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Sylvain Béal, Emmanuelle Lebeuf, Kevin Techer

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CRESE
30, avenue de l'Observatoire
25009 Besançon
France
http://crese.univ-fcomte.fr/



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# Allocation rules for network games with local considerations

Sylvain Béal[b], Emmanuelle Lebeuf[a,*], Kevin Techer[b]

[a] *Université Marie et Louis Pasteur, F-25000 Besançon, France*
[b] *Université Marie et Louis Pasteur, CRESE (UR 3190), F-25000 Besançon, France*

**Abstract**

We introduce a new allocation rule for network games that combines a local component and a global component. The local component depends only on the links incident to each player, whereas the global component allocates a surplus equally among the members of each connected component. We characterize this allocation rule by three classical axioms together with a new axiom, Fairness under Neighborhood Restriction, which requires that two adjacent players experience the same payoff variation when the network is restricted to their local neighborhoods, that is, to the sets of links incident to each player. We also examine an alternative allocation rule that differs only in its global component, distributing the surplus within each connected component in proportion to players' degrees in the network.

*Keywords:* Network games, Fairness under Neighborhood Restriction, Neighborhood Equal Surplus Division, axiomatic characterization

*JEL codes:* C71

## 1. Introduction

Network games constitute a class of cooperative games introduced by Jackson and Wolinsky (1996). A network game consists of an undirected graph representing a network, in which nodes correspond to players and links represent bilateral relationships between them, together with a characteristic function that assigns a value to every subnetwork. One of the main advantages of this framework is that the worth generated by a connected coalition of players depends on the way they are connected within the network. This

*Corresponding author

*Email addresses:* `sylvain.beal@umlp.fr` (Sylvain Béal), `emmanuelle.lebeuf@edu.umlp.fr` (Emmanuelle Lebeuf), `kevin.techer@umlp.fr` (Kevin Techer)

*URL:* `https://sites.google.com/site/bealpage/` (Sylvain Béal), `https://sites.google.com/view/kevintecher/research` (Kevin Techer)

contrasts with the framework of Myerson (1977), where a communication graph is added to a classical cooperative game but does not affect the characteristic function.

Most allocation rules proposed for network games build upon the principles underlying the Shapley value (Shapley, 1953) for classical cooperative games and the Myerson value (Myerson, 1977), its extension to cooperative games with a communication graph. Notable examples include the Myerson value for network games introduced by Jackson and Wolinsky (1996), together with its variant based on a flexible efficiency requirement (Jackson, 2005), the position value characterized in Slikker (2007), and the outside-option value introduced by Belau (2016). More egalitarian allocation rules have also been proposed, such as the Egalitarian Rule (Jackson, 2005), which divides the value of the network equally among all players, and the Component-Wise Egalitarian Value (Slikker, 2007), which applies the same principle separately within each connected component.[1] All these allocation rules derive from allocation rules originally developed in other branches of cooperative game theory.

This article pursues two objectives. First, we introduce an allocation rule specifically designed for network games, with no counterpart in either classical cooperative games or cooperative games with a communication graph. Second, we would like this allocation rule to have an egalitarian flavor while simultaneously taking into account the players' local performances. To this end, we introduce the axiom of Fairness under Neighborhood Restriction, which requires that any two adjacent players experience the same payoff variation when the network is restricted to their local environments, namely, for each player, the set of its incident links. This axiom aligns with numerous existing axioms in the literature that rely on payoff differences. The most well-known such axiom is Balanced contributions introduced in Myerson (1977) for cooperative games with a communication graph and called equal bargaining power in Jackson and Wolinsky (1996), which requires that deleting a link yields the same payoff variation for the two adjacent players. Other axioms in this category for network games can be found in Slikker (2007) and Belau (2016), among others. We show that Fairness under Neighborhood Restriction, together with three standard axioms uniquely characterizes a novel allocation rule. Component Efficiency requires that each connected component receives a total payoff equal to the value generated by its links. Two-Player Symmetry requires the two players of a two-player connected component, who are symmetric with respect to both the network and

[1]This literature also includes Borkotokey et al. (2015), which studies another marginalistic value, and Borkotokey et al. (2025), which considers convex combinations between the Myerson value and the Component-Wide Egalitarian Value, among others.

value creation, to receive identical payoffs. Finally, Component Independence requires that the allocation within a connected component be unaffected by links belonging to other components.

We call the resulting allocation rule the Neighborhood Equal Surplus Division rule. For a given network, the corresponding allocation is determined in two steps. First, each player receives a local payoff equal to the value generated by the links in its neighborhood minus one half of the sum of the values generated by each of these links taken individually. Second, within each connected component, each player gets an equal share of what was not yet distributed of the value achieved by all the links in the component. Thus, the allocation naturally decomposes into a local part, reflecting each player's immediate neighborhood, and a global surplus part at the scale of each connected component.

Although the Neighborhood Equal Surplus Division rule is specific to network games, it bears some resemblance to the Equal Surplus Division rule for classical cooperative games. This similarity arises from the aforementioned two-stage structure of the allocation. We further illustrate this connection through an application to a minimal-cost network connection problem, where the Neighborhood Equal Surplus Division Rule closely parallels the Equal Surplus Division rule.

One may argue that the axiom of Fairness under Neighborhood Restriction is not fully equitable because it ignores the relative positions of adjacent players within the network. We therefore introduce a weighted variant, Weighted Fairness under Neighborhood Restriction, which requires payoff variations to be equalized proportionally to the players' degrees. Replacing Fairness under Neighborhood Restriction with its weighted version yields a characterization of the Weighted Neighborhood Equal Surplus Division rule, in which the surplus within each connected component is allocated proportionally to players' degrees. More generally, this result suggests a parametric family of fairness axioms that characterizes a corresponding family of allocation rules. Interestingly, some members of this family are no longer linear in the characteristic function.

The remainder of this article is organized as follows. Section 2 introduces the notation and basic definitions. Section 3 presents the axiomatic analysis and the characterization results. Section 4 applies the proposed allocation rule to the minimal-cost network connection problem. Section 5 concludes. The logical independence of the axioms used in the characterizations is established in the Appendix.

## 2. Notation

Throughout this article, we consider a fixed and finite player set $N = \{1, \ldots, n\}$. A network on $N$ is a set of links $g$, i.e., $g \subseteq \{\{i, j\} \subseteq N, i \neq j\}$. Let $G^N$ be the set of all networks on $N$. To save on notation, we write $ij$ instead of $\{i, j\}$ and $i$ instead of $\{i\}$. If $ij \in g$, we say that $i$ and $j$ are neighbors in $g$. For each coalition $S$ of players, we denote by $g[S]$ the network on $N$ including only the links among players in $S$, that is

$$g[S] = \{ij \in g : i \in S, j \in S\}.$$

For each player $i \in N$, $L_i(g) = \{ij \in g : j \in N \setminus i\}$ stands for the neighborhood of $i$ in $g$. A path between $i$ and $j$ in $g$ is a sequence of players $(i_1, i_2, \ldots, i_k)$ such that $i_1 = i$, $i_k = j$ and, for each $q \in \{1, \ldots, k-1\}$, $i_q i_{q+1} \in g$. Two players are connected in $g$ if there is a path between them. A component of $g$ is a maximally connected set of players. We denote by $N/g$ the set of components induced by $g$, which is a partition of $N$. Let $C_i(g)$ be the unique component in $g$ containing player $i$, $N_i(g) = \{j \in N : ij \in g\}$ the set of $i$'s neighbors in $g$, and $N_i^+(g) = i \cup N_i(g)$. A value function is a function $v : G^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. For each network $g \in G^N$, $v(g)$ is the total value generated by the players of $N$ in the network structure $g$. Let $V^N$ be the set of all value functions. A pair $(g, v) \in G^N \times V^N$ is called a network game. An allocation rule is a function $f : G^N \times V^N \to \mathbb{R}^N$ which assigns to each network game $(g, v)$ an allocation $f(g, v)$ specifying the payoff $f_i(g, v)$ obtained by each player $i \in N$ for their participation in game $(g, v)$.

## 3. Axiomatic study

In this section, we introduce several axioms in order to characterize an allocation rule for network games. We invoke classical properties as well as new ones. We begin with the main new axiom.

Our first axiom highlights the importance of the neighborhood of each player. In a hypothetical situation in which the network is restricted, for each player, to its neighborhood, we impose that the payoff variation compared to the original situation is the same for any pair of neighbors. In other words, any two adjacent players are affected similarly by the restriction of the network to their respective neighborhoods.

**Fairness under neighborhood restriction ($FNR$).** For each $(g, v) \in G^N \times V^N$ and for each $ij \in g$, it holds that

$$f_i(g, v) - f_i(L_i(g), v) = f_j(g, v) - f_j(L_j(g), v).$$

This axiom is one of numerous axioms relying on payoff differences in the literature on cooperative games. As such, it can be compared to Balanced contributions (Myerson, 1977; Jackson and Wolinsky, 1996), which relies on a global approach in the sense that all links but the deleted one are still part of the network. Therefore, the payoff difference for two adjacent players is measured at the global scale of the network. To the contrary, Fairness under neighborhood restriction evaluates the payoff differences at a local scale, since only the neighborhood of each adjacent player is taken into account in the counterfactual situations. Fairness under neighborhood restriction formalizes the notion of embeddedness advanced by Granovetter (1985). Economic action and distributive justice criteria do not unfold in a social vacuum, but are instead anchored in proximate interpersonal relationships. The player's direct neighborhood serves as their social frame of reference, making them less sensitive to structural changes in distant components.

Our second new axiom is called Two-player component symmetry. It requires that two players obtain the same payoff if they belong to a two-player component.

**Two-player component symmetry ($TCS$).** For each $(g, v) \in G^N \times V^N$, and each component $C \in N/g$ with $C = ij$, it holds that

$$f_i(g, v) = f_j(g, v).$$

The interpretation is intuitive. Given that the two players hold identical positions within their pair, and because the remainder of the network affects both players identically (since none of them has a link with any other player), they should receive identical payoffs. This is the minimal requirement which is satisfied by most of the allocation rules in the literature such as the Myerson value, the Position value and the Component-wise egalitarian value. A notable exception is the kappa value in Belau (2016).

The two other axioms are classical and are also satisfied by the Myerson value, the Position value and the Component-wise egalitarian value. Component efficiency requires that the total payoff distributed to the players in a component equals the value generated by the links of this component in the network. This axiom was introduced by Myerson

(1977) and adapted to network games by Jackson and Wolinsky (1996) for component additive value functions.

**Component Efficiency** $(CE)$**.** For each $(g, v) \in G^N \times V^N$, and for each component $C \in N/g$, it holds that

$$\sum_{i \in C} f_i(g, v) = v(g[C]).$$

In this article, we do not impose that value functions are component additive. As a consequence, the total value distributed to all players can be different from the total value of a network if it is not connected. As such, we adopt the interpretations of Myerson (1977) which considers the network as a communication device and believes that communication is necessary for cooperation. Note that, Component efficiency is called Component balanced in Jackson and Wolinsky (1996).

Our final axiom is called Component independence. It states that the payoff of a player should not depend on the links among players that do not belong to its component.

**Component independence** $(CI)$**.** For each $(g, v) \in G^N \times V^N$, for each component $C \in N/g$, and for all $i$ in $C$, it holds that

$$f_i(g, v) = f_i(g[C], v)$$

An axiom with the same name and a similar flavor is used in van den Brink (2009). We can remark that Component independence and Component efficiency are not related to each other. In particular, Component efficiency allows the possibility that the payoff allocation within a component depends on the rest of the network, while this is not possible under Component independence.

In the result below, we prove that there is a single allocation rule satisfying these four axioms.

**Proposition 1.** *The unique allocation rule satisfying Component efficiency (CE), Two-player component symmetry (TCS), Fairness under neighborhood restriction (FNR) and Component independence (CI) is the Neighborhood Equal Surplus Division,* $NESD$*, de-*

*fined for each $g \in G^N \times V^N$ and each player $i \in N$ as:*

$$NESD_i(g,v) = v(L_i(g)) - \sum_{k \in N_i(g)} \frac{v(ik)}{2} + \frac{1}{|C_i(g)|}\left[v(g[C_i(g)]) - \sum_{j \in C_i(g)} \left( v(L_j(g)) - \sum_{k \in N_j(g)} \frac{v(jk)}{2} \right)\right]. \qquad (1)$$

Before proving this proposition, we would like to interpret the definition of the Neighborhood Equal Surplus Division and to make some remarks.

The payoff of a player is composed of two blocks. The first one (first line) is the difference between the value of its neighborhood and half of the sum of the individual value of each of its links. This first term establishes player $i$'s baseline payoff based on the performance of its (immediate) neighborhood. By definition, the value of the neighborhood $v(L_i(g))$ incorporates the value generated by all the local links. However, each link $ik$ is a shared asset, half of which rightfully belongs to neighbor $k$. By deducting this neighbor's share, the first term isolates the net value that goes to player $i$. The second block (second line) guarantees component budget balance after the initial local baseline is distributed. Specifically, the term in brackets computes the residual surplus of the component, defined as the total value generated by all its links minus the sum of the local payoffs of its members. This surplus is then divided equally among all members of the component.

It should be noted that the payoff (1) admits a slightly simplified formulation:

$$NESD_i(g,v) = v(L_i(g)) - \sum_{k \in N_i(g)} \frac{v(ik)}{2} + \frac{1}{|C_i(g)|}\left[v(g[C_i(g)]) - \sum_{j \in C_i(g)} v(L_j(g)) + \sum_{kr \in g[C_i(g)]} v(kr)\right].$$

Furthermore, we would like to point out a property of the $NESD$ value for certain networks, which will be useful on several occasions.

**Remark 1.** Consider a network $g \in G^N$ and a player $i \in N$ such that the connected component $C_i(g)$ induces a star network centered on $i$, i.e., $g[C_i(g)] = L_i(g)$. Then the surplus term of the NESD rule vanishes. To see this, for player $i$, the surplus component according to $NESD_i(g,v)$ is:

$$v(g[C_i(g)]) - \sum_{j \in C_i(g)} \left( v(L_j(g)) - \sum_{k \in N_j(g)} \frac{v(jk)}{2} \right).$$

Given that $L_i(g) = g[C_i(g)]$ and that $L_j(g) = ij$ for any peripheral player $j \in N_i(g)$, separating the payoff of the central player $i$ from those of the peripheral players in the previous expression yields:

$$v(g[C_i(g)]) - \left( v(g[C_i(g)]) - \sum_{k \in C_i(g) \setminus i} \frac{v(ik)}{2} \right) - \sum_{j \in C_i(g) \setminus i} \frac{v(ij)}{2} = 0,$$

as desired. Therefore, in a star component centered at $i$, player $i$'s payoff simplifies to the local base allocation:

$$NESD_i(g, v) = v(L_i(g)) - \sum_{k \in C_i(g) \setminus i} \frac{v(ik)}{2}.$$

Finally, note that this property is obviously not affected by the coefficient $1/|C_i(g)|$ multiplying the surplus term in $NESD_i(g, v)$.

We are now ready to prove Proposition 1.

*Proof.* (Proposition 1)

[Existence] We first show that the $NESD$ rule satisfies the four axioms. It is clear from formula (1) that $NESD$ satisfies Component efficiency and Component independence.

*Fairness under neighborhood restriction..* Consider any $(g, v) \in G^N \times V^N$ and $ij \in g$. By definition, since $i$ and $j$ belong to the same component, the surplus term cancels out by subtraction:

$$NESD_i(g, v) - NESD_j(g, v) = v(L_i(g)) - \sum_{k \in N_i(g)} \frac{v(ik)}{2} - \left( v(L_j(g)) - \sum_{k \in N_j(g)} \frac{v(jk)}{2} \right).$$

Using Remark 1, we get:

$$NESD_i(L_i(g), v) = v(L_i(g)) - \sum_{k \in N_i(g)} \frac{v(ik)}{2}$$

and

$$NESD_j(L_j(g), v) = v(L_j(g)) - \sum_{k \in N_j(g)} \frac{v(jk)}{2}.$$

We conclude that $NESD_i(g, v) - NESD_j(g, v) = NESD_i(L_i(g), v) - NESD_j(L_j(g), v)$, as desired.

*Two-player component symmetry.*. Consider any $(g, v) \in G^N \times V^N$ and a component $C = ij$. Therefore $L_i(g) = L_j(g) = g[C] = ij$. This implies that:

$$NESD_i(g, v) = v(L_i(g)) - \frac{v(ij)}{2} + \frac{1}{2}\left[v(g[C]) - \left(v(L_i(g)) - \frac{v(ij)}{2}\right) - \left(v(L_j(g)) - \frac{v(ji)}{2}\right)\right],$$

which simplifies to:

$$NESD_i(g, v) = v(L_i(g)) - \frac{v(ij)}{2}$$

and finally:

$$NESD_i(g, v) = \frac{1}{2} v(g[C]).$$

Similarly for $j$, we obtain $NESD_j(g, v) = v(g[C])/2$, which implies that the $TCS$ axiom is therefore satisfied.

[Uniqueness] Consider an arbitrary allocation rule $f$ satisfying the four axioms. We prove that $f$ is uniquely determined in any network game $(g, v) \in G^N \times V^N$. So pick any such $(g, v)$. We consider three cases depending on the size of a component $C \in N/g$ of a network.

**Case 1.** $|C| = 1$, i.e. $C = i$ for some $i \in N$. Then,

$$f_i(g, v) \stackrel{\text{CI}}{=} f_i(g[C], v) \stackrel{\text{CE}}{=} v(g[C]) = v(\emptyset) = 0 = NESD_i(g, v).$$

**Case 2.** $|C| = 2$, i.e. $C = ij$ for a pair of distinct players $ij$. Then, $TCS$ and $CE$ imply that,

$$f_i(g, v) = f_j(g, v) = \frac{v(ij)}{2} = \text{NESD}_i(g, v). \tag{2}$$

**Case 3.** Consider now $|C| \geq 3$. We distinguish two subcases depending on the structure of the component.

**Subcase 3.1.** Assume that $g[C]$ is a star. Without loss of generality, we denote by $i$ the player at the center of the star. Since $g[C] = \{ij : j \in C \setminus i\}$, $FNR$ implies, for each $j \in C \setminus i$, that:

$$f_i(g, v) - f_i(L_i(g), v) = f_j(g, v) - f_j(L_j(g), v). \tag{3}$$

From $CI$, for each $j \in C \setminus i$, we have:

$$f_j(g, v) = f_j(g[C], v), \tag{4}$$

For player $i$, note that $g[C] = L_i(g)$, which implies: $f_i(g, v) \stackrel{\text{CI}}{=} f_i(g[C], v) = f_i(L_i(g), v)$. Combined with (3), we get, for each $j \in C \setminus i$,

$$0 = f_j(g, v) - f_j(L_j(g), v). \tag{5}$$

For each player $j \in C \setminus i$, $L_j(g) = ij$. Together with (2) and (5) we get:

$$f_j(g,v) \overset{(5)}{=} f_j(L_j(g),v) = f_j(ij,v) \overset{(2)}{=} \frac{v(ij)}{2} = NESD_j(g,v).$$

Using the previous equalities, $CI$ and $CE$ we can write that:

$$f_i(g,v) \overset{\text{CI}}{=} f_i(g[C],v) \overset{\text{CE}}{=} v(g[C]) - \sum_{j \in N_i(g)} \frac{v(ij)}{2} \overset{\text{Remark 1}}{=} NESD_i(g,v).$$

This concludes Subcase 1.

**Subcase 3.2.** Assume that $g[C]$ is not a star. For each $i \in C$, remark that $L_i(g) \subsetneq g[C]$. For each link $ij \in g[C]$, $FNR$ yields:

$$f_i(g,v) - f_j(g,v) = f_i(L_i(g),v) - f_j(L_j(g),v). \tag{6}$$

Since $f$ and $NESD$ satisfy $FNR$, (6) can therefore be rewritten as:

$$\begin{aligned} f_i(g,v) - f_j(g,v) &\overset{FNR}{=} f_i(L_i(g),v) - f_j(L_j(g),v) \\ &= NESD_i(L_i(g),v) - NESD_j(L_j(g),v) \\ &\overset{FNR}{=} NESD_i(g,v) - NESD_j(g,v), \end{aligned} \tag{7}$$

where the second equality comes from the fact that $f_i(L_i(g),v) = NESD_i(L_i(g),v)$ for each $i \in C$ due to Subcase 3.1 (or Case 2 if $|N_i(g)| = 1$). Let us denote by $e^{(ij)}$ equality (7) in which $i$ and $j$ are on the left and right of the payoff differences respectively. For each $k \in N_j(g) \setminus i$, summing $e^{(ij)}$ et $e^{(jk)}$ implies that:

$$f_i(g,v) - f_k(g,v) = NESD_i(g,v) - NESD_k(g,v).$$

Replicating the previous step, for each player $i \in C$ and each player $j \in C \setminus i$, we obtain equality (7) for all pairs of distinct players in $C$, including those that are not linked to each other in the network. Summing on all players of $C$ except $i$ yields:

$$\sum_{j \in C \setminus i} e^{(ij)} \iff \sum_{j \in C \setminus i} \big(f_i(g,v) - f_j(g,v)\big) = \sum_{j \in C \setminus i} \big(NESD_i(g,v) - NESD_j(g,v)\big),$$

or equivalently

$$\sum_{j \in C} (f_i(g,v) - f_j(g,v)) = \sum_{j \in C} (NESD_i(g,v) - NESD_j(g,v)).$$

Using $CE$ for $f$ and $NESD$, the previous equality becomes:

$$|C| f_i(g,v) - v(g[C]) = |C| NESD_i(g,v) - v(g[C]).$$

Thus, $f_i(g,v) = NESD_i(g,v)$. Since player $i$ has been chosen arbitrarily in $C$ we proved that $f_i(g,v) = NESD_i(g,v)$ for each $i \in C$. The proof is complete. $\square$

The logical independence of the axioms invoked in Proposition 1 is relegated to the appendix.

In the rest of this section, we focus on a weighted version of Fairness under neighborhood restriction. The idea is to implement the same principle but instead of an equal payoff variation, the axiom below requires the payoff variations to be proportional to the degree of the incident players.

**Weighted fairness under neighborhood restriction ($WNFR$).** For each $(g, v) \in G^N \times V^N$ and each $ij \in g$ :

$$\frac{f_i(g,v) - f_i(L_i(g),v)}{|L_i(g)|} = \frac{f_j(g,v) - f_j(L_j(g),v)}{|L_j(g)|}. \tag{8}$$

This axiom implies that two neighbors pool the changes at the global scale of a network while reflecting the local relative centrality of each of them in their neighborhood. Replacing Fairness under neighborhood restriction by this weighted variant in Proposition 1 leads to the characterization of a new allocation rule.

**Proposition 2.** *The unique allocation rule satisfying Component efficiency (CE), Two-player component symmetry (TCS), Weighted fairness under neighborhood restriction (WFNR) and Component independence (CI) is the Weighted Neighborhood Equal Surplus Division, $WNESD$, defined for each $g \in G^N \times V^N$ and each player $i \in N$ as:*

$$\begin{aligned} WNESD_i(g,v) &= v(L_i(g)) - \sum_{k \in N_i(g)} \frac{v(ik)}{2} \\ &\quad + \frac{|L_i(g)|}{2|g[C_i(g)]|}\left[v(g[C_i(g)]) - \sum_{j \in C_i(g)} \left( v(L_j(g)) - \sum_{k \in N_j(g)} \frac{v(jk)}{2} \right)\right]. \end{aligned} \tag{9}$$

The only difference between $WNESD_i(g, v)$ and $NESD_i(g, v)$ is that the surplus is now shared within the component of player $i$ in proportion to the degrees of the component members.

*Proof.* [Existence] We first show that the $WNESD$ rule satisfies the four axioms. It is clear from (9) that $WNESD$ satisfies $CE$ and $CI$. Regarding $WFNR$, consider any $ij \in g$. We have to show that:

$$\frac{WNESD_i(g,v)}{|L_i(g)|} - \frac{WNESD_j(g,v)}{|L_j(g)|} = \frac{WNESD_i(L_i(g),v)}{|L_i(g)|} - \frac{WNESD_j(L_j(g),v)}{|L_j(g)|}.$$

For player $i$ in $(g, v)$ we have:

$$\frac{WNESD_i(g,v)}{|L_i(g)|} = \frac{v(L_i(g)) - \sum_{k\in N_i(g)} \frac{v(ik)}{2}}{|L_i(g)|} + \frac{1}{2|g[C]|}\left[v(g[C_i(g)]) - \sum_{j\in C_i(g)}\left(v(L_j(g)) - \sum_{k\in N_j(g)} \frac{v(jk)}{2}\right)\right].$$

A similar expression is obtained for $j$, which highlights that both surpluses are associated with the same coefficient. Hence, the two surpluses cancel when we compute the difference.

$$\frac{WNESD_i(g,v)}{|L_i(g)|} - \frac{WNESD_j(g,v)}{|L_j(g)|} = \frac{v(L_i(g)) - \sum_{k\in N_i(g)} \frac{v(ik)}{2}}{|L_i(g)|} - \frac{v(L_j(g)) - \sum_{k\in N_j(g)} \frac{v(jk)}{2}}{|L_j(g)|}. \tag{10}$$

Applying once again Remark 1, we get

$$WNESD_i(L_i(g), v) = v(L_i(g)) - \sum_{k\in N_i(g)} \frac{v(ik)}{2},$$

and

$$WNESD_j(L_j(g), v) = v(L_j(g)) - \sum_{k\in N_j(g)} \frac{v(jk)}{2},$$

which implies that the difference $WNESD_i(L_i(g), v) - WNESD_j(L_j(g), v)$ coincides with (10) and proves that $WNESD$ satisfies $WFNR$.

Regarding $TCS$, the argument are similar to those developped for $NESD$. For a two-player component $C = ij$, we have that $|L_i(g)| = |L_j(g)| = 1$. Hence, the only change is to observe that:

$$\frac{|L_i(g)|}{2|g[C]|} = \frac{|L_j(g)|}{2|g[C]|} = \frac{1}{2}.$$

Thus, $WNESD$ satisfies TCS.

[Uniqueness] Consider an arbitrary allocation rule $f$ satisfying the four axioms. We prove that $f$ is uniquely determined in any network game $(g, v) \in G^N \times V^N$ by considering the same three cases as for the proof of Proposition 1. Furthermore, cases 1 and 2 are identical since they do not necessitate $FNR$ or $WFNR$.

**Case 3.** Consider $|C| \geq 3$. We distinguish two subcases depending on the structure of the component.

**Subcase 3.1.** Assume that $g[C]$ is a star. Without loss of generality, we denote by $i$ the player at the center of the star. Since $g[C] = \{ij; j \in C \setminus i\}$, $WFNR$ implies for each $j \in C \setminus i$ that:

$$\frac{f_i(g,v) - f_i(L_i(g),v)}{|L_i(g)|} = \frac{f_j(g,v) - f_j(L_j(g),v)}{|L_j(g)|}. \tag{11}$$

As in the proof of Proposition 1, $L_i(g) = g[C]$ implies that

$$f_i(g,v) \overset{\text{CI}}{=} f_i(g[C],v) = f_i(L_i(g),v).$$

This means that the left-hand side of (11) is null. As a consequence the same steps as in the proof of Proposition 1 implies that for each $j \in C \setminus i$:

$$f_j(g) = \frac{v(ij)}{2} = WNESD_j(g,v),$$

and:

$$f_i(g,v) = v(L_i(g)) - \sum_{k \in N_i(g)} \frac{v(ik)}{2} = WNESD_i(g,v),$$

where the last equality comes from the fact that Remark 1 also applies to $WNESD$.
**Subcase 3.2.** Assume that $g[C]$ is not a star. The steps are similar to the corresponding subcase to the proof of Proposition 1. Combining $WFNR$ and Subcase 3.1, we get for each $ij \in g[C]$:

$$\frac{f_i(g,v) - WNESD_i(L_i(g),v)}{|L_i(g)|} = \frac{f_j(g,v) - WNESD_j(L_j(g),v)}{|L_j(g)|}.$$

This equality can be easily extended to any pairs of distinct players in $C$. Summing on the population of $C$, and using $CE$ for both $f$ and $WNESD$, we obtain that $f_i(g,v) = WNESD_i(g,v)$ for each $i \in C$. This concludes the proof. □

## 4. Network connection problems

In this section, we illustrate the form of our allocation rules in the context of a minimal network cost connection problem. Such a problem involves a cost vector $c$ such that, for each $ij \subseteq N$, $c(ij)$ denotes the cost of establishing a connection between players $i$ and $j$. For a set of players (or nodes) $N$, a pair $(g,c)$, where $g$ is a network on $N$ and $c$ is a cost vector, is called a minimal cost network connection problem. The objective is to maintain connectivity among the players in $N$ as depicted by $g$ at minimum cost. This requires to avoid considering the most costly links that are not necessary to connectivity. Since $g$ need not be connected, connectivity is required only within each connected component induced by $g$. Formally, we seek a subnetwork $g' \subseteq g$ such that $N/g' = N/g$ (connectivity preservation) and that minimizes

$$\sum_{ij \in g'} c(ij).$$

We refer to such a network $g'$ as an optimal network. This problem is closely related to the well-known minimum cost spanning tree problem (see Bergantiños and Vidal-Puga, 2021), since the resulting optimal network is a forest on the original set of connected components. It differs, however, in that there is no distinguished source node to which all other nodes must be connected.

Every minimal cost network connection problem induces a network game $(g, v_c)$ by applying the above optimization to every subnetwork of $g$. Formally, for each $g' \subseteq g$,

$$v_c(g') = \min_{g'' \subseteq g' : N/g'' = N/g'} \sum_{ij \in g''} c(ij).$$

To the best of our knowledge, this is the first work to model network connection problems using network games. In contrast, the existing literature on minimum cost spanning tree problems has relied exclusively on the classical cooperative game framework.

Applying $NESD$ to the network game $(g, v_c)$ yields a particularly intuitive allocation, obtained in two steps. First, each player pays half of the cost of each of its incident links. Second, within each connected component, the cost savings, i.e., the total cost of the links that are not included in the optimal network, are refunded equally among the members of the component.[2] More precisely, for each connected component $C \in N/g$, let $g^*(C)$ denote an optimal set of links connecting the players in $C$. Then, for every $(g, v_c)$, every $C \in N/g$, and every $i \in C$,

$$NESD_i(g, v_c) = \sum_{j \in N_i(g)} \frac{c(ij)}{2} - \frac{1}{|C|} \sum_{jk \in g[C] \setminus g^*(C)} c(jk). \tag{12}$$

Replacing $NESD$ with $WNESD$ only changes the proportions according to which the cost savings are redistributed among the players.

To see how (12) is obtained, it is enough to remark that for each $i \in N$,

$$v_c(L_i(g)) - \sum_{j \in N_i(g)} \frac{v_c(ij)}{2} = \sum_{j \in N_i(g)} c(ij) - \sum_{j \in N_i(g)} \frac{c(ij)}{2} = \sum_{j \in N_i(g)} \frac{c(ij)}{2},$$

since $L_i(g)$ is a cycle-free network in which preserving connectivity requires to keep all existing links.

[2] Although the optimal network need not be unique, the resulting allocation is independent of the particular optimal network selected.

## 5. Conclusion

As concluding remarks, we allude to a generalization of our approach. More specifically, $NESD$ and $WNESD$ are two instances of a more general family of allocation rules. All the members of this family share to the players their local payoffs and differ on how to split the surplus attributed to the players in a component. More formally, we consider a weight function $\omega$ which assigns to each network game $(g, v)$ and each player $i \in N$ a strictly positive weight $\omega(g, v, i)$. For a network game $(g, v)$ and a player $i$, the payoff associated to each weight function is:

$$NESD_i^{\omega}(g,v) = v(L_i(g)) - \sum_{k \in N_i(g)} \frac{v(ik)}{2}$$
$$+ \frac{\omega(g,v,i)}{\sum_{j \in C_i(g)} \omega(g,v,j)} \left[ v(g[C_i(g)]) - \sum_{j \in C_i(g)} \left( v(L_j(g)) - \sum_{k \in N_j(g)} \frac{v(jk)}{2} \right) \right].$$

Propositions 1 and 2 can be adapted to characterize this larger class of weighted allocation rules by modifying $FNR$ and $WFNR$ into $\omega$-$FNR$ axiom. The only additional constraint is to require that $\omega(g, v, i) = \omega(g, v, j)$ for any two-player component $ij$, in order that $NESD^{\omega}$ satisfies $TCS$. This is an inherent property of almost all network centrality measures. Two examples are the following. For each network $g$ and each player $i$, let $\omega(g, v, i)$ be the Katz centrality of player $i$ in network $g$ (Katz, 1953). In this case, $NESD^{\omega}$ remains linear in the characteristic function as it is independent of $v$. The second example is to set $\omega(g, v, i) = v(L_i(g))$. In this second case, $NESD^{\omega}$ is no longer linear in the characteristic function $v$. Two others allocation rules belonging to this family are exhibited in the Appendix to show the logical independence of our axioms.

## 6. Appendix

*Logical Independence of the Axioms*

To demonstrate that our axiomatic systems are non-redundant, we establish the logical independence of the axioms invoked in each proposition, by a series of counterexamples.

The Component-Wise Egalitarian Value $f^0$, defined for each for each network game $(g, v) \in G^N \times V^N$ and for each player $i \in N$, as:

$$f_i^0(g,v) = \frac{v(g[C_i(g)])}{|C_i(g)|},$$

satisfies:

- Component efficiency,
- Component independence,
- Two-player component symmetry.

However, it satisfies neither Fairness under neighborhood restriction nor Weighted fairness under neighborhood restriction.

Let $\gamma \in \mathbb{R}^N_{++}$. The value $f^1$, defined for each network game $(g, v) \in G^N \times V^N$ and for each player $i \in N$, as:

$$f^1_i(g,v) = v(L_i(g)) - \sum_{k \in N_i(g)} \frac{\gamma_k}{\gamma_i + \gamma_k} v(ik) \\ + \frac{1}{|C_i(g)|}\left[v(g[C_i(g)]) - \sum_{j \in C_i(g)} \left(v(L_j(g)) - \sum_{k \in N_j(g)} \frac{\gamma_k}{\gamma_j + \gamma_k} v(jk)\right)\right],$$

satisfies:

- Component efficiency,
- Component independence,
- Fairness under neighborhood restriction.

However, it does not satisfy Two-player component symmetry.

Let $\delta \in \mathbb{R}^N_{+}$. The value $f^2$, defined for each network game $(g, v) \in G^N \times V^N$ and for each player $i \in N$, as:

$$f^2_i(g,v) = v(L_i(g)) - \sum_{k \in N_i(g)} \frac{\delta_k}{\delta_i + \delta_k} v(ik) \\ + \frac{|N_i(g)|}{2|g[C_i(g)]|}\left[v(g[C_i(g)]) - \sum_{j \in C_i(g)} \left(v(L_j(g)) - \sum_{k \in N_j(g)} \frac{\delta_k}{\delta_j + \delta_k} v(jk)\right)\right],$$

satisfies:

- Component efficiency,
- Component independence,
- Weighted fairness under neighborhood restriction.

However, it does not satisfies Two-player component symmetry.

The value $f^3$, defined for each network game $(g, v) \in G^N \times V^N$ and for each player $i \in N$, as:

$$f_i^3(g, v) = 0,$$

satisfies:

- Component independence,
- Two-player component symmetry,
- Fairness under neighborhood restriction and Weighted fairness under neighborhood restriction.

However, it does not satisfies Component efficiency.

For each network $g \in G^N$, let $p(g) \in \mathbb{R}_+^N$ be a weight system defined as:

- $p_i(g) = \frac{1}{|C_i(g)|}$ if $g[C_i(g)]$ is not a star or if $|g[C_i(g)]| = 1$,
- $p_i(g) = \frac{|N/g|+1}{|N/g|+|C_i(g)|}$ if $i$ is the center of a star with more than two players,
- $p_i(g) = \frac{1}{|N/g|+|C_i(g)|}$ if $i$ is not the center of a star with more than two players.

The value $f^4$, defined for each network game $(g, v) \in G^N \times V^N$ and for each player $i \in N$, as:

$$f_i^4(g, v) = v(L_i(g)) - \sum_{k \in N_i(g)} \frac{v(ki)}{2} + p_i(g) \left[ v(g[C_i(g)]) - \sum_{j \in C_i(g)} v(L_j(g)) + \sum_{kr \in g[C_i(g)]} v(kr) \right],$$

satisfies:

- Component efficiency,
- Two-player component symmetry,
- Fairness under neighborhood restriction.

However, it does not satisfies Component independence.

For each network $g \in G^N$, let $q(g) \in \mathbb{R}_+^N$ be a weight system defined as:

- $q_i(g) = \frac{|N_i(g)|}{2|g[C]|}$ if $g[C]$ is not a star or if $|g[C_i(g)]| = 1$,
- $q_i(g) = \frac{|N/g|+1}{|N/g|+|C|}$ if $i$ is the center of a star with more than two players,
- $q_i(g) = \frac{1}{|N/g|+|C|}$ if $i$ is not the center of a star with more than two players.

The value $f^5$, defined for each network game $(g, v) \in G^N \times V^N$ and for each player $i \in N$, as:

$$f_i^5(g, v) = v(L_i(g)) - \sum_{k \in N_i(g)} \frac{v(ki)}{2} + q_i(g) \left[ v(g[C_i(g)]) - \sum_{j \in C_i(g)} v(L_j(g)) + \sum_{kr \in g[C_i(g)]} v(kr) \right],$$

satisfies:

- Component efficiency,
- Two-player component symmetry,
- Weighted fairness under neighborhood restriction.

However, it does not satisfies Component independence.

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