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A weighted mechanism for minority voting in sequential voting*

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Abstract

We propose a weighted minority voting mechanism within a two-round sequential voting process, in which all individuals retain their voting rights in the second round but with different weights depending on the first-round outcome. In a utilitarian framework where individuals have a given utility function that depends on the outcomes of each round, first-round winners are identified and vote with reduced weight in the second round, while losers retain full weight. By giving greater weight to first-round losers, this design ensures that first-round winners continue to contribute to the final decision without dominating it, thereby mitigating repeated disadvantages for losers. We then compare the expected aggregate utility of society across different levels of second-round weight assigned to first-round losers, including both the simple majority rule – where all voters carry equal weight in both rounds – and the limiting case of minority voting where first-round losers receive no weight in the second round. To do so, we analyze two models: one in which individual utility derives solely from material payoffs, and another in which a form of harmony is considered, whereby individuals incur a utility loss if others repeatedly belong to the losing minority. This analysis allows us to assess how strategic behavior affects the effectiveness of the proposed mechanism.

Keywords: Voting, Minority Voting, Simple Majority, Utilitarianism, Harmony.
JEL classification: C72, D70, D71, D72.

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1 Introduction

The world is changing at a rapid pace, and the challenges facing democracy today are numerous. Across the globe, traditional democratic systems are struggling to meet citizens' needs and expectations, leading to growing distrust, particularly as minorities are systematically marginalized. Indeed, around the world, grassroots movements are emerging to demand a fairer, more inclusive, and more participatory democracy. For example, the Black Lives Matter movement seeks greater recognition and stronger representation for Black communities. Climate activists push for sustainable and responsible policies. Women's rights advocates fight for gender equality and increased political representation for women, among other goals. These demands for reform and greater inclusion highlight a fundamental issue: the way decisions are made and the extent to which all voices are genuinely heard.

Alongside grassroots efforts, institutional and academic debates have also addressed the limitations of existing voting mechanisms. For instance, the European Union and the United Nations have explored reforms to enhance representativeness and inclusivity in decision-making bodies (see, e.g., United Nations Secretary-General, 2021). In the academic sphere, scholars such as Arrow (1951) highlighted the impossibility of designing a perfect voting system, while more recent contributions (see, e.g., List and Goodin, 2001; Mackie, 2003; Raducha et al., 2023; Rau and Stokes, 2025) have addressed broader challenges of collective decision-making, including issues of legitimacy, resilience, fairness, efficiency, and inclusiveness. These discussions underscore the need for innovative mechanisms to overcome the shortcomings of traditional voting rules.

In this vein, we refer the reader to Gersbach (2024), which surveys several innovative voting rules designed to improve democratic decision-making. Among them, Assessment Voting (see, e.g., Gersbach, 1995, 2000, 2015; Gersbach et al., 2021) is a two-round procedure where a randomly selected Assessment Group votes first, their choices are revealed, and then the rest of the electorate decides whether to participate, with the final outcome determined by the aggregate vote (ties resolved by a coin toss). Building on this framework, Pendular Voting (see, e.g., Gersbach, 2024) introduces a third, intermediate option after the first-stage vote, allowing the full electorate to choose between the status quo, the original proposal, and the compromise alternative. Complementary to these sequencing mechanisms, Storable Votes[1] provides a system for repeated binary decisions in which voters accumulate and strategically allocate votes across issues, thereby capturing preference intensity, enhancing minority protection, and often improving welfare relative to standard voting.

Along these lines, many contributions in social choice theory and political science have proposed innovative mechanisms and voting rules aimed at enhancing minority protection. Democratic systems still grapple with the problem of the *tyranny of the majority*, where the majority pursues exclusively its own objectives at the expense of minority interests. For this reason, constitutional democracies also incorporate the minority principle – the idea that majority au-

[1] This mechanism was introduced in the literature by Casella (2012) as "Storable Votes" and, independently, by Hortala-Vallve (2012) as "Qualitative Voting". Note that Jackson and Sonnenschein (2007) generalized this approach to a wide range of problems, including voting, bargaining, the allocation of indivisible objects, and related settings.

thority should be limited to protect individuals or groups from excessively harmful outcomes. The literature on this topic has grown rapidly in recent years, combining both theoretical advances and empirical investigations.[2] Our interest in the paper at hand focuses on *Minority Voting* (henceforth MV), a mechanism introduced for sequential voting that operates in two stages. In the first round, all individuals vote under a simple majority rule (henceforth SM); in the second round, only the losers of the first round are allowed to vote. The goal is to give voice to the minority by granting them exclusive decision-making power in the second stage, thereby promoting inclusiveness and limiting majority dominance.

The MV mechanism has been explored in various contexts. In Gersbach (2009), the rule is applied to public project decisions: the first round determines whether the project is adopted, and the second round allows the minority to decide how it will be financed. The paper shows that MV, by granting the losing minority exclusive rights to decide on the financing of a public project approved by the majority, avoids inefficient projects and redistribution schemes, ensures that only Pareto-improving projects are adopted, and generally achieves higher welfare compared to SM. Fahrenberger and Gersbach (2010) extend this idea to sequential voting on two projects with long-term consequences, showing that MV improves collective welfare compared to repeated majority voting, especially when voters are risk-averse. Fahrenberger and Gersbach (2012) extend the model by incorporating social preferences for harmony, whereby individuals care not only about their own payoffs but also about whether others repeatedly belong to the losing minority. The paper shows that when preferences for harmony are sufficiently strong – meaning individuals value avoiding repeated disadvantage for others – MV outperforms SM in terms of aggregate welfare, as it mitigates repeated minority disadvantage while balancing material payoffs.

While effective in countering majority dominance, we argue that MV may go too far: by granting exclusive power to the minority in the second round, it risks replacing the *tyranny of the majority* with a form of *tyranny of the minority*. Thus, we propose a new mechanism – Weighted Minority Voting (henceforth WMV) – that offers a third path. This extension ensures that the majority retains influence, but no longer holds a veto. The core idea is to allow all voters to participate in both rounds and then always vote on both projects, but to reduce the voting weight of first-round winners in the second round according to a parameter $\alpha \in [0, 1]$, while minority members keep their full voting weight (i.e., 1). This approach positions α as a middle ground between exclusion and domination, i.e., between MV and SM.

Our contribution builds on a utilitarian framework to study the impact of the weight parameter α on strategic voting incentives and aggregate welfare in two (related) models. The first focuses solely on material payoffs: individuals care only about their own gains from the outcomes of both voting rounds. The second, inspired by Fahrenberger and Gersbach (2012), introduces harmony preferences: individuals incur a utility loss when others repeatedly belong to the losing

[2] Let us give, without claiming exhaustivity, two recent examples that illustrate this trend. Kizilkaya and Kempe (2025) propose the k-Approval Veto rule, a flexible class of voting systems that provides a level of minority protection proportional to k, the number of approved candidates, while controlling overall welfare distortion. Engelmann et al. (2023) provide experimental evidence showing how different voting mechanisms can strengthen minority protection without undermining collective efficiency.

minority. In both settings, first-round winners vote in the second round with reduced weight, while first-round losers retain full weight. By assigning greater weight to the losers, the mechanism ensures that first-round winners still contribute to the final decision without dominating it, mitigating repeated disadvantages for losers.[3] We compare strategic voting behavior and the expected aggregate utility of society across different levels of α, including the limiting case of both SM and MV. Our results show that specific values of α often provide the best balance between limiting strategic manipulation and enhancing collective welfare. They also show that incorporating harmony concerns significantly broadens the range of α for which the mechanism outperforms both SM and MV.

The structure of the paper is as follows. Section 2 introduces the basic framework, the objective functions, and the alternative WMV mechanism, among others. Section 3 presents our main findings in the two related models. Finally, Section 4 concludes and outlines avenues for future research. All proofs are provided in the appendix.

2 The framework

2.1 The voting process

We consider a committee of w (with $w \geq 3$, w odd) individuals who vote sequentially on two proposals, Project 1 and Project 2, through a two-ballot process.[4] Each individual – also referred to as a voter, participant, or agent – is referred to as "she" throughout the paper for simplicity. The two projects are indexed by $x \in \{1, 2\}$. In each ballot, participants cast "Yes" or "No" votes. Project 1 is submitted to a vote in the first round and is adopted if it obtains a simple majority of "Yes" votes. The outcome of the first round splits the voters into winners (the majority, composed of w_2 individuals) and losers (the minority, composed of w_1 individuals), where $w = w_1 + w_2$.

As mentioned in the previous section, we consider WMV, a mechanism that allows majority members to retain some influence in the second round through a reduced voting weight α, thereby striking a balance between avoiding a "tyranny of the majority" and a "tyranny of the minority". More exactly, first-round winners vote in the second round with a reduced weight $\alpha \in [0, 1]$, while losers retain full weight 1. Project 2 is adopted if the cumulative weight of the "Yes" votes strictly exceeds the majority threshold $q = \frac{w_1 + \alpha w_2}{2}$; it is rejected if below q, and a tie-breaking rule (a fair coin toss) is applied in the case of exact equality, where the project is

[3] Recent work by Lackner (2020) introduces the framework of perpetual voting, which explicitly accounts for decision histories in repeated collective choice settings. By taking past outcomes into consideration, this approach aims at achieving long-term temporal fairness, ensuring that minorities obtain a proportional share of favorable decisions over time, without relying on strategic vote storage or the specification of utility functions. In this sense, the underlying spirit of perpetual voting is closely aligned with our approach: both seek to mitigate the systematic disadvantage of minorities in sequential decision-making by embedding fairness considerations directly into the voting rule.

[4] Project 1 and Project 2 can be completely different. For example, Project 1 might involve building a daycare center, while Project 2 could focus on constructing a garden. Alternatively, the two projects may be linked – for instance, Project 1 could be an infrastructure initiative, and Project 2 might concern voting on how to finance it. In such case, the interdependence between the two projects may affect the resolution. The latter case is beyond the scope of this paper.

accepted or rejected with probability $\frac{1}{2}$. The parameter α controls the influence of first-round winners in the second round: (i) When α is close to 0, the system approximates MV, maximizing protection for the minority but increasing the incentive for strategic behavior among majority voters who seek to retain full voting weight. (ii) When α is close to 1, the system resembles SM, reducing the likelihood of strategic behavior. By reducing the second-round weight of first-round winners, our mechanism increases the chances that minority members can overturn the outcome, lowering the probability that some individuals lose in both rounds. Finally we assume that in our voting mechanism, abstention is not allowed,[5] meaning that each voter is required to cast a vote in both the first and second rounds.

2.2 The utility function

We extend the two-round voting framework initially proposed by Fahrenberger and Gersbach (2012). Our analysis explores two settings: one where voters are driven solely by material payoffs, and another where voters' preferences also reflect social cohesion through a harmony component that values the success of others across ballots. For clarity and in order to avoid unnecessary repetition, we introduce the general utility specification that combines both components, although our initial results concentrate on material payoffs alone. We assume that individual i's utility is given by

$$u_i = z_i + \sum_{\substack{j=1 \\ j\neq i}}^{w} b(\delta_j)\,\delta_j. \tag{1}$$

It consists of two parts:

(i) Material Payoff: The utility derived from the projects is represented by $z_i = a_1 z_{i1} + a_2 z_{i2}$, where z_{ix} denotes the individual material payoff that individual i receives if the project $x \in \{1, 2\}$ is implemented. The committee's decision is represented by the indicator variable a_x, where $a_x = 0$ indicates that the status quo is maintained, and $a_x = 1$ indicates that project x is adopted. We assume that z_{ix} is distributed independently and uniformly on the interval $[-1, 1]$. We assume that the material utility of the status quo is normalized to zero. The first-round material payoffs of all individuals are commonly known, whereas those of the second round are private information.[6]

In the model, individuals are divided into subgroups according to both their material payoffs and their voting behavior. First, *project winners* are those who derive a non-negative material payoff from the adopted project x, that is, $z_{ix} \geq 0$; *project losers* are those with a negative material payoff, $z_{ix} < 0$. The committee also splits into *majority* and *minority* groups based on numerical dominance of first-round material payoff types (non-negative or negative). The majority represents the numerically dominant group: it consists of members with $z_{ix} \geq 0$ if they are more numerous, or those with $z_{ix} < 0$ if negative-payoff members are dominant. The minority is always the less numerous group with opposing

[5] In any case, it will become clearer in the model below that abstention is always weakly dominated and can therefore be disregarded in what follows.

[6] A detailed discussion of this aspect will be presented in Section 2.3, among other key issues.

payoffs. Simultaneously, we classify individuals based on their material payoffs from the implemented decision. For a given project x, *voting winners* are those with non-negative payoffs ($z_{ix} \geq 0$) if the project is adopted ($a_x = 1$), or negative payoffs ($z_{ix} < 0$) if the project is rejected ($a_x = 0$). Conversely, *voting losers* have $z_{ix} < 0$ when $a_x = 1$ or $z_{ix} \geq 0$ when $a_x = 0$. Crucially, these classifications are distinct. A majority member may strategically vote and then follow the minority (for instance, by voting against her payoff interest to retain full weight voting in the second round), creating potential mismatches between first-round material and voting group affiliations. This strategic behavior explains why the individuals in the majority (minority) not necessarily coincide with the group of voting winners (losers).

(ii) Harmony payoff: The term $+\sum_{\substack{j=1 \\ j \neq i}}^{w} b(\delta_j)\,\delta_j$,[7] represents the part of the utility function that captures preferences for harmony. It consists of two components:

1. The variable $\delta_j = \delta_{j1} + \delta_{j2}$, where $\delta_{jx} \in \{\delta_L, 0\}$ and $\delta_L < 0$, indicates whether individual j lost or won in each ballot. Specifically, $\delta_{jx} = \delta_L$ if the individual loses in ballot x, and $\delta_{jx} = 0$ if the individual wins. We assume that $\delta_L = \frac{-y}{w}$, where $y > 0$ represents the degree of aversion to disharmony in society, and w denotes the number of voters. The parameter y captures how strongly voters care about the repeated exclusion of others: the higher y, the greater the utility loss an individual suffers when observing that other members of the committee lose multiple times. Importantly, since δ_L is inversely proportional to the committee size w, the individual contribution to the harmony payoff becomes smaller as the committee grows, reflecting that interpersonal concerns may weaken in larger groups. In other words, the threshold δ_L expresses how the discomfort caused by the repeated exclusion of others scales with the sensitivity of society (y) and the number of voters (w).
2. The function $b(\delta_j)$ is defined as follows:

$$b(\delta_j) = \begin{cases} 1, & \text{if } \delta_j < \delta_L \\ 0, & \text{otherwise.} \end{cases}$$

Hence, there is a negative impact on individual i's utility only when another individual j loses twice. It is worth mentioning that the parameter α influences the likelihood of repeated losses, which directly affects the preferences of harmony. When α is close to 0, the first-round winners have very limited influence in the second round. This strengthens the minority and increases their probability of winning in the second round, thereby reducing the probability that individuals lose both rounds. As a result, disharmony is minimized,

[7] As in the original model by Fahrenberger and Gersbach (2012), the harmony term is restricted to $\sum_{j\neq i} b(\delta_j)\,\delta_j$, thereby excluding the individual's own component. One could, however, consider an extension where the full sum $\sum_{j=1}^{w} b(\delta_j)\,\delta_j$ is used, so that voters also experience disutility from their own repeated exclusion. While such a modification would slightly depart from the original setup, it would remain consistent with the idea that disutility from harmony arises whenever a voter fails to retain both ballots, without altering the core structure or welfare implications of the model.

which increases the aggregate utility. Conversely, when α is close to 1, the decision-making process tends to favor the same majority in both rounds, leading to more repeated losers and higher disharmony. Hence, lower values of α better meet the needs of both individuals and the group in terms of harmony.

We present a simple example to illustrate the idea of strategic voting in our framework. As previously noted and shown in this example, strategic voting may arise even in the basic setting where individuals' utility depends solely on material payoff. Naturally, a more complete example, with a utility function including both components, could also be constructed.

Example 1 *Consider a committee of five voters deciding sequentially on two projects under the WMV mechanism with* $\alpha = 0.6$*. The material payoffs of the voters with respect to the first project are known and given by the list:* $u = (0.7,\ 0.2,\ 0.4,\ 0.1,\ -0.6)$*. The project winners are voters 1, 2, 3, and 4 (since they derive non-negative material payoffs from the adoption of Project 1), while the project loser is voter 5. Under sincere voting, voters 1, 2, 3, and 4 (the majority) support the project, whereas voter 5 (the minority) votes against it. The outcome of the first round is therefore the* adoption *of the project by 4 votes against 1. According to the WMV rule, each project winner receives a reduced weight of* $\alpha = 0.6$ *in the second round, while the project loser keeps full weight* 1*. The total voting weight is thus* 3.4*, yielding a majority threshold of* 1.7*. In this configuration, voter 5 gains significant influence, as her weight alone is almost equivalent to that of two other voters combined. Now suppose that voter 1, anticipating the second round, decides to vote strategically against the project, despite having a positive utility from its adoption. The vote tally becomes 3 in favor (voters 2, 3, and 4) and 2 against (voters 1 and 5). The project is still* adopted *by majority rule. However, the voting winners are now voters 2, 3, and 4, while the voting losers are voters 1 and 5. Thus, voter 1 moves from the winning coalition to the losing coalition without altering the collective outcome. Consequently, instead of being assigned weight* $\alpha = 0.6$ *in the second round, voter 1 retains full weight* 1*. This example illustrates that a voter may cast a strategic ballot in the first round in order to remain among the losers and preserve full voting weight in the second round, while the project outcome remains unchanged. Notice, however, that if more than one voter from the majority were to deviate strategically, the outcome of the first round could be overturned, and then the strategy fails since Project 1 would be rejected and no utility from that project would be assigned to voters with a positive payoff.*

As mentioned earlier, we will later show that our model characterizes the conditions under which strategic voting arises both in the setting described in Example 1 – where utility consists solely of material payoff – and in the case where harmony is introduced. We will also show how the value of α influences both the probability of a second-round victory and the incentives for sincere versus strategic voting. Before doing so, and to set the stage for this comparison, we first turn our attention to some properties of the framework.

2.3 Key considerations

- Payoffs: In the paper at hand, we assume that the payoffs from the first project (z_{i1}) are common knowledge, while those from the second project (z_{i2}) are privately observed after the first voting round. This assumption is theoretically plausible for several reasons: First, it matches most standard dynamic game-theoretic models, where early decisions are made with full information, but future outcomes are uncertain. This reflects real-life situations where first-stage proposals are openly discussed and understood by all, while later decisions involve more uncertain or personal effects that are harder to predict. Second, we consider this assumption to be broadly consistent with the way many real institutions operate in practice. In parliaments or company boards, for example, the first vote often concerns a big-picture decision (like agreeing on a general reform), where everyone's views are known. Later votes (such as deciding how to fund the reform) are more detailed, and how they affect each person is not always clear. This difference makes the first round strategic: people may vote not only for today but also thinking about how the second round will unfold.

- Equilibrium voting behavior: We search for the two-decision voting game's perfect Bayesian Nash equilibria. We eliminate weakly dominated strategies in order to eliminate implausible voting behaviors. Our analysis proceeds by backward induction. Starting from the second round, we evaluate each individual's probability of belonging to a winning coalition, which depends on her voting weight (either 1 or α), and these probabilities are later used to compute the expected aggregate utilities.

3 Results

3.1 General insights

The first result of our paper is given in Lemma 1, which provides a formal characterization of the voting dynamics under WMV. By establishing the exact probabilities that a weight-1 voter or weight-α belongs to a winning coalition in the second round, it will serve later as the analytical foundation for examining strategic incentives and welfare implications. This result is particularly important as it allows us to compare WMV across different levels of α: from SM when $\alpha = 1$, to MV when $\alpha = 0$, as well as intermediate values of α.

Lemma 1 *Consider a weighted voting system with w_1 weight-1 voters and w_2 weight-α voters, where $0 \leq \alpha \leq 1$. Each voter votes independently "Yes" or "No" with probability $\frac{1}{2}$, contributing her respective weight to the total. The majority threshold is given by the quota: $q = \frac{w_1 + \alpha w_2}{2}$. If the total weight of "Yes" votes is strictly greater than q, the decision is accepted. If the total weight of "No" votes is strictly greater than q, the decision is rejected. If the total weight of "Yes" votes is exactly equal to q, a random tie-breaking rule is applied (a fair coin toss) where the decision is accepted/rejected with probability $1/2$.*

The probability that a given weight-1 voter belongs to a winning coalition in the second round is

$$P_1(w_1, w_2) = \sum_{x=0}^{w_1-1} \sum_{\substack{y=0 \\ x+\alpha y > q-1}}^{w_2} \binom{w_1 - 1}{x} \binom{w_2}{y} \left(\frac{1}{2}\right)^{w-1} + \frac{1}{2} \sum_{\substack{x=0 \\ x+\alpha y = q-1}}^{w_1-1} \sum_{y=0}^{w_2} \binom{w_1 - 1}{x} \binom{w_2}{y} \left(\frac{1}{2}\right)^{w-1}.$$

The probability that a given weight-α voter belongs to a winning coalition in the second round is

$$P_2(w_1, w_2) = \sum_{x=0}^{w_1} \sum_{\substack{y=0 \\ x+\alpha y > q-\alpha}}^{w_2-1} \binom{w_1}{x} \binom{w_2 - 1}{y} \left(\frac{1}{2}\right)^{w-1} + \frac{1}{2} \sum_{\substack{x=0 \\ x+\alpha y = q-\alpha}}^{w_1} \sum_{y=0}^{w_2-1} \binom{w_1}{x} \binom{w_2 - 1}{y} \left(\frac{1}{2}\right)^{w-1}.$$

The proof of Lemma 1 is given in the Appendix. Note that $P_1(w_1, w_2)$ and $P_2(w_1, w_2)$ coincide when $\alpha = 1$, as all individuals vote with equal weight in the second round. Furthermore, our probability expressions generalize those given in Lemma 2 of Fahrenberger and Gersbach (2010). In particular, when $\alpha = 0$, our formula for $P_1(w_1, w_2)$ matches theirs when we identify our w_1 with their w, which denotes the number of voters in the second round in their model. In this limiting case, the value of $P_2(w_1, w_2)$ is equal to $1/2$, reflecting the fact that voters with weight $\alpha = 0$ have no influence on the decision and are included in winning coalitions purely at random (i.e., by coin toss). Finally, when $\alpha = 1$, their total population size N corresponds to our $w = w_1 + w_2$, and both $P_1(w_1, w_2)$ and $P_2(w_1, w_2)$ again coincide with their expression.

Note that the probability $P_1(w_1, w_2)$ is decomposed into two parts with distinct behaviors. The first part corresponds to the sum over all configurations satisfying the strict inequality $x+\alpha y > q-1$, and the second part corresponds to the sum over all configurations satisfying the tie condition $x + \alpha y = q - 1$. The probability $P_1(w_1, w_2)$ exhibits a nuanced relationship with the weight parameter α. The general tendency is a decrease, and this behavior is driven by the linear increase of the quota $q = \frac{w_1+\alpha w_2}{2}$, which raises the bar for a coalition to be considered winning. Specifically, the condition $x + \alpha y > q - 1$ becomes increasingly demanding as α rises, reducing the number of (x, y) configurations and consequently lowering $P_1(w_1, w_2)$.

The limiting cases offer useful benchmarks and clarify the role of the weight α. When α approaches zero, the quota simplifies to $q = \frac{w_1}{2}$, which is strictly lower than any other quota for any $\alpha > 0$, given fixed w_1 and w_2. Only the w_1 weight-1 voters participate in the vote, and all are symmetric. Since the threshold for acceptance is lower, a randomly selected weight-1 voter tends to have a higher probability of belonging to a winning coalition. Conversely, when α approaches one, all voters carry equal weight, and the quota becomes $q = \frac{w_1+w_2}{2}$. In this setting, the distinction between voter groups disappears and the total number of weight-1 voters increases, making it harder for a given individual – particularly one of the original w_1 voters – to belong to a coalition that exceeds the threshold. The symmetry among all $w = w_1+w_2$ voters implies that the individual probability of inclusion in a winning coalition is diluted. As a result, $P_1(w_1, w_2)$ reaches minimal values as α tends to one.[8] However, the discrete nature of the summations in $P_1(w_1, w_2)$ introduces important exceptions to the monotonic trend described above: local

[8]Note that, in the special case where $w_2 = 0$, the outcome becomes entirely independent of α, since no voter's weight is affected by it. The probability $P_1(w_1, w_2)$ is then fully determined by the configurations formed by the w_1 weight-1 voters alone, reproducing the baseline structure of simple majority voting.

plateaus may appear as α crosses critical thresholds where the set of winning coalitions remains unchanged. Such plateaus arise because small changes in α may not immediately alter the set of integer-valued configurations (x, y) satisfying the winning condition, thus leaving the overall probability temporarily unchanged until a new configuration crosses the threshold. Figure 1 provides a graphical illustration of the behavior of the probability $P_1(w_1, w_2)$ as a function of the weight parameter α for various values of w_1 and w_2.

A concrete example with $w_1 = 8$ and $w_2 = 11$ illustrates this phenomenon clearly: as α increases from 0 to 1, the probability $P_1(w_1, w_2)$ continues to overall decrease from 0.637 to 0.593. Contrary to the general decreasing trend, a plateau occurs when α increases, for example, from 0.306 to 0.363, as the value of $P_1(w_1, w_2)$ remains constant at 0.633. In contrast, the two bottom graphs present the same analysis for a committee with higher values of w_1 and w_2. Clearly, as the committee size increases, the graphs reveal a much smoother and nearly strictly decreasing pattern for the total probability $P_1(w_1, w_2)$, with fewer and less visible plateaus.

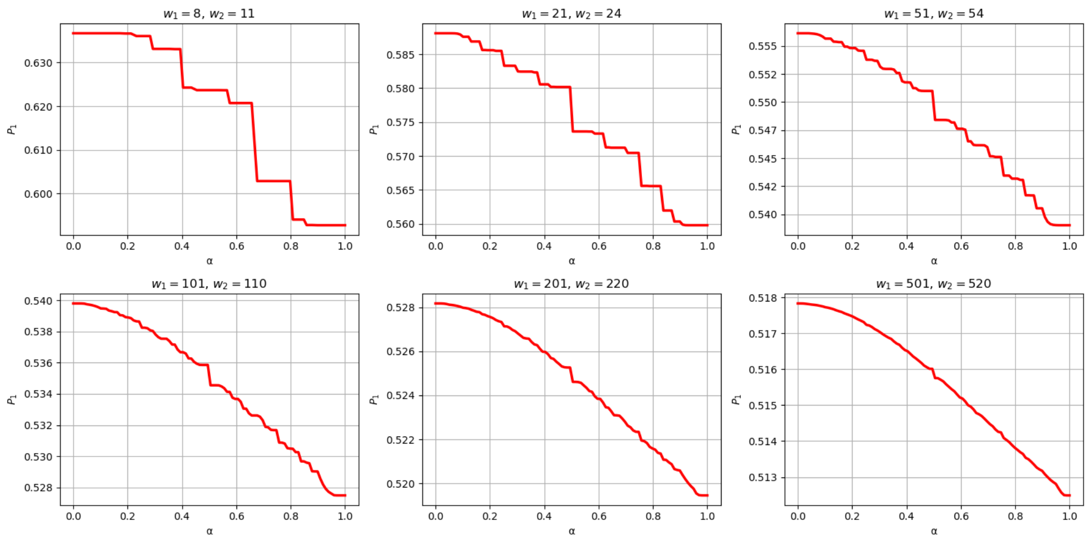

Figure 1: Analysis of the variation of $P_1(w_1, w_2)$ across varying weight parameters α for selected values of w_1 and w_2.

In contrast to $P_1(w_1, w_2)$, the probability $P_2(w_1, w_2)$ generally exhibits an opposite pattern, reflecting the different roles and incentives of voters in the second round. Its overall trend is increasing with respect to the weight parameter α, with minimal values as α tends to zero and maximal values as α tends to one. As α increases, the quota becomes more demanding, but so does the individual influence of weight-α voters, enhancing their chances of belonging to a winning coalition. However, this pattern is not strictly smooth. As in the case of $P_1(w_1, w_2)$, the discrete nature of the underlying vote configurations induces local irregularities. In particular, plateaus may emerge when small changes in α do not immediately affect the set of (x, y) configurations satisfying the winning condition. These discontinuities reflect the same threshold effects seen previously, and they highlight the importance of integer-valued jumps in the underlying probability mass. Figure 2 illustrates these features through a set of graphs and values constructed for $P_2(w_1, w_2)$ in parallel to those for $P_1(w_1, w_2)$.

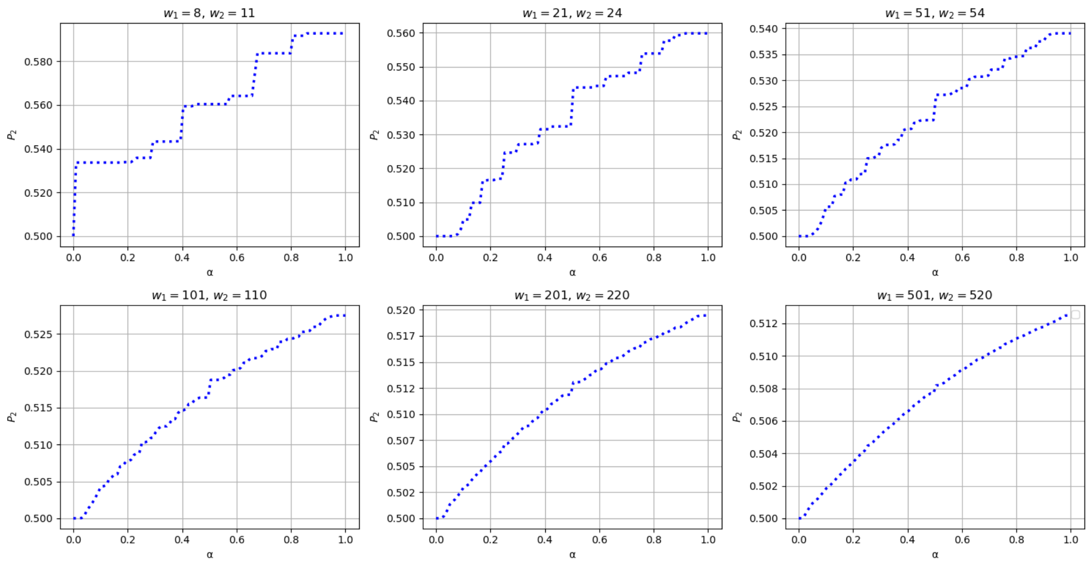

Figure 2: Analysis of the variation of $P_2(w_1, w_2)$ across varying weight parameters α for selected values of w_1 and w_2.

Having derived the probabilities that voters of different weights belong to a winning coalition in the second round, we now turn to the strategic implications of these results. In particular, we investigate whether voters especially those in the majority have incentives to vote strategically in the first ballot in order to retain their full voting weight in the second round. Clearly, minority voters have no such incentive: if they vote strategically in the first round (i.e., contrary to their true preference), they risk being classified as part of the majority and thus having their voting weight reduced in the second round. Therefore, minority voters always have an incentive to vote sincerely. In contrast, majority voters might behave differently as explained in Example 1. If a majority voter anticipates that the majority group is sufficiently large to win the first ballot without her support, she may choose to vote against the project strategically in order to retain her full weight in the second round. This raises the question: Under what conditions would such a strategy be beneficial? We then define the probabilities that a majority individual is part of a winning coalition under both strategic and sincere voting behaviors. This leads us to the following lemma.

Lemma 2 *Let m be the size of the minority group. Suppose that $k \in \left\{0, \ldots, \frac{w-3}{2} - m\right\}$ denotes the number of other majority members who vote strategically in the first round.*[9] *Consider now a majority individual i who also votes strategically in that first ballot (a weight-1 voter). Then, the total number of strategic majority voters is $k+1$, the corresponding quota is $q_1 = \frac{(m+k+1)+\alpha\left(w-(m+k+1)\right)}{2}$, and the probability that i belongs to a winning coalition in the second*

[9] Note that the upper bound $k \leq \frac{w-1}{2} - 1 - m$ (or $k \leq \frac{w-3}{2} - m$) ensures that the number of strategic deviations remains below the level at which the first-round outcome would be overturned. If too many majority members were to deviate, the result would be reversed, making the deviation ineffective.

round when is: $P_1\big(m+(k+1),\, w-(m+k+1)\big)$*, where and*

$$P_1(m+(k+1), w-(m+k+1)) = \sum_{x=0}^{m+k} \sum_{\substack{y=0 \\ x+\alpha y > q_1-1}}^{w-(m+k+1)} \binom{m+k}{x}\binom{w-(m+k+1)}{y}\left(\frac{1}{2}\right)^{w-1} + \frac{1}{2}\sum_{\substack{x=0 \\ x+\alpha y = q_1-1}}^{m+k} \sum_{y=0}^{w-(m+k+1)} \binom{m+k}{x}\binom{w-(m+k+1)}{y}\left(\frac{1}{2}\right)^{w-1}.$$

Let $P_2(m+k, w-(m+k))$ *be the probability that a sincere majority individual (weight-*α *voter) belongs to a winning coalition in the second round, given that* k *majority members voted strategically, where* $q_2 = \frac{m+k+\alpha(w-(m+k))}{2}$*:*

$$P_2(m+k, w-(m+k)) = \sum_{x=0}^{m+k} \sum_{\substack{y=0 \\ x+\alpha y > q_2-\alpha}}^{w-(m+k)-1} \binom{m+k}{x}\binom{w-(m+k)-1}{y}\left(\frac{1}{2}\right)^{w-1} + \frac{1}{2}\sum_{\substack{x=0 \\ x+\alpha y = q_2-\alpha}}^{m+k} \sum_{y=0}^{w-(m+k)-1} \binom{m+k}{x}\binom{w-(m+k)-1}{y}\left(\frac{1}{2}\right)^{w-1}.$$

The proof of Lemma 2 is given in the Appendix. The probabilities P_1 and P_2 in Lemma 2 generalize those of Lemma 1, reducing to the latter when $k = 0$ (i.e., when all voters behave sincerely). When α=0, i.e., the voting rule becomes MV, the quota simplifies to $q_1 = \frac{m+k+1}{2}$ and it is straightforward to show that the expression for P_1 corresponds to the probability of being in a winning coalition in the second round in Fahrenberger and Gersbach (2012). It is also straightforward to show that in this case P_2 converges to 1/2, reflecting the neutral effect of weight-α voters who lose all influence in the second round since α is null.

A graphical analysis is provided in Figure 3 for selected values of w, m, k, and α. Plotting P_1 and P_2 as functions of k (with fixed w, m, and α) reveals a general downward trend. Intuitively, as the number of strategic voters k increases, more individuals retain full weight 1 in the second round, which dilutes the influence of any single voter – regardless of whether she has weight 1 or weight α.[10] Similarly, and as previously discussed, when α increases (for fixed w, m, and k), the probability P_1 tends to decrease while P_2 tends to increase. This reflects the fact that sincere voters gain more influence as their voting weight increases, whereas the relative advantage of retaining full weight 1 through strategic voting becomes less pronounced. As in Lemma 1, the discrete nature of coalition formation induces local discontinuities in both P_1 and P_2. Small variations in α or k may not immediately affect these probabilities until certain thresholds are crossed – thresholds that change the set of winning configurations (x, y).

[10] We will comment in more detail later on this variation of P_1 as k increases. We will also show that, given the discrete nature of our probabilities, P_1 can even increase in some cases when k rises while the other parameters remain fixed. This case occurs for instance, as shown in the middle bottom part of Figure 3, when $\alpha = 0.75$ and k increases from 1 to 2.

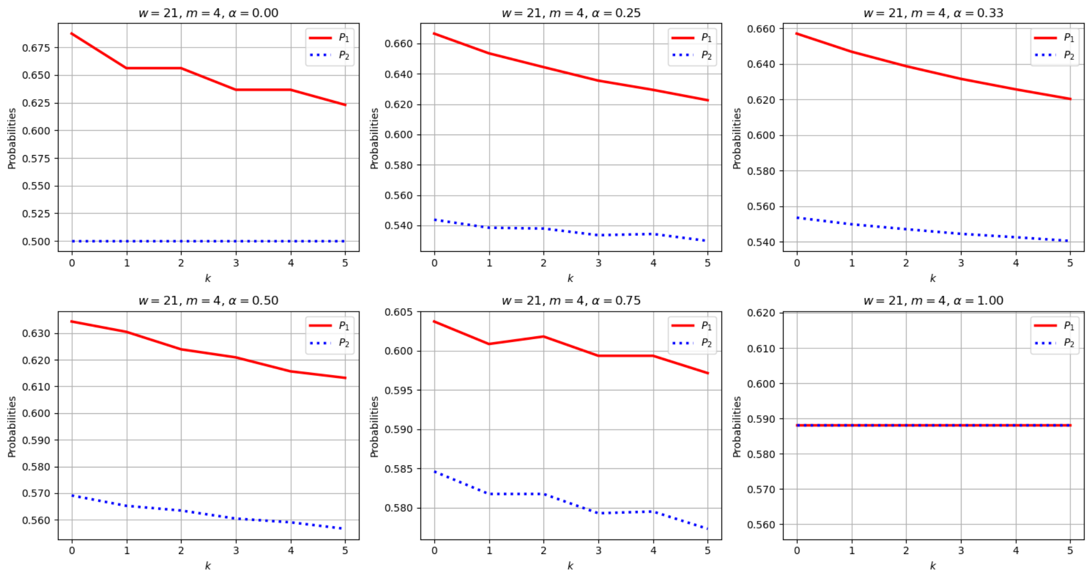

Figure 3: Analysis of the variation of $P_1(m+(k+1), w-(m+k+1))$ and $P_2(m+k, w-(m+k))$ for $w = 21$ and $m = 4$ across varying parameters k and α.

Let us mention that when $\alpha = 0$, it is easy to show that $P_1 > P_2$ for all values of k and m. This strict inequality highlights a strong incentive for majority individuals to vote strategically in the first round in order to retain full influence in the second round. In contrast, when $\alpha = 1$, all voters – whether they voted sincerely or strategically – have the same weight in the second round. Consequently, the two probabilities coincide, $P_1 = P_2$ for all k and m, eliminating any strategic advantage. In this fully symmetric setting, strategic behavior becomes irrelevant, and voters are indifferent between sincere and strategic voting. More generally, we will show later that $P_1 \geq P_2$ for all admissible values of m, k, and w (with w odd in our framework). This means that, regardless of the composition of the electorate and the level of strategic behavior, the probability that a majority individual who voted strategically in the first round (and thus retains weight 1) belongs to a winning coalition in the second round is always at least as high as that of a sincere majority voter (whose weight is α).

3.2 Utility grounded in material payoffs

We now characterize the equilibrium behavior under our WMV mechanism. As stated in the introduction, we start with the case where individuals care exclusively about their material payoffs, without considering the harmony component. This is precisely the focus of Lemma 3, which addresses the second ballot, and Proposition 1, which characterizes the first ballot behavior.

Lemma 3 *In the second ballot, sincere voting is a weakly dominant strategy under WMV.*

The proof of Lemma 3 is given in the Appendix. A brief outline of the proof is as follows. In the second round, payoffs for project 2 are private information, and each voter simply aims for her preferred outcome. Given any fixed profile of votes by the others, casting a sincere ballot is never worse and may be strictly better, especially when the voter is pivotal. This ensures that no one gains from deviating from sincere behavior.

Having clarified the logic of behavior in the second period, we now turn to Proposition 1, which describes equilibrium behavior in the first round.

Proposition 1 *Consider WMV with $\alpha \in [0,1]$ and an odd number of individuals w, where voters' utility depends solely on material payoffs. The following optimal voting behavior emerges in the first ballot:*

(i) All minority individuals (of size m) vote sincerely in the first round.

(ii) The majority optimally divides into: Exactly $\frac{w+1}{2}$ members voting sincerely to form a minimal winning coalition. The remaining $\frac{w-1}{2} - m$ members voting strategically against their true preferences.

In addition, for any majority voter, voting strategically in the first round is weakly dominant over sincere voting. Specifically:

$$\mathbb{E}u_i^{strat} \geq \mathbb{E}u_i^{sinc},$$

where the inequality is strict unless $\nexists (x,y) \in \{0,\ldots,m+k\} \times \{0,\ldots,w-(m+k+1)\}$ such that $x + \alpha y \in [q_1 - 1,\ q_2 - \alpha]$, with q_1 and q_2 are as defined in Lemma 2.

The proof of Proposition 1 is given in the Appendix. Points (ii) and the last statement of Proposition 1 might at first seem contradictory, but this is not the case. Point (ii) describes an equilibrium in which only part of the majority votes strategically, because if all of them did so, the majority alternative might lose in the first round, which is not optimal for the majority. The last statement focuses on individual incentives: if a member of the majority assumes that the others vote as described in (ii), then for that individual, voting strategically is optimal. In other words, the $\frac{w+1}{2}$ sincere votes ensure victory in the first round, while the other members benefit from the strategy without changing the outcome. The proposition therefore combines both a collective analysis and an individual one, and there is no contradiction as long as we consider that the subgroup of $\frac{w+1}{2}$ acts to secure the majority victory, while the others ($\frac{w-1}{2} - m$) optimize their utility given this behavior.

Note also that the condition $\mathbb{E}u_i^{\text{strat}} = \mathbb{E}u_i^{\text{sinc}}$ holds only in the very specific case where no voting configuration (x,y) allows a strategic vote to influence the outcome. That is, when the quantity $x + \alpha y$ systematically falls outside the critical interval $[q_1 - 1,\ q_2 - \alpha]$. This implies that strategic voting offers no advantage when the institutional thresholds – namely α, q_1, and q_2 – are such that individual actions cannot affect the final outcome. In such cases, manipulation becomes entirely ineffective, and voters are indifferent between voting sincerely or strategically. Of course, the trivial case is when $\alpha = 1$. Mathematically, in this case we have $q_1 = q_2 = \frac{w}{2}$, so the interval $[q_1 - 1,\ q_2 - \alpha]$ reduces to a single point. Since $x + \alpha y$ cannot exactly equal this fixed point (w is odd) for any integer pair (x,y), the condition for equality is automatically satisfied. Beyond this trivial situation, Figure 4 illustrates how the probabilities P_1 and P_2 evolve as functions of the weight parameter α, for different values of k, with fixed values $w = 21$

and $m = 4$. The plots show that the equality $P_1 = P_2$ may occur for specific threshold values of α, typically close to 1. In our example, the equality holds for $\alpha \geq 0.837$ when $k = 0$ and $k = 2$, and for $\alpha \geq 0.878$ when $k = 3$ and $k = 5$, up to a precision of 10^{-3}. This confirms our analytical insight: in these specific configurations, voters have no incentive to manipulate the outcome, as their strategic behavior yields no gain over sincere voting.

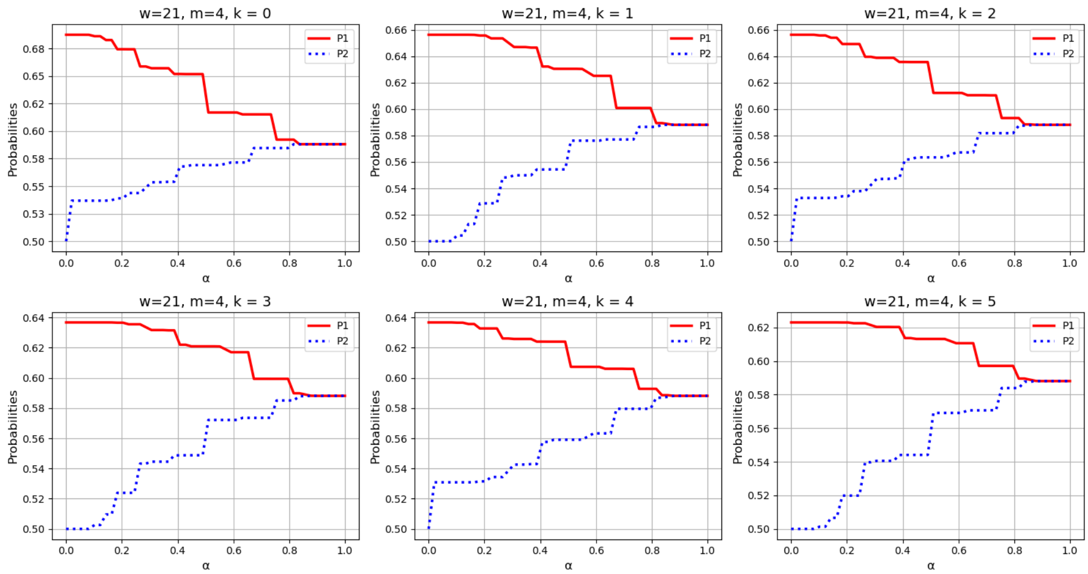

Figure 4: Analysis of the possible equality between $P_1(m + (k + 1), w - (m + k + 1))$ and $P_2(m + k, w - (m + k))$ for $w = 21$ and $m = 4$ across varying parameters k and α.

We continue our analysis of the model without harmony but we now focus on the expected aggregate utility under WMV by considering two different weight parameters, α and α'. This allows us to study how changes in the voting weight of majority members in the second ballot affect the expected aggregate welfare and strategic behavior. We adopt an utilitarian criterion: WMV with parameter α is said to outperform that with parameter α' if the expected aggregate utility under α is strictly greater than the one under α'. To compute the expected aggregate utility, we rely on the probability that an individual i belongs to a winning coalition in the second ballot, under both strategic and sincere voting, as defined in Lemma 1 and Lemma 2. We focus on the equilibrium described by (i) and (ii) in Proposition 1. Recall that $P_1^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ is the probability that individual i, with weight 1 belongs to a winning coalition in the second round under WMV[11] and $P_2^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ is the probability that individual i, with weight α, belongs to a winning coalition in the second round. When we consider an alternative value α', the same expressions of probabilities apply but with α' instead of α. Taking into account the structure of our model, it is worthwhile noting that we calculate the expected aggregate welfare after the first ballot has taken place, yet before the second-project payoffs are revealed. We now establish our main result regarding the expected aggregate utility without harmony.

In Proposition 2 we consider the configuration where exactly $\frac{w-1}{2}$ voters keep full weight and

[11] We use the superscripts α and α' in the notation of the probabilities to indicate that these are computed under the WMV mechanisms defined by the parameters α and α', respectively.

$\frac{w+1}{2}$ voters have reduced weight α. This represents the worst-case scenario for first-round losers: when the number of full-weight voters is maximal, each has the least individual influence, which maximizes their probability of losing again in the second round. Consequently, comparing two weight parameters α and α' under this setting provides a conservative test: if WMV with α outperforms WMV with α' here, it will do so in any situation with fewer weight-1 voters.

Proposition 2 *Consider WMV with an odd number of voters w, where individuals' utility is determined solely by material payoffs. Take $w_1 = \frac{w-1}{2}$ and $w_2 = \frac{w+1}{2}$. WMV with parameter α outperforms that with parameter α' if:*

$$P_1^{\alpha}(w_1, w_2) - P_1^{\alpha'}(w_1, w_2) > \frac{w-m}{m}\left(P_2^{\alpha'}(w_1, w_2) - P_2^{\alpha}(w_1, w_2)\right).$$

The proof of Proposition 2 is given in the Appendix. The term $P_1^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2}) - P_1^{\alpha'}(\frac{w-1}{2}, \frac{w+1}{2})$ captures the effect on the probability that a first-round loser belongs to a winning coalition in the second round, when the weight assigned to first-round winners increases from α' to α. Conversely, $P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right) - P_2^{\alpha'}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)$ captures the effect on the probability that a first-round winner remains in a winning coalition in the second round under the same increase in weight. The coefficient $\frac{w-m}{m}$ reflects the relative size of the majority of size $w-m$ and the minority of size m. Altogether, the inequality in Proposition 2 expresses a dominance condition: we say that WMV with parameter α outperforms that with α' when the improvement in the second-round influence of first-round losers outweighs the loss in the influence of first-round winners, after adjusting for group sizes. This provides a criterion for preferring one rule over another based on its overall effect on coalition formation.

It is worth noting that a closer examination reveals that the comparison between two levels of expected social welfare under different weights α and α', as captured by the condition in Proposition 2, can also be interpreted as a comparison of the expression

$$m \cdot P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right) + (w-m) \cdot P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right) \tag{2}$$

for the two values of α and α'. Although, as previously discussed, the probability $P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)$ generally decreases and $P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)$ generally increases as α increases, their weighted sum given in (2) turns out to be non-decreasing in α. This implies that the optimal value of α for maximizing social welfare is $\alpha = 1$, which corresponds to SM. However, and quite interestingly, we can identify an infinite number of values of α – in addition to $\alpha = 1$ – for which the expected social welfare is also maximized. Table 1 reports these values for small committee sizes. For instance, when $w = 21$, the welfare-maximizing values of α lie in the interval $[0.859,\ 1]$. Choosing a value of α strictly less than 1 within this interval allows us to mitigate both majority and minority tyranny, while still aiming to maximize collective welfare. Indeed, selecting a value of α strictly less than 1 within the identified interval represents a deliberate compromise between two extremes: the risk of majority dominance when α is close to 1, and the potential for minority overrepresentation when α is too low. This intermediate choice helps to attenuate the effects of both majority and minority tyranny, thereby promoting a more balanced and equitable decision-making process. At the same time, it enables us to preserve high levels of collective welfare, as the interval is precisely characterized by values of α that maximize expected aggregate utility.

w	The weight α maximizing W^α	w	The weight α maximizing W^α
3	[0.505 ; 1]	5	[0.677 ; 1]
7	[0.758 ; 1]	9	[0.808 ; 1]
11	[0.838 ; 1]	21	[0.859 ; 1]
31	[0.879 ; 1]	41	[0.899 ; 1]
51	[0.909 ; 1]	101	[0.929 ; 1]

Table 1: Intervals of α maximizing the social welfare W^α for different values of w

3.3 Utility enriched by harmony

We now turn to the second part of the paper, which explores a richer behavioral setting. While the first part of the paper focused exclusively on individuals who care only about their material payoff, we now consider a scenario in which individuals are also averse to social disharmony. Specifically, in addition to maximizing their own material payoffs, individuals experience disutility when others suffer two consecutive losses across both voting rounds – these individuals are referred to as double losers. To formally incorporate this social concern into the model, we reintroduce the harmony component into the utility function, as previously defined and discussed in Subsection 2.2. This extended framework allows us not only to analyze how the interplay between material incentives and harmony aversion shapes both individual behavior and collective outcomes, but also to compare the performance of different voting rules, as in the first part of the paper.

Let us recall the complete utility function: $u_i = z_i + \sum_{j=1}^{w} b(\delta_j)\,\delta_j$. To prepare for the comparison, we first record how the loss parameter δ_j is determined, as it depends on the stage of the voting procedure and the information available. In the first ballot, project payoffs are common knowledge, so the losing parameter that individual i assigns to j after the first ballot is given by:

$$\delta_{j1} = \begin{cases} \delta_L, & \text{if } (z_{j1} \geq 0 \wedge a_1 = 0) \text{ or } (z_{j1} < 0 \wedge a_1 = 1), \\ 0, & \text{otherwise.} \end{cases}$$

In the second ballot (ex post), once the vote has taken place and the payoffs z_{i2} are realized and privately observed and since under WMV individuals observe others' votes a_{j2}. Thus

$$\delta_{j2} = \delta_L \iff a_{j2} \neq a_2.$$

Ex ante for the second ballot, before z_{i2} is realized, let $P_1\left(\frac{w-1}{2}, \frac{w+1}{2}\right)$ denote the probability that a minority member wins. We do not take majority members into consideration because they have $\delta_{j1} = 0$, and therefore do not contribute to the harmony payoff. For this reason, we focus only on the minority members, who have $\delta_{j1} = \delta_L$. Then:

$$\mathbb{E}[b(\delta_L + \delta_{j2})(\delta_L + \delta_{j2})] = \Big(1 - P_1\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\Big)\, 2\delta_L.$$

As the first part of the paper, Lemma 3 remains valid in the harmony setting. Since second-project payoffs are private information, each voter knows only his own payoff realization and has no way to infer how others will be affected by the final decision. In particular, an individual cannot predict which members of the committee risk becoming double losers, nor how his own vote could modify that probability. As a consequence, the harmony component of utility is unaffected by any strategic deviation in the second round. The only part of expected utility that a voter can influence is therefore his material payoff from project 2. This brings the problem back to a standard private-value binary choice in which sincere voting is a weakly dominant strategy. All voters with the right to vote thus choose their preferred alternative, knowing that their behavior neither protects nor harms others in expected terms. This feature plays a central role for the welfare comparison because it implies that the probability of winning in the second round depends solely on the number of voters who retain voting rights, but not on any strategic pattern of behavior at that stage.

Now, strategic voting behavior may emerge in the first round as individuals can choose to intentionally lose in that stage in order to retain the full weight of their vote in the second round. While this strategy may improve their material payoff, it can also reduce their harmony payoff. Equilibrium voting behavior under WMV is shaped by three key factors: the material payoff obtained from the first round, the incentive to preserve full voting weight in the second round, and the expected harmony payoff. As will become clear from the analysis of first-round incentives, majority members face a trade-off between preserving their voting right and limiting the risk that minority members become double losers. This generates a critical level of disharmony aversion beyond which strategic voting is no longer profitable. We denote this cutoff by $y^v(k)$, and its explicit expression will be derived formally in Proposition 3. For ease of reference, it can already be written as

$$y^v(k) = \frac{w\,[P_1(m+k+1,\, w-(m+k+1)) - P_2(m+k,\, w-(m+k))]}{4m\,[P_1(m+k,\, w-(m+k)) - P_1(m+k+1,\, w-(m+k+1))]}.$$

Notice that when $\alpha = 0$, i.e., when the voting rule reduces to MV, it is straightforward to verify that the expression for $P_1(m+k+1, w-(m+k+1))$ and $P_1(m+k, w-m-k)$ coincides with the probability of being in a winning coalition in the second round as derived in Fahrenberger and Gersbach (2012). In this case, it is also immediate to show that $P_2(m+k,\, w-(m+k))$ converges to $1/2$, reflecting the neutral effect of weight-α voters, who lose all influence in the second round since $\alpha = 0$. Using these probabilities, we recover exactly the threshold y^v reported in Fahrenberger and Gersbach (2012).

We will show in Propositions 3 and 4 that the equilibrium strategy hinges on the comparison between $y^v(k)$ and y which measures the intensity of harmony aversion. A proper understanding of this result requires a closer examination of the function y^v. The plots in Figure 5 illustrate the highly irregular behavior of $y^v(k)$ in the example $w = 51$ and $m = 5$, evaluated for several values of α and for $k \in \{1, \dots, \frac{w-3}{2} - m\}$. This irregularity is not specific to this numerical case. Similar patterns arise for many other combinations of w, m, and α.

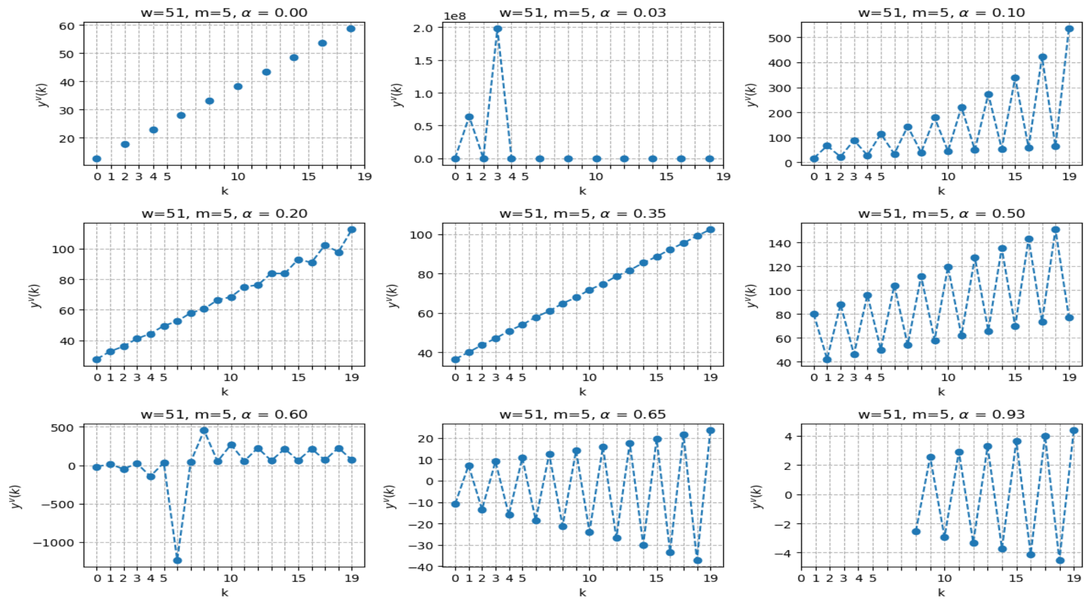

Figure 5: Behavior of $y^v(k)$ for $w = 51$ and $m = 5$, evaluated across a range of values of α and $k \in \{1, \ldots, \frac{w-3}{2} - m\}$.

A first observation is that, for some parameter values, the function $y^v(k)$ is not defined because its denominator becomes equal to zero. That is,

$$y^v(k) \text{ is not defined} \iff P_1(m+k+1,\, w-(m+k+1)) \;=\; P_1(m+k,\, w-(m+k)).$$

This occurs, for instance, whenever $m+k$ is even under $\alpha = 0$, which is consistent with Fahrenberger and Gersbach (2012). Further indeterminacies arise for several small values of k, typically forming an interval when α is close to one, as illustrated in Figure 5 for $\alpha = 0.93$. Similar phenomena are observed for several large values of k when α is close to zero, as shown in Figure 5 for $\alpha = 0.03$. The indeterminate behavior of $y^v(k)$ in these regions calls for a more detailed examination, which will be provided in a separate proposition.

Beyond these indeterminacies, the plots show that $y^v(k)$ may also take negative values. Since Proposition 1 ensures that the numerator of $y^v(k)$ is always nonnegative and strictly positive in almost all cases (except for the specific configurations mentioned at the end of that proposition), the sign of $y^v(k)$ is entirely determined by its denominator. Hence,

$$y^v(k) < 0 \iff P_1(m+k+1,\, w-(m+k+1)) \;>\; P_1(m+k,\, w-(m+k)).$$

In other words, $y^v(k) < 0$ holds exactly when the probability P_1 increases as the number of strategic voters rises from k to $k+1$. This situation indeed arises, as illustrated for instance in the middle bottom panel of Figure 3, where with $w = 21$, $m = 4$ and $\alpha = 0.75$, the probability increases when k moves from 1 to 2. This also explains all negative values appearing in Figure 5.

More generally, Figure 5 shows that $y^v(k)$ may be increasing, decreasing, or oscillatory depending on the parameter set. It may also display sharp local fluctuations instead of any stable monotone pattern. Moreover, its maximum does not necessarily occur at $k = \frac{w-3}{2} - m$, and its minimum does not always occur at $k = 0$. This diversity of shapes highlights the strong sensitivity of $y^v(k)$ to the underlying parameters and motivates our approach in characterizing equilibrium behavior in Propositions 3 and 4.

Let us first consider the case in which the function $y^v(k)$ is well defined. In this situation, the denominator in the expression of $y^v(k)$ does not vanish, so that both the sign and the magnitude of $y^v(k)$ can be meaningfully interpreted. This allows us to analyze how variations in the number of strategic voters k affect the relative incentives captured by $y^v(k)$, and to determine under which conditions these incentives support a given equilibrium behavior.

Proposition 3 *Suppose a society with an odd number of voters w has preferences for harmony with parameter $\delta_L = -y/w$, so that each individual's utility includes both material and harmony payoffs. Suppose that the function $y^v(k)$ is well defined for every admissible value of $k \in \{0, \ldots, \frac{w-3}{2} - m\}$. The following first-ballot behaviors are optimal under WMV.*

(i) *All minority members vote sincerely.*

(ii) *If $y^v(0) \leq 0$ or if $y^v(0) > 0$ and $y \geq y^v(0)$, then there exists an equilibrium such that all majority members vote sincerely.*

(iii) *Suppose that $y^v(0) > 0$ and $y < y^v(0)$. If $\exists k \in \{1, \ldots, \frac{w-3}{2} - m\}$ such that $y \geq y^v(k)$, take $k^* = \min\{\, k : y \geq y^v(k)\,\}$. The equilibrium then consists of k^* strategic majority members and $w - m - k^*$ sincere majority members. If $\nexists k \in \{1, \ldots, \frac{w-3}{2} - m\}$ such that $y \geq y^v(k)$, the equilibrium then consists of $\frac{w-1}{2} - m$ strategic majority members and $\frac{w+1}{2}$ sincere majority members.*

(iv) *If $y^v(0) \leq 0$ or if $y^v(0) > 0$, $y \geq y^v(0)$ and $y^v(k-1) < y < y^v(k)$ for some $k \in \{1, \ldots, \frac{w-3}{2} - m\}$, then there may exist an additional equilibrium such that:*

- *If $\hat{k}$ $(< k)$ majority members vote strategically and $y > y^v(\hat{k} - 1)$, the equilibrium has $k^* + 1$ strategic majority members and $w - m - k^* - 1$ sincere majority members, where $k^* = \min\{k \in \{1, \ldots, \hat{k} - 2\} : y \leq y^v(k)\}$; if no such k^* exists, all majority members are sincere. If instead $y \leq y^v(\hat{k} - 1)$, the equilibrium has k^* strategic majority members and $w - m - k^*$ sincere majority members, where $k^* = \min\{k \in \{\hat{k}, \ldots, k - 2\} : y \geq y^v(k)\}$; if no such k^* exists, the equilibrium consists of $k - 1$ strategic majority members and $w - m - k + 1$ sincere majority members.*
- *If $\hat{k}$ $(> k)$ majority members vote strategically and $y > y^v(\hat{k} - 1)$, the equilibrium has $k^* + 1$ strategic majority members and $w - m - k^* - 1$ sincere majority members, where $k^* = \max\{k \in \{k + 1, \ldots, \hat{k} - 2\} : y \leq y^v(k)\}$; if no such k^* exists, the equilibrium consists of $k + 1$ strategic majority members and $w - m - k - 1$ sincere majority members. If instead $y \leq y^v(\hat{k} - 1)$, the equilibrium has k^* strategic majority members and $w - m - k^*$ sincere majority members, where $k^* = \min\{k \in \{\hat{k}, \ldots, \frac{w-3}{2} - m\} : y \geq y^v(k)\}$; if no such k^* exists, the equilibrium has $\frac{w+1}{2}$ sincere majority members and $\frac{w-1}{2} - m$ strategic majority members.*

The proof is given in the Appendix. Table 2 clarifies point (ii) of Proposition 3 by showing how the sign of $y^v(0)$ governs strategic behavior in the first round. When $y^v(0) < 0$, strategic incentives vanish immediately. For $\alpha = 0.55$ for instance, we have $y^v(0) = -8.57$, which implies that even without any concern for harmony, a majority member has no profitable deviation. The equilibrium therefore settles on fully sincere voting. When $y^v(0) \geq 0$, the value becomes a threshold determining the minimal harmony level y needed to eliminate strategic deviations. For $\alpha = 0.50$ for instance, $y^v(0) = 80.12$, so as long as society's preference for harmony remains below this value, a majority voter may still find a deviation profitable. Once y reaches or exceeds this threshold, no deviation is attractive. Overall, Table 2 shows how small variations in α may shift the system from immediate sincerity ($\alpha = 0.55$) to a situation where only sufficiently strong harmony concerns can sustain sincere behavior ($\alpha = 0.50$). Notice that the entries marked with a dash in Table 2 correspond to values of α for which $y^v(0)$ is not defined. This indeterminacy arises exactly when the probabilities $P_1(m + k, w - m - k)$ and $P_1(m + k + 1, w - (m + k + 1))$ coincide, causing the denominator of $y^v(0)$ to vanish. In such cases, the usual comparison between sincere and strategic incentives cannot be applied. These situations require a separate analysis, which is provided in Proposition 4.

α	$y^v(0)$	α	$y^v(0)$
0.00	12.75	0.55	-8.57
0.05	12.79	0.60	-20.05
0.10	16.30	0.65	-10.86
0.15	22.55	0.70	4.25
0.20	27.63	0.75	48.88
0.25	32.86	0.80	-7.65
0.30	28.86	0.85	-2.55
0.35	36.53	0.90	–
0.40	34.05	0.95	–
0.45	8.05	1.00	–
0.50	80.12		

Table 2: Numerical values of $y^v(k = 0)$ across a range of values of α with $w = 51$ and $m = 5$.

A further remark is needed concerning point (iii) of Proposition 3. For several parameter configurations, the function $y^v(k)$ exhibits marked oscillations with pronounced local minima. To illustrate, take $\alpha = 0.67$ with $w = 51$ and $m = 5$. We obtain: $y^v(0) = 72.27$, $y^v(1) = 92.89$, $y^v(2) = 67.56$, $y^v(3) = 99.36$, $y^v(4) = 69.11$, $y^v(5) = 104.36$. Although the sequence shows an overall upward drift, the minimum is reached at $k = 2$. This has direct implications for point (iii). If $y < 67.56$, then no k satisfies $y \geq y^v(k)$, so no cutoff k^* exists and the equilibrium moves to the extreme case described in point (iii). If instead $67.56 < y < 72.27$, then the first integer k satisfying $y \geq y^v(k)$ is $k^* = 2$, and the equilibrium is accordingly determined at this level such that $k^* = 2$ majority members vote strategically and $w - k^* = 49$ vote sincerely (including the minority members). This example highlights that the identification of k^* crucially depends on the local minima of the function $y^v(k)$, and that small variations in α may generate substantial shifts in equilibrium behavior.

We now present a numerical example that illustrates how the non-monotonicity of $y^v(k)$ and the initial configuration of strategic voters jointly determine the equilibrium outcomes described in point (iv) of Proposition 3. To clarify the intuition underlying our proof, we consider the case where the initial number of strategic majority voters satisfies $\hat{k} > k$. Consider a committee of size $w = 51$, with $m = 5$ minority members, and let the weight parameter be $\alpha = 0.5$. In this setting, we obtain $y^v(0) = 80$, $y^v(5) = 50.03$, $y^v(6) = 104$, $y^v(7) = 53.98$, $y^v(8) = 111.87$, and $y^v(9) = 57.90$. We focus on the case where the harmony parameter is $y = 85$, which satisfies $y \geq y^v(0) = 80$. Since $y^v(5) = 50.03 < 85 < 104 = y^v(6)$, the condition $y^v(k-1) < y < y^v(k)$ is satisfied for $k = 6$. Suppose that initially $\hat{k} = 10$ majority members vote strategically. Because $y = 85 > y^v(9) = 57.90$, we are in the first subcase: one of the ten strategic voters has an incentive to deviate to sincere voting. However, since $y = 85 \leq y^v(8) = 111.87$, an additional majority member has no incentive to deviate and vote sincerely. Hence, $k^* = 8$ is the largest admissible value satisfying the inequality. As a result, the equilibrium features $k^* + 1 = 9$ majority members voting strategically and $w - m - 9 = 37$ majority members voting sincerely, in addition to the five minority members who always vote sincerely. Importantly, even though the process starts with a larger number of strategic voters ($\hat{k} = 10$), the equilibrium involves fewer of them (9).

Now consider an example corresponding to the second subcase, where $y \leq y^v(\hat{k} - 1)$ and $\hat{k} = 10$ majority members vote strategically (with $\hat{k} > k$). Using $\alpha = 0.24$, the relevant thresholds are $y^v(6) = 32.20$, $y^v(7) = 175.44$, $y^v(8) = 37.58$, $y^v(9) = 187.32$, $y^v(10) = 43.07$, $y^v(11) = 198.43$, $y^v(12) = 48.67$, $y^v(13) = 208.84$, $y^v(14) = 54.39$, $y^v(15) = 218.67$, etc. Choose $y = 42$, which satisfies $y \geq y^v(0) = 16.40$ and $y^v(6) = 32.20 < 42 < 175.44 = y^v(7)$, so that $k = 7$. Since $y = 42 \leq y^v(9) = 187.32$ (where $\hat{k} - 1 = 9$), none of the $\hat{k} = 10$ strategic voters has an incentive to deviate to sincere voting. However, a sincere majority voter may have an incentive to deviate to strategic voting. The process therefore continues until we find the smallest integer $k^* = \min\left\{k \in \left\{\hat{k} = 10, \dots, \frac{w-3}{2} - m\right\} : y \geq y^v(k)\right\}$. We check sequentially. For $k = 10$, we have $y^v(10) = 43.07 > 42$, so the condition fails. This implies that, in addition to these 10 individuals, an eleventh majority member has an incentive to deviate and vote strategically. For $k = 11$, $y^v(11) = 198.43 > 42$, and the condition again fails, meaning that, in addition to these 11 individuals, a twelfth majority member has an incentive to deviate and vote strategically. In fact, all thresholds $y^v(k)$ for $k \geq 10$ exceed 42. Therefore, no k^* satisfies $y \geq y^v(k^*)$. This implies that all remaining majority members have an incentive to vote strategically, up to the maximal level compatible with the profitability of the deviation. Equivalently, in accordance with point (iv), the equilibrium consists of $\frac{w+1}{2}$ sincere majority voters and $\frac{w-1}{2} - m$ strategic majority voters.

While Proposition 3 characterizes equilibrium behavior when the threshold function $y^v(k)$ is well defined, it remains to analyze the cases in which $y^v(k)$ is undefined. This occurs precisely when the denominator of $y^v(k)$ vanishes, i.e., when $P_1(m + k, w - (m + k)) = P_1(m + k + 1, w - (m + k + 1))$ for some $k \in \left\{0, \dots, \frac{w-3}{2} - m\right\}$. In such cases, the marginal effect of an additional strategic voter on the first-round winning probability is null, so that the usual incentive comparison underlying Proposition 3 is no longer informative. To streamline notation, we henceforth write $P_1 = P_1(m+k+1,\, w-(m+k+1))$, $P_1^{\text{sinc}} = P_1(m+k,\, w-(m+k))$, and $P_2 = P_2(m+k,\, w-(m+k))$. Proposition 4 provides a complete characterization of first-ballot equilibria under WMV in these cases, relying on a sequential equilibrium selection procedure.

Proposition 4 *Suppose a society with an odd number of voters w has preferences for harmony with parameter $\delta_L = -y/w$, so that each individual's utility includes both material and harmony payoffs. Suppose that the function $y^v(k)$ is undefined for some admissible values of $k \in \{0, \dots, \frac{w-3}{2} - m\}$. The first-ballot optimal behaviors under WMV are found as follows:*

(i) *All minority members vote sincerely.*

(ii) *If $y^v(0)$ is well-defined and $y^v(0) \leq 0$ or $y^v(0) > 0$ and $y \geq y^v(0)$, then there exists an equilibrium such that all majority members vote sincerely.*

(iii) *1) If $y^v(0)$ is undefined and $P_1 = P_2 = P_1^{\text{sinc}}$ for $k = 0$, then all majority members vote sincerely.*

2) If $y^v(0)$ is well-defined, $y^v(0) > 0$, and $y < y^v(0)$, or if $y^v(0)$ is undefined and $P_1 = P_1^{\text{sinc}} \neq P_2$ for $k = 0$, then one majority member votes strategically. Then, set $k = 1$ and proceed to (iv).

(iv) *For $k = 1, 2, \dots, \frac{w-3}{2} - m$, repeat the following procedure sequentially:*

1) If $y^v(k)$ is well-defined and $y \geq y^v(k)$, then the equilibrium consists of k strategic majority members and $w-m-k$ sincere majority members. Otherwise, if $y < y^v(k)$, then one additional majority member votes strategically. Proceed to evaluate the case for $k+1$ using this same procedure.

2) If $y^v(k)$ is undefined and $P_1 = P_2 = P_1^{\text{sinc}}$ for k, then the equilibrium consists of k strategic majority members and $w-m-k$ sincere majority members. Otherwise, if $P_1 = P_1^{\text{sinc}} \neq P_2$ for k, or if $y^v(k)$ is well-defined and $y < y^v(k)$ then one additional majority member votes strategically. Proceed to evaluate the case for $k+1$ using this same procedure.

(v) *If $y^v(0)$ is defined and satisfies $y^v(0) \leq 0$ or $y^v(0) > 0$ and $y \geq y^v(0)$, or if $y^v(0)$ is undefined and $P_1 = P_1^{sinc} = P_2$ holds for $k = 0$, then additional equilibria may exist in which a specific number of majority voters vote strategically. The exact condition and the number of strategic voters depend on the initial number of majority members who vote strategically in the first ballot and on the comparison between y and the values of $y^v(k)$ (when defined), or on the equalities among P_1, P_1^{sinc}, and P_2 (when $y^v(k)$ is undefined).*

The proof is provided in the Appendix. Note that when $\alpha = 1$, both the numerator and the denominator of the expression vanish for every $k \in \{0, \dots, \frac{w-3}{2} - m\}$, this case is trivial because setting $\alpha = 1$ removes any meaningful trade-off between sincere and strategic voting. Note also that when $\alpha = 0$, the equilibria identified in Fahrenberger and Gersbach (2012) become special cases within our framework. The situation can be summarized as follows: For $y^v(0) \leq y < y^v\left(\frac{w-3}{2} - m\right)$, multiple equilibria arise, whereas for $y \geq y^v\left(\frac{w-3}{2} - m\right)$, the equilibrium is unique.

For intermediate α values, Proposition 4 addresses the situation in which $y^v(k)$ is not defined because its denominator vanishes for some admissible k. In this case the usual comparison based on $y^v(k)$ cannot be applied, and a distinct analysis is required. It is important to note that indeterminacies in $y^v(k)$ can occur in several distinct forms, as illustrated in Figure 5: for $\alpha = 0.93$ indeterminacies appear over an interval of consecutive k values before the function becomes well-defined again; and for $\alpha = 0.03$ the function is well-defined for small k and then enters a region where indeterminacies alternate. Proposition 4 covers all these cases, as well as any other patterns that may occur.

We now present numerical examples that illustrate how the interplay between undefined thresholds $y^v(k)$ and the initial configuration of strategic voters determines equilibrium outcomes according to Proposition 4. To clarify the intuition underlying our analysis, we consider three distinct cases that demonstrate the sequential adjustment procedure.

First, consider the case where $y^v(0)$ is undefined with $P_1 = P_2 = P_1^{\text{sinc}}$. Let $\alpha = 0.88$, $w = 51$, and $m = 5$. With $P_1 = P_2$, no individual voter has an incentive to deviate from sincere voting, as switching to strategic voting yields no additional material benefit while incurring harmony costs. Consequently, all 5 minority voters and all 46 majority voters vote sincerely.

Second, consider a more complex scenario where thresholds alternate between defined and undefined values. For $\alpha = 0.02$, $w = 51$, $m = 5$, and $y = 10$, we obtain the following pattern: $y^v(0) = 12.75$, $y^v(1)$ is undefined with $P_1 = P_1^{\text{sinc}} \neq P_2$, $y^v(2) = 17.85$, $y^v(3)$ is undefined with $P_1 = P_1^{\text{sinc}} \neq P_2$, $y^v(4) = 22.95$, and so on. Since $y^v(0) = 12.75 > 0$ and $y = 10 < y^v(0)$, the first majority member votes strategically. The maximum admissible k is $k = (w-3)/2 - m = 19$. Now we trace the sequential procedure step by step, where all undefined thresholds correspond to the case where $P_1 = P_1^{\text{sinc}} \neq P_2$. Beginning with $k = 1$ strategic voter: for $k = 1$, $y^v(1)$ is undefined with $P_1 \neq P_2$, so one additional majority member votes strategically, bringing the total to 2 strategic voters. For $k = 2$, $y^v(2) = 17.85$ is defined and since $y = 10 < 17.85 = y^v(2)$, one additional majority member votes strategically, bringing the total to 3. For $k = 3$, $y^v(3)$ is undefined with $P_1 \neq P_2$, so another majority member votes strategically, bringing the total to 4. For $k = 4$, $y^v(4) = 22.95$ is defined and since $y = 10 < 22.95 = y^v(4)$, one more majority member votes strategically, bringing the total to 5. This pattern continues systematically: at each odd k (undefined threshold with $P_1 \neq P_2$) we add one strategic voter, and at each even k (defined threshold) we compare y with $y^v(k)$, and since $y = 10$ is always less than the defined thresholds (which start at 12.75 and increase to 58.65), we again add one strategic voter. The process continues unabated because neither stopping condition is met: we never encounter an undefined threshold with $P_1 = P_2 = P_1^{\text{sinc}}$, and we never find a defined threshold $y^v(k)$ such that $y \geq y^v(k)$. The sequential procedure therefore continues until $k = 19$, the maximum admissible value. The final equilibrium consists of $\frac{w-1}{2} - m = 20$ strategic majority voters and $w - m - 20 = 26$ sincere majority voters. All 5 minority members vote sincerely. The previous example considers a situation in which strategic voting emerges from an initially sincere majority; we now turn to a case where the initial configuration already involves more strategic voters than the relevant threshold k, and examine how undefined values of $y^v(k)$ affect the subsequent adjustment process.

Consider the parameters $w = 51$, $m = 5$, $\alpha = 0.95$, and $y = 4$. For these values, the threshold function $y^v(k)$ is undefined for $k = 0$ to $k = 12$ when $P_1 = P_1^{\text{sinc}} = P_2$. It remains undefined but with $P_1 = P_1^{\text{sinc}} \neq P_2$ for $k = 13$ and $k = 14$, and finally becomes defined for the remaining values up to $k = 19$. Specifically, $y^v(15) = -3.15$ and $y^v(16) = 41.22$, with the actual value $y = 4$ satisfying $y^v(15) = -3.15 < y < y^v(16) = 4.22$. Suppose initially $\hat{k} = 13$ majority members vote strategically (with $\hat{k} < k$), so that $y^v(\hat{k} - 1) = y^v(12)$ is undefined with $P_1 = P_1^{\text{sinc}} = P_2$. This implies that none of the $\hat{k}$ strategic voters has an incentive to switch to sincere voting. We then examine whether sincere majority voters have an incentive to become strategic. Since $y^v(13)$ is undefined with $P_1 = P_1^{\text{sinc}} \neq P_2$, one sincere voter deviates to strategic voting, increasing the number of strategic voters to $k = 14$. At $k = 14$, $y^v(14)$ is also undefined under the same condition ($P_1 = P_1^{\text{sinc}} \neq P_2$), so another sincere voter deviates, yielding $k = 15$ strategic voters. At $k = 15$, $y^v(15) = -3.15$ is defined and satisfies $y > y^v(15)$, which stops the iterative process. Therefore, the equilibrium consists of 15 strategic majority voters and $w - m - 15 = 31$ sincere majority voters, with all minority members voting sincerely. This example illustrates how the sequential adjustment procedure accommodates undefined thresholds and systematically determines the equilibrium number of strategic voters.

Having characterized the equilibrium voting behavior under harmony preferences, we now turn to the welfare comparison between WMV mechanisms with different weight parameters. When preferences for harmony are introduced, the welfare comparison between two weight parameters α and α' must account for both material payoffs and the disutility from repeated losses. We again consider the configuration $w_1 = \frac{w-1}{2}$, $w_2 = \frac{w+1}{2}$, which corresponds to the worst-case scenario in terms of disharmony: with the maximal number of weight-1 voters, each minority member faces the highest probability of losing a second time, thereby generating the largest expected harmony loss. Evaluating the threshold y^* under this extreme setting provides a conservative benchmark: if WMV with α yields higher aggregate welfare than WMV with α' when disharmony is most severe, it will also do so when the disharmony concern is weaker (i.e., when w_1 is smaller).

Proposition 5 *Consider WMV with an odd number of voters w, where each individual's utility is determined by both material payoffs and harmony payoffs. Take $P_1 = P_1(\frac{w-1}{2}, \frac{w+1}{2})$ and $P_2 = P_2(\frac{w-1}{2}, \frac{w+1}{2})$. WMV with parameter α outperforms that with parameter α' if the degree of disharmony aversion satisfies $y > y^*$, where*

$$y^* = \frac{w[(w-1)(P_1^{\alpha'} - P_1^{\alpha}) + (w+1)(P_2^{\alpha'} - P_2^{\alpha})]}{8m(w-1)(P_1^{\alpha} - P_1^{\alpha'})}. \tag{3}$$

This result shows that incorporating harmony preferences affects both individual incentives and the normative comparison between voting rules. When society becomes increasingly sensitive to the possibility that some individuals may lose in both rounds, this concern introduces an additional welfare cost. As this cost grows, the appeal of a rule that strengthens the influence of the minority in the second round naturally increases. The threshold y^* captures precisely the point at which this concern becomes strong enough for WMV with parameter α to yield a higher level of social welfare than WMV with parameter α'. Whenever $y > y^*$, the reduction in the probability of repeated losses more than compensates for the change in the voting weight allocated to first-round winners.

To illustrate this mechanism, consider two committee sizes, $w = 51$ and $w = 101$, each with a minority of size $m = 5$. In both cases, suppose that the number of strategic voters reaches its maximal admissible value, which corresponds to the upper bound of the equilibrium range described in the model. Taking WMV with $\alpha' = 0$ as the reference rule, we compute, for several alternative values of α, the critical threshold y^* provided by Proposition 5. The resulting values for both committee sizes, reported in Table 3, show how the degree of harmony aversion determines which WMV mechanism is socially preferable. Each entry should be read as follows: for a given α, WMV with parameter α yields higher social welfare than the benchmark mechanism with parameter $\alpha' = 0$ whenever the degree of disharmony aversion satisfies $y > y^*$. For very low values of α, the threshold y^* is high in both committee sizes, indicating that only a very strong concern for harmony would justify adopting such strongly minority-favoring rules. As α increases, the threshold y^* gradually decreases and eventually stabilizes at very small values. When α becomes large, first-round winners retain substantial influence in the second round, which reduces the incentive for strategic behavior and lowers the likelihood of repeated losses. In this regime, the mechanism with a larger α almost always dominates the reference rule, regardless of the committee size, because even minimal harmony aversion is sufficient to offset the small remaining differences in second-round influence. Overall, Table 3 highlights the central role of the weight parameter α in determining the social welfare ranking of WMV mechanisms across different committee sizes.

$w = 51$; $m = 5$		$w = 101$; $m = 5$	
α	y^*	α	y^*
0.1	22239.57	0.1	45.83
0.2	41.05	0.2	20.43
0.3	12.32	0.3	14.12
0.4	7.46	0.4	10.60
0.5	5.52	0.5	8.19
0.6	4.38	0.6	6.76
0.7	2.96	0.7	5.93
0.8	2.83	0.8	4.83
0.9	2.23	0.9	3.99
1.0	2.21	1.0	3.75

Table 3: Critical values y^* determining when WMV with parameter α outperforms WMV with parameter $\alpha' = 0$.

4 Conclusion

We have introduced a weighted minority voting mechanism that modifies the second round of a sequential voting process by adjusting the influence of first-round winners. This design creates a continuum between simple majority and minority voting while preserving participation by all voters. Our analysis shows how strategic incentives, welfare outcomes, and harmony concerns evolve as the weight parameter varies. We identify conditions under which sincere behavior arises, characterize equilibrium manipulation patterns, and compare aggregate welfare for different values of the weight.

Possible extensions to our model include introducing asymmetric information by allowing voters to hold heterogeneous beliefs about the number of strategic voters k, which could significantly affect their incentives and lead to different equilibrium behaviors. Another natural direction is to explore how the choice of tie-breaking rule influences the outcomes, as our current results rely on a specific tie-breaking mechanism (a fair coin toss); examining alternative rules would help assess the robustness of our findings. Further extensions could also be explored. One avenue is to allow voters to abstain at a cost, or to make abstention informative, which may generate new strategic patterns in both rounds. Another is to consider correlated material payoffs across voters, which would introduce richer coalition structures and possibly shift the distribution of double losers. A dynamic extension with more than two rounds could reveal propagation effects in voting weights and show how early decisions shape long-run equilibria.

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Appendix: Proofs

Proof of Lemma 1. To compute the probability that a weight-1 voter belongs to a winning coalition, we take the perspective of that specific voter. The probability is computed by considering all possible coalitions that would still be winning if this voter were included. Removing this voter leaves a system with $w_1 - 1$ weight-1 voters and w_2 weight-α voters. The remaining voters must form a coalition that is *almost* winning – more precisely, a coalition that, without the individual i, has a total weight strictly greater than $q - 1$. Indeed, if their weight is strictly less than $q-1$, then even adding the voter i will not achieve a majority. If their total weight exceeds $q-1$, the individual i simply reinforces an already winning coalition, which means that she is not pivotal. But if the coalition weight is exactly $q-1$, then the individual i becomes pivotal, turning the tie into a winning vote with probability $\frac{1}{2}$. Therefore, the threshold $x+\alpha y > q-1$ captures the condition under which the individual i contributes to a winning coalition. The remaining voters must form a winning coalition on their own and vote in the same way, meaning their total weight must strictly exceed $q-1$. The number of ways to choose x weight-1 voters from the $w_1 - 1$ is given by $\binom{w_1-1}{x}$, and the number of ways to choose y weight-α voters from w_2 is $\binom{w_2}{y}$. Since each voter votes independently, the probability of a specific configuration occurring is $\left(\frac{1}{2}\right)^{w-1}$. Summing over all valid configurations gives:

$$P_1(w_1, w_2) = \frac{1}{2^{w-1}} \sum_{\substack{x=0, \\ x+\alpha y > q-1}}^{w_1-1} \sum_{y=0}^{w_2} \binom{w_1 - 1}{x} \binom{w_2}{y}.$$

When a coalition reaches exactly the threshold $q-1$, adding our weight-1 voter i creates a tie, which is resolved randomly with a probability of $1/2$ for acceptance/rejection. Following the same combinatorial reasoning, the contribution of these cases is given by:

$$\frac{1}{2^w} \sum_{\substack{x=0, \\ x+\alpha y = q-1}}^{w_1-1} \sum_{y=0}^{w_2} \binom{w_1 - 1}{x} \binom{w_2}{y}.$$

Combining both contributions, we obtain the complete expression for $P_1(w)$.
For a weight-α voter, the same logic applies. Removing this voter leaves w_1 weight-1 voters and $w_2 - 1$ weight-α voters. The condition for the remaining voters to be a winning coalition when this voter votes in the same way as the coalition is that their total weight strictly exceeds $q-\alpha$. The computation gives:

$$P_2(w_1, w_2) = \frac{1}{2^{w-1}} \sum_{\substack{x=0, \\ x+\alpha y > q-\alpha}}^{w_1} \sum_{y=0}^{w_2-1} \binom{w_1}{x} \binom{w_2 - 1}{y}.$$

When the coalition reaches exactly $q-\alpha$, adding this voter results in a tie, which is resolved with a probability of $1/2$ for acceptance/rejection. The contribution of these cases is thus:

$$\frac{1}{2^w} \sum_{\substack{x=0, \\ x+\alpha y = q-\alpha}}^{w_1} \sum_{y=0}^{w_2-1} \binom{w_1}{x} \binom{w_2 - 1}{y}.$$

Summing these terms, we obtain the complete expression for $P_2(w_1, w_2)$, which completes the proof. ■

Proof of Lemma 2. We note that $P_1(m+(k+1), w-(m+k+1))$ represents the probability that a majority individual i, who voted strategically in the first ballot (and thus holds full weight in the second round), belongs to a winning coalition in the second round, given that $k+1$ majority members (including i) voted strategically. In this setting, there are $m+k+1$ voters with weight 1 in the second round: the m minority members and the $k+1$ strategic majority voters. The remaining $w-(m+k+1)$ voters – those from the majority who voted sincerely – have weight α. Since we compute the probability that voter i is part of a winning coalition from her point of view, we exclude her from the summation: the remaining weight-1 voters are thus of size $m+k$, and the remaining weight-α voters are still of size $w-(m+k+1)$. The quota in this setting is given by $q_1 = \frac{(m+k+1)+\alpha(w-(m+k+1))}{2}$, which corresponds to half of the total weight (including i). The condition $x+\alpha y > q_1 - 1$ ensures that the coalition excluding i already meets or exceeds the quota once i joins, making her vote pivotal.

We now focus on the expression $P_2(m+k, w-(m+k))$, which represents the probability from the point of view of a majority individual i who voted sincerely in the first ballot and thus retains only a weight-α in the second round. The k majority members who voted strategically join the group of weight-1 voters, so this group has size $m+k$. The remaining majority members who voted sincerely form the group of weight-α voters, of size $w-(m+k)-1$. As before, we exclude individual i from the summation since we are evaluating the probability that she is part of a winning coalition from her point of vue. The quota for success in the second round is given by $q_2 = \frac{m+k+\alpha(w-(m+k))}{2}$, which corresponds to half the total weight, including i. Because i has weight α, the coalition without her must exceed the threshold $q_2 - \alpha$ for her to be pivotal. The first term of the expression captures the configurations where this inequality is strictly satisfied. The second term handles the tie case, where the total weight equals exactly $q_2 - \alpha$ and the outcome is determined by a fair random draw, each side winning with probability $\frac{1}{2}$. ■

Proof of Lemma 3. In the second ballot, let S denote the total Yes-weight contributed by all voters except voter i. The proposal is accepted whenever the total Yes-weight exceeds $q = \frac{w_1+\alpha w_2}{2}$. Voter i has weight 1 if she belongs to the group of weight-1 voters, and weight α if she belongs to the group of weight-α voters.

Suppose first that i prefers acceptance. If she votes No, the total Yes-weight remains S. If she votes Yes, the total becomes $S+1$ (if i has weight 1) or $S+\alpha$ (if i has weight α). A simple case analysis shows that a sincere Yes-vote is never worse and sometimes strictly better: (i) If $S > q$, both actions lead to acceptance. (ii) If $S = q$, voting No yields acceptance with probability 1/2, while voting Yes ensures acceptance with probability 1. (iii) If $q-1 < S < q$ (for a weight-1 voter) or $q-\alpha < S < q$ (for a weight-α voter), voting No leads to rejection whereas voting Yes leads to acceptance. (iv) If $S = q-1$ (weight 1) or $S = q-\alpha$ (weight α), a No-vote leads to rejection whereas a Yes-vote reaches the tie, accepted with probability 1/2, which is strictly better. (v) If $S < q-1$ (or $S < q-\alpha$ for a weight-α voter), neither action can reach the quota, so both actions yield rejection. Thus, whenever i prefers acceptance, voting Yes weakly maximizes her expected payoff for every possible value of S, and is strictly better whenever she is pivotal.

By symmetry, if i prefers rejection, voting No is weakly optimal: it strictly improves the outcome on the pivotal region $S \in [q-1, q]$ for weight-1 voters or $S \in [q-\alpha, q]$ for weight-α voters, and yields the same payoff elsewhere. Since sincere voting weakly dominates any deviation in every possible state of the world, it is a weakly dominant strategy in the second ballot. ■

Proof of Proposition 1. (i) Follows directly from the structure of WMV: if a minority individual votes strategically and ends up on the winning side, she loses her minority status and therefore her full weight of vote in the second round, her overall influence. Thus, voting sincerely is always optimal for minority individuals.

(ii) Suppose without loss of generality that the majority is in favor of the adoption of the project while the minority is against it. In the first round, the project is adopted if it receives at least $\frac{w+1}{2}$ "Yes" votes. Given that minority members always vote sincerely (as shown in point (i)), exactly m votes against the project come from the minority. To reach the required majority, we therefore need exactly $\frac{w+1}{2}$ sincere votes in favor of the project from majority members. If fewer than $\frac{w+1}{2}$ majority members vote sincerely, the project is rejected, which leads to an undesirable outcome. If more than $\frac{w+1}{2}$ vote sincerely in favor of the project, the project is still accepted, but some of these individuals lose the opportunity to keep their full voting weight in the second round (by deviating and then voting in the same way as the minority, i.e., against the project), which is also not optimal. Therefore, the optimal strategy for the majority group is to have exactly $\frac{w+1}{2}$ of its members vote sincerely in favor of the project. The remaining $w - m - \frac{w+1}{2} = \frac{w-1}{2} - m$ majority members can then vote strategically without affecting the outcome of the first round. By doing so, they keep full voting weight in the second round and increase their expected payoff. No majority member wants to deviate from this plan as long as the others stick to it.

We now turn to the proof of the final statement of the proposition. If an individual i from the majority votes sincerely in the first ballot, her expected utility is given by:

$$\mathbb{E}u_i^{\text{sinc}} = a_1 z_{i1} + \frac{1}{2}\left(P_2(m+k, w-(m+k)) - \frac{1}{2}\right)$$

The second term represents the expected material utility from the second ballot for individual i who voted sincerely in the first ballot and thus votes with weight α, which explains the presence of $P_2(m+k, w-(m+k))$. The factor $\frac{1}{2}$ captures the fact that belonging to a winning coalition does not necessarily mean that the project aligns with the voter's preferences – only half of the time, the coalition votes in the direction that matches her material interest (i.e., project adoption when $z_{i2} > 0$, or rejection when $z_{i2} < 0$). We recall that the individual payoff in the second round z_{i2} follows a uniform distribution over the interval $[-1, 1]$, that is,

$$\int_{-1}^{0} z_{i2}\, dz_{i2} = -\frac{1}{2} \quad \text{and} \quad \int_{0}^{1} z_{i2}\, dz_{i2} = \frac{1}{2}.$$

If the individual i votes strategically, her expected utility becomes:

$$\mathbb{E}u_i^{\text{strat}} = a_1 z_{i1} + \frac{1}{2}\left(P_1(m+k+1, w-(m+k+1)) - \frac{1}{2}.\right)$$

The second term represents the expected material utility from the second ballot for individual

i that voted strategically in the first ballot in order to retain the full weight of vote in the second ballot, which explains $P_1(m+k+1, w-(m+k+1))$. A majority voter will choose to vote strategically in the first ballot if the expected utility under strategic voting exceeds the one under sincere voting. That is, the individual prefers to retain her full weight of vote in the second round, provided that this strategy increases her probability of belonging to a winning coalition, formally

$$\mathbb{E}u_i^{\text{strat}} > \mathbb{E}u_i^{\text{sinc}}.$$

Expanding this inequality, we obtain:

$$P_1(m+k+1, w-(m+k+1)) > P_2(m+k, w-(m+k))$$

Let us consider $\Delta P = P_1(m+k+1; w-(m+k+1)) - P_2(m+k, w-(m+k)) > 0$, $q_1 = \frac{m+k+1+\alpha(w-(m+k+1))}{2}$, and $q_2 = \frac{m+k+\alpha(w-(m+k))}{2}$. We can notice that $q_2 - \alpha = q_1 - 1 + \frac{1-\alpha}{2}$. Since $\alpha \in [0,1]$, we have $\frac{1-\alpha}{2} \geq 0$, which implies $q_1 - 1 \leq q_2 - \alpha$. Now, let us consider the conditions within the summations. If $x + \alpha y > q_2 - \alpha$, then $x + \alpha y > q_1 - 1 + \frac{1-\alpha}{2}$. The difference ΔP is given as follows:

$$\begin{aligned}
\Delta P = \left(\frac{1}{2}\right)^{w-1} \Bigg[& \Bigg(\sum_{x=0}^{m+k} \sum_{\substack{y=0 \\ x+\alpha y > q_1 - 1}}^{w-(m+k+1)} \binom{m+k}{x} \binom{w-(m+k+1)}{y} \\
& + \frac{1}{2} \sum_{x=0}^{m+k} \sum_{\substack{y=0 \\ x+\alpha y = q_1 - 1}}^{w-(m+k+1)} \binom{m+k}{x} \binom{w-(m+k+1)}{y} \Bigg) \\
& - \Bigg(\sum_{x=0}^{m+k} \sum_{\substack{y=0 \\ x+\alpha y > q_2 - \alpha}}^{w-(m+k)-1} \binom{m+k}{x} \binom{w-(m+k)-1}{y} \\
& + \frac{1}{2} \sum_{x=0}^{m+k} \sum_{\substack{y=0 \\ x+\alpha y = q_2 - \alpha}}^{w-(m+k)-1} \binom{m+k}{x} \binom{w-(m+k)-1}{y} \Bigg) \Bigg].
\end{aligned}$$

We can simplify ΔP as follows:

$$\begin{aligned}
\Delta P = \left(\frac{1}{2}\right)^{w-1} \Bigg[& \sum_{x=0}^{m+k} \sum_{\substack{y=0 \\ q_1 - 1 < x+\alpha y \leq q_2 - \alpha}}^{w-(m+k+1)} \binom{m+k}{x} \binom{w-(m+k+1)}{y} \\
& + \frac{1}{2} \sum_{x=0}^{m+k} \sum_{\substack{y=0 \\ x+\alpha y = q_1 - 1}}^{w-(m+k+1)} \binom{m+k}{x} \binom{w-(m+k+1)}{y} \\
& - \frac{1}{2} \sum_{x=0}^{m+k} \sum_{\substack{y=0 \\ x+\alpha y = q_2 - \alpha}}^{w-(m+k)-1} \binom{m+k}{x} \binom{w-(m+k)-1}{y} \Bigg].
\end{aligned}$$

We can further simplify ΔP as follows:

$$\Delta P = \left(\frac{1}{2}\right)^{w-1} \Bigg[\sum_{x=0}^{m+k} \sum_{\substack{y=0 \\ q_1-1<x+\alpha y<q_2-\alpha}}^{w-(m+k+1)} \binom{m+k}{x}\binom{w-(m+k+1)}{y}$$

$$+ \frac{1}{2}\Bigg(\sum_{x=0}^{m+k} \sum_{\substack{y=0 \\ x+\alpha y=q_1-1}}^{w-(m+k+1)} \binom{m+k}{x}\binom{w-(m+k+1)}{y}$$

$$+ \sum_{x=0}^{m+k} \sum_{\substack{y=0 \\ x+\alpha y=q_2-\alpha}}^{w-(m+k)-1} \binom{m+k}{x}\binom{w-(m+k)-1}{y}\Bigg)\Bigg].$$

Since the factor $\left(\frac{1}{2}\right)^{w-1}$ is strictly positive, the sign of ΔP is entirely determined by the expression inside the brackets. Each term in the summation is a product of combination numbers, which are all non-negative. It follows that $\Delta P \geq 0$, and $\Delta P = 0$ if there exists no integer pair (x, y) that satisfies the conditions of at least one of the three double summations. That is, $\Delta P = 0$ if there exists no integer pair

$$(x, y) \in \{0, \ldots, m+k\} \times \{0, \ldots, w-(m+k+1)\},$$

such that

$$x + \alpha y \in [q_1 - 1,\ q_2 - \alpha].$$

(An example of such a case will be given just after Proposition 1). This completes the proof. ■

Proof of Proposition 2. We begin with the case where the first project is adopted in the first round, i.e., $a_1 = 1$. We assume that the equilibrium described in Proposition 1 holds, where strategic behavior may arise among majority individuals in the first ballot, while minority voters have no incentive to vote strategically at this stage. We note that these majority members, together with the minority, are indexed from 1 to $\frac{w-1}{2}$. Consequently, the first $\frac{w-1}{2}$ individuals retain their full voting weight in the second round, with a winning probability of $P_1^{\alpha}(\frac{w-1}{2},\ \frac{w+1}{2})$ in the second round. In contrast, the remaining $\frac{w+1}{2}$ individuals have a α-weighted vote in the second ballot and therefore a winning probability of $P_2^{\alpha}(\frac{w-1}{2},\ \frac{w+1}{2})$. Recall that the quota required for acceptance in the second round is defined as

$$q = \frac{\frac{w-1}{2} + \alpha\frac{w+1}{2}}{2}.$$

We then obtain

$$
\begin{aligned}
W^{\alpha} = \sum_{i=1}^{\frac{w-1}{2}} \Bigg[& \frac{1}{2} P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\left(z_{i1} + \int_0^1 z_{i2}\, dz_{i2}\right) + \frac{1}{2}\left(1 - P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\right) z_{i1} \\
& + \frac{1}{2} P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right) z_{i1} + \frac{1}{2}\left(1 - P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\right)\left(z_{i1} + \int_{-1}^0 z_{i2}\, dz_{i2}\right)\Bigg] \\
+ \sum_{i=\frac{w+1}{2}}^{w} \Bigg[& \frac{1}{2} P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\left(z_{i1} + \int_0^1 z_{i2}\, dz_{i2}\right) + \frac{1}{2}\left(1 - P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\right) z_{i1} \\
& + \frac{1}{2} P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right) z_{i1} + \frac{1}{2}\left(1 - P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\right)\left(z_{i1} + \int_{-1}^0 z_{i2}\, dz_{i2}\right)\Bigg].
\end{aligned}
$$

Our assumptions imply: $\int_{-1}^{0} z_{i2}\, dz_{i2} = -\frac{1}{2}$ and $\int_0^1 z_{i2}\, dz_{i2} = \frac{1}{2}$. As a consequence,

$$
\begin{aligned}
W^{\alpha} = \sum_{i=1}^{\frac{w-1}{2}} \Bigg[& \frac{1}{2} P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\left(z_{i1} + \frac{1}{2}\right) + \frac{1}{2}\left(1 - P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\right) z_{i1} \\
& + \frac{1}{2} P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right) z_{i1} + \frac{1}{2}\left(1 - P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\right)\left(z_{i1} - \frac{1}{2}\right)\Bigg] \\
+ \sum_{i=\frac{w+1}{2}}^{w} \Bigg[& \frac{1}{2} P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\left(z_{i1} + \frac{1}{2}\right) + \frac{1}{2}\left(1 - P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\right) z_{i1} \\
& + \frac{1}{2} P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right) z_{i1} + \frac{1}{2}\left(1 - P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\right)\left(z_{i1} - \frac{1}{2}\right)\Bigg].
\end{aligned}
$$

In the first sum of W^{α}, from $i = 1$ to $\frac{w-1}{2}$ (weight-1 voters), the material payoff is divided into four cases: (i) having positive z_{i2} with probability $\frac{1}{2}$ and winning with probability $P_1^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 1$); (ii) having positive z_{i2} with probability $\frac{1}{2}$ and losing with probability $1 - P_1^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 0$); (iii) having negative z_{i2} with probability $\frac{1}{2}$ and winning with probability $P_1^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 0$); and finally, (iv) having negative z_{i2} with probability $\frac{1}{2}$ and losing with probability $1 - P_1^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 1$). In the second sum of W^{α}, from $i = \frac{w+1}{2}$ to w (weight-α voters), the material payoff is also divided into four cases: (i) having positive z_{i2} with probability $\frac{1}{2}$ and winning with probability $P_2^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 1$);(ii) having positive z_{i2} with probability $\frac{1}{2}$ and losing with probability $1 - P_2^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 0$); (iii) having negative z_{i2} with probability $\frac{1}{2}$ and winning with probability $P_2^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 0$); and finally, (iv) having negative z_{i2} with probability $\frac{1}{2}$ and losing with probability $1 - P_2^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 1$).

The equation can be rearranged to

$$W^{\alpha} = \sum_{i=1}^{w} z_{i1} + m\left[\frac{1}{4}\left(2P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)-1\right)\right] + (w-m)\left[\frac{1}{4}\left(2P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)-1\right)\right].$$

Similarly, the expression for WMV with the parameter α' is given by:

$$W^{\alpha'} = \sum_{i=1}^{w} z_{i1} + m\left[\frac{1}{4}\left(2P_1^{\alpha'}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)-1\right)\right] + (w-m)\left[\frac{1}{4}\left(2P_2^{\alpha'}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)-1\right)\right].$$

Using the previously derived expressions for W^{α} and $W^{\alpha'}$ yields the following welfare difference:

$$\begin{aligned} W^{\alpha} - W^{\alpha'} = &\left(\sum_{i=1}^{w} z_{i1} + m\left[\frac{1}{4}\left(2P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)-1\right)\right]\right. \\ &\left.+(w-m)\left[\frac{1}{4}\left(2P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)-1\right)\right]\right) \\ &-\left(\sum_{i=1}^{w} z_{i1} + m\left[\frac{1}{4}\left(2P_1^{\alpha'}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)-1\right)\right]\right. \\ &\left.+(w-m)\left[\frac{1}{4}\left(2P_2^{\alpha'}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)-1\right)\right]\right). \end{aligned}$$

By simplifying the expression above, we observe that the two sums over $\sum_{i=1}^{w} z_{i1}$ cancel out. This yields the following form of the welfare difference:

$$\begin{aligned} W^{\alpha} - W^{\alpha'} = \frac{1}{2}\Bigg[&m\left(P_1^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right) - P_1^{\alpha'}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\right) \\ &+ (w-m)\left(P_2^{\alpha}\left(\frac{w-1}{2}, \frac{w+1}{2}\right) - P_2^{\alpha'}\left(\frac{w-1}{2}, \frac{w+1}{2}\right)\right)\Bigg]. \end{aligned} \tag{4}$$

In order for $W^{\alpha} - W^{\alpha'}$ to be strictly positive, the following condition must be satisfied:

$$P_1^{\alpha}(w_1, w_2) - P_1^{\alpha'}(w_1, w_2) > \frac{w-m}{m}\left(P_2^{\alpha'}(w_1, w_2) - P_2^{\alpha}(w_1, w_2)\right).$$

This completes the proof of the case where $a_1 = 1$. Note that in the case where $a_1 = 0$, the first project is rejected, and no material payoff is realized from it. Therefore, the expected welfare remains unchanged under both voting mechanisms, as it depends only on second-round outcomes and associated probabilities. Consequently, Proposition 2 continues to hold under $a_1 = 0$, and the comparison between the two voting schemes remains valid. ■

Proof of Proposition 3.

Point (i): The argument follows the same line as in the proof of point (i) of Proposition 1.

All minority members vote sincerely in the first ballot, since voting strategically would reduce their weight in the second round without any material gain. This reasoning also extends when harmony considerations are present: a deviation would not only weaken their weight in the second round, but could also generate additional disutility from disharmony.

For majority members, we first show under what condition they vote strategically, let $k \in \left\{0, \ldots, \frac{w-3}{2} - m\right\}$ be the number of strategic majority voters (other than individual i). The expected utility of a sincere voter i includes the material payoff from the first round and the harmony payoff associated with the probability that the minority wins in the second round. For a strategic voter i, the first-round material payoff remains unchanged, but the voting weight in the second round increases to 1, which changes the probability of being in a winning coalition and therefore modifies both the material and harmony components of the utility. The expected utility from sincere voting is given by

$$\begin{aligned}\mathbb{E}u_i^{\text{sinc}} = a_1 z_{i1} + \tfrac{1}{2}\big(P_2(m+k,\, w-(m+k)) - \tfrac{1}{2}\big) \\ +2m\,\delta_L\left(1 - P_1(m+k,\, w-(m+k))\right),\end{aligned}$$

while strategic voting yields

$$\begin{aligned}\mathbb{E}u_i^{\text{strat}} = a_1 z_{i1} + \tfrac{1}{2}\big(P_1(m+k+1,\, w-(m+k+1)) - \tfrac{1}{2}\big) \\ +2m\,\delta_L\left(1 - P_1(m+k+1,\, w-(m+k+1))\right).\end{aligned}$$

Using $\delta_L = \frac{-y}{w}$, we compare both expressions. By simplifying the difference and isolating y, strategic voting is optimal when

$$y < \frac{w\left(P_1(m+k+1, w-(m+k+1)) - P_2(m+k, w-(m+k))\right)}{4m\left(P_1(m+k, w-(m+k)) - P_1(m+k+1, w-(m+k+1))\right)} := y^v(k)$$

In other words, when k voters vote strategically: (a) a sincere voter has an incentive to deviate to strategic voting if $y < y^v(k)$; (b) a sincere voter has no incentive to deviate to strategic voting if $y \geq y^v(k)$; (c) a strategic voter has no incentive to deviate to sincere voting if $y \leq y^v(k)$; and (d) a strategic voter has an incentive to deviate to sincere voting if $y > y^v(k)$.

Point (ii): According to the first case of point (ii), since the model assumes $y > 0$, we immediately obtain $y > y^v(0)$. Hence $y < y^v(0)$ can never hold, and no majority member has an incentive to deviate. Therefore an equilibrium exists in which all majority members vote sincerely. Now consider the second case: By definition of $y^v(0)$, we have $Eu_i^{\text{sinc}} \geq Eu_i^{\text{strat}}$ whenever $y \geq y^v(0)$. Hence no majority member finds it profitable to deviate, and sincere voting by all majority members again forms an equilibrium. This proves part (ii).

Point (iii): Assume $y^v(0) > 0$ and $y < y^v(0)$, so that a single majority member strictly prefers to deviate from sincere voting when all others vote sincerely. Consider now the case where $k \geq 1$ majority members vote strategically. By definition of $y^v(k)$, a $(k+1)$-th majority member finds it profitable to deviate if $y < y^v(k)$. Hence, for every k such that $y < y^v(k)$, strategic voting strictly dominates sincere voting for any majority member not yet deviating. Therefore such configurations cannot form an equilibrium. If there exists a smallest integer

$k^* \in \{1, \ldots, \frac{w-3}{2} - m\}$ such that $y \geq y^v(k^*)$, then when exactly k^* individuals deviate, no remaining majority member has an incentive to deviate further. Thus the profile with k^* strategic majority voters and $w - m - k^*$ sincere majority voters constitutes an equilibrium. If no such k exists, i.e. $y < y^v(k)$ for all $k \in \{1, \ldots, \frac{w-3}{2} - m\}$, then every majority member strictly prefers to deviate as long as this does not overturn the first-round outcome. The maximal number of strategic deviations compatible with the acceptance of project 1 is $k = \frac{w-1}{2} - m$. At this point, no further deviation is feasible, and no strategic voter has an incentive to switch back to sincere voting. Hence this configuration forms an equilibrium.

Point (iv): Suppose that $y \geq y^v(0)$ and that y satisfies $y^v(k-1) < y < y^v(k)$ for some $k \in \left\{1, \ldots, \frac{w-3}{2} - m\right\}$. First, assume that $\hat{k}$ $(< k)$ majority members vote strategically. Two subcases arise: (1) If $y > y^v(\hat{k}-1)$: A strategic voter among the $\hat{k}$ has an incentive to deviate to sincere voting, and the other strategic voters may have the same incentive. Deviations continue until we find the smallest $k^* \in \{1, \ldots, \hat{k} - 1\}$ such that $y \leq y^v(k^*)$. If such a k^* exists, the equilibrium has $k^* + 1$ majority members voting strategically and $w - m - k^* - 1$ majority members voting sincerely; otherwise, if such a k^* does not exist, the equilibrium has 0 strategic voters (all sincere). (2) If $y \leq y^v(\hat{k} - 1)$: No strategic majority voter among the $\hat{k}$ wants to deviate. However, a sincere voter may have an incentive to deviate to strategic voting, and the process continues until we find the first $k^* \in \{\hat{k}, \ldots, k-2\}$ such that $y \geq y^v(k^*)$. If such a k^* exists, the equilibrium has k^* strategic majority voters and $w - m - k^*$ sincere majority voters; otherwise, we will certainly find $k^* = k-1$ such that $y > y^v(k-1)$ according to our hypothesis, in which case the equilibrium consists of $k-1$ strategic voters and $w - m - k + 1$ sincere voters.

Second, assume that $\hat{k}$ $(> k)$ majority members vote strategically. Again two subcases appear: (1) If $y > y^v(\hat{k}-1)$: A strategic voter among the $\hat{k}$ has an incentive to deviate to sincere voting, and the remaining $\hat{k} - 1$ strategic voters may follow, initiating a sequence of deviations. This process continues until we find the largest integer $k^* = \max\{k \in \{k+1, \ldots, \hat{k} - 2\} : y \leq y^v(k)\}$. If such a k^* exists, the equilibrium consists of $k^* + 1$ strategic majority voters and $w - m - k^* - 1$ sincere majority voters. If no such k^* exists, we proceed to the final value of k. Since by hypothesis $y < y^v(k)$, the condition $y \leq y^v(k)$ holds, resulting in an equilibrium with $k+1$ strategic majority voters and $w-m-k-1$ sincere majority voters. (2) If $y \leq y^v(\hat{k}-1)$: None of the $\hat{k}$ strategic voters may have an incentive to deviate to sincere voting. However, a sincere voter may have an incentive to deviate to strategic voting, potentially followed by others. The process continues until we find the smallest integer $k^* = \min\{k \in \{\hat{k}, \ldots, \frac{w-3}{2} - m\} : y \geq y^v(k)\}$. If such a k^* exists, the equilibrium consists of k^* strategic majority voters and $w - m - k^*$ sincere majority voters. If no such k^* exists, the equilibrium consists of $\frac{w+1}{2}$ sincere majority voters and $\frac{w-1}{2} - m$ strategic majority voters. This completes the proof of the proposition. ∎

Proof of Proposition 4.

Point (i): The argument follows the same line as in the proof of point (i) of Propositions 1. For majority members, we already know from the proof of Proposition 3 that strategic voting is optimal when

$$y < \frac{w\left(P_1(m+k+1, w-(m+k+1)) - P_2(m+k, w-(m+k))\right)}{4m\left(P_1(m+k, w-(m+k)) - P_1(m+k+1, w-(m+k+1))\right)} := y^v(k).$$

Recall that in Proposition 3 we established the conditions under which voters have incentives to deviate between sincere and strategic voting. In particular, when k voters vote strategically: (a) a sincere voter has an incentive to deviate to strategic voting if $y < y^v(k)$; (b) a sincere voter has no incentive to deviate to strategic voting if $y \geq y^v(k)$; (c) a strategic voter has no incentive to deviate to sincere voting if $y \leq y^v(k)$; and (d) a strategic voter has an incentive to deviate to sincere voting if $y > y^v(k)$.

Point (ii): The argument follows the same line as in the proof of point (ii) of Propositions 3.

Point (iii): First, (iii)-1, when $y^v(0)$ is undefined, all three probabilities coincide for $k = 0$, that is, $P_1 = P_1^{\text{sinc}} = P_2$, Hence, $\mathbb{E}u_i^{\text{strat}} = \mathbb{E}u_i^{\text{sinc}}$, and no majority member has an incentive to deviate from sincere voting. Second, (iii)-2, if $P_1 = P_1^{\text{sinc}} \neq P_2$, the numerator $(P_1 - P_2) \neq 0$ while the denominator is zero, implying $\mathbb{E}u_i^{\text{strat}} > \mathbb{E}u_i^{\text{sinc}}$. Alternatively, when $y^v(0)$ is well defined and satisfies $y^v(0) > 0$ with $y < y^v(0)$, the condition for strategic voting is met. Consequently, one majority member votes strategically, and we set $k = 1$ for the sequential procedure in point (iv).

Point (iv): We proceed sequentially for $k = 1, 2, \ldots, \frac{w-3}{2} - m$. At each step k, we consider whether an additional majority member would prefer to vote strategically given that k majority members are already voting strategically. When $y^v(k)$ is undefined ($P_1 = P_1^{\text{sinc}}$ for k), two subcases occur. If $P_1 = P_1^{\text{sinc}} = P_2$ for k, all probabilities are equal, so no further deviation is profitable. The equilibrium consists of k strategic majority members and $w - m - k$ sincere majority members. If $P_1 = P_1^{\text{sinc}} \neq P_2$ for k, the marginal material gain is positive. One additional majority member votes strategically, and we proceed to evaluate $k+1$. When $y^v(k)$ is well-defined, we compare y with $y^v(k)$. If $y \geq y^v(k)$, the condition for strategic voting is not satisfied for an additional voter. The equilibrium consists of k strategic majority members and $w - m - k$ sincere majority members. If $y < y^v(k)$, an additional majority member prefers strategic voting. We add one strategic voter and proceed to evaluate $k+1$. The procedure continues until either a stopping condition is met or we exhaust all admissible k.

Point (v): The logic follows that of Proposition 3 (iv), but at each step we must check whether the threshold $y^v(k)$ is defined or not. Consider an initial configuration with $\hat{k} < k$ strategic voters. First examine whether these strategic voters wish to deviate to sincere voting. They do not wish to deviate if $y^v(\hat{k}-1)$ is defined and $y \leq y^v(\hat{k}-1)$, or if $y^v(\hat{k}-1)$ is undefined (with either $P_1 = P_1^{\text{sinc}} = P_2$ for $k = \hat{k} - 1$, so $\mathbb{E}u_i^{\text{strat}} = \mathbb{E}u_i^{\text{sinc}}$, or $P_1 = P_1^{\text{sinc}} \neq P_2$, implying $\mathbb{E}u_i^{\text{strat}} > \mathbb{E}u_i^{\text{sinc}}$) . In this case, a filling process may occur: one additional sincere voter may have an incentive to become strategic if, for a given number k of strategic voters, $y^v(k)$ is defined with $y < y^v(k)$ or if $y^v(k)$ is undefined with $P_1 = P_1^{\text{sinc}} \neq P_2$. The filling stops at the smallest $k \in \{\hat{k}, \ldots, k-2\}$ where $y^v(\hat{k})$ is undefined with $P_1 = P_1^{\text{sinc}} = P_2$ (no further incentive to add strategists) or where $y \geq y^v(\hat{k})$ (if $y^v(\hat{k})$ is defined). Denote this k by k^*; the equilibrium then has k^* strategic voters (if such a k^* exists) or, if no k^* is found, $k-1$ strategic voters (since by hypothesis $y > y^v(k-1)$). Conversely, if $y^v(\hat{k}-1)$ is defined and $y > y^v(\hat{k}-1)$, a strategic voter may wish to deviate, initiating an emptying process. This emptying continues as long as, for a given number k of strategic voters, $y^v(k)$ is defined and $y > y^v(k)$. It stops as soon as $y^v(k)$ becomes undefined or $y \leq y^v(k)$. Let k^* be the smallest $k' \leq \hat{k}-1$ where this stopping condition is met; the equilibrium then consists of k^*+1 strategic majority voters (if $y^v(k^*)$ is defined), with the remaining majority members voting sincerely. The case $\hat{k} > k$ is treated analogously, following the same emptying/filling procedure as in the second subcase of Proposition 3 (iv), while carefully accounting at each step for whether $y^v(k)$ is defined. Thus all cases are covered by adapting the reasoning of Proposition 3 (iv) to accommodate the possible undefinedness of $y^v(k)$. This completes the proof of the proposition. ∎

Proof of Proposition 5. We focus on the situation where the first project is accepted in the first round, that is $a_1 = 1$. The same conclusion still holds when $a_1 = 0$, and the comparison between the two voting procedures remains unchanged. We note that the majority members, together with the minority, are indexed from 1 to $\frac{w-1}{2}$. Consequently, the first $\frac{w-1}{2}$ individuals retain their full voting weight in the second round, with a winning probability of $P_1^\alpha(\frac{w-1}{2}, \frac{w+1}{2})$ in the second round. In contrast, the remaining $\frac{w+1}{2}$ individuals have a α-weighted vote in the second ballot and therefore a winning probability of $P_2^\alpha(\frac{w-1}{2}, \frac{w+1}{2})$. Recall that the quota required for acceptance in the second round is defined as $q = \frac{\frac{w-1}{2}+\alpha\frac{w+1}{2}}{2}$. We then obtain

$$\begin{aligned}
W^\alpha = \sum_{i=1}^{\frac{w-1}{2}} \Bigg[& \frac{1}{2} P_1^\alpha\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) \left(z_{i1} + \int_0^1 z_{i2}\, dz_{i2} \right) + \frac{1}{2}\left(1 - P_1^\alpha\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right) z_{i1} \\
& + \frac{1}{2} P_1^\alpha\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) z_{i1} + \frac{1}{2}\left(1 - P_1^\alpha\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right) \left(z_{i1} + \int_{-1}^0 z_{i2}\, dz_{i2} \right) \\
& + \sum_{\substack{j=1 \\ j\neq i}}^{m} \mathbb{E}\left[b(\delta_L + \delta_{j2})(\delta_L + \delta_{j2})\right] \Bigg] + \sum_{i=\frac{w+1}{2}}^{w} \Bigg[\frac{1}{2} P_2^\alpha\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) \left(z_{i1} + \int_0^1 z_{i2}\, dz_{i2} \right) \\
& + \frac{1}{2}\left(1 - P_2^\alpha\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right) z_{i1} + \frac{1}{2} P_2^\alpha\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) z_{i1} \\
& + \frac{1}{2}\left(1 - P_2^\alpha\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right) \left(z_{i1} + \int_{-1}^0 z_{i2}\, dz_{i2} \right) + \sum_{j=1}^{m} \mathbb{E}\left[b(\delta_L + \delta_{j2})(\delta_L + \delta_{j2})\right] \Bigg].
\end{aligned}$$

Our assumptions imply

$$\int_{-1}^0 z_{i2}\, dz_{i2} = -\frac{1}{2}, \qquad \int_0^1 z_{i2}\, dz_{i2} = \frac{1}{2},$$

and

$$\mathbb{E}\left[b(\delta_L + \delta_{j2})(\delta_L + \delta_{j2})\right] = 2\delta_L \left(1 - P_1^\alpha\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right) \quad \text{for all } j = 1, \ldots, m.$$

Then,

$$
\begin{aligned}
W^{\alpha} = \sum_{i=1}^{\frac{w-1}{2}} \Bigg[& \frac{1}{2} P_1^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) \left(z_{i1} + \frac{1}{2}\right) + \frac{1}{2}\left(1 - P_1^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right) z_{i1} \\
& + \frac{1}{2} P_1^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) z_{i1} + \frac{1}{2}\left(1 - P_1^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right) \left(z_{i1} - \frac{1}{2}\right) \\
& + \sum_{\substack{j=1 \\ j \neq i}}^{m} \mathbb{E}\left[b(\delta_L + \delta_{j2})(\delta_L + \delta_{j2})\right] \Bigg] + \sum_{i=\frac{w+1}{2}}^{w} \Bigg[\frac{1}{2} P_2^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) \left(z_{i1} + \frac{1}{2}\right) \\
& + \frac{1}{2}\left(1 - P_2^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right) z_{i1} + \frac{1}{2} P_2^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) z_{i1} \\
& + \frac{1}{2}\left(1 - P_2^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right) \left(z_{i1} - \frac{1}{2}\right) + \sum_{j=1}^{m} \mathbb{E}\left[b(\delta_L + \delta_{j2})(\delta_L + \delta_{j2})\right] \Bigg].
\end{aligned}
$$

In the first sum of W^{α}, from $i = 1$ to $\frac{w-1}{2}$ (weight-1 voters), the material payoff is divided into four cases: (i) having positive z_{i2} with probability $\frac{1}{2}$ and winning with probability $P_1^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 1$); (ii) having positive z_{i2} with probability $\frac{1}{2}$ and losing with probability $1 - P_1^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 0$); (iii) having negative z_{i2} with probability $\frac{1}{2}$ and winning with probability $P_1^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 0$); and finally, (iv) having negative z_{i2} with probability $\frac{1}{2}$ and losing with probability $1 - P_1^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 1$). In the second sum of W^{α}, from $i = \frac{w+1}{2}$ to w (weight-α voters), the material payoff is also divided into four cases: (i) having positive z_{i2} with probability $\frac{1}{2}$ and winning with probability $P_2^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 1$); (ii) having positive z_{i2} with probability $\frac{1}{2}$ and losing with probability $1 - P_2^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 0$); (iii) having negative z_{i2} with probability $\frac{1}{2}$ and winning with probability $P_2^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 0$); and finally, (iv) having negative z_{i2} with probability $\frac{1}{2}$ and losing with probability $1 - P_2^{\alpha}(\frac{w-1}{2}, \frac{w+1}{2})$ (i.e., $a_2 = 1$). That is, the equation can be rearranged to

$$
\begin{aligned}
W^{\alpha} = & \sum_{i=1}^{w} z_{i1} + m \left[\frac{1}{4}\left(2P_1^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) - 1\right) + 2(m-1)\delta_L\left(1 - P_1^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right) \right] \\
& + \left(\frac{w-1}{2} - m\right) \left[\frac{1}{4}\left(2P_1^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) - 1\right) + 2m\delta_L\left(1 - P_1^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right) \right] \\
& + \frac{w+1}{2} \left[\frac{1}{4}\left(2P_2^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) - 1\right) + 2m\delta_L\left(1 - P_1^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right) \right]. \\
= & \sum_{i=1}^{w} z_{i1} + \frac{w-1}{8}\left(2P_1^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) - 1\right) + \frac{w+1}{8}\left(2P_2^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right) - 1\right) \\
& - \frac{2m(w-1)}{w} y \left(1 - P_1^{\alpha}\left(\tfrac{w-1}{2}, \tfrac{w+1}{2}\right)\right).
\end{aligned}
$$

Similarly, the expression for WMV with the parameter α' is given by:

$$\begin{aligned} W^{\alpha'} = & \sum_{i=1}^{w} z_{i1} + \frac{w-1}{8}\Big(2P_1^{\alpha'}\big(\tfrac{w-1}{2}, \tfrac{w+1}{2}\big) - 1\Big) + \frac{w+1}{8}\Big(2P_2^{\alpha'}\big(\tfrac{w-1}{2}, \tfrac{w+1}{2}\big) - 1\Big) \\ & - \frac{2m(w-1)}{w}\, y \left(1 - P_1^{\alpha'}\big(\tfrac{w-1}{2}, \tfrac{w+1}{2}\big)\right). \end{aligned}$$

Using the previously derived expressions for W^{α} and $W^{\alpha'}$, we now consider the difference $W^{\alpha} - W^{\alpha'}$, which can be simplified as follows:

$$\begin{aligned} W^{\alpha} - W^{\alpha'} = & \frac{w-1}{4}\Big(P_1^{\alpha}\Big(\tfrac{w-1}{2}, \tfrac{w+1}{2}\Big) - P_1^{\alpha'}\Big(\tfrac{w-1}{2}, \tfrac{w+1}{2}\Big)\Big) \\ & + \frac{w+1}{4}\Big(P_2^{\alpha}\Big(\tfrac{w-1}{2}, \tfrac{w+1}{2}\Big) - P_2^{\alpha'}\Big(\tfrac{w-1}{2}, \tfrac{w+1}{2}\Big)\Big) \\ & + \frac{2m(w-1)}{w}\, y \left(P_1^{\alpha}\Big(\tfrac{w-1}{2}, \tfrac{w+1}{2}\Big) - P_1^{\alpha'}\Big(\tfrac{w-1}{2}, \tfrac{w+1}{2}\Big)\right). \end{aligned}$$

Rearranging terms, the inequality $W^{\alpha} - W^{\alpha'} > 0$ is clearly equivalent to the condition:

$$y \geq \frac{w\left[(w-1)\left(P_1^{\alpha'} - P_1^{\alpha}\right) + (w+1)\left(P_2^{\alpha'} - P_2^{\alpha}\right)\right]}{8m(w-1)\left(P_1^{\alpha} - P_1^{\alpha'}\right)} \;:=\; y^{*}.$$

■