null paid player with null unpaid players. For each game $(N, v, \{P, U\}) \in \mathbf{GB}$, if $i \in N$ is null in $(N, v, \{P, U\})$ and all unpaid players are null in (N, v), it holds that

$$f_i((N, v, \{P, U\})) = 0.$$

The following axiom requires that a null paid player gets a null payoff in a situation where unpaid players are not productive without the collaboration of some paid players.

Null paid player with stand-alone unproductive unpaid players. For each $(N, v, \{P, U\}) \in$ **GB** such that the subgame (U, v) is the null game, and each null paid player $i \in P$ in $(N, v, \{P, U\})$, it holds that:

$$f_i(N, v, \{P, U\}) = 0.$$

Null paid player with stand-alone unproductive unpaid players imposes that a null paid player gets a zero payoff if, in addition, all unpaid players are null. If all unpaid players are null in $(N, v, \{P, U\})$, then no coalition of unpaid players is productive. The converse implication does not hold: even if all coalitions of unpaid players are unproductive, then these players may not be null, for instance if they enjoy positive interactions with paid players. Therefore, Null paid player with stand-alone unproductive unpaid players implies Null paid player with null unpaid player, but not the way around.

The following axiom adapts Coalition strategic equivalence for values on **G** and requires that a paid player should not be affected whenever a TU-game with unpaid players is added in which this player and each unpaid players are null.

Coalitional strategic equivalence for changes of paid players. For each pair of TU-games with unpaid players $(N, v, \{P, U\}), (N, w, \{P, U\}) \in$ **GB** and paid player $i \in P$ such that each $j \in U \cup \{i\}$ is a null player in $(N, w, \{P, U\})$, it holds that:

$$f_i(N, v + w, \{P, U\}) = f_i(N, v, \{P, U\}).$$

The next axiom adapts Marginality for values on **G** by imposing the same condition both on one paid player and at the same time to each unpaid player.

Marginality with respect to the unpaid players. For each pair of TU-games with unpaid players $(N, v, \{P, U\}), (N, w, \{P, U\}) \in$ **GB**, each $i \in P$ such that

$$v(S \cup \{k\}) - v(S) = w(S \cup \{k\}) - w(S), \quad \forall k \in U \cup \{i\}, \forall S \subseteq N \setminus \{k\},$$

it holds that:

$$f_i(N, v, \{P, U\}) = f_i(N, w, \{P, U\}).$$

The next axiom applies the principle of Balanced contributions in situations where all unpaid players are null players in the TU-game with unpaid players.

Balanced contributions for paid players with null unpaid players. For each TU-game with unpaid players $(N, v, \{P, U\}) \in$ **GB** such that all unpaid players are null in $(N, v, \{P, U\})$, and each pair $\{i, j\} \subseteq P$, it holds that:

$$f_i(N, v, \{P, U\}) - f_i(N \setminus \{j\}, v, \{P \setminus \{j\}, U\}) = f_j(N, v, \{P, U\}) - f_j(N \setminus \{i\}, v, \{P \setminus \{j\}, U\}).$$

We conclude this list with two new axioms. The first one requires that paid players are equally affected by a change in any coalition involving some unpaid players, ceteris paribus. The second one requires that if the status of an unpaid player changes so that it is now one of the paid players, then this change must affect each paid player of the original situation in the same way.

Equal impact of changes in a coalition containing an unpaid player. For any two TU-games with unpaid players $(N, v, \{P, U\}) \in \mathbf{GB}$ and $(N, w, \{P, U\}) \in \mathbf{GB}$ such that $v(S) \neq w(S)$ for some nonempty $S \subseteq N$ such that $S \cap U \neq \varnothing$ and $v(T) = w(T)$ for each other coalition $T \subseteq N$, and any pair of paid players $\{i, j\} \subseteq P$, it holds that:

$$f_i(N, v, \{P, U\}) - f_i(N, w, \{P, U\}) = f_j(N, v, \{P, U\}) - f_j(N, w, \{P, U\}).$$

Equal impact of promoting an unpaid player. For each $(N, v, \{P, U\}) \in \mathbf{GB}$, each pair of paid players $\{i, j\} \subseteq P$ and each unpaid player $k \in U$, it holds that:

$$f_i(N, v, \{P, U\}) - f_i(N, v, \{P \cup \{k\}, U \setminus \{k\}\}) = f_j(N, v, \{P, U\}) - f_j(N, v\{P \cup \{k\}, U \setminus \{k\}\}).$$

Lemma 1 below shows the equivalence Coalition strategic equivalence for changes of paid players and Marginality with respect to unpaid players.

Lemma 1. *Coalition strategic equivalence for changes of paid players is equivalent to Marginality with respect to unpaid players on* **GB**.

Proof. Let f be a value on **GB** that satisfies Coalition strategic equivalence for changes of paid players. Let $(N, v, \{P, U\}), (N, w, \{P, U\}) \in \mathbf{GB}$ as hypothesized, and $i \in P$. Pick any $k \in U \cup \{i\}$. Observe that for each $S \subseteq N$, $S \ni k$, $\Delta_S(v) = \Delta_S(w)$. Thus, one can write

$$w = v + \sum_{S \subseteq P \setminus \{i\}} \Delta_S(w - v) u_S.$$

Because i is a null player in each $(N, \Delta_S(w - v)u_S)$, $S \subseteq P \setminus \{i\}$, successive applications of Coalition strategic equivalence for changes of paid players yields that $f_i(N, v, \{P, U\}) = f_i(N, w, \{P, U\})$. Therefore, f satisfies Marginality with respect to unpaid players.
To show the converse implication, let f be a value on **GB** that satisfies Marginality with respect to unpaid players. Consider $(N, v, \{P, U\}) \in \mathbf{GB}$, $i \in P$ and $(N, w, \{P, U\}) \in \mathbf{GB}$ such that each $k \in U \cup \{i\}$ is null in $(N, w, \{P, U\})$. It follows that each $k \in U \cup \{i\}$ is such that

$$(v + w)(S \cup \{k\}) - (v + w)(S) = v(S \cup \{k\}) - v(S).$$

By Marginality with respect to unpaid players, one concludes that $f_i(N, v, \{P, U\}) = f_i(N, v + w, \{P, U\})$. Therefore f satisfies Coalition strategic equivalence for changes of paid players. This concludes the proof. ■

We conclude this section with a series of remarks on the values constructed from the two-step procedure and some of the axioms in the above list.

Remark 2. *Assume that a value f on* **GB** *is defined through the two-step procedure (5)-(6).*

1. *If $f^{(1)}$ and $f^{(2)}$ satisfy Efficiency, then f satisfies Efficiency for paid players.*
2. *If $f^{(1)}$ and $f^{(2)}$ satisfy Additivity, then f satisfies Additivity.*
3. *If $f^{(1)}$ and $f^{(2)}$ satisfy Equal treatment of equal players, then f satisfies Equal treatment of equal paid players.*
4. *If $f^{(1)}$ satisfies Balanced contributions and the Null player axiom and $f^{(2)}$ satisfies Covariance, then f satisfies Balanced contributions for paid players with null unpaid players.*
5. *If $f^{(1)}$ satisfies the Null player axiom and $f^{(2)}$ satisfies Covariance, then f satisfies Null paid player with null unpaid players. This comes from the fact that $f^{(2)}(P, \mathbf{0}_P)$ is the null payoff vector.*
6. *If $f^{(1)}$ satisfies Marginality and $f^{(2)}$ satisfies Covariance, then f satisfies Marginality with respect to the unpaid players. Assume that for $(N, v, \{P, U\})$ and $(N, w, \{P, U\})$ of* **GB***, paid player $i \in P$ and each unpaid player $k \in U$ satisfies the condition of Marginality with respect to the unpaid players. We have that i and each k have the same contributions in both TU-games with unpaid players. By Marginality applied to $k \in U$, one gets $f_k^{(1)}(N, v, \{P, U\}) = f_k^{(1)}(N, w, \{P, U\})$ so that $f_U^{(1)}(N, v, \{P, U\}) = f_U^{(1)}(N, w, \{P, U\})$, and by Marginality applied to $i \in P$, it holds that $f_i^{(1)}(N, v, \{P, U\}) = f_i^{(1)}(N, w, \{P, U\})$. Using the above equalities and Covariance of $f^{(2)}$, one obtains*

$$\begin{aligned} f_i(N, v, \{U, P\}) &= f_i^{(1)}(N, v, \{U, P\}) + f^{(2)}(f_U^{(1)}(N, v, \{U, P\})u_P) \\ &= f_i^{(1)}(N, w, \{U, P\}) + f^{(2)}(f_U^{(1)}(N, w, \{U, P\})u_P) \\ &= f_i(N, w, \{U, P\}), \end{aligned}$$

   as desired.
7. *We say that $f^{(1)}$ satisfies Partition invariance if $f_i^{(1)}(N, v, \{P, U\})$ is independent of the partition $\{P, U\}$. Assume that $f^{(1)}$ satisfies Partition invariance and $f^{(2)}$ satisfies Covariance and Equal treatment of equal players. Then f satisfies Equal impact of promoting an unpaid player. Indeed, by Covariance and Equal treatment of equal players there is a constant c_P such that*

$$f_i(N, v, \{P, U\}) = f_i^{(1)}(N, v, \{P, U\}) + c_P, \quad \forall i \in P.$$

   The same argument applies when $P \cup \{k\}$ and $U \setminus \{k\}$, $k \in U$, so that

$$f_i(N, v, \{P \cup \{k\}, U \setminus k\}) = f_i^{(1)}(N, v, \{P \cup \{k\}, U \setminus k\}) + c_{P \cup k}, \quad \forall i \in P.$$

   By Partition invariance,

$$f_i^{(1)}(N, v, \{P, U\}) = f_i^{(1)}(N, v, \{P \cup \{k\}, U \setminus k\}),$$

so that the difference

$$f_i(N, v, \{P, U\}) - f_i(N, v, \{P \cup \{k\}, U \setminus k\}) = c_P - c_{P\cup k}$$

does not depend on $i \in P$, as stated.

5. Axiomatic analysis

In this section, we propose different combinations of axioms from the list of axioms presented in section 4 to arrive at the characterisation of three values defined using the two-step procedure (5)-(6).

Proposition 2.

1. *There is at most one value on* **GB** *that satisfies Efficiency for paid players and Balanced contributions for paid players.*
2. *There is at most one value on* **GB** *that satisfies Efficiency for paid players, Additivity, Equal treatment of equal paid players and Null paid player with stand-alone unproductive unpaid players.*

Proof. Point 1. The proof follows similar arguments as the proof in Myerson (1980) for values on **G**, so it is omitted.

Point 2. Pick any value f satisfying the four axioms of the statement of point 2. Because f satisfies Additivity, from (1), it is enough to show that $f(N, cu_S, \{U, P\})$ is uniquely determined for each nonempty $S \subseteq N$ and each scalar $c \in \mathbb{R}$. If $c = 0$, then $f(N, cu_S, \{U, P\}) = \mathbf{0}_P$ by Additivity of f. So let $c \neq 0$ and pick any nonempty $S \subseteq N$. Two cases are considered.
Case (a): $S \subseteq U$. Note that all paid players are null and so equal players in $(N, cu_S, \{U, P\})$ and the subgame $(S, u_S) \neq \mathbf{0}_S$. From Efficiency and Equal treatment of equal paid players, we get $f_i(N, cu_S, \{U, P\}) = c/p$ for each $i \in P$.
Case (b): $S \cap P \neq \varnothing$. Each $i \in P \setminus S$ is a null player in $(N, cu_S, \{U, P\})$. In addition, for each $T \subseteq U$, it holds that $(cu_S)(T) = 0$ since $S \cap P \neq \varnothing$, that is the subgame $(S, u_S) = \mathbf{0}_S$. Hence, applying Null paid player with stand-alone unproductive unpaid players yields that $f_i(N, cu_S, \{U, P\}) = 0$ for each $i \in P \setminus S$. Next, any pair of players in $S \cap P$ are equal players in (N, cu_S). Thus, Efficiency and Equal treatment of equal paid players imply that $f_i(N, cu_S, \{U, P\}) = c/s_p$ for each $i \in S \cap P$, where s_p denotes the cardinality of the set $S \cap P$. From **Case (a)-(b)**, the proof of point 2 is complete. ∎

Using the procedure (5)-(6), define the P-**priority value** f^P on **GB** as follows:

$$f^P(N, v, \{U, P\}) = f^{P,(2)}(P, v_{f^{P,(1)}(N,v,\{U,P\})}), \tag{7}$$

where $f^{P,(2)} = Sh$, and

$$f_i^{P,(1)}(N,v,\{U,P\}) = \sum_{\substack{S\subseteq N:\\ S\ni i}} \frac{\Delta_S(v)}{s_p},\ \forall i\in P, \quad \text{and} \quad f_i^{P,(1)}(N,v,\{U,P\}) = \sum_{\substack{S\subseteq U:\\ S\ni i}} \frac{\Delta_S(v)}{s},\ \forall i\in U, \tag{8}$$

where s_p denotes the cardinality of the set $S\cap P$.

Note that $f_i^{P,(1)}(N,v,\{U,P\}) = Sh_i(U,v)$ for each $i\in U$ so that by Efficiency of Sh,

$$f_U^{P,(1)}(N,v,\{U,P\}) = v(U).$$

Next, using Covariance, Equal treatment of equal players and Efficiency of the Shapley value in the second step, f^P rewrites as:

$$\begin{aligned} f_i^P(N,v,\{U,P\}) &= Sh_i\Bigg(P, \sum_{i\in P}\Bigg(\sum_{\substack{S\subseteq N:\\ S\ni i}} \frac{\Delta_S(v)}{s_p}\Bigg)u_{\{i\}} + v(U)u_P\Bigg) \\ &= \sum_{\substack{S\subseteq N:\\ S\ni i}} \frac{\Delta_S(v)}{s_p} + \frac{v(U)}{p}, \quad \forall i\in P \end{aligned} \tag{9}$$

The P-**priority value** f^P allocates the dividend of each coalition equally among its paid members and if a coalition contains no paid player, the associated dividend is split equally among all unpaid players. The interpretation is as follows. In the first step, members of P have priority over members of U in claiming the dividend of a coalition containing members of P and U. On the other hand, coalitions formed solely by members of U share the dividends equally among themselves. In other words, cooperation between members of P and U results in a sharing of dividends among the paid players. In the second step, the Shapley value is applied to the quasi-additive TU-game on P so that paid players receive their payoff from the first step plus an equal share of the total payoff $v(U)$ obtained by the members of U in the first step.

Proposition 3. *The P-priority value f^P is the unique value for TU-games with unpaid players on* **GB** *that satisfies (1) Efficiency for paid players and Balanced contributions for paid players or (2) Efficiency for paid players, Additivity, Equal treatment of equal paid players and Null paid player with stand-alone unproductive unpaid players.*

Proof. By Proposition 2, it suffices to show the f^P satisfies the axioms of points 1 and 2 of the statement of the proposition.

Efficiency for paid players. $f^{P,(1)}$ and $f^{P,(2)}$ are both Efficient values so that f^P is Efficient for paid players by point 1 of Remark 2. To see that $f^{P,(1)}$ is Efficient, note that the members of P share equally the dividend of each coalition S such that $S\cap P\neq\varnothing$ so that they collect in total:

$$f_P^{P,(1)}(N,v,\{P,U\}) = \sum_{\substack{S\subseteq N:\\ S\cap P\neq\varnothing}} \Delta_S(v),$$

while the members of U collect in total,

$$f_U^{P,(1)}(N,v,\{P,U\}) = \sum_{\substack{S\subseteq N:\\ S\subseteq U, S\neq\varnothing}} \Delta_S(v).$$

It follows that:

$$f_P^{P,(1)}(N,v,\{P,U\}) + f_U^{P,(1)}(N,v,\{P,U\}) = \sum_{\substack{S\subseteq N:\\ S\neq\varnothing}} \Delta_S = v(N),$$

as desired.

Balanced contributions for paid players Pick any $(N,v,\{P,U\}) \in \mathbf{GB}$ and any pair of distinct paid players $\{i,j\} \subseteq P$. We have:[4]

$$\begin{aligned} f_i^P(N,v,\{P,U\}) - f_i^P(N\setminus\{j\},v,\{P\setminus\{j\},U\}) &= \sum_{\substack{S\subseteq N:\\ S\supseteq\{i,j\}}} \frac{\Delta_S(v)}{s_p} + \frac{v(U)}{p} - \frac{v(U)}{p-1}, \\ &= f_j^P(N,v,\{P,U\}) - f_j^P(N\setminus\{i\},v,\{P\setminus\{i\},U\}). \end{aligned}$$

Additivity. $f^{P,(1)}$ and $f^{P,(2)}$ are both additive values so that f^P is Additive by point 2 of Remark 2.

Equal treatment of equal paid players. By expression (9) of f^P, $f^{P,(2)} = Sh$. The result follows from point 1 of Proposition 1.

Null paid player with stand-alone unproductive unpaid players. It must be clear that $f^{P,(1)}$ satisfies the Null player axiom. In particular, $v(U) = 0$ in $(N,v,\{P,U\})$ implies that

$$v_{f^{P,(1)}}(N,v,\{P,U\}) = \sum_{j\in P} f_j^{P,(1)}(N,v,\{P,U\})u_{\{i\}}.$$

Thus, if $i \in P$ is a null player in $(N,v,\{P,U\})$, $f_i^{P,(2)}(P,v,\{P,U\}) = f_i^{(1)}(N,v,\{P,U\}) = 0$ by Covariance of $f^{P,(2)} = Sh$. This concludes the proof of Proposition 3. ■

The next three new combinations of axioms result in at most one value in **GB**.

Proposition 4.

1. *There is at most one value on* **GB** *that satisfies Efficiency for paid players, Additivity, Weak equal treatment of equal paid players and Null paid player with null unpaid players.*

[4]Note that we can not use point 4 of Remark 2 because $f^{P,(1)}$ do not satisfy Balanced contributions.

2. *There is at most one value on* **GB** *that satisfies Efficiency for paid players, Equal impact of changes in a coalition containing an unpaid player and Balanced contributions for paid players with null unpaid players.*
3. *There is at most one value on* **GB** *that satisfies Efficiency for paid players, Weak equal treatment of equal paid players, Coalition strategic equivalence for changes of paid players (or Marginality with respect to unpaid players)*[5]*, and Equal impact of changes in a coalition containing an unpaid player.*

Proof. Point 1. Assume there is a value f on **GB** that satisfies Efficiency for paid players, Additivity, Equal treatment of restricted equal paid players and Null paid player with null unpaid players. Because f satisfies Additivity, from (1), it is enough to show that $f(N, cu_S, \{P, U\})$ is uniquely determined for all nonempty coalition $S \subseteq N$ and all scalar $c \in \mathbb{R}$. If $c = 0$, then $f(N, cu_S, \{P, U\}) = \mathbf{0}_P$ by Additivity of f. So let $c \neq 0$ and pick any nonempty coalition $S \subseteq N$. Two cases are considered.

Case (a): $S \cap U \neq \varnothing$. All players in P are equal players in (P, cu_S). By Weak equal treatment of equal paid players and Efficiency for paid players f is uniquely determined:

$$f_i(N, cu_S, \{P, U\}) = \frac{c}{p}, \quad \forall i \in P.$$

Case (b): $S \cap U = \varnothing$. Hence, $S \subseteq P$. Each $j \in (P \setminus S) \cup U$ is null in $(N, cu_S, \{P, U\})$ so that one can apply Null paid player with null unpaid players to get $f_i(N, cu_S, \{P, U\}) = 0$ for each $i \in P \setminus S$. Because $S \subseteq P$, any two (paid) players in S are equal in (P, cu_S). Invoking Weak equal treatment of equal paid players and Efficiency for paid players yields

$$f_i(N, cu_S, \{P, U\}) = \frac{c}{s}, \quad \forall i \in P,$$

so that f is uniquely determined.

From **Case (a)-(b)**, the proof of point 1 is complete.

Point 2. Pick any value f on **GB** satisfying Efficiency for paid players, Equal impact of changes in a coalition containing an unpaid player and Balanced contributions for paid players with null unpaid players. Pick any $(N, v, \{P, U\}) \in$ **GB**. We prove that $f(N, v, \{P, U\})$ is uniquely determined by induction on the number $\delta(v)$ of coalitions involving some unpaid players and with a non-zero Harsanyi dividend.

$$\delta(v) = \text{card}\{S \subseteq N : S \cap U \neq \varnothing, \Delta_S(v) \neq 0\}.$$

[5]See Lemma 1.

Initial step: $\delta(v) = 0$. Remark that each player $i \in U$ is a null player in $(N, v, \{P, U\})$. Proceeding as in Myerson (1980), Balanced contributions for paid players with null unpaid players and Efficiency for paid players ensure that f is uniquely determined.
Induction hypothesis: Assume that $f(N, v, \{P, U\})$ is uniquely determined for each $(N, v, \{P, U\})$ such that $\delta(v) \leq k$ with $0 \leq k \leq \delta(v) < 2^p(2^u - 1)$.
Induction step: $\delta(v) = k + 1$. Choose any $S \subseteq N$ such that $S \cap U \neq \emptyset$ and $\Delta_S(v) \neq 0$ and consider any $T \subseteq N$ such that $S \subseteq T$. Obviously, $T \cap U \neq \emptyset$. From Equal impact of changes in a coalition containing an unpaid player, for each pair of paid players $\{i, j\} \subseteq P$, one obtains:

$$f_i(N, v, \{P, U\}) - f_i(N, v - \Delta_S(v)1_T, \{P, U\}) = f_j(N, v, \{P, U\}) - f_j(N, v - \Delta_S(v)1_T, \{P, U\}) \tag{10}$$

Repeating this operation successively for each coalition T such that $S \subseteq T$ and summing over all the resulting equalities, one gets:

$$f_i(N, v, \{P, U\}) - f_i\big(N, v - \sum_{\substack{T \subseteq N: \\ T \supseteq S}} \Delta_S(v)1_T, \{P, U\}\big) = f_j(N, v, \{P, U\}) - f_j\big(P, U, v - \sum_{\substack{T \subseteq N: \\ T \supseteq S}} \Delta_S(v)1_T, \{P, U\}\big),$$

which is equivalent to

$$f_i(N, v, \{P, U\}) - f_i(N, v - \Delta_S(v)u_S, \{P, U\}) = f_j(N, v, \{P, U\}) - f_j(N, v - \Delta_S(v)u_S, \{P, U\}) \tag{11}$$

Since $\Delta_S(v - \Delta_v(S)u_S) = 0$ and $\Delta_T(v - \Delta_v(S)u_S) = \Delta_T(v)$ for each $T \neq S$, one obtains $\delta(v - \Delta_v(S)u_S) = \delta(v) - 1 = k$. Therefore, the induction hypothesis implies that $f(N, v - \Delta_S(v)u_S, \{P, U\})$ is uniquely determined. Summing both sides of (11) over all elements j of P and using Efficiency for paid players, yields

$$p\big(f_i(N, v, \{P, U\}) - f_i(N, v - \Delta_S(v)u_S, \{P, U\}\big) = v(N) - (v - \Delta_S(v)u_S)(N),$$

that is,

$$f_i(N, v, \{P, U\}) = f_i(N, v - \Delta_S(v)u_S, \{P, U\}\big) + \frac{\Delta_S(v)}{p},$$

which makes $f_i(N, v, \{P, U\})$ uniquely determined. This completes the induction step.

Point 3. Let f be any value on **GB** that satisfies Efficiency for paid players, Weak equal treatment of equal paid players, Coalition strategic equivalence for changes of paid players, and Equal impact of changes in a coalition containing an unpaid player. As above one proceeds by induction on $\delta(v)$ to prove that $f(N, v, \{P, U\})$, $(N, v, \{P, U\}) \in$ **GB** is uniquely determined.
Initial step: $\delta(v) = 0$. It holds that $v = \sum_{S \subseteq P} \Delta_S(v)u_S$ that coincides with (P, v). So one can proceed as in Chun (1989) to conclude that the combination of Efficiency for paid

players, Equal treatment for restricted equal paid players, Coalition strategic equivalence for changes of paid players determines $f(N, v, \{P, U\})$ uniquely.
Induction hypothesis: Assume that $f(N, v, \{P, U\})$ is uniquely determined for each $(N, v, \{P, U\})$ such that $\delta(v) \leq k$ with $0 \leq k \leq \delta(v) < 2^p(2^u - 1)$.
Induction step: The induction step is identical to the induction step of point 2 above and is not replicated here. ■

Using the procedure (5)-(6), define the U-**priority value** f^U on **GB** as follows:

$$f^U(N, v, \{U, P\}) = f^{U,(2)}(P, v_{f^{U,(1)}}), \tag{12}$$

where $f^{U,(2)} = Sh$, and

$$f_i^{U,(1)}(N, v, \{U, P\}) = \sum_{\substack{S \subseteq P: \\ S \ni i}} \frac{\Delta_S(v)}{s}, \ \forall i \in P, \quad \text{and} \quad f_i^{U,(1)}(N, v, \{U, P\}) = \sum_{\substack{S \subseteq N: \\ S \ni i}} \frac{\Delta_S(v)}{s_u}, \ \forall i \in U, \tag{13}$$

where s_u denotes the cardinality of the set $S \cap U$.

Note that $f_i^{U,(1)}(N, v, \{U, P\}) = Sh_i(P, v)$ for each $i \in P$ so that by Efficiency of Sh,

$$f_P^{U,(1)}(N, v, \{U, P\}) = v(P).$$

Next, using Covariance, Equal treatment of equal players and Efficiency of the Sh value in the second step, f^U writes as:

$$f_i^U(N, v, \{U, P\}) = Sh_i(P, v) + \frac{\left(\sum_{i \in U} \sum_{\substack{S \subseteq N: \\ S \ni i}} \frac{\Delta_S(v)}{s_u} \right)}{p} \tag{14}$$

The U-**priority value** f^U allocates the dividend of each coalition equally among its unpaid members and if a coalition contains no unpaid player, the associated dividend is split among all paid players. This is the opposite situation to that described for the P-priority value f^P. In the first step, members of U have priority over members of P in claiming the dividend of a coalition containing members of P and U. On the other hand, coalitions formed solely by members of P share the dividends equally among themselves. In other words, cooperation between members of P and U in the first step results in a sharing of dividends among the unpaid players. In the second step, the Shapley value is applied to the quasi-additive TU-game on P so that paid players receive their payoff from the first step plus an equal share of the total payoff obtained by the members of U in the first step. As for $f^{P,(1)}$, $f^{U,(1)}$ if an Efficient coalitional value so that

$$\sum_{i \in U} \sum_{\substack{S \subseteq N: \\ S \ni i}} \frac{\Delta_S(v)}{s_u} = v(N) - v(P).$$

Therefore, the U-priority value f^U rewrites as :

$$f_i^U(N, v, \{U, P\}) = Sh_i(P, v) + \frac{v(N) - v(P)}{p}, \quad \forall i \in P \tag{15}$$

This two-step procedure leads to a value that assigns to each paid player its Shapley value in the subgame (P, v) and shares the incremental contribution of all unpaid players equally among paid players.

The next result provides three alternative characterizations of the U-priority value.

Proposition 5. *The U-priority value f^U as defined in (15) is the unique value on* **GD** *that satisfies (1) Efficiency for paid players, Additivity, Weak equal treatment of equal paid players and Null paid player with null unpaid players, (2) Efficiency for paid players, Equal impact of changes in a coalition containing an unpaid player and Balanced contributions for paid players with null unpaid players or (3) Efficiency for paid players, Weak equal treatment of equal paid players, Coalition strategic equivalence for changes of paid players (or Marginality with respect to unpaid players) and Equal impact of changes in a coalition containing an unpaid player.*

Proof. In view of Proposition 4, it suffices to show that f^U satisfies the axioms contained in points 1-3 of the statement of the proposition.
Efficiency of paid players, Additivity, Null paid player with null unpaid players: it suffices to proceed in a similar way as in the proof of Proposition 3 to conclude.
Weak equal treatment of equal paid players: consider any $(N, v, \{P, U\})$ in **GB**, assume that two players i and j in P are equal players in (P, v). By equal treatment of equal players of Sh, $Sh_i(P, v) = Sh_j(P, v)$ and so by (15), $f_i^U(N, v, \{U, P\}) = f_j^U(N, v, \{U, P\})$.

Equal impact of changes in a coalition containing an unpaid player: consider any two TU-games with unpaid players $(N, v, \{P, U\})$ and $(N, w, \{P, U\}) \in$ **GB** such that $v(S) \neq w(S)$ for some nonempty $S \subseteq N$, $S \cap U \neq \emptyset$, and $v(T) = w(T)$ for each other coalition $T \neq S$. Because $(P, v) = (P, w)$, it holds that $Sh_i(P, v) = Sh_i(P, w)$ for each $i \in P$. Therefore, by (15), for each $i \in P$, we have

$$f_i^U(N, v, \{P, U\}) - f_i^U(N, w, \{P, U\}) = \frac{v(N) - v(P)}{p} - \frac{w(N) - w(P)}{p},$$

which does not depend on $i \in P$, from which Equal impact of changes in a coalition containing an unpaid player follows.
Balanced contributions for paid players with null unpaid players: consider any TU-game with unpaid players $(N, v, \{P, U\}) \in$ **GB**. If all unpaid players are null players, then

this implies that $v(N) = v(P)$ and in turn, for each $i \in P$, that $f_i^U(N, v, \{P, U\}) = Sh_i(P, v)$. As a consequence, f^U immediately inherits Balanced contributions for paid players with null unpaid players from the fact that the Shapley value satisfies Balanced contributions (see point 2 of Proposition 1).

Coalition strategic equivalence for changes of paid players: let $(N, v, \{P, U\})$ and $(N, w, \{P, U\}) \in \mathbf{GB}$ and $i \in P$ be as hypothesized by Coalition strategic equivalence for changes of paid players. By the definition (15) of f^U, it holds that:

$$f_i^U(N, v + w, \{P, U\}) - f_i^U(N, v, \{P, U\}) = Sh_i(P, v + w) - Sh_i(P, v) + \frac{w(N) - w(P)}{p}.$$

Because player $i \in P$ is null in $(N, w, \{P, U\})$ by hypothesis, it is also null in (P, w). Thus, $Sh_i(N, v + w, \{P, U\}) = Sh_i(P, v)$ by Coalitional strategic equivalence of Sh (see point 3 of Proposition 1). Finally, because each player $k \in U$ is null in $(N, w, \{P, U\})$ by hypothesis, it holds that $w(N) = w(P)$. One concludes that $f_i^U(N, v + w, \{P, U\}) - f^U(N, v, \{P, U\}) = 0$ as desired. ∎

Finally, one last combination of axioms leads to uniqueness.

Proposition 6. *There is at most one value on* $\mathbf{GB}$ *that satisfies Efficiency for paid players, Equal impact of promoting an unpaid player and Balanced contributions for paid players with null unpaid players.*

Proof. Consider any f on $\mathbf{GB}$ satisfying Efficiency for paid players, Equal impact of promoting an unpaid player and Balanced contributions for paid players with null unpaid players. We proceed by induction on the cardinality u of U to show that f is uniquely determined.

Initial step: U is such that $u = 0$, that is $(N, v, \{P, U\})$ reduces to the TU-game $(P, v) \in \mathbf{G}$, where $P = N$. Then, Balanced contributions for paid players with null unpaid players vacuously reduces to the axiom of Balanced contributions and Efficiency for paid players reduces to the axiom of Efficiency. From Myerson (1980), conclude that $f_i(P, v)$, $i \in P$, is uniquely determined.

Induction hypothesis: Assume that $f(N, v, \{P, U\})$ is uniquely determined for each $(N, v, \{P, U\}) \in \mathbf{GB}$ such that $u \leq q$, where $0 \leq q < n - 1$.

Induction step: Consider any $(N, v, \{P, U\}) \in \mathbf{GB}$ such that $u = q + 1$. Applying Equal impact of promoting an unpaid player, for each pair $\{i, j\} \subseteq P$ and each $k \in U$, one obtains

$$f_i(N, v, \{P, U\}) - f_j(N, v, \{P, U\}) = f_i(N, v, (P \cup \{k\}, U \setminus \{k\}\}) - f_j(N, v, \{P \cup \{k\}, U \setminus \{k\}), \tag{16}$$

By the induction hypothesis, note that the right hand side of (16) is uniquely determined. Summing (16) on $j \in P$ and using Efficiency for paid players, one obtains:

$$pf_i(N,v,\{P,U\})-v(N) = pf_i(N,v,\{P\cup\{k\},U\setminus\{k\}\})-(v(N)-f_k(N,v,\{P\cup\{k\},U\setminus\{k\}\}),$$

that is

$$f_i(N,v,\{P,U\}) - v(N) = f_i(N,v,\{P \cup \{k\},U \setminus \{k\}\}) + \frac{f_k(N,v,\{P \cup \{k\},U \setminus \{k\}\})}{p},$$

from which one concludes that f is uniquely determined. ■

Using the procedure (5)-(6), define the **Redistribution value** f^R on **GB** as follows:

$$f^R(N,v,\{U,P\}) = f^{R,(2)}(P,v_{f^{R,(1)}}), \tag{17}$$

where $f^{R,(2)} = Sh$, and $f^{R,(1)} = Sh$, so that

$$f_i^R(N,v,\{U,P\}) = Sh_i(N,v,\{P,U\}) + \frac{1}{p}\left(\sum_{j\in U} Sh_j(N,v,\{P,U\})\right), \quad \forall i \in P \tag{18}$$

Therefore $f^{(1)}$ assigns to each player its Shapley value in the first step and $f^{(2)}$ redistributes the total payoff obtained by U in the first step equally among the paid players.

Unlike f^P and f^U, the first step of the procedure treats the members of P in the same way as the members of U, without establishing a priority relationship between these two groups. The second step is identical for f^P, f^U, and f^R.

The last result of this section provides a characterization of the Redistribution value.

Proposition 7. *The Redistribution value f^R is the unique value on* **GB** *that satisfies Efficiency for paid players, Equal impact of promoting an unpaid player and Balanced contributions for paid players with null unpaid players.*

Proof. In view of Proposition 6, it suffices to show that f^R satisfies the axioms of the statement of the proposition. It is obvious that f^R satisfies Efficiency for paid players Balanced contributions for paid players with null unpaid players, given that Sh satisfies Efficiency, the Null player axiom and Balanced contributions. Equal impact of promoting an unpaid player follows from point 7 of Remark 2. ■

Remark that the final equation of the proof of Proposition 6 yields a somehow recursive (w.r.t. the distribution of players between P and U) formula for f^R: for each $(N,v,\{P,U\}) \in$ **GB**, each $i \in P$ and each $k \in U$,

$$f_i^R(N,v,\{P,U\}) = f_i^R(N,v,\{P \cup \{k\},U \setminus \{k\}\}) + \frac{1}{p} f_k^R(N,v,\{P \cup \{k\},U \setminus \{k\}\}).$$

Our characterizations can be summarized by the following recap chart, in which a "⊕" symbol means that the axiom is invoked in the corresponding characterization, a "+" symbol means is satisfied but not invoked in a characterization and a "−" symbol means that the axiom is not satisfied. The number of the corresponding proposition is indicated for the multiple characterizations of the P-priority and U-priority values.

Axioms	f^P		f^U			f^R
	$P3(1)$	$P3(2)$	$P5(1)$	$P5(2)$	$P5(3)$	
Efficiency for paid players	⊕	⊕	⊕	⊕	⊕	⊕
Additivity	+	⊕	⊕	+	+	+
Equal treatment of equal paid players	+	⊕	−	−	−	+
Equal treatment for restricted equal paid players	−	−	⊕	+	⊕	−
Null paid player with stand-alone unproductive unpaid players	+	⊕	−	−	−	−
Null paid player with null unpaid players	+	+	⊕	+	+	+
Equal impact of changes in a coalition containing an unpaid player	−	−	+	⊕	⊕	−
Balanced contributions for paid players	⊕	+	−	−	−	−
Balanced contributions for paid players with null unpaid players	+	+	+	⊕	+	⊕
Equal impact of promoting an unpaid player	−	−	−	−	−	⊕
Coalition strategic equivalence for changes of paid players/Marginality with respect to the unpaid players	+	+	+	+	⊕	+

6. Additional content

6.1. Formulation as the Shapley value of an alternative reduced games

The quasi-additive TU-games constructed in the second-step of our procedure can be considered as reduced games on the set of paid players obtained from the original TU-game with unpaid players and the value $f^{(1)}$.[6] In this subsection, we present an alternative

[6]Reduced games considered here are different from reduced games proposed in the literature on the axioms of consistency (see Thomson, 2011, for instance). In this literature, the leaving players take payoffs with them. In our framework, the leaving (unpaid) players get nothing.

reduced-game approach in which the reduced game does not depend on $f^{(1)}$. We show that this alternative viewpoint is suitable to compare the P-priority and U-priority values: these values can be written as the Shapley value of these reduced games.

More specifically, from each TU-game with unpaid players $(N, v, \{P, U\})$, construct the reduced game (P, v^P) such that $v^P(\emptyset) = 0$ and for each nonempty $S \subseteq P$,

$$v^P(S) = \sum_{T \subseteq S} \sum_{K \subseteq U} \Delta_{T \cup K}(v) \tag{19}$$

Equivalently, remark that

$$v^P(S) = v(S) + \sum_{T \subseteq S} \sum_{K \subseteq U : K \neq \emptyset} \Delta_{T \cup K}(v), \tag{20}$$

which leads to the interpretation that $v^P(S)$ is $v(S)$ plus a claim by S of all dividends that its members can create by cooperating with the members of U. The next result proves that the P-priority value is the Shapley value of this reduced game.

Proposition 8. *For each game $(N, v, \{P, U\})$, it holds that $f^P(N, v, \{P, U\}) = Sh(P, v^P)$.*

As a preliminary result, we start by exhibiting the Harsanyi dividends of (P, v^P).

Lemma 2. *For each game (P, v^P) and each nonempty coalition $S \subseteq P$ of cardinality s, it holds that:*

$$\Delta_S(v^P) = \Delta_S(v) - (-1)^s v(U) + \sum_{K \subseteq U : K \neq \emptyset} \Delta_{K \cup S}(v) \tag{21}$$

Proof. We proceed by induction on the cardinality of coalition S.

Initialization. Assume that $s = 1$, i.e., $S = \{i\}$ for some $i \in P$. From (20), we have

$$\begin{aligned}
\Delta_{\{i\}}(v^P) &= v^P(\{i\}) \\
&= v(\{i\}) + \sum_{\substack{K \subseteq U: \\ K \neq \emptyset}} \Delta_K(v) + \sum_{\substack{K \subseteq U: \\ K \neq \emptyset}} \Delta_{K \cup \{i\}}(v) \\
&= \Delta_{\{i\}}(v) + v(U) + \sum_{\substack{K \subseteq U: \\ K \neq \emptyset}} \Delta_{K \cup \{i\}}(v) \\
&= \Delta_{\{i\}}(v) - (-1)^1 v(U) + \sum_{\substack{K \subseteq U: \\ K \neq \emptyset}} \Delta_{K \cup \{i\}}(v),
\end{aligned}$$

Induction hypothesis. Assume that $\Delta_S(v^P)$ is given by (21) for all nonempty $S \subseteq P$ such that $s \leq k$, $1 \leq k < p$.

Induction step. Consider any $S \subseteq P$ of cardinality $s = k + 1$. By definition,

$$\Delta_S(v^P) = v^P(S) - \sum_{T \subsetneq S} \Delta_T(v^P).$$

From (20), the induction hypothesis and (21), this equality can be rewritten as

$$\begin{aligned}
v^P(S) = & \left[v(S) + \sum_{T\subseteq S}\sum_{\substack{K\subseteq U:\\ K\neq\varnothing}} \Delta_{K\cup T}(v)\right] - \sum_{T\subsetneq S}\left[\Delta_T(v) - (-1)^t v(U) + \sum_{\substack{K\subseteq U:\\ K\neq\varnothing}} \Delta_{K\cup T}(v)\right] \\
= & \Delta_S(v) + \sum_{\substack{K\subseteq U:\\ K\neq\varnothing}} \Delta_{K\cup S}(v) + \sum_{T\subsetneq S}(-1)^t v(U) \\
= & \Delta_S(v) + \sum_{\substack{K\subseteq U:\\ K\neq\varnothing}} \Delta_{K\cup S}(v) + v(U)\sum_{T\subseteq S}(-1)^t - (-1)^s v(U) \\
= & \Delta_S(v) + \sum_{\substack{K\subseteq U:\\ K\neq\varnothing}} \Delta_{K\cup S}(v) + v(U)\sum_{t=0}^{s}\binom{s}{t}(-1)^t - (-1)^s v(U) \\
= & \Delta_S(v) + \sum_{\substack{K\subseteq U:\\ K\neq\varnothing}} \Delta_{K\cup S}(v) - (-1)^s v(U),
\end{aligned}$$

where the final equality comes from the Binomial theorem. This concludes the proof. ■

Proof. (Proposition 8) For each $(N, v, \{P, U\}) \in \mathbf{GB}$ and each $i \in P$, we have

$$Sh_i(P, v^P) = \sum_{\substack{S\subseteq P:\\ S\ni i}} \frac{\Delta_S(v^P)}{s}.$$

Using (21) from Lemma 2 and the notation s_p for $|S \cap P|$, we have

$$\begin{aligned}
& Sh_i(P, v^P) \\
= & \sum_{\substack{S\subseteq P:\\ S\ni i}} \frac{\Delta_S(v)}{s} + \sum_{\substack{S\subseteq P:\\ S\ni i}}\sum_{\substack{K\subseteq U:\\ K\neq\varnothing}} \frac{\Delta_{K\cup S}(v)}{s} - \sum_{\substack{S\subseteq P:\\ S\ni i}} (-1)^s \frac{v(U)}{s} \\
= & \sum_{\substack{S\subseteq N:\\ S\ni i}} \frac{\Delta_S(v)}{s_p} - v(U)\sum_{\substack{S\subseteq P:\\ S\ni i}} (-1)^s \frac{1}{s} \\
= & \sum_{\substack{S\subseteq N:\\ S\ni i}} \frac{\Delta_S(v)}{s_p} - v(U)\sum_{s=1}^{p}\sum_{\substack{S\subseteq P:\\ S\ni i, |S|=s}} (-1)^s \frac{1}{s} \\
= & \sum_{\substack{S\subseteq N:\\ S\ni i}} \frac{\Delta_S(v)}{s_p} - v(U)\sum_{s=1}^{p}\binom{p-1}{s-1}(-1)^s \frac{1}{s} \\
= & \sum_{\substack{S\subseteq N:\\ S\ni i}} \frac{\Delta_S(v)}{s_p} - v(U)\sum_{s=1}^{s}\binom{s}{s}(-1)^s \frac{1}{p} \\
= & \sum_{\substack{S\subseteq N:\\ S\ni i}} \frac{\Delta_S(v)}{s_p} - \frac{v(U)}{p}\left[\sum_{s=0}^{p}\binom{p}{s}(-1)^s - \binom{p}{0}(-1)^0\right] \\
= & \sum_{\substack{S\subseteq N:\\ S\ni i}} \frac{\Delta_S(v)}{s_p} - \frac{v(U)}{p}\left[0 - 1\right] \\
= & \sum_{\substack{S\subseteq N:\\ S\ni i}} \frac{\Delta_S(v)}{s_p} + \frac{v(U)}{p} \\
= & f_i^P(N, v, \{P, U\}),
\end{aligned}$$

where the seventh equality is once again due to the Binomial theorem. ∎

A similar reduced game, closely related to (20), that leads to $f^U(N, v, \{P, U\})$ is (P, v^U) where, for each nonempty $S \subseteq P$,

$$v^U(S) = v(S) + \sum_{T \subseteq N} \sum_{K \subseteq U : K \neq \varnothing} \Delta_{T \cup K}(v).$$

The only change compared to (20) is that the sum on subsets of S is replaced by the sum on subsets on N. Remark that the second component of $v^U(S)$ is independent of S and can be rewritten as $\sum_{T \subseteq N} \Delta_T - \sum_{T \subseteq P} \Delta_T$ or, equivalently, as $v(N) - v(P)$. Hence (P, v^U) is the sum of (P, v) and a constant symmetric TU-game in which the grand coalition P has a worth equal to $v(N) - v(P)$, which implies that, for each $i \in P$,

$$\begin{aligned} Sh_i(P, v^U) &= Sh_i(P, v) + \frac{v(N) - v(P)}{p} \\ &= f_i^U(N, v, \{P, U\}). \end{aligned}$$

6.2. Relationship with the Priority value

Our three values are already described by a unified procedure in which they all make use of the Shapley value in the second step. In this subsection, we show more explicitly that they can also be derived by relying on the Priority value (Béal et al., 2022) for TU-games with a priority structure in the first step for suitable choices of a priority structure between the elements of P and those of U.

A **priority structure** on a player set N is a **partially ordered set** or **poset** $\succcurlyeq$ on N, *i.e.* a reflexive, antisymmetric and transitive binary relation. The relation $i \succcurlyeq j$ means that i has priority over j. The poset $(N, \succcurlyeq^0)$ containing no priority relation among pair of distinct players is called the **trivial poset**. The **subposet** $(S, \succcurlyeq^S)$ of $(N, \succcurlyeq)$ induced by S is defined as follows: for each $i \in S$ and $j \in S$, $i \succcurlyeq^S j$ if $i \succcurlyeq j$. A player i is a **priority player** in $(S, \succcurlyeq)$ if, for $\in S$, the relation $j \succcurlyeq i$ implies $i = j$. Denote by $M(S, \succcurlyeq)$ the nonempty subset of priority players in $(S, \succcurlyeq)$.

The **Priority value** PV distributes the dividend of each coalition equally among its priority players:

$$PV_i(N, v, \succcurlyeq) = \sum_{S \subseteq N : M(S, \succcurlyeq) \ni i} \frac{\Delta_S(v)}{|M(S, \succcurlyeq)|} \quad \forall i \in N. \tag{22}$$

The proposition below establishes that $f^{P,(1)}$ is obtained from the Priority value when the paid players have priority over unpaid players. By reversing all priorities, one gets $f^{U,(1)}$. Finally, in absence of priority among the players, the Priority value coincides with the Shapley value, leading to $f^{R,(1)}$.

Proposition 9. *For each game* $(N, v, \{P, U\}) \in \mathbf{GB}$, *it holds that* *(i)* $f^{P,(1)}(N,v,\{P,U\}) = PV(N,v,\succcurlyeq^P)$*, where* $\succcurlyeq^P$ *is such that* $[i \succcurlyeq^P j] \Longleftrightarrow [i \in P, j \in U]$*;*

(ii) $f^{U,(1)}(N,v,\{P,U\}) = PV(N,v,\succcurlyeq^U)$*, where* $\succcurlyeq^U$ *is such that* $[j \succcurlyeq^U i] \Longleftrightarrow [j \in U, i \in P]$*;*

(iii) $f^{R,(1)}(N,v,\{P,U\}) = PV(N,v,\succcurlyeq^0)$.

The proof of this results follow directly from the definitions (8) and (13) of $f^{P,(1)}$ and $f^{U,(1)}$, the fact that $f^{R,(1)} = Sh$ and Proposition 3 in Béal et al. (2022).

6.3. Logical independence of the axioms

The axioms invoked in the characterization results are logically independent as illustrated by the following counter-examples:

- The null value defined, for each $(N,v,\{P,U\}) \in \mathbf{GB}$ and each $i \in N$, as $f_i^0(N,v,\{P,U\}) = 0$ satisfies:

  - Additivity,
  - Equal treatment of equal paid players,
  - Equal treatment for restricted equal paid players,
  - Null paid player with stand-alone unproductive unpaid players,
  - Null paid player with null unpaid players,
  - Equal impact of changes in a coalition containing an unpaid player,
  - Balanced contributions for paid players with null unpaid players,
  - Coalition strategic equivalence for changes of paid players,
  - Equal impact of promoting an unpaid player

  However, f^0 does not satisfy Efficiency for paid players.

- Let $\gamma \in \mathbb{R}^N_{++}$. The value f^γ defined, for each $(N,v,\{P,U\}) \in \mathbf{GB}$ and each $i \in P$, as

$$f_i^\gamma(N,v,\{P,U\}) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{\gamma_i}{\sum_{i \in S \setminus U} \gamma_j} \Delta_S(v) + \frac{v(U)}{|P|}$$

  satisfies:

  - Efficiency for paid players,
  - Additivity (when γ is exogenous),
  - Null paid player with null unpaid players,
  - Null paid players with stand-alone unproductive unpaid players,

– Coalition strategic equivalence for changes of paid players.

However, f^γ does not satisfy Equal treatment of equal paid players, Equal treatment of restricted equal paid players, Balanced contribution for paid players with null unpaid players. If one consider an endogenous weight system γ^v with $\gamma_i^v = v(i)^2 + 1$ for all $i \in N$, then the value f^{γ^v} satisfies all the previously listed axioms but Additivity.

- Let $w \in \mathbb{R}^N_{++}$ and consider the corresponding positively weighted Shapley value Sh^w. The value f^w defined, for each $(N, v, \{P, U\}) \in \mathbf{GB}$ and each $i \in P$, as

$$f_i^w(N, v, \{P, U\}) = Sh_i^w(P, v) + \frac{v(N) - v(P)}{|P|},$$

satisfies :

  – Efficiency for paid players,

  – Additivity (when w is exogenous),

  – Null paid player with null unpaid players,

  – Equal impact of changes in a coalition containing an unpaid player,

  – Coalition strategic equivalence for changes of paid players.

However, f^w does not satisfy Equal treatment for equal paid players, nor Equal treatment for restricted equal paid players, nor Balanced contributions for paid players with null unpaid players, nor Equal impact of promoting an unpaid player. If one consider an endogenous weight system w^v with $w_i^v = v(i)^2 + 1$ for all $i \in N$, then the value f^{w^v} satisfies all the previously listed axioms but Additivity.

- The equal division value for paid players defined, for each $(N, v, \{P, U\}) \in \mathbf{GB}$ and each $i \in P$, as

$$f_i^{ED}(N, v, \{P, U\}) = \frac{v(N)}{|P|}$$

satisfies:

  – Efficiency for paid players,

  – Additivity,

  – Equal treatment for restricted equal paid players,

  – Equal treatment of equal paid players,

  – Equal impact for changes in a coalition containing an unpaid player,

  – Equal impact of promoting an unpaid player.

However, f^{ED} does not satisfy Balanced contributions for paid players, Balanced contributions for paid players with null unpaid players, nor Coalition strategic equivalence for changes of paid players, nor Null paid player with null unpaid player, nor Null paid player with stand-alone unproductive unpaid players.

7. Conclusion

We believe that there are at least three ways to extend our work.

Firstly, as noted in footnote 3, it is possible to consider more refined structures than a mere bipartition between paid and unpaid players. Subdividing each set into subcategories would, for instance, make it possible to distinguish different priority relations among paid and unpaid players, depending on the subgroups to which they belong.

Secondly, this article has focused on extensions of the Shapley value to the framework of cooperative games with unpaid players. The literature on allocation rules for cooperative games studies many of types of allocation rules. For instance, more egalitarian values and their (convex) combinations with the Shapley value have received considerable attention in recent years. See van den Brink (2007); Ju et al. (2007); Casajus and Huettner (2014) among others. A natural extension of our work would therefore be to adapt these allocation rules to our framework.

Thirdly, it would be valuable to examine practical applications. In certain contexts, the purely marginalistic P-priority value is particularly relevant, whereas in others the more solidaristic U-priority and Redistribution values appear more appropriate, especially in cases where resources must be reallocated to reduce inequalities among paid players.

These extensions are left for future work.

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