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Abstract

When voters rank all alternatives from best to worst, an alternative is said to be socially acceptable if the number of voters who rank it among their most preferred half of the alternatives is at least as large as the number of voters who rank it among their least preferred half of the alternatives (Mahajne and Volij, 2018, *Social Choice and Welfare* 51:223-233). It has also been established that there exists a unique scoring rule that always selects a subset of socially acceptable alternatives. This notion of social acceptability has since been extensively studied in the social choice literature. We study a generalized version of this principle based on asymmetric thresholds: for integers p and q such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$, an alternative is socially p/q -acceptable whenever the number of voters who rank it among their top p positions is at least as large as the number of voters who rank it among their bottom q positions. We provide a complete characterization of the scoring rules that always select socially p/q -acceptable alternatives. We show that such a rule exists if

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and only if $p \geq q$, and that this rule is unique when $p = q$. For any pair (p, q) , we further give a complete characterization of every admissible scoring rule, together with a monotonicity property describing how the admissible class varies with p and q . Finally, we apply our results to several classical scoring rules, including the Borda, plurality, antiplurality, and k -approval rules, identifying the conditions under which they satisfy the proposed acceptability property.

Keywords: Voting, scoring rules, social acceptability, asymmetric thresholds.

JEL classification: D71, D72.

1 Introduction

How should we identify the “best” alternative¹ in an election? This question is one of the central concerns of social choice theory and has given rise to a vast literature devoted to the analysis and design of voting rules (Arrow, 1951). When voters rank all alternatives from best to worst, several notions of a “best” alternative naturally emerge from the literature. A first and simple idea is the *plurality winner*, namely the alternative that receives the largest number of first-place votes. A more demanding concept is that of the *Condorcet winner* (Condorcet, 1785), defined as an alternative that defeats every other alternative in pairwise majority comparisons. In contrast with the plurality winner, the Condorcet winner takes into account the relative comparison between alternatives across the entire electorate and is often viewed as a natural majoritarian benchmark. These notions play a fundamental role in the study of collective decision-making, as they serve as reference points for the evaluation and construction of voting rules. More generally, social choice theory proposes many competing criteria intended to capture different interpretations of what it means for an alternative to be socially desirable or collectively acceptable.

This paper focuses on the notion of *social acceptability* introduced by Mahajne and Volij (2018). When voters rank all alternatives from best to worst, an alternative is said to be socially acceptable if the number of voters who place it in the top half of their ranking is at least as large as the number of voters who place it in the bottom half. This criterion distinguishes between alternatives that are broadly tolerated by the electorate and alternatives that generate substantial rejection. An alternative may therefore be socially acceptable even without being ranked first by many voters, provided that it is not placed near the bottom by too many individuals. Conversely, an alternative that receives strong support from a minority but is ranked very low by a large fraction of voters may fail to

¹We use the term “alternative” throughout the paper for consistency, although the term “candidate” or other terms could be used interchangeably in the electoral context. Likewise, we use the neutral pronoun “it” when referring to an alternative.

be socially acceptable. Unlike the plurality criterion or the Condorcet criterion, social acceptability evaluates the balance between favorable and unfavorable positions in voters' rankings and therefore favors alternatives with broad support and limited opposition within the electorate.

Several recent contributions have studied voting rules related to the notion of social acceptability. In particular, [Mahajne and Volij \(2018\)](#) show that among all scoring rules, the unique rule that always selects only socially acceptable alternatives is the half accepted-half rejected (HAHR) rule. Under this rule, each voter assigns $+1$ to the alternatives ranked in the top half of her preference ordering and -1 to those ranked in the bottom half; when the number of alternatives is odd, the median-ranked alternative receives a score of 0. Alternatives with non-negative aggregate scores are then selected.² This result provides a normative justification for the HAHR rule and shows how the notion of social acceptability can be implemented through a simple voting procedure based on partial preference information.

It is worth noting that the HAHR rule is also closely related to several approval and disapproval voting procedures studied in the recent literature. These include the 1-best-1-worst rule of [García-Lapresta et al. \(2010\)](#), in which voters indicate only their most and least preferred alternatives, the disapproval rule of [Alcantud and Laruelle \(2014\)](#), where voters specify approved and disapproved alternatives, and the single-approval multiple-rejection (SAMR) rules introduced by [Baharad and Nitzan \(2016\)](#). More generally, approval and disapproval procedures based on partial evaluations have been comparatively analyzed in [Gonzalez et al. \(2019\)](#). Some of these procedures belong to the broader family of flexible scoring rules introduced by [Brams and Fishburn \(1978\)](#), in which voters are not required to report complete preference rankings. Although these rules differ in their formal structure, they all aim at selecting alternatives that are broadly acceptable while avoiding alternatives that face substantial opposition within the electorate.

Building on the notion of social acceptability introduced in earlier work, [Mahajne and Volij \(2019\)](#) study the relationship between the Condorcet winner and socially acceptable alternatives. They show that a Condorcet winner need not be socially acceptable in general by exhibiting preference profiles in which a Condorcet winner is ranked in the bottom half by a majority of voters. However, they prove that under several classical preference restrictions - including single-peaked preferences, single-dipped preferences, and the single-crossing property - every Condorcet winner is socially acceptable. They further introduce the notion of regular preference domains, namely domains in which every Condorcet winner

²Strictly speaking, HAHR is therefore not a scoring rule in the sense of the argmax-based rules f_s studied in this paper: rather than selecting the alternative(s) with maximal score, HAHR selects every alternative whose score is non-negative.

is socially acceptable whenever it exists, and characterize maximal families of preferences satisfying this property. More generally, they investigate the relationship between social acceptability and q -Condorcet winners,³ showing in particular that a q -Condorcet winner may fail to be socially acceptable unless the threshold q is sufficiently large or preferences belong to suitable restricted domains.

Diss et al. (2023) study the relationship between the majoritarian compromise rule introduced by Sertel (1986) and further analyzed by Sertel and Yilmaz (1999), and the principle of social acceptability. They show that the majoritarian compromise rule always selects a socially acceptable alternative when the number of alternatives is even, and they provide a necessary and sufficient condition for this property to hold when the number of alternatives is odd. They further prove that under several classical domain restrictions - including single-peaked, single-caved, and single-crossing preferences - every majoritarian compromise winner is socially acceptable regardless of the numbers of voters and alternatives.

The notion of social acceptability has also motivated recent probabilistic analyses of voting rules under well-known assumptions on the distribution of individual preferences. In particular, Awde et al. (2022) study the likelihood that some classical voting procedures elect socially unacceptable alternatives under the Impartial Culture and the Impartial Anonymous Culture assumptions. Focusing on the plurality rule, the antiplurality rule, the Borda rule, and their runoff versions, they show that the propensity to elect socially unacceptable alternatives strongly depends on the voting rule under consideration.

Diss and Mahajne (2020) extend the concept of social acceptability to multi-winner elections by studying the social acceptability of sets of elected alternatives, called committees. Focusing on Condorcet committees - namely committees in which every elected member defeats every non-member in pairwise majority comparisons - they show that such committees may nevertheless be partly or even completely socially unacceptable in general. They further prove that stronger positive results can be obtained under restricted preference domains such as single-peaked and single-caved preferences, where Condorcet committees satisfy stronger forms of social acceptability. They also provide probabilistic evaluations under the Impartial Anonymous Culture assumption, showing that Condorcet committees can be significantly exposed to social unacceptability in general settings.

In this paper, we propose a generalization of the notion of social acceptability. Existing approaches rely on a symmetric comparison between the top half and the bottom half of voters' rankings. Such a framework implicitly assumes that positive and negative evaluations should be treated symmetrically, in the sense that support and rejection are assessed

³A q -Condorcet winner is an alternative that is preferred to each of the other alternatives in pairwise majority comparisons by a fraction q of the total number of voters, with $\frac{1}{2} \leq q < 1$.

using thresholds of identical size. However, many collective decision-making situations involve asymmetric concerns. In some contexts, avoiding strongly rejected alternatives may be viewed as more important than rewarding highly supported ones. In other settings, an alternative may be considered acceptable only if it enjoys sufficiently broad support, even when opposition remains limited. More generally, voters and institutions may attach different importance to acceptance and rejection, making a symmetric treatment too restrictive to capture the diversity of collective choice situations. To account for such asymmetries, we introduce two independent thresholds, p and q , and define an alternative to be socially p/q -acceptable whenever the number of voters who rank the alternative among their top p positions is at least as large as the number of voters who rank the alternative among their bottom q positions. This generalized framework makes it possible to capture a broader range of normative requirements and to distinguish between different notions of electoral legitimacy. It also encompasses the standard notion of social acceptability as the special case $p = q = \lfloor \frac{m}{2} \rfloor$.

Our main contribution is to provide a complete characterization of the scoring rules that always select socially p/q -acceptable alternatives and identify the precise conditions under which asymmetric acceptability requirements can be satisfied. We show that the class of admissible scoring rules for a given pair (p, q) is non-empty if and only if $p \geq q$, and that it reduces to a single rule when $p = q$, namely the rule that assigns score 1 to the top p positions, $\frac{1}{2}$ to the intermediate positions, and 0 to the bottom p positions. We also establish a monotonicity property showing how the admissible class grows as the support threshold increases or the rejection threshold decreases. We then apply our characterization to several classical voting rules - including the Borda, plurality, antiplurality, and k -approval rules - and determine exactly for which values of p and q these rules always select socially p/q -acceptable alternatives. Our results show that introducing asymmetric thresholds substantially enriches the analysis of acceptability and reveals important differences between scoring rules in the way they balance electoral support and rejection.

The remainder of this paper is structured as follows. Section 2 presents the formal framework, including the definition of social acceptability and the key concepts used throughout the analysis. Section 3 states our main theoretical results and applies them to several well-known scoring rules. Section 4 offers concluding remarks and discusses possible directions for future research. The proofs are provided in the Appendix.

2 Basic framework

Consider a non-empty set $A = \{a_1, \dots, a_m\}$ of m alternatives and a non-empty set $N = \{1, \dots, n\}$ of n voters, indexed by j , with $m \geq 3$ and $n \geq 2$. Each voter is assumed to

have a linear order $\succ$ on A , ranking all alternatives from most desirable to least desirable. A preference profile can be represented by a matrix $R = (r_{ij})$, where $1 \leq i \leq m$ and $1 \leq j \leq n$; rows correspond to alternatives and columns to voters. Each entry r_{ij} belongs to $\{1, \dots, m\}$ and indicates that alternative a_i is ranked at position $\alpha = r_{ij}$ by voter j . For simplicity of notation, alternatives may occasionally be denoted by a, b, c , etc. A social choice rule maps each preference profile to a non-empty subset of A .

Scoring rules are defined by a score vector $s = (s_1, s_2, \dots, s_m)$, where the scores satisfy $s_1 \geq s_2 \geq \dots \geq s_m$ and $s_1 > s_m$. Under such rules, each voter assigns points to alternatives according to their positions in the ranking, and alternatives are ranked by the total scores they receive across all voters. Well-known examples include the plurality rule with score vector $(1, 0, \dots, 0)$, the antiplurality rule with score vector $(1, \dots, 1, 0)$, the Borda rule with score vector $(m-1, m-2, \dots, 1, 0)$, and the k -approval rule with score vector $(\underbrace{1, \dots, 1}_{k \text{ times}}, \underbrace{0, \dots, 0}_{m-k \text{ times}})$.

Throughout the paper, we focus on normalized score vectors, that is, sequences satisfying $s_1 \geq s_2 \geq \dots \geq s_m$, $s_1 = 1$ and $s_m = 0$. For a score vector s , we denote by $\Delta_s = \sum_{i=1}^m s_i$ the total sum of scores. We also use the notation f_s to denote any scoring rule with score vector $s = (s_1, \dots, s_m)$, defined by $f_s = \arg \max_{a \in A} \text{Score}(a)$, where $\text{Score}(a)$ denotes the total score assigned to alternative a under the scoring rule associated with s .

We now introduce the notions of social acceptability and generalized social p/q -acceptability that constitute the basis of our analysis. As noted in the introduction, an alternative is socially acceptable if the number of voters who rank it in the first half of their preference ordering is at least as large as the number of voters who rank it in the second half of their preference ordering, and an alternative is socially p/q -acceptable if the number of voters who rank it among their first p positions is at least as large as the number of voters who rank it among their last q positions. Formally, we propose the following two definitions, where $\lfloor z \rfloor$ denotes the integer part of z and $|Z|$ denotes the cardinality of a finite set Z :

Definition 1. (*Mahajne and Volij, 2018*) An alternative $a_i \in A$ is socially acceptable if

$$\left| \left\{ j \in N : r_{ij} \leq \left\lfloor \frac{m}{2} \right\rfloor \right\} \right| \geq \left| \left\{ j \in N : r_{ij} > m - \left\lfloor \frac{m}{2} \right\rfloor \right\} \right|.$$

Definition 2. Let (p, q) be a pair of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$. An alternative $a_i \in A$ is socially p/q -acceptable if

$$\left| \left\{ j \in N : r_{ij} \leq p \right\} \right| \geq \left| \left\{ j \in N : r_{ij} > m - q \right\} \right|.$$

Notice that if $p = q$, we simply speak of social p -acceptability. In addition, if $p = q = \lfloor \frac{m}{2} \rfloor$, then social p/q -acceptability coincides with the standard acceptability requirement. The next example illustrates the previous definitions.

Example 1. Let $m = 5$ and $n = 4$. Consider the following preference profile, where rows correspond to alternatives $a_1, \dots, a_5$ and columns to voters $1, \dots, 4$:

a_1	a_2	a_1	a_1
a_2	a_3	a_4	a_4
a_3	a_5	a_2	a_5
a_4	a_4	a_3	a_2
a_5	a_1	a_5	a_3

For the standard notion of social acceptability (i.e., $p = q = 2$), alternatives a_1 , a_2 , and a_4 are socially acceptable, whereas a_3 and a_5 are not. If we move to the asymmetric thresholds $p = 1$ and $q = 2$, the criterion becomes more demanding on the support side. In this case, alternative a_4 , which was socially acceptable under the symmetric criterion, is no longer socially $1/2$ -acceptable since it is never ranked in first position while it appears twice among the last two positions. Hence only alternatives a_1 and a_2 remain socially $1/2$ -acceptable. This example illustrates how varying the thresholds p and q may substantially modify the set of socially acceptable alternatives.

It is worth emphasizing that, although each preference profile involves a fixed and finite number of voters n , we do not impose any upper bound on the size of the electorate. Our analysis therefore allows the number of voters to be arbitrarily large, while remaining finite for each individual profile. This distinction is important for the characterization results developed in the next section. In particular, establishing the necessity of certain conditions may require constructing preference profiles with a sufficiently large number of voters, whereas the corresponding sufficiency results hold independently of the electorate size, for every $n \geq 2$.

3 Results

This section is divided into two main parts. We begin by studying social p/q -acceptability and deriving necessary and sufficient conditions under which scoring rules always select socially p/q -acceptable alternatives. We then investigate the existence, uniqueness, and monotonicity properties of these rules, before applying our general results to several classical scoring rules, including the Borda, plurality, antiplurality, and k -approval rules. As already mentioned in the Introduction, all proofs are provided in the Appendix.

3.1 General results

We first state the sufficiency result.

Proposition 1. *Let f_s be a scoring rule with score vector $s = (s_1, \dots, s_m)$, and let (p, q) be a pair of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$. The scoring rule f_s always selects socially p/q -acceptable alternatives if the following condition holds:*

$$(C) \quad \begin{cases} s_{p+1} \leq \frac{\Delta_s}{m}, \\ s_{m-q+1} < \frac{\Delta_s}{m}, \\ m s_{m-q+1} \leq 2\Delta_s - m. \end{cases}$$

The conditions are summarized by a set of three inequalities, denoted (C), relating the scores at positions $p+1$ and $m-q+1$ to the average score $\frac{\Delta_s}{m}$. To interpret these inequalities, recall that for a normalized score vector we have $s_1 = 1$ and $s_m = 0$. The first inequality $s_{p+1} \leq \frac{\Delta_s}{m}$ imposes that the weight assigned to the $(p+1)$ -th position (the first position *outside* the top p ranks) does not exceed the average. This controls the contribution of voters who rank an alternative in an intermediate position (from $p+1$ to $m-q$); they cannot artificially inflate the alternative's score. The second inequality $s_{m-q+1} < \frac{\Delta_s}{m}$ is more restrictive: it requires that the first position among the worst q ranks (position $m-q+1$) receives a weight strictly below the average. Consequently, all positions from $m-q+1$ to m (the last q positions) have scores strictly less than the average. Hence a voter who ranks an alternative in these last q positions contributes strictly less than the average score. The third inequality, $m s_{m-q+1} \leq 2\Delta_s - m$ (equivalently $m + m s_{m-q+1} \leq 2\Delta_s$), controls the combined influence of the highest score ($s_1 = 1$) and the best score among the last q positions (s_{m-q+1}). It guarantees that even if an alternative receives many first-place votes together with many votes in the bottom q positions, the total contribution per voter, when aggregated over m alternatives, does not exceed twice the total score sum Δ_s . Together, these conditions ensure that a non-acceptable alternative scores strictly below the average of the other alternatives.

For the special case $p = q = \lfloor \frac{m}{2} \rfloor$, condition (C) is satisfied by the HAHHR rule associated with [Mahajne and Volij \(2018\)](#). Indeed, after normalization, the alternatives ranked in the top half receive score 1, the alternatives ranked in the bottom half receive score 0, and the median-ranked alternative (when m is odd) receives a score of $\frac{1}{2}$. Hence, the average score satisfies $\frac{\Delta_s}{m} = \frac{1}{2}$. Moreover, since the first position outside the top half is either the median position (when m is odd) or the first position in the bottom half (when m is even), we have $s_{p+1} \leq \frac{1}{2}$. Similarly, the first position belonging to the bottom half receives a score

of 0, so that $s_{m-q+1} = 0 < \frac{\Delta_s}{m}$. Finally, the third inequality of condition (C) becomes $m s_{m-q+1} \leq 2 \Delta_s - m$, that is, $0 \leq 0$, which also holds. Therefore, all inequalities in condition (C) are satisfied.⁴

We now turn to the converse direction. To show that condition (C) is also necessary, we need to construct, for each violation of (C), a preference profile in which a non-acceptable alternative wins. In these constructions, we must arrange the preferences of the $m - 1$ alternatives other than the distinguished alternative a so that their total scores are as balanced as possible. For the sequel, we will need the following balancing lemma.

Lemma 1. *Let $s = (s_1, \dots, s_m)$ be a normalized score vector, and let i, j be two indices such that $1 \leq i < j \leq m$. Let α, β, γ be non-negative integers and let $l = \alpha + \beta + \gamma$. Then, there exists a matrix M with $m - 1$ rows and l columns, divided into blocks $M = [M^{(1)}, M^{(2)}, M^{(3)}]$ of sizes α, β, γ respectively (any of which may be empty) such that:*

- *each column in the first block contains exactly one occurrence of each of the m scores except s_1 ,*
- *each column in the second block contains exactly one occurrence of each of the m scores except s_i ,*
- *each column in the third block contains exactly one occurrence of each of the m scores except s_j , and*
- $\Delta(M) = \max_{1 \leq r \leq m-1} R_r(M) - \min_{1 \leq r \leq m-1} R_r(M) \leq 1$, *where for each $r \in \{1, \dots, m-1\}$, $R_r(M)$ is the sum of the entries in row r .*

In the lemma, the matrix M has $m - 1$ rows (representing the $m - 1$ alternatives other than a distinguished alternative a) and $l = \alpha + \beta + \gamma$ columns (each column corresponds to one voter). The columns are divided into three blocks. For every voter in the first block (α voters), alternative a is ranked first; therefore the remaining $m - 1$ scores that the voter distributes are exactly all scores except s_1 , i.e., the multiset $\{s_2, \dots, s_m\}$ appears once in the column. In the second block (β voters), alternative a is placed at a fixed rank i ($1 \leq i < j \leq m$), so the column contains every score except s_i . In the third block (γ voters), alternative a is placed at a lower rank j , so the column contains every score except s_j . The entries in each column can be permuted arbitrarily; the goal is to choose these permutations so that the row sums $R_r(M)$ (the total score of the r -th other alternative)

⁴As discussed in Footnote 2, the verification of condition (C) concerns the argmax-based scoring rule associated with the HAHR score vector, rather than the original HAHR rule, which selects alternatives with non-negative aggregate scores. By Theorem 1, this argmax-based rule always selects socially acceptable alternatives.

are as equal as possible. The lemma guarantees that such permutations exist making the spread $\Delta(M) = \max R_r - \min R_r$ at most 1.

With this balancing lemma at hand, we can now show that condition (C) is not only sufficient but also necessary. More precisely, we assume that condition (C) fails and, for each possible violation, construct a preference profile in which a distinguished alternative a is not socially p/q -acceptable but nevertheless wins the election. In each construction, we will place a in specific positions (depending on which inequality is violated) and use the lemma to arrange the remaining $m - 1$ alternatives so that their total scores differ by at most 1. This ensures that if the score of a is at least equal to $\frac{W}{m-1} + 1$ (where W is the sum of the scores of the other alternatives), then a is a winner. The lemma applies for any choice of indices i and j ; in particular, the symmetric case $p = q$ is covered by taking $i = p + 1$ and $j = m - p + 1$ (and more generally, i and j can be 1, $p + 1$, or $m - q + 1$ as needed in the constructions that follow). The following proposition makes this precise by formally stating the necessity result: if condition (C) is not satisfied, then an alternative that is not socially p/q -acceptable can be a winner.

Proposition 2. *Let f_s be a scoring rule with score vector $s = (s_1, \dots, s_m)$, and let (p, q) be a pair of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$. If condition (C) is not satisfied, then there exists a profile for which an alternative that is not socially p/q -acceptable is a winner.*

Propositions 1 and 2 establish that condition (C) is both sufficient and necessary for a scoring rule to always select socially p/q -acceptable alternatives. Combining these two results yields the following characterization theorem.

Theorem 1. *Let f_s be a scoring rule with score vector $s = (s_1, \dots, s_m)$, and let (p, q) be a pair of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$. The scoring rule f_s always selects socially p/q -acceptable alternatives if and only if condition (C) holds:*

$$(C) \quad \begin{cases} s_{p+1} \leq \frac{\Delta_s}{m}, \\ s_{m-q+1} < \frac{\Delta_s}{m}, \\ m s_{m-q+1} \leq 2 \Delta_s - m. \end{cases}$$

Notice that for the special case $p = q$, the framework becomes symmetric, since the thresholds used to define support and rejection coincide. In this situation, the position $m - q + 1$ is replaced by $m - p + 1$, so that condition (C) depends only on the single parameter p . The three inequalities then describe exactly when a scoring rule always selects alternatives for which the number of voters ranking them among the top p positions is at least as large as the number of voters ranking them among the bottom p positions.

Hence, the theorem provides a complete characterization of the scoring rules that always select socially p -acceptable alternatives, thereby recovering the standard symmetric notion of acceptability as a special case of our more general p/q -framework.

Theorem 1 characterizes, for a given pair (p, q) , the scoring rules that always select socially p/q -acceptable alternatives. A natural question is whether such rules exist at all. For each pair (p, q) , let

$$\mathcal{S}(p, q) = \{s = (s_1, \dots, s_m) : s \text{ is a normalized score vector satisfying condition (C)}\}$$

denote the set of admissible scoring vectors satisfying condition (C). We also write $\bar{s} = \frac{\Delta_s}{m}$ for the average score. Since $s_1 = 1$ and $s_m = 0$, it follows that $0 < \bar{s} < 1$.

Proposition 3. *Let (p, q) be a pair of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$. There exists a normalized score vector s satisfying condition (C) if and only if $p \geq q$. Equivalently, $\mathcal{S}(p, q) \neq \emptyset$ if and only if $p \geq q$.*

Proposition 3 establishes that no scoring rule can guarantee social p/q -acceptability when $q > p$. Intuitively, condition (C) imposes two competing requirements: the average score must be sufficiently high, while the scores assigned to positions beyond the top p must remain sufficiently low. These requirements become incompatible when the number of rejection positions exceeds the number of support positions. Conversely, when $p \geq q$, at least one admissible scoring rule always exists, and an explicit construction is provided in the proof of Proposition 3.

We now consider the symmetric case $p = q$. While Proposition 3 guarantees the existence of admissible scoring rules in this case, it does not establish whether several such rules may exist. The following proposition shows that the symmetry between the support and rejection thresholds imposes a stronger restriction: there is exactly one admissible scoring rule, whose score vector is uniquely determined.

Proposition 4. *Let p be an integer such that $1 \leq p \leq \lfloor \frac{m}{2} \rfloor$. A normalized score vector s satisfies condition (C) for the pair (p, p) if and only if*

$$s = (\underbrace{1, \dots, 1}_p, \underbrace{\frac{1}{2}, \dots, \frac{1}{2}}_{m-2p}, \underbrace{0, \dots, 0}_p).$$

In particular, $\mathcal{S}(p, p)$ is a singleton.

The unique scoring rule identified in Proposition 4 has a symmetric score vector, satisfying $s_r + s_{m+1-r} = 1$ for every rank r . This symmetry provides a simple interpretation of the rule. Consider an alternative a receiving x top- p votes, y bottom- p votes, and

$z = n - x - y$ votes in the remaining positions. Its total score is $x + \frac{1}{2}z = \frac{1}{2}(n + x - y)$. Thus, the rule ranks alternatives according to their net support $x - y$, which is precisely the quantity determining social p/p -acceptability. Moreover, since each voter ranks exactly p alternatives among the top p positions and p among the bottom p positions, the sum of net supports across all alternatives is zero. Consequently, at least one alternative has non-negative net support, ensuring that the rule always selects socially p/p -acceptable alternatives.

Proposition 4 extends the uniqueness result of Mahajne and Volij (2018), which was established for the standard notion of social acceptability corresponding to $p = q = \lfloor m/2 \rfloor$. Our result shows that uniqueness is not specific to this particular choice of thresholds, but holds for every admissible value of $p = q$. In each case, the unique scoring rule assigns a score of 1 to the top p positions, a score of $1/2$ to the intermediate positions, and a score of 0 to the bottom p positions. In particular, this rule coincides with the HAHR rule characterized by Mahajne and Volij (2018). Thus, the uniqueness of the HAHR rule is a special case of a more general uniqueness result for symmetric support and rejection thresholds.

We now examine a particular family of scoring rules that assigns a score of 1 to the top p positions, a score of 0 to the bottom q positions, and a common score to all intermediate positions. We aim to determine precisely which values of this intermediate score guarantee social p/q -acceptability.

Definition 3. Let (p, q) be a pair of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$ and let $\alpha \in [0, 1]$. The p -best/ q -worst rule with intermediate weight α , denoted s^α , is the normalized score vector defined by

$$s^\alpha = \begin{cases} 1 & \text{if } 1 \leq r \leq p, \\ \alpha & \text{if } p+1 \leq r \leq m-q, \\ 0 & \text{if } m-q+1 \leq r \leq m. \end{cases}$$

This family includes several well-known scoring rules as special cases. In particular, when $\alpha = 0$, all positions beyond the top p receive a score of 0, so the rule coincides with the p -approval rule. Another important special case arises when $p = q = \lfloor m/2 \rfloor$ and $\alpha = 1/2$, which corresponds to the HAHR rule characterized by Mahajne and Volij (2018). More generally, varying α allows us to examine how the weight assigned to intermediate positions affects the ability of a scoring rule to guarantee social p/q -acceptability. The next proposition provides a complete characterization of the values of α for which this requirement is satisfied.

Proposition 5. *Let (p, q) be a pair of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$ and $p \geq q$. If $p + q < m$, the rule s^α satisfies condition (C) if and only if*

$$\frac{m - 2p}{2(m - p - q)} \leq \alpha \leq \frac{p}{p + q},$$

and this interval is a non-empty subinterval of $[0, 1]$. If $p + q = m$, the rule s^α reduces to the p -approval rule for every α , and it always satisfies (C).

Each of the two bounds on α comes from the first and third inequalities of condition (C). The upper bound $\frac{p}{p+q}$ is the first inequality, $s_{p+1} \leq \frac{\Delta_s}{m}$: the weight given to the intermediate positions cannot exceed the average score of the rule, so these positions cannot be rewarded more than an average position. The lower bound $\frac{m-2p}{2(m-p-q)}$ is the third inequality, which, for a rule with $s_{m-q+1} = 0$, amounts to $\Delta_s \geq \frac{m}{2}$: the average score must be at least one half, so the rule cannot be too severe towards the intermediate positions. The interval is therefore non-empty precisely because $p \geq q$: the larger the support block is relative to the rejection block, the wider the interval. When $p = q$, the two bounds coincide and equal $\frac{1}{2}$, which is the rule defined in Proposition 4. The value $\alpha = 0$ belongs to the interval if and only if $p \geq \frac{m}{2}$, that is $p = \frac{m}{2}$ (with m even), in which case s^0 is the p -approval rule.

Propositions 3–5 establish the conditions under which $\mathcal{S}(p, q)$ is non-empty, identify the case in which it contains a unique scoring rule, and characterize a particular family of admissible scoring rules. We now extend these results by providing a complete description of $\mathcal{S}(p, q)$ for any pair (p, q) satisfying $p \geq q$ and $p + q < m$. The main observation is that condition (C) depends only on three quantities: $a := s_{p+1}$, $b := s_{m-q+1}$, and the total score Δ_s . Letting $L = m - p - q$ denote the number of positions between the top p and bottom q ranks, the remaining scores are constrained only by the monotonicity of the score vector and their contribution to Δ_s . This observation allows us to characterize all admissible scoring vectors, rather than restricting attention to a particular family.

Theorem 2. *Let (p, q) be a pair of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$ and $p \geq q$.*

1. For $p + q < m$,

- If $q \geq 2$, then a normalized score vector s with $s_{p+1} = a$ and $s_{m-q+1} = b$ can be completed into a vector satisfying condition (C) if and only if*

$$0 \leq b \leq \frac{p - q}{p - q + 2} \quad \text{and} \quad \frac{(m - 2p) + b(m - 2q + 2)}{2L} \leq a \leq \frac{p + (q - 1)b}{p + q}.$$

- If $q = 1$, then a completion exists if and only if a lies in the interval above evaluated at $b = 0$.*

2. For $p + q = m$, condition (C) holds if and only if s is the p -approval rule.

Theorem 2 shows that, for a given pair (p, q) , the admissible rules are governed by only two scores: the score $a = s_{p+1}$ of the first position outside the support block, and the score $b = s_{m-q+1}$ of the first position inside the rejection block. The bound $b \leq \frac{p-q}{p-q+2}$ means that a rule may give some credit to the positions that voters reject only if the support block is larger than the rejection block, and the larger the difference $p - q$, the more credit it may give. When $p = q$, no such credit is allowed, since $b = 0$. Moreover, the range allowed for a increases with b : a rule that is more generous towards the rejection block must also be more generous towards the intermediate positions. When b takes its largest possible value, this range reduces to a single value, as in Example 2.

Theorem 2 also allows us to recover two of the previous results as special cases. First, setting $b = 0$ reduces the bounds on a to

$$\frac{m - 2p}{2(m - p - q)} \leq a \leq \frac{p}{p + q},$$

which is precisely the interval obtained in Proposition 5 for the family of scoring rules with zero scores in the bottom q positions. Second, when $p = q$, the upper bound on b becomes zero, so $b = 0$. In this case, the lower and upper bounds on a both reduce to $1/2$, and the admissible score vector is uniquely determined, as established in Proposition 4. Thus, Theorem 2 provides a common framework for these earlier results, while extending the characterization to scoring rules beyond the particular family considered in Proposition 5. The following example illustrates the characterization established in Theorem 2 for a particular choice of p and q .

Example 2. Consider $m = 6$, $p = 3$, and $q = 2$. In this case, $L = m - p - q = 1$, and Theorem 2 gives $0 \leq b \leq 1/3$. Taking $b = 1/3$, the lower and upper bounds on a coincide, yielding $a = 2/3$. We therefore obtain the admissible score vector

$$s = \left(1, 1, 1, \frac{2}{3}, \frac{1}{3}, 0\right).$$

Indeed, $\Delta_s = 4$, so the first and third inequalities of condition (C) are satisfied with equality, while the second inequality holds strictly. Moreover, this vector does not belong to the family s^α introduced in Definition 3, since the scores assigned to the bottom two positions are not both zero. This example illustrates that $\mathcal{S}(p, q)$ contains admissible scoring rules beyond those identified in Proposition 5.

Beyond providing a complete characterization of admissible scoring rules, Theorem 2 also reveals some important properties of the set $\mathcal{S}(p, q)$. Since condition (C) consists

of linear inequalities in the components of the score vector, together with the normalization and monotonicity requirements, $\mathcal{S}(p, q)$ is a convex set. In particular, any convex combination of admissible scoring vectors remains admissible. The theorem also clarifies the relationship between the scores assigned to intermediate and rejection positions. The bound $b \leq (p - q)/(p - q + 2)$ shows that the score assigned to the first position of the rejection block must be zero when $p = q$, whereas positive values are possible when $p > q$. Moreover, both bounds on a increase with b , indicating that assigning higher scores to rejection positions requires correspondingly higher scores at the beginning of the intermediate block. When b reaches its maximum admissible value, the lower and upper bounds on a coincide, as illustrated in Example 2. Thus, the characterization also highlights how the scores assigned to different ranking positions are constrained by the requirement of social p/q -acceptability.

We next examine how the set of admissible scoring rules changes as the support and rejection thresholds vary. The following proposition establishes a monotonicity property relating these sets for different values of p and q .

Proposition 6. *If a scoring rule f_s always selects socially p/q -acceptable alternatives, it also always selects socially p'/q' -acceptable alternatives for every $p' \geq p$ and $q' \leq q$ (with $1 \leq p', q' \leq \lfloor \frac{m}{2} \rfloor$).*

Proposition 6 establishes that increasing the support threshold p makes it easier for an alternative to receive sufficient support, while decreasing the rejection threshold q reduces the number of positions counted as rejections. Both changes therefore make social p/q -acceptability less restrictive. As a result, whenever $p' \geq p$ and $q' \leq q$, we have $\mathcal{S}(p, q) \subseteq \mathcal{S}(p', q')$. Thus, the sets of admissible scoring rules are nested, with more demanding acceptability requirements corresponding to smaller sets. This monotonicity property is also consistent with Proposition 3, which identifies $p = q$ as the boundary between the existence and non-existence of admissible scoring rules.

3.2 Applications to classical scoring rules

For the normalized Borda rule, the score assigned to an alternative ranked at position r is given by $s_r = \frac{m-r}{m-1}$, so that the scores decrease linearly from 1 for the top-ranked alternative to 0 for the bottom-ranked alternative. The total sum of scores distributed by each voter is $\Delta_s = \sum_{r=1}^m \frac{m-r}{m-1} = \frac{m}{2}$, and therefore the average score per alternative is $\frac{\Delta_s}{m} = \frac{1}{2}$. Because the Borda rule rewards all ranking positions in a gradual way, alternatives may accumulate substantial total scores even when they are not ranked among the top positions by many voters. The next proposition determines exactly for which values of p and q the Borda rule nevertheless always selects socially p/q -acceptable alternatives.

Proposition 7. *For the Borda rule and for any pair (p, q) of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$ with $m \geq 3$, the rule always selects socially p/q -acceptable alternatives if and only if $p = \lfloor \frac{m}{2} \rfloor$ and $q = 1$. For all other admissible values of p and q , the rule may select alternatives that are not socially p/q -acceptable.*

The symmetric case corresponds to the situation where the thresholds for support and rejection coincide, that is, $p = q$. Applying Proposition 7 under this restriction imposes simultaneously $p = \lfloor \frac{m}{2} \rfloor$ and $q = 1$. Since $p = q$, we necessarily obtain $p = 1$, which in turn implies $\lfloor \frac{m}{2} \rfloor = 1$. This occurs only when $m = 3$. We therefore obtain the following corollary.

Corollary 1. *The Borda rule always selects socially p -acceptable alternatives if and only if $(m, p) = (3, 1)$.*

For the plurality rule, the score vector is given by $s_1 = 1$ and $s_r = 0$ for every $r \geq 2$. Hence, each voter assigns one point only to her top-ranked alternative, while all remaining alternatives receive score 0. The total sum of scores distributed by each voter is therefore $\Delta_s = 1$, so that the average score per alternative is $\frac{\Delta_s}{m} = \frac{1}{m}$. Because the plurality rule takes into account only first-place votes, an alternative may obtain the highest total score despite being ranked among the bottom positions by a large proportion of voters. In particular, the rule completely ignores the information contained in intermediate and lower ranks. The next proposition shows that, as a consequence, the plurality rule does not always select socially p/q -acceptable alternatives for any admissible pair (p, q) of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$. The symmetric case $p = q$ follows immediately as a particular instance of this more general impossibility result.

Proposition 8. *For the plurality rule and for any admissible pair (p, q) of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$ with $m \geq 3$, the rule does not always select socially p/q -acceptable alternatives.*

For the antiplurality rule, the score vector is given by $s_r = 1$ for every $r \leq m - 1$ and $s_m = 0$. Hence, each voter assigns one point to all alternatives except the alternative ranked last, which receives a score of 0. The total sum of scores distributed by each voter is therefore $\Delta_s = m - 1$, and the average score per alternative is $\frac{\Delta_s}{m} = \frac{m-1}{m}$. Unlike the plurality rule, the antiplurality rule rewards almost all ranking positions equally and penalizes only the last position. As a consequence, an alternative may accumulate a very large total score even without receiving strong support from the electorate. In particular, an alternative systematically ranked just above the last position can receive score 1 from every voter while never appearing among the top p positions. The next proposition shows that

this feature prevents the antiplurality rule from always selecting socially p/q -acceptable alternatives. The symmetric case $p = q$ follows immediately as a particular instance of this more general impossibility result.

Proposition 9. *For the antiplurality rule and for any admissible pair (p, q) of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$ with $m \geq 3$, the rule does not always select socially p/q -acceptable alternatives.*

For the k -approval rule, the score vector is given by $s_r = 1$ if $r \leq k$ and $s_r = 0$ otherwise, where $1 \leq k \leq m - 1$. Hence, each voter assigns one point to the alternatives ranked among the top k positions and zero to all remaining alternatives. The total sum of scores distributed by each voter is therefore $\Delta_s = k$, and the average score per alternative is $\frac{\Delta_s}{m} = \frac{k}{m}$. Because the k -approval rule distinguishes only between approved and non-approved alternatives, its compatibility with social p/q -acceptability strongly depends on the approval threshold k . In particular, if too many positions are approved, alternatives with weak top support may nevertheless accumulate large scores, whereas if too few positions are approved, the rule behaves similarly to plurality. The next proposition shows that the unique compatible case corresponds to the half-approval rule when the number of alternatives is even, where exactly half of the alternatives are approved, the parameter p equals this threshold, and the parameter q does not exceed it.

Proposition 10. *For the k -approval rule and for any admissible pair (p, q) of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$, the rule always selects socially p/q -acceptable alternatives if and only if m is even, $k = \frac{m}{2}$ and $p = \frac{m}{2}$.*

The symmetric case corresponds to the situation where $p = q$. Applying Proposition 10 under this restriction imposes simultaneously $p = \frac{m}{2}$ and $q = \frac{m}{2}$, which is possible only when m is even. We therefore obtain the following corollary.

Corollary 2. *The k -approval rule always selects socially p -acceptable alternatives if and only if m is even and $k = p = \frac{m}{2}$.*

Notice that the results obtained for classical scoring rules are consistent with the general existence condition established in Proposition 3. In particular, the Borda rule satisfies social p/q -acceptability only when $q = 1$ and $p = \lfloor m/2 \rfloor$, while the half-approval rule satisfies this requirement when m is even, $p = m/2$, and $q \leq m/2$. In both cases, the admissible pairs satisfy $p \geq q$, as required by Proposition 3. However, these results also show that this condition, although sufficient to guarantee the existence of at least one admissible scoring rule, does not ensure that a particular scoring rule satisfies social p/q -acceptability. Indeed, the plurality and antiplurality rules fail to satisfy this requirement for every admissible pair

(p, q) , including those for which $p \geq q$. More generally, the results illustrate that the ability of a scoring rule to guarantee social p/q -acceptability depends not only on the relative sizes of the support and rejection blocks, but also on the distribution of scores across ranking positions.

4 Conclusion

This paper extends the notion of social acceptability introduced by [Mahajne and Volij \(2018\)](#) by allowing for asymmetric thresholds in the assessment of electoral support and rejection. An alternative is socially p/q -acceptable if the number of voters ranking it among their top p positions is at least as large as the number ranking it among their bottom q positions. This framework encompasses the standard notion of social acceptability as the special case $p = q = \lfloor m/2 \rfloor$.

We provide a complete characterization of the scoring rules that always select socially p/q -acceptable alternatives, through necessary and sufficient conditions on their score vectors. Beyond this characterization, we establish that such rules exist if and only if $p \geq q$, and that the admissible scoring rule is unique when $p = q$. We also characterize the set of admissible scoring vectors more generally, showing that it forms a convex set and identifying how the scores assigned to intermediate and rejection positions are constrained. Furthermore, we establish a monotonicity property: the set of admissible scoring rules expands when the support threshold increases or the rejection threshold decreases. We also apply our characterizations to several classical scoring rules. The Borda rule satisfies the social p/q -acceptability requirement only when $p = \lfloor m/2 \rfloor$ and $q = 1$, while the plurality and antiplurality rules fail for every admissible pair (p, q) . Among k -approval rules, only the half-approval rule, with an even number of alternatives, satisfies the requirement for suitable values of p and q .

Several directions for future research naturally arise from our framework.

- A first direction for future research concerns the relationship between social p/q -acceptability and Condorcet-based voting rules. In particular, it would be interesting to examine whether Condorcet extensions, such as the Kemeny and Copeland rules, can guarantee the selection of socially acceptable alternatives for suitable values of p and q . This question is particularly relevant given the majoritarian nature of the Condorcet criterion, which relies on pairwise comparisons between alternatives. We also intend to investigate the relationship between our acceptability criteria and q -Condorcet winners, defined as alternatives that defeat every other alternative in pairwise comparisons by at least a fraction q of the voters, with $q \in [1/2, 1)$. Es-

establishing conditions under which these concepts are compatible, or identifying cases in which such compatibility is impossible, would provide further insights into the relationship between social acceptability and majority-based voting procedures.

- Another possible direction would be to extend our analysis to approval-disapproval procedures and multi-winner elections. In the former setting, voters can express both positive and negative evaluations of alternatives without necessarily providing complete preference rankings. It would be interesting to examine how our acceptability criteria could be adapted to such procedures and whether similar characterization results could be obtained. In multi-winner elections, the main challenge would be to extend these criteria from individual alternatives to committees, taking into account both the support and rejection received by their members. This would also raise the question of whether a committee should be considered socially acceptable when all its members satisfy the acceptability requirement, or whether a weaker collective condition would be more appropriate (see, for instance, [Diss and Mahajne, 2020](#)).
- Another direction for future research concerns the probabilistic performance of scoring rules that do not satisfy our theoretical conditions. Although such rules cannot guarantee the selection of socially p/q -acceptable alternatives for every preference profile, they may still select acceptable alternatives with high probability. It would therefore be interesting to determine the probability that a scoring rule selects at least one non-acceptable alternative and to examine how this probability varies with the parameters p and q . This analysis could be carried out under standard probabilistic models of preferences, such as the Impartial Culture and Impartial Anonymous Culture, drawing on the analytical and computational approaches developed in the literature on voting paradoxes and probabilistic evaluations of voting rules (see, for instance, [Gehrlein and Lepelley, 2011, 2017](#); [Diss and Merlin, 2021](#)). Such an approach would complement our theoretical characterization by providing a probabilistic assessment of social acceptability.
- Another promising direction for future research is to extend social p/q -acceptability by allowing support and rejection to carry different weights. While the thresholds p and q determine the sizes of the support and rejection blocks, the current definition treats the two counts symmetrically. A natural generalization would introduce an additional parameter $\lambda > 0$, requiring the number of voters ranking an alternative among their top p positions to be at least λ times the number ranking it among their bottom q positions. The original notion of social p/q -acceptability would then be recovered when $\lambda = 1$. It would be interesting to investigate how this additional

parameter affects the characterization of admissible scoring rules and whether the results established in this paper can be extended to this more general setting.

Declaration of Competing Interests

The authors declare that they have no competing interests.

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Appendix

Proof of Proposition 1. Assume that condition (C) holds, i.e.,

$$s_{p+1} \leq \frac{\Delta_s}{m}, \quad m \cdot s_{m-q+1} < \Delta_s, \quad m + m \cdot s_{m-q+1} \leq 2\Delta_s.$$

Let R be any preference profile with n voters, and let a be an alternative that is not socially p/q -acceptable. We prove that a cannot be a winner.

Denote by x the number of voters who rank a among their first p positions, and by y the number of voters who rank a among their last q positions. Let $z = n - x - y$ be the number of voters who rank a in the remaining positions, i.e., between $p + 1$ and $m - q$. Because a is not p/q -acceptable, by definition we have $x < y$, so $y - x > 0$.

We now bound $\text{Score}(a)$ from above. A voter who places a among the first p positions can give at most the highest possible score $s_1 = 1$. For a voter who places a among the last q positions, the best score she can assign is the score of the first of those positions, namely s_{m-q+1} (since scores are decreasing: $s_{m-q+1} \geq s_{m-q+2} \geq \dots \geq s_m$). Every remaining voter ranks a between positions $p + 1$ and $m - q$; the maximum score that a voter can assign is therefore s_{p+1} . Consequently,

$$\text{Score}(a) \leq x \cdot 1 + y \cdot s_{m-q+1} + z \cdot s_{p+1}.$$

Denote the right-hand side by V ; thus $\text{Score}(a) \leq V$. Multiply this inequality by m :

$$m \text{Score}(a) \leq mV = mx + my s_{m-q+1} + mz s_{p+1}.$$

To exploit the fact that $y = x + (y - x)$, rewrite the first two terms:

$$mx + my s_{m-q+1} = mx(1 + s_{m-q+1}) + m(y - x)s_{m-q+1}.$$

Hence,

$$mV = mx(1 + s_{m-q+1}) + m(y - x)s_{m-q+1} + mz s_{p+1}.$$

Now we apply the three inequalities of condition (C). The third inequality (after factoring) gives $m(1 + s_{m-q+1}) \leq 2\Delta_s$; multiplying by $x \geq 0$ yields $mx(1 + s_{m-q+1}) \leq 2x\Delta_s$. The first inequality gives $m s_{p+1} \leq \Delta_s$; multiplying by $z \geq 0$ yields $mz s_{p+1} \leq z\Delta_s$. The second inequality is $m s_{m-q+1} < \Delta_s$; because $y - x > 0$, we obtain $m(y - x)s_{m-q+1} < (y - x)\Delta_s$. Substituting these bounds into the expression for mV we obtain:

$$mV < 2x\Delta_s + (y - x)\Delta_s + z\Delta_s = (x + y + z)\Delta_s = n\Delta_s,$$

Therefore, $mV < n\Delta_s$. Since $m \text{Score}(a) \leq mV$, we have $m \text{Score}(a) < n\Delta_s$.

Let W be the sum of the scores of all alternatives except a . The total sum of all scores is $n\Delta_s = \text{Score}(a) + W$, so the strict inequality becomes $m \text{Score}(a) < \text{Score}(a) + W$, i.e., $(m - 1)\text{Score}(a) < W$. Thus,

$$\text{Score}(a) < \frac{W}{m - 1}.$$

Hence, the score of a is strictly smaller than the average score of the other $m - 1$ alternatives. Consequently, at least one of those alternatives has a score strictly greater than $\text{Score}(a)$. Therefore, a cannot be a winner. We have thus proved that every winner of the scoring rule f_s is socially p/q -acceptable. ■

Proof of Lemma 1. We construct the desired matrix column by column, using a simple rule that keeps the row sums as balanced as possible. Consider an arbitrary column c . Its $m - 1$ entries (the scores assigned to the $m - 1$ alternatives other than the distinguished one) form a multiset of values taken from the score vector s . Because the scores are normalized, each entry lies in the interval $[0, 1]$. Sorting these entries in non-decreasing order, we obtain

$$x_1^{(c)} \leq x_2^{(c)} \leq \dots \leq x_{m-1}^{(c)}.$$

Clearly, $0 \leq x_1^{(c)} \leq x_{m-1}^{(c)} \leq 1$, so the spread of the column satisfies

$$x_{m-1}^{(c)} - x_1^{(c)} \leq 1. \quad (1)$$

Now suppose we have already placed the first t columns. Let

$$u_1^{(t)} \leq u_2^{(t)} \leq \dots \leq u_{m-1}^{(t)}$$

be the current row sums sorted in increasing order. For the next column we decide how to distribute its entries among the rows. The rule is:

Assign the largest entry $x_{m-1}^{(c)}$ to the row that currently has the smallest sum $u_1^{(t)}$, the second largest entry $x_{m-2}^{(c)}$ to the row with the second smallest sum $u_2^{(t)}$, and so on. Finally, the smallest entry $x_1^{(c)}$ goes to the row that currently has the largest sum $u_{m-1}^{(t)}$.

Consequently, after adding this column the new row sums become

$$v_r = u_r^{(t)} + x_{m-r}^{(c)} \quad (r = 1, 2, \dots, m-1).$$

We now prove that the spread of the new sums does not exceed the maximum of the previous spread and the spread of the column. More precisely, we claim

$$\max_{1 \leq r \leq m-1} v_r - \min_{1 \leq r \leq m-1} v_r \leq \max \left\{ u_{m-1}^{(t)} - u_1^{(t)}, x_{m-1}^{(c)} - x_1^{(c)} \right\}. \quad (2)$$

To verify this, take any two indices r, q with $r < q$. The difference between the corresponding new sums is

$$v_r - v_q = (u_r^{(t)} - u_q^{(t)}) + (x_{m-r}^{(c)} - x_{m-q}^{(c)}).$$

Because the u 's are sorted increasingly, $u_r^{(t)} \leq u_q^{(t)}$ and therefore $u_r^{(t)} - u_q^{(t)} \leq 0$. Because the x 's are sorted increasingly and $m-r > m-q$ (since $r < q$), we have $x_{m-r}^{(c)} \geq x_{m-q}^{(c)}$ and thus $x_{m-r}^{(c)} - x_{m-q}^{(c)} \geq 0$. Hence, $v_r - v_q$ is the sum of a non-positive number and a non-negative number; its sign is not fixed, but we can bound it from above by ignoring the non-positive part:

$$v_r - v_q \leq x_{m-r}^{(c)} - x_{m-q}^{(c)} \leq x_{m-1}^{(c)} - x_1^{(c)}.$$

Now, consider the symmetric difference $v_q - v_r$. Using the same expressions,

$$v_q - v_r = (u_q^{(t)} - u_r^{(t)}) + (x_{m-q}^{(c)} - x_{m-r}^{(c)}) = (u_q^{(t)} - u_r^{(t)}) - (x_{m-r}^{(c)} - x_{m-q}^{(c)}).$$

Here, $u_q^{(t)} - u_r^{(t)} \geq 0$ and $x_{m-r}^{(c)} - x_{m-q}^{(c)} \geq 0$. By dropping the non-positive term we obtain

$$v_q - v_r \leq u_q^{(t)} - u_r^{(t)} \leq u_{m-1}^{(t)} - u_1^{(t)}.$$

Thus, for every pair (r, q) we have

$$|v_r - v_q| \leq \max\{u_{m-1}^{(t)} - u_1^{(t)}, x_{m-1}^{(c)} - x_1^{(c)}\}.$$

The spread $\max v_r - \min v_r$ is exactly the maximum of $|v_r - v_q|$ over all pairs; therefore it also satisfies the same bound. This establishes (2).

Now we proceed by induction. At the very beginning, before any column is placed, all row sums are zero, so the initial spread is 0. For each new column, the spread can only increase to at most the maximum of the previous spread and the column's spread. Because every column's spread is at most 1 by (1), after processing all l columns the final spread $\Delta(M)$ satisfies

$$\Delta(M) \leq 1.$$

Hence we have constructed a matrix M (by the described ordering inside each column) such that $\Delta(M) \leq 1$. Consequently, the minimum possible value of $\Delta(M)$ over all choices of column orderings is also at most 1. This completes the proof. $\blacksquare$

Proof of Proposition 2. Suppose that condition (C) is not satisfied. We construct a preference profile in which a distinguished alternative a is a winner even though a is not socially p/q -acceptable. In each of the three cases below, the preferences of the $m - 1$ alternatives other than a are arranged by means of the balancing Lemma 1, so that their total scores differ by at most 1. Let W denote the sum of these $m - 1$ scores. Whenever

$$(m - 1) \text{Score}(a) - W \geq m - 1,$$

the alternative a is a winner.

We distinguish three mutually exclusive cases, according to the first inequality of (C) that fails, examined in the order (C_2) , (C_1) , (C_3) , where (C_1) , (C_2) , (C_3) denote respectively the first, second, and third inequality of condition (C).

Case A: (C_2) fails, that is, $s_{m-q+1} \geq \frac{\Delta_s}{m}$.

Subcase A1: $s_{m-q+1} > \frac{\Delta_s}{m}$ strictly. Since $m s_{m-q+1} - \Delta_s > 0$, choose

$$t = \max \left\{ 2, \left\lceil \frac{m - 1}{m s_{m-q+1} - \Delta_s} \right\rceil \right\},$$

so that $t \geq 2$ and $t(m s_{m-q+1} - \Delta_s) \geq m - 1$. Consider a profile with $n = t$ voters, all ranking a in position $m - q + 1$. Since $m - q + 1 > p$, no voter ranks a among her first p positions ($x = 0$), while all t voters rank her among the last q ($y = t \geq 2$); hence $x < y$ and a is not acceptable. With $\text{Score}(a) = t s_{m-q+1}$ and $W = t(\Delta_s - s_{m-q+1})$,

$$(m - 1)\text{Score}(a) - W = t(m s_{m-q+1} - \Delta_s) \geq m - 1,$$

so a is a winner.

Subcase A2: $s_{m-q+1} = \frac{\Delta_s}{m}$ exactly. Since $m - \Delta_s > 0$ (because $\Delta_s < m$), choose $t = \lceil (m - 1)/(m - \Delta_s) \rceil$. Consider a profile with $n = 2t + 1$ voters: t rank a first, $t + 1$ rank a at position $m - q + 1$. Since $p \geq 1$ and $m - q + 1 \geq 2$, we have $x = t < t + 1 = y$, so a is not acceptable. With $\text{Score}(a) = t + (t + 1)\frac{\Delta_s}{m}$ and $W = t(\Delta_s - 1) + (t + 1)\frac{m-1}{m}\Delta_s$, one computes

$$(m - 1)\text{Score}(a) - W = t(m - \Delta_s) \geq m - 1,$$

so a is a winner.

Case B: (C_2) holds but (C_1) fails, that is, $s_{m-q+1} < \frac{\Delta_s}{m}$ and $s_{p+1} > \frac{\Delta_s}{m}$.

This configuration forces $p + q < m$: if $p + q = m$ then $p + 1 = m - q + 1$, so $s_{p+1} = s_{m-q+1}$, which cannot simultaneously satisfy (C_2) and violate (C_1) . Hence $p + 1 < m - q + 1$ and the two positions are distinct, so Case B cannot occur when $p + q = m$, and Cases A and B are mutually exclusive.

Since $m s_{p+1} - \Delta_s > 0$, choose

$$t = \left\lceil \frac{(m - 1) - (m s_{m-q+1} - \Delta_s)}{m s_{p+1} - \Delta_s} \right\rceil.$$

Since (C_2) holds, $m s_{m-q+1} - \Delta_s < 0$, so the numerator exceeds $m - 1 > 0$ and $t \geq 1$. Consider a profile with $n = t + 1$ voters: one ranks a at $m - q + 1$, the other t rank a at $p + 1$. Then $x = 0 < 1 \leq y$, so a is not acceptable. One computes

$$(m - 1)\text{Score}(a) - W = t(m s_{p+1} - \Delta_s) + (m s_{m-q+1} - \Delta_s) \geq m - 1,$$

so a is a winner.

Case C: (C_1) and (C_2) hold but (C_3) fails, that is, $m s_{m-q+1} > 2\Delta_s - m$.

Since $m s_{m-q+1} + m - 2\Delta_s > 0$, choose

$$t = \left\lceil \frac{(m - 1) - (m s_{m-q+1} - \Delta_s)}{m s_{m-q+1} + m - 2\Delta_s} \right\rceil,$$

with $t \geq 1$ as in Case B, since (C_2) again gives a positive numerator. Consider a profile with $n = 2t+1$ voters: t rank a first, $t+1$ rank a at $m-q+1$. Since $p \geq 1$ and $m-q+1 \geq 2$, $x = t < t+1 = y$, so a is not acceptable. Using $m - \Delta_s = (m s_{m-q+1} + m - 2\Delta_s) - (m s_{m-q+1} - \Delta_s)$, one computes

$$(m-1)\text{Score}(a) - W = t(m s_{m-q+1} + m - 2\Delta_s) + (m s_{m-q+1} - \Delta_s) \geq m-1,$$

so a is a winner.

In each case we have exhibited a profile in which a non-acceptable alternative wins, which completes the proof. ■

Proof of Theorem 1. The theorem follows directly from Proposition 1 (sufficiency) and Proposition 2 (necessity). Let f_s be a scoring rule with score vector $s = (s_1, \dots, s_m)$, and let (p, q) be a pair of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$.

Suppose first that condition (C) holds. Then, Proposition 1 implies that every winning alternative is socially p/q -acceptable. Therefore, the scoring rule f_s always selects socially p/q -acceptable alternatives.

Conversely, if (C) does not hold, Proposition 2 then provides, for each possible violation of (C) (including the borderline case, treated as Subcase A2 in its proof) a preference profile in which an alternative a that is not socially p/q -acceptable is nevertheless a winner. Consequently, the scoring rule f_s cannot always select socially p/q -acceptable alternatives. ■

Proof of Proposition 3. Necessity ($p \geq q$). Let s satisfy (C) , and write $\bar{s} = \Delta_s/m$. Split the m positions into the top block $(1, \dots, p)$, the middle block $(p+1, \dots, m-q)$ (possibly empty), and the bottom block $(m-q+1, \dots, m)$. By monotonicity, $s_i \leq 1$ for $i \leq p$; by the first inequality of (C) and monotonicity, $s_i \leq s_{p+1} \leq \bar{s}$ for $p+1 \leq i \leq m-q$; and by the third inequality of (C) (rewritten as $s_{m-q+1} \leq 2\bar{s} - 1$) together with monotonicity, $s_i \leq s_{m-q+1} \leq 2\bar{s} - 1$ for $m-q+1 \leq i \leq m$. Summing,

$$\Delta_s \leq p + (m-p-q)\bar{s} + q(2\bar{s} - 1) = (p-q) + (m-p+q)\bar{s}.$$

Since $\Delta_s = m\bar{s}$ and $m - (m-p+q) = p-q$, this rearranges to $0 \leq (p-q)(1-\bar{s})$. As $\bar{s} < 1$, we conclude $p \geq q$.

Sufficiency ($p \geq q \Rightarrow$ existence). Suppose $p \geq q$ and let $c = \frac{p}{p+q} \in (0, 1)$. If $p+q = m$, take $s = e_p$, the p -approval rule: then $\Delta_s = p$, $s_{p+1} = s_{m-q+1} = 0$, and the third inequality of (C) reads $m \leq 2p$, which holds since $m = p+q \leq 2p$. If $p+q < m$, consider $s = s^c$

(Definition 3). Then $\Delta_s = p + (m - p - q)c = \frac{mp}{p+q} = mc$, so $\bar{s} = c$. The first inequality of (C), $s_{p+1} = c \leq \bar{s}$, holds with equality. The second, $s_{m-q+1} = 0 < c$, holds since $p \geq 1$. The third, $ms_{m-q+1} \leq 2\Delta_s - m$, reads $0 \leq m(2c - 1) = \frac{m(p-q)}{p+q}$, which holds since $p \geq q$. Hence $s^c \in \mathcal{S}(p, q)$. $\blacksquare$

Proof of Proposition 4. Sufficiency. This is the rule $s^{1/2}$ of Definition 3 with $q = p$: $\Delta_s = \frac{m}{2}$, so $\bar{s} = \frac{1}{2}$; the first inequality of (C) holds with equality ($s_{p+1} = \frac{1}{2} = \bar{s}$) if $m > 2p$ and with strict inequality if $m = 2p$ ($s_{p+1} = 0 < \bar{s}$). The second holds strictly ($s_{m-p+1} = 0 < \frac{1}{2}$) and the third holds with equality ($m = 2\Delta_s$).

Necessity. Let s satisfy (C) for (p, p) , and write $\bar{s} = \Delta_s/m$. As in the proof of Proposition 3 (with $q = p$), monotonicity together with (C) gives, for every position i ,

$$s_i \leq 1 \quad (1 \leq i \leq p), \quad s_i \leq \bar{s} \quad (p+1 \leq i \leq m-p), \quad s_i \leq 2\bar{s} - 1 \quad (m-p+1 \leq i \leq m),$$

where the middle range is empty when $m = 2p$. Summing these m termwise upper bounds gives

$$p + (m - 2p)\bar{s} + p(2\bar{s} - 1) = m\bar{s} = \Delta_s,$$

so the sum of the upper bounds equals the actual sum Δ_s exactly, an identity that holds whether or not the middle block is empty, since an empty sum contributes 0 on both sides when $m = 2p$. Because each termwise inequality already has the form $s_i \leq (\text{bound}_i)$, equality of the totals forces $s_i = \text{bound}_i$ for every i . In particular $s_m = 2\bar{s} - 1$, and since $s_m = 0$ by normalization, $\bar{s} = \frac{1}{2}$. Substituting back yields the announced vector. $\blacksquare$

Proof of Proposition 5. Suppose $p + q < m$, so that $\Delta_{s^\alpha} = p + (m - p - q)\alpha$. The second inequality of (C), $0 < \bar{s}^\alpha$, holds since $\Delta_{s^\alpha} \geq p \geq 1$. The first inequality, $\alpha \leq \bar{s}^\alpha$, is equivalent, after clearing denominators, to $(p + q)\alpha \leq p$, i.e.

$$\alpha \leq \frac{p}{p+q}. \tag{1}$$

The third inequality, $0 \leq 2\Delta_{s^\alpha} - m$, is equivalent to

$$\alpha \geq \frac{m - 2p}{2(m - p - q)}. \tag{2}$$

Since $p \leq \lfloor \frac{m}{2} \rfloor$, we always have $m - 2p \geq 0$, with equality exactly when m is even and $p = \frac{m}{2}$, in which case (2) reduces to $\alpha \geq 0$. The interval defined by (1)–(2) is non-empty precisely when $\frac{m-2p}{2(m-p-q)} \leq \frac{p}{p+q}$, i.e. (both denominators being positive) $(m - 2p)(p + q) \leq 2p(m - p - q)$, which simplifies to $m q \leq m p$, that is, $p \geq q$ — which holds by hypothesis.

Both endpoints lie in $[0, 1]$: $\frac{p}{p+q} \in (0, 1)$ since $p, q \geq 1$, and the lower bound is non-negative as noted above.

If $p + q = m$, the middle block of s^α is empty, so s^α coincides with the p -approval rule for every $\alpha \in [0, 1]$; that this rule belongs to $\mathcal{S}(p, q)$ when $p \geq q$ was shown in the proof of Proposition 3. $\blacksquare$

Proof of Theorem 2.

1. For $p + q < m$.

In this case $L \geq 1$. Split the m ranks into the top block $(1, \dots, p)$, the middle block $(p+1, \dots, m-q)$, of size L , and the bottom block $(m-q+1, \dots, m)$, of size q . Fix $a = s_{p+1}$ and $b = s_{m-q+1}$.

Top block: $s_1 = 1$ is fixed; $s_2, \dots, s_p$ are free subject only to $a \leq s_p \leq \dots \leq s_2 \leq 1$. The block's sum ranges from $1 + (p-1)a$ to p .

Middle block: $s_{p+1} = a$ is fixed; $s_{p+2}, \dots, s_{m-q}$ are free subject to $b \leq s_{m-q} \leq \dots \leq s_{p+2} \leq a$. The block's sum ranges from $a + (L-1)b$ to La .

Bottom block: $s_{m-q+1} = b$ and $s_m = 0$ are fixed; the remaining $q-2$ values (if $q \geq 2$) are free subject to $0 \leq s_{m-1} \leq \dots \leq s_{m-q+2} \leq b$. The block's sum ranges from b to $b(q-1)$. When $q = 1$, the block consists of the single position $m-q+1 = m$, so $b = s_{m-q+1} = s_m = 0$ identically, and the block's sum is 0 regardless of the bounds above.

Summing the three maxima, the largest total attainable for given (a, b) is

$$\Delta_s^{\max} = p + La + (q-1)b. \quad (3)$$

A completion satisfying (C) exists if and only if some attainable Δ_s satisfies both $\Delta_s \geq ma$ (the first inequality of (C), rearranged) and $\Delta_s \geq \frac{m(1+b)}{2}$ (the third inequality, rearranged); since the attainable values form an interval up to $\Delta_s^{\max}$, this holds if and only if $\Delta_s^{\max}$ itself meets both thresholds. Substituting (3) into $\Delta_s^{\max} \geq ma$ and using $m-L = p+q$ gives, after rearranging,

$$a \leq \frac{p + (q-1)b}{p+q}. \quad (4)$$

Substituting (3) into $\Delta_s^{\max} \geq \frac{m(1+b)}{2}$ gives, after rearranging,

$$a \geq \frac{(m-2p) + b(m-2q+2)}{2L}. \quad (5)$$

Conversely, if a satisfies both (4) and (5), choosing the top, middle, and bottom blocks at their maximal values yields a vector with $\Delta_s = \Delta_s^{\max}$ satisfying the first and third inequalities of (C); the second inequality then follows automatically, since $\Delta_s \leq m - 1$ for any normalized score vector, which forces $m s_{m-q+1} < \Delta_s$ whenever the third inequality holds. This proves the equivalence for fixed b .

It remains to determine the admissible range of b itself: the interval (5)–(4) is non-empty precisely when

$$\frac{(m - 2p) + b(m - 2q + 2)}{2L} \leq \frac{p + (q - 1)b}{p + q}.$$

Clearing denominators (both positive) and simplifying using $m - L = p + q$ reduces this to $m(p - q) \geq bm(p - q + 2)$, that is,

$$b \leq \frac{p - q}{p - q + 2},$$

which together with $b \geq 0$ gives the stated range for $q \geq 2$. For the pair (a, b) to be extendable to a monotone normalized vector we must have $b \leq a \leq 1$, which is implied by the displayed bounds. When $p = q$ this range collapses to $b = 0$, consistent with Proposition 4. As noted above, when $q = 1$ the bottom block collapses to the single fixed position $s_m = 0$, so $b = 0$ identically rather than merely bounded by $\frac{p-q}{p-q+2}$; the interval for a is then given by (5)–(4) evaluated at $b = 0$.

2. For $p + q = m$.

In this case the middle block is empty and $p + 1 = m - q + 1$, so $a = b$ necessarily. Furthermore, $p + q = m$ and $p = q$ imply $p = q = \frac{m}{2}$ and thanks to Proposition 4, the vector reduces to the p -approval rule regardless of the values chosen elsewhere, and Proposition 3 (sufficiency part) shows this rule satisfies (C) precisely when $p \geq q$.

■

Proof of Proposition 6. Let $s \in \mathcal{S}(p, q)$. Since s is non-increasing, $s_{p'+1} \leq s_{p+1}$ whenever $p' \geq p$. Hence, the first inequality of (C) continues to hold at p' , since

$$s_{p'+1} \leq s_{p+1} \leq \frac{\Delta_s}{m}.$$

Likewise, whenever $q' \leq q$, we have $m - q' + 1 \geq m - q + 1$, and therefore $s_{m-q'+1} \leq s_{m-q+1}$.

Thus, the second inequality of (C) continues to hold at q' , since

$$s_{m-q'+1} \leq s_{m-q+1} < \frac{\Delta_s}{m}.$$

Finally, the third inequality of (C) also continues to hold at q' , since

$$m s_{m-q'+1} \leq m s_{m-q+1} \leq 2\Delta_s - m.$$

Since Δ_s is unchanged, all three inequalities of (C) remain satisfied at (p', q') . Therefore, $s \in \mathcal{S}(p', q')$. $\blacksquare$

Proof of Proposition 7. We apply the characterization of Theorem 1. Condition (C) becomes

$$s_{p+1} \leq \frac{1}{2}, \quad m s_{m-q+1} < \frac{m}{2}, \quad m + m s_{m-q+1} \leq m.$$

The third inequality simplifies to $s_{m-q+1} \leq 0$, forcing $s_{m-q+1} = 0$, which for the Borda rule means $m - q + 1 = m$, i.e., $q = 1$. With $q = 1$, the second inequality is automatic. The first inequality $s_{p+1} \leq \frac{1}{2}$ becomes $\frac{m-p-1}{m-1} \leq \frac{1}{2}$, which simplifies to $p \geq \frac{m-1}{2}$. Since $p \leq \lfloor \frac{m}{2} \rfloor$, this forces $p = \lfloor \frac{m}{2} \rfloor$. Conversely, for $p = \lfloor \frac{m}{2} \rfloor$ and $q = 1$ the three inequalities hold (with equality when m is odd). Hence, the Borda rule always selects socially p/q -acceptable alternatives if and only if $p = \lfloor \frac{m}{2} \rfloor$ and $q = 1$. $\blacksquare$

Proof of Proposition 8. Applying Theorem 1, condition (C) for the plurality rule becomes

$$s_{p+1} \leq \frac{1}{m}, \quad m s_{m-q+1} < 1, \quad m + m s_{m-q+1} \leq 2.$$

Since the plurality score vector satisfies $s_1 = 1$ and $s_r = 0$ for all $r \geq 2$, and because $p + 1 \geq 2$, we have $s_{p+1} = 0$. Hence, the first inequality becomes $0 \leq \frac{1}{m}$, which always holds. Moreover, because $q \leq \lfloor \frac{m}{2} \rfloor$ and $m \geq 3$, we have $m - q + 1 \geq m - \lfloor \frac{m}{2} \rfloor + 1 \geq 2$, implying $s_{m-q+1} = 0$. Therefore, the second inequality reduces to $0 < 1$, which is always satisfied. However, the third inequality becomes $m \leq 2$, which is impossible for every $m \geq 3$. Consequently, condition (C) is never satisfied for any admissible pair (p, q) of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$. That is, the plurality rule does not always select socially p/q -acceptable alternatives by Theorem 1. $\blacksquare$

Proof of Proposition 9. Applying Theorem 1, condition (C) for the antiplurality rule becomes

$$s_{p+1} \leq \frac{m-1}{m}, \quad m s_{m-q+1} < m-1, \quad m + m s_{m-q+1} \leq 2(m-1).$$

Since $p \leq \lfloor \frac{m}{2} \rfloor$ and $m \geq 3$, we necessarily have $p+1 \leq m-1$, implying $s_{p+1} = 1$. Hence, the first inequality of condition (C) becomes $1 \leq \frac{m-1}{m}$, which is equivalent to $m \leq m-1$, an impossibility for every $m \geq 2$. Consequently, condition (C) is never satisfied for any admissible pair (p, q) of integers such that $1 \leq p, q \leq \lfloor \frac{m}{2} \rfloor$. That is, the antiplurality rule does not always select socially p/q -acceptable alternatives by Theorem 1. ■

Proof of Proposition 10. Applying Theorem 1, condition (C) for the k -approval rule becomes

$$s_{p+1} \leq \frac{k}{m}, \quad m s_{m-q+1} < k, \quad m + m s_{m-q+1} \leq 2k.$$

Since the scores are only 0 or 1, and because $\frac{k}{m} < 1$ whenever $k \leq m-1$, the first inequality forces $s_{p+1} = 0$; otherwise we would obtain $1 \leq \frac{k}{m}$, which is impossible. Hence, $p+1 > k$, that is, $p \geq k$. Similarly, the second inequality forces $s_{m-q+1} = 0$, since $s_{m-q+1} = 1$ would imply $m < k$, impossible because $k \leq m-1$. Therefore, $m-q+1 > k$, which is equivalent to $q \leq m-k$. With $s_{m-q+1} = 0$, the third inequality reduces to $m \leq 2k$, that is, $k \geq \frac{m}{2}$. Collecting these restrictions, we obtain $p \geq k$, $q \leq m-k$, $k \geq \frac{m}{2}$. Since by definition $p \leq \lfloor \frac{m}{2} \rfloor$, the inequality $p \geq k$ implies $k \leq \lfloor \frac{m}{2} \rfloor$. Combining this with $k \geq \frac{m}{2}$ yields $\lfloor \frac{m}{2} \rfloor \geq k \geq \frac{m}{2}$, which is possible only if m is even. Consequently, $k = \frac{m}{2}$. From $p \geq k$ and $p \leq \frac{m}{2}$, we deduce $p = \frac{m}{2}$. Finally, $q \leq m-k = \frac{m}{2}$. Conversely, if m is even, $k = p = \frac{m}{2}$, and $q \leq \frac{m}{2}$, then $s_{p+1} = 0$, $s_{m-q+1} = 0$, and the third inequality becomes $m \leq m$, which is satisfied. Hence, condition (C) holds. By Theorem 1, the k -approval rule always selects socially p/q -acceptable alternatives. ■