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POPULAR ADMISSIBILITY AND PARLIAMENTARY STABILITY IN SEQUENTIAL SPATIAL VOTING GAMES

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Abstract

Popular consultations and citizens' assemblies can restrict the set of politically admissible alternatives before parliamentary decision-making takes place. We study a two-stage spatial voting model in which a first-stage admissibility restriction determines the domain over which a parliamentary voting game operates. The sequential core consists of admissible alternatives that no winning parliamentary coalition can block through another admissible alternative. At the coalition level, the no-blocking problem is a constrained Euclidean Pareto-efficiency problem. Applying the projection characterization of Carrizosa et al. (Operations Research Letters, 14(5):291–295, 1993.) represents the sequential core as the intersection of the Euclidean projections of the convex hulls of winning coalitions' ideal points onto the admissible domain. This yields invariance to standard blocking conventions, a barycentric support interpretation, and preservation of unrestricted core points under projection, while localizing additional stability to the boundary of a full-dimensional admissibility domain. Our main existence results exploit the effective dimension of the projected no-blocking loci. When these loci are convex and lie in a common affine space of dimension r , an $(r + 1)$ -fold intersection property of winning coalitions guarantees a nonempty sequential core. For symmetric quota rules, a quota above $r/(r + 1)$ of the assembly is sufficient, and this bound is exact for the corresponding coalition-intersection property. Under the leading core-based admissibility rule, the first-stage domain is a polytope. Around almost every exterior point, a whole neighbourhood is projected affinely onto one proper face, so any parliamentary profile in that neighbourhood inherits the lower-dimensional stability condition. In the plane, this reduces the sufficient quota from two-thirds to a strict majority, even when the unrestricted majority core is empty.

Keywords: Popular admissibility; sequential spatial voting; parliamentary stability; sequential core; spatial core; Euclidean projection; multidimensional voting.

JEL Classification: D71; D72; D78; C62; C71.

1. Introduction

Collective decisions are often made in stages. Before a parliament, council, or committee votes on a final policy, some alternatives may already have been ruled out by public consultation, constitutional requirements, prior agreements, political commitments, or procedures designed to identify socially acceptable options. Citizens' assemblies and other participatory institutions provide a visible example of this sequencing. Their role need not be to replace representative institutions or to select the final policy themselves. They may instead shape the set of proposals that representatives are politically able, or institutionally entitled, to consider at the final stage (Fishkin, 2018; Gersbach, 2017, 2024; Gersbach et al., 2026; OECD, 2020).

This distinction between admissibility and final selection matters for stability. In a spatial voting model, an alternative is stable when no decisive coalition can move unanimously to another policy that all of its members prefer. Stability is therefore defined relative to a set of feasible challenges. A prior restriction can leave parliamentary preferences and voting rights unchanged while altering which coalitionary moves are available. Formal political theory has long emphasized that procedures affect outcomes not only through the identity of the decisive coalition, but also through the alternatives that can be brought to a vote: agenda control, structure-induced equilibrium, veto arrangements, and multi-stage institutions all operate by restricting or sequencing political moves (Banks, 1985; Diermeier and Myerson, 1999; Romer and Rosenthal, 1978; Shepsle, 1979; Tsebelis, 1995). The setting considered here isolates one such mechanism: the first stage restricts the policy space over which parliamentary dominance is evaluated. Unlike agenda-setting models, the object is not proposal selection; unlike standard bicameral models, the two stages do not impose separate voting requirements on the same proposal; and popular and parliamentary inputs are not aggregated through a common voting rule or weighting scheme. Parliamentary ideal points and winning coalitions remain unchanged, allowing the effect of admissibility itself to be identified without conflating it with changes in preferences or voting power. The relevant question is consequently not only which alternatives survive the first stage, but also how the geometry of the surviving set changes the collection of deviations that winning coalitions can use against a proposed policy.

The issue becomes particularly important in multidimensional voting. Representative bodies need not reproduce the preference geometry of the underlying population, whether because of electoral rules, districting, apportionment, political selection, or the aggregation of heterogeneous constituency interests (Carey and Hix, 2011; Gallagher, 1991; Lijphart, 1990; Monroe, 1994; Samuels and Snyder, 2001). Independently of this representational gap, multidimensional majority voting faces a familiar stability problem. Under Euclidean preferences, simple majority generically fails to produce a nonempty core when the policy space has at least two dimensions, except under restrictive configurations of ideal points (Davis and Hinich, 1966; Plott, 1967). Stronger quotas can restore core existence by imposing enough intersection among winning coalitions (Nakamura, 1979; Schofield et al., 1988). Experimental evidence also suggests that core predictions are informative when a majority-rule equilibrium exists, while procedural details become especially consequential when such an equilibrium is absent (Fiorina and Plott, 1978).

Formally, the first stage is represented by a popular spatial structure and an admissibility rule. The popular core is the leading specification and gives a direct interpretation to the admissible set as the alternatives that withstand sufficiently broad popular opposition. The parliamentary stage then

asks which of these admissible alternatives are immune to feasible unanimous improvement by every winning coalition. The sequential core is thus a parliamentary stability concept conditional on a prior social admissibility requirement. The two stages perform different functions: the first limits the political menu, while the second determines whether the remaining menu contains a policy that is stable under the parliamentary rule. This formulation keeps the roles of the two stages distinct: a citizens' assembly, referendum-related consultation, constitutional procedure, or prior bargaining body may impose a demanding condition for an alternative to remain admissible even when the final parliamentary decision is taken by ordinary majority. The model therefore permits a situation in which broad agreement is required to define the range of acceptable policies, while a less demanding coalition rule governs final parliamentary stability inside that range.

A key geometric link comes from constrained multiobjective location theory. Carrizosa et al. (1993) study a finite set of demand points and characterize the efficient and weakly efficient locations in a nonempty closed convex feasible set. Under Euclidean distance, both efficiency sets coincide with the closest-point projection of the convex hull of the demand points onto the feasible set. For a parliamentary coalition, the members' ideal points play the role of demand points, while admissible policies play the role of feasible locations. The coalition's no-blocking set is therefore exactly one of these constrained efficiency sets. This characterization is the geometric input imported from location theory. The new voting analysis starts from its simultaneous application to all winning coalitions and studies when the resulting projected loci must intersect. While constrained efficiency has been characterized in location theory (Carrizosa et al., 1993), its implications for parliamentary stability under a prior admissibility restriction have not been studied.

The projection formula also gives a useful interpretation of coalition stability. Every point of a projected coalition hull is Pareto-undominated for that coalition once common deviations are restricted to admissible alternatives, and it can equivalently be obtained as the closest admissible policy to a weighted barycenter of the coalition members' ideal points. The associated weights are coalition-specific certificates of support rather than voting weights: their role is to identify a location in the coalition hull whose projection is the policy under consideration. Carrizosa et al. further show that weak and ordinary efficiency coincide in the Euclidean constrained problem, so the sequential core is invariant to two standard conventions for coalition blocking. Their characterization of properly efficient points also identifies the part of a coalition's stable set that can be supported by strictly positive weights, and their ellipsoidal extension covers a common positive-definite reweighting of the underlying policy dimensions.

Once each winning coalition has been assigned its projected no-blocking locus, the problem becomes one of simultaneous intersection. This is where the voting rule re-enters. The sequential core is the intersection of these loci, and the dimension relevant for that intersection need not be the original policy dimension. The main existence theorem therefore starts from the affine dimension of the projected coalition loci, provided that these sets are convex and lie in a common affine subspace. If that effective dimension is r , the existence of a common member in every collection of at most $r + 1$ winning coalitions guarantees a nonempty sequential core by Helly's theorem (Danzer et al., 1963; Helly, 1923). Equivalently, if $\nu(W)$ denotes the Nakamura number of the parliamentary simple game, the sufficient coalition condition is $\nu(W) > r + 1$. The classical intersection argument can thus be applied in the effective dimension of the projected coalition loci, which may be smaller than the ambient policy dimension. A common projected face is an important specialization of this result. When the projection of every winning-coalition hull coincides with the convex hull of the projected

ideal points and all these sets lie on a face of dimension r , the face supplies both convexity and a common affine space. For symmetric quota rules, the corresponding condition is $\rho > rn/(r+1)$; this inequality is exactly the condition for $(r+1)$ -intersection of the symmetric winning-coalition family.

Under a prior admissibility restriction, the parliamentary requirement can therefore be governed by an effective dimension smaller than the ambient one, so that the first stage supports stability by altering the geometry on which the coalition rule operates. In the leading core-based specification this geometry is generated by the popular profile and influence structure themselves: whenever the popular core is nonempty, it is a polytope whose faces and normal regions determine how parliamentary coalition hulls are transformed.

For a polytopal admissibility domain, the common-face configuration arises locally around almost every exterior point. Euclidean projection is affine on a neighbourhood contained in one normal region and maps that neighbourhood onto a proper face. Hence any parliamentary profile contained in such a neighbourhood satisfies the common-face structure; convergence toward the exterior point is one sufficient way of entering it after a finite threshold. In an ambient space of dimension k , the selected face has dimension at most $k-1$, so k -fold intersection of winning coalitions is sufficient for eventual sequential stability, and a symmetric quota above $(k-1)n/k$ guarantees this property.

The local formulation makes clear that representatives need not converge or coincide for stabilization to occur. It is enough that their ideal points lie simultaneously in one suitable normal region. In the plane, the projected constraints then lie on a common edge or vertex. Properness of the parliamentary simple game suffices, and every strict-majority quota satisfies this requirement. Nearby nonaligned parliamentary profiles may still have an empty unrestricted majority core, so sequential stability can arise while the unconstrained game remains unstable.

The paper also distinguishes four effects of the first stage: preservation, when an unrestricted parliamentary core point is already admissible; relocation, when an inadmissible unrestricted core point is projected to a sequentially stable policy; localization, when disjointness forces sequential stability to the admissible boundary; and creation, when an empty unrestricted core is replaced by a nonempty sequential core because the projected winning-coalition loci intersect. For a full-dimensional admissible set, sequential and unrestricted stability coincide in the interior, so localization and genuinely new stability are necessarily boundary phenomena. This classification links the preservation and projection results directly to the later dimension-reduction mechanism.

The analysis brings together several strands of social choice and political economy. It relates to multi-stage settings in which earlier collective decisions constrain later ones (Levmore, 1992). The paper addresses the long-standing problem of instability in multidimensional voting from a different angle than enlargements or refinements of the core (McKelvey, 1986; Miller, 1980; Saari and Asay, 2010; Wuffle et al., 1989): the object of interest remains the parliamentary core, but the feasible set on which coalitions can challenge a policy is determined at an earlier stage. The contribution of this paper lies in combining the distinction between admissibility and final selection with the geometric characterization of coalition stability, an effective-dimension existence theorem, and a local projection mechanism that links admissibility restrictions to parliamentary stability.

The rest of the paper is organized as follows. Section 2 recalls the spatial voting framework, the unrestricted core, and Euclidean projection. Section 3 introduces the two-stage model and defines

the sequential core. [Section 4](#) links coalition-specific stability to constrained Euclidean efficiency and derives the projection, support, preservation, and boundary results. [Section 5](#) develops the effective-dimension theorem, its common-face and quota implications, the local normal-region mechanism, the convergence corollaries, and the planar interpretation. [Section 6](#) concludes. Proofs are collected in the Appendix.

2. Preliminaries

This section introduces the notation and the spatial voting benchmark used throughout the paper. Alternatives and individual ideal points are represented in a Euclidean policy space, with preferences determined by distance. We then recall the unrestricted core and the coalition-hull characterization on which the later comparison is built. Finally, we introduce Euclidean projection onto a nonempty closed convex set. These two geometric objects, coalition hulls and projections, allow the analysis to distinguish the dimension of the original policy space from the dimension that remains relevant after admissibility is imposed.

Let $n \geq 2$ be an integer, and let $N = \{1, \dots, n\}$ be a finite set of agents. A coalition is any subset of N , and 2^N denotes the family of all coalitions. More generally, for any finite set E , $|E|$ denotes its cardinality and 2^E its power set. For any positive integer k , the set of alternatives is the Euclidean space $\mathbb{R}^k$, endowed with the Euclidean norm $\|\cdot\|$ and the associated distance d , where $d(x, y) = \|x - y\| = (\sum_{\ell=1}^k (x_\ell - y_\ell)^2)^{1/2}$. For any positive real number ρ , $\lfloor \rho \rfloor$ denotes the integer part of ρ , that is, the greatest integer less than or equal to ρ .

2.1. Spatial Voting Games

The decision-making environment consists of k quantifiable dimensions, each corresponding to one coordinate of the Euclidean space $\mathbb{R}^k$. An alternative assigns one value to each of these dimensions. Since no alternative is ruled out a priori, the set of all possible alternatives is identified with the whole space $\mathbb{R}^k$. Each voter $i \in N$ is associated with an alternative that represents the voter's most preferred position. This alternative is called the ideal point, or bliss point, of voter i , and is denoted by $q^i \in \mathbb{R}^k$. The profile of ideal points is denoted by $q = (q^i)_{i \in N} \in (\mathbb{R}^k)^N$. When no confusion can arise, we simply write (q^i) .

A spatial voting game is a triple $v = (N, W, q)$, where W is a nonempty subset of $2^N \setminus \{\emptyset\}$. The family W is interpreted as the set of winning, or decisive, coalitions. Throughout, W is assumed to be monotonic: for all $S, T \subseteq N$, if $S \in W$ and $S \subseteq T$, then $T \in W$. Thus, a coalition that is already decisive cannot lose decisiveness by adding voters. Since W is nonempty, there exists $S \in W$; and since $S \subseteq N$, monotonicity implies that $N \in W$.

An important special case is given by symmetric quota rules. Let $\rho \in \{1, \dots, n\}$. The associated family of winning coalitions is $W_\rho = \{S \subseteq N : |S| \geq \rho\}$. A coalition is therefore decisive if and only if its cardinality reaches the quota. Since decisiveness depends only on coalition size, all voters have the same formal voting weight. This class includes simple majority when $\rho = \lfloor n/2 \rfloor + 1$ and unanimity when $\rho = n$.

This formulation separates two ingredients used repeatedly below. The profile q describes the geometry of parliamentary preferences, whereas W describes the institutional rule by which coalitions become decisive. The same configuration of ideal points can therefore have different stability properties under different coalition rules, and the same voting rule can behave differently as the ideal-point geometry changes. The sequential model adds a third object, the admissible policy set, while leaving these two parliamentary ingredients intact. This makes it possible to identify whether a change in stability comes from the voting rule, from preferences, or from restrictions on the moves available to winning parliamentary coalitions.

2.2. Euclidean Preferences and the Core

Given a spatial voting game $v = (N, W, q)$, preferences are induced by Euclidean distance from ideal points. For $i \in N$ and $x, y \in \mathbb{R}^k$, we write $y \succ_i x$ if and only if $\|y - q^i\| < \|x - q^i\|$. Thus, voter i strictly prefers y to x whenever y is closer to the voter's ideal point than x . For $S \in W$, we write $y \succ_S x$ when $y \succ_i x$ for every $i \in S$. The alternative y then represents a unanimous improvement over x for the decisive coalition S . We write $y \succ x$ when there exists $S \in W$ such that $y \succ_S x$. We denote by $\neg\Delta$ the negation of a statement Δ . Thus, $\neg(y \succ x)$ means that y does not dominate x . The core of v is the set of undominated alternatives:

$$C(v) = \left\{ x \in \mathbb{R}^k : \forall y \in \mathbb{R}^k, \neg(y \succ x) \right\}.$$

An alternative in $C(v)$ is stable because no decisive coalition can unanimously move from it to another alternative that all its members strictly prefer. For any $S \subseteq N$, let $\text{Conv}(S) = \text{conv}\{q^i : i \in S\}$ denote the convex hull of the ideal points of the members of coalition S . Under Euclidean preferences, the core admits the geometric characterization (Martin et al., 2021; Saari, 2014)

$$C(v) = \bigcap_{S \in W} \text{Conv}(S).$$

Hence, stability is equivalent to belonging to the convex hull of every decisive coalition. If an alternative lies outside the hull of some decisive coalition, the members of that coalition have a common direction of improvement. The core is therefore obtained by intersecting the separate unanimity constraints generated by all winning coalitions. This decomposition is especially useful here because the first-stage admissibility restriction will transform each coalition constraint separately before the parliamentary rule requires them to hold simultaneously.

In one dimension, with simple majority and an odd number of voters, the core is nonempty and reduces to the ideal point of the median voter. This benchmark illustrates the stabilizing role of order on a line. In multidimensional policy spaces, this order structure disappears. For $k \geq 2$, the core is generically empty under simple majority (Davis and Hinich, 1966; Plott, 1967).¹

¹Several alternative solution concepts have been developed to address the instability associated with core emptiness in multidimensional spatial voting. Among them are the yolk (McKelvey, 1986; Miller, 2015), the strong point (Owen and Shapley, 1989), the finagle point (Wuffle et al., 1989), the Simpson-Kramer point (Kramer, 1977), the uncovered set (Miller, 1980, 1983), the ϵ -core (Salant and Goodstein, 1990; Tovey, 2010; Wooders, 1983), the finesse point (Saari, 2006; Saari and Asay, 2010), the heart (Schofield, 1995), the soul (Austen-Smith, 1996), the center of power (Godfrey, 2005; Owen and Shapley, 1989), and the δ -core (Martin et al., 2021).

Core nonemptiness can be recovered by imposing more demanding quota requirements. In a symmetric quota spatial voting game, a convenient sufficient condition for nonemptiness in $\mathbb{R}^k$ is $\rho > \frac{k}{k+1}n$. This condition makes decisive coalitions sufficiently intersecting to guarantee a nonempty spatial core (Nakamura, 1979; Schofield et al., 1988). The requirement becomes stronger as the policy space becomes higher-dimensional. For $k = 1$, it corresponds to a quota strictly above one half. For $k = 2$, it requires more than two thirds of the electorate. For $k = 3$, the corresponding threshold is above three fourths of the electorate.

This bound provides the unrestricted benchmark for the results below. It links the ambient dimension k to the amount of overlap required among winning coalitions. The later effective-dimension theorem retains the same intersection logic but applies it after projection, where the relevant affine dimension may be smaller than k . The comparison therefore isolates a precise way in which a prior admissibility stage can alter a quota condition sufficient for stability while leaving parliamentary voting rights unchanged.

2.3. Projection onto Closed Convex Sets

Finally, we recall the Euclidean projection theorem (Bauschke and Combettes, 2017; Deutsch, 2001) and introduce the topological notation used throughout the paper. For any set $E \subseteq \mathbb{R}^k$, we denote by $\text{Int}(E)$ its interior and by ∂E its boundary, both taken with respect to the Euclidean topology of the ambient space $\mathbb{R}^k$. The Euclidean inner product is denoted by $\langle \cdot, \cdot \rangle$.

Let $D \subseteq \mathbb{R}^k$ be a nonempty closed convex set. For every $z \in \mathbb{R}^k$, there exists a unique point $\Pi_D(z) \in D$, called the Euclidean projection of z onto D , such that

$$\|z - \Pi_D(z)\| = \min_{y \in D} \|z - y\|, \quad \text{or equivalently} \quad \Pi_D(z) = \arg \min_{y \in D} \|z - y\|.$$

By construction, $\Pi_D(z)$ is the point of D nearest to z , and $\text{dist}(z, D) = \min_{y \in D} \|z - y\| = \|z - \Pi_D(z)\|$. The projection is characterized by the variational inequality $\langle z - \Pi_D(z), y - \Pi_D(z) \rangle \leq 0$ for every $y \in D$; equivalently, if $p \in D$, then $p = \Pi_D(z)$ if and only if $\langle z - p, y - p \rangle \leq 0$ for every $y \in D$. This characterization will be used repeatedly below. Geometrically, the vector from $\Pi_D(z)$ to z is normal to D at $\Pi_D(z)$, forming a right or obtuse angle with every feasible direction from $\Pi_D(z)$ into D . Hence, if $z \notin D$, then $\Pi_D(z) \in \partial D$, while if $z \in D$, then $\Pi_D(z) = z$. The projection map is nonexpansive, $\|\Pi_D(z) - \Pi_D(z')\| \leq \|z - z'\|$ for all $z, z' \in \mathbb{R}^k$, and in particular continuous. For any set $A \subseteq \mathbb{R}^k$, we extend the notation by writing $\Pi_D(A) = \{\Pi_D(a) : a \in A\}$.

In one dimension, when $D = [a, b]$, the projection has the simple form

$$\Pi_D(z) = \begin{cases} a, & \text{if } z < a, \\ z, & \text{if } z \in [a, b], \\ b, & \text{if } z > b. \end{cases}$$

Projection will be used below to associate points or sets with their closest representatives in a given nonempty closed convex domain. Its interpretation is that of a minimal Euclidean correction: an object located outside the domain is not chosen freely inside the domain, but is moved to the

nearest feasible point. In the sequential voting model, this operation will translate an unrestricted parliamentary position into its closest counterpart inside the admissibility domain selected before parliamentary deliberation begins.

3. Two-Stage Sequential Spatial Voting Games

This section formalizes the two-stage voting environment. The first stage determines which policies remain admissible, while the second stage applies a parliamentary coalition rule to those policies. The ordering matters because a parliamentary deviation that would defeat an alternative in the unrestricted game may no longer be available after the first stage. A planar illustration then displays the basic configurations that the general results will distinguish.

3.1. Popular Admissibility and the First-Stage Domain

The popular chamber represents the social layer through which collective concerns become politically visible. It is modeled as a representative subset of the population. We take this representative subset as given.²

Let m be a positive integer, and let $M = \{1, \dots, m\}$ denote the set of agents in the popular chamber. Each agent $j \in M$ is endowed with an ideal point $p^j \in \mathbb{R}^k$. The associated profile is denoted by $p = (p^j)_{j \in M} \in (\mathbb{R}^k)^M$. When no confusion can arise, we simply write (p^j) .

Popular influence is represented by a nonempty family $I \subseteq 2^M \setminus \{\emptyset\}$. Each coalition $S \in I$ is called a popular coalition. Such a coalition is interpreted as a group whose positions, needs, or constraints are sufficiently relevant to affect the perimeter of collective deliberation. The family I may represent a symmetric quota rule, unanimity, or a more general pattern of popular influence. It identifies which popular groups enter the admissibility test, while the profile p locates their preferred policies and the rule $\mathcal{D}$ translates these ingredients into the feasible political domain. No monotonicity assumption is imposed on I : the first stage only uses the collection of popular coalitions that enter the admissibility construction. This differs from the parliamentary family W , whose monotonicity has a direct interpretation in terms of decisiveness.

The triple $V = (M, I, p)$ is called a popular spatial structure. For $x, y \in \mathbb{R}^k$ and $j \in M$, we write $y >_j x$ if $\|y - p^j\| < \|x - p^j\|$. For $S \in I$, we write $y >_S x$ if $y >_j x$ for every $j \in S$, and $y > x$ when there exists $S \in I$ such that $y >_S x$.³

An *admissibility rule* is a mapping $\mathcal{D}$ that associates with each popular spatial structure V a nonempty closed convex set $\mathcal{D}(V) \subseteq \mathbb{R}^k$. Throughout the paper, we write $D = \mathcal{D}(V)$ for the first-stage admissibility domain: the set of alternatives that remain available for parliamentary deliberation after the popular stage. At the parliamentary stage, the effect of V is summarized by the geometry of this domain. Sections 4 and 5 therefore formulate their results conditional on D , which allows

²The selection of representative subsets of the population is studied in the literature on survey and electoral sampling (Cochran, 1977; Kish, 1965; Stoker and Bowers, 2002), as well as in work on deliberative polling, mini-publics, citizens' assemblies, and sortition (Fishkin, 2018; Flanigan et al., 2021; Gąsiorowska, 2023; OECD, 2020).

³The symbols $>_j, >_S, >$ are reserved for the popular stage, while $\succ_i, \succ_S, \succ$ denote preferences and dominance used in parliamentary deliberation.

the same parliamentary analysis to be applied to different first-stage procedures while retaining the geometry generated by each procedure.

A central specification is generated directly by popular dominance. The popular core is

$$C(V) = \left\{ x \in \mathbb{R}^k : \forall y \in \mathbb{R}^k, \neg(y \succ x) \right\} = \bigcap_{S \in I} \text{conv}\{p^j : j \in S\}.$$

The core-based specification is used on the class of popular spatial structures for which $C(V) \neq \emptyset$, and then $\mathcal{D}(V) = C(V)$. If the popular core is empty, this particular specification does not generate an admissible domain; the sequential model can still be applied once another admissibility rule producing a nonempty closed convex set has been specified.

Because M is finite, the family $I \subseteq 2^M$ is finite and each coalition hull $\text{conv}\{p^j : j \in S\}$ is a polytope; hence a nonempty popular core is itself a compact polytope. The polytopal results developed below therefore apply directly to this leading specification. The broader closed-convex formulation keeps the second-stage analysis modular: the popular profile and influence structure determine the geometry of D , while the parliamentary rule determines which projected coalition constraints must intersect. In the core-based specification, changes in p or I can change the active faces and normal regions of D , and thereby the effective dimension relevant for parliamentary stability.

3.2. Parliamentary Deliberation within the Popular Domain

The second stage is parliamentary. It is described by a standard spatial voting game $v = (N, W, q)$ on $\mathbb{R}^k$, where N is the finite set of parliamentary representatives, $W \subseteq 2^N \setminus \{\emptyset\}$ is the family of winning parliamentary coalitions, and $q = (q^i)_{i \in N} \in (\mathbb{R}^k)^N$ is the profile collecting the ideal points of all parliamentary representatives.

Fix an admissibility rule $\mathcal{D}$ and let $D = \mathcal{D}(V)$. A two-stage sequential spatial voting game is an ordered pair $G = (V, v)$. The first stage fixes the admissibility domain D , and the second stage determines which alternatives in D can be defeated by winning parliamentary coalitions. Parliamentary stability is therefore evaluated within the domain generated by the first stage. The sequential terminology refers to this institutional ordering of admissibility followed by parliamentary choice.

For $x, y \in D$ and $i \in N$, we write $y \succ_i x$ if $\|y - q^i\| < \|x - q^i\|$. For $S \in W$, we write $y \succ_S x$ if $y \succ_i x$ for every $i \in S$. Finally, $y \succ x$ means that there exists $S \in W$ such that $y \succ_S x$.⁴

The sequential core is the set of admissible alternatives that cannot be defeated by any admissible parliamentary deviation:

$$C(G) = \{x \in D : \forall y \in D, \neg(y \succ x)\}.$$

Throughout this paper, the term *sequential core* refers specifically to this two-stage notion: parliamentary stability conditional on the admissibility domain selected at the first stage.

⁴The symbols $\succ_i, \succ_S, \succ$ are the same as in the unrestricted parliamentary game because individual preferences and the winning-coalition rule are unchanged; in the sequential game, candidate alternatives and parliamentary deviations are simply restricted to D .

The popular stage determines the domain D within which parliamentary choice takes place. Both candidate alternatives and the deviations that may defeat them must belong to D . Parliamentary preferences and winning coalitions are those of v , while the first-stage domain determines which parliamentary deviations are feasible. An alternative belongs to $C(G)$ precisely when it is admissible and no winning parliamentary coalition can unanimously improve upon it through another alternative that remains available for parliamentary choice. The sequential core should therefore be read as a joint compatibility requirement. Popular admissibility alone does not imply parliamentary stability, and parliamentary stability in the unrestricted space does not by itself imply admissibility. The solution consists of policies that satisfy the first requirement and remain unblocked once the second-stage deviations are restricted accordingly.

3.3. A Guiding Illustration of the Sequential Model

For the illustration, we use the core-based admissibility rule $\mathcal{D}(V) = C(V)$. Consider a parliamentary chamber with three representatives, $N = \{1, 2, 3\}$, operating under simple majority. Thus a coalition is winning whenever it contains at least two representatives, and the parliamentary game is $v = (N, W, q)$ with $q = (q^1, q^2, q^3)$.⁵

Parliamentary deliberation is preceded by a popular stage with six members, $M = \{1, \dots, 6\}$, and ideal-point profile $p = (p^j)_{j \in M}$. Popular influence is generated by a $5/6$ quota, so the admissible set is $D = C(V) = \bigcap_{S \in I} \text{conv}\{p^j : j \in S\}$. In dimension two, this quota is above the usual $2/3$ sufficient bound for nonempty spatial cores under symmetric rules. The example therefore gives the first stage a well-defined nonempty core while allowing the parliamentary stage to operate under simple majority. The small chambers are chosen only to keep the geometry visible. What matters is the contrast between a demanding admissibility filter and a parliamentary rule that may have an empty unrestricted core.

Begin with a parliamentary profile for which the unrestricted core is nonempty. With three representatives and simple majority in dimension two or higher, the three parliamentary ideal points must be aligned. Otherwise, the three segments joining these points form the sides of a triangle, with each pair of segments intersecting at a vertex that does not belong to the third segment. Hence, the three segments have no common intersection. The intersection of the three two-member coalition hulls is then the intermediate ideal point on that line. Denote this median parliamentary position by q^* , so that $C(v) = \{q^*\}$. Exact alignment is used here only to illustrate stability inherited from the unrestricted core; the boundary-stabilization results below apply to nonaligned parliamentary profiles as well.

⁵Each representative ideal point q^i summarizes the representative's political position. It may combine personal preferences, constituency interests, and a judgment about collective fairness.

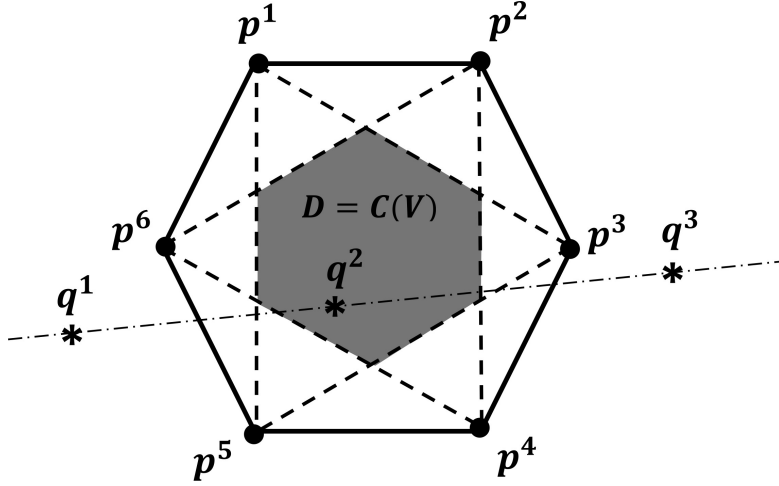


Figure 1: Popular admissibility and median stability. The shaded region is $D = C(V)$; dashed segments show the relevant popular-coalition hulls.

Figure 1 displays the case in which the median parliamentary position is already admissible. The shaded region is the popular core $D = C(V)$; the dashed segments depict the convex hulls associated with the relevant popular coalitions. The three parliamentary ideal points are aligned and $q^* = q^2 \in D$. The same policy therefore satisfies popular admissibility and unrestricted parliamentary stability at the second stage of the procedure.

The figure also indicates the questions that matter once the two stages are combined. If the parliamentary core point lies outside D , one needs to determine whether admissibility destroys stability or moves it to another policy. If the parliamentary ideal points are not aligned, the unrestricted majority core is empty and no median policy is available. The central issue is then whether the boundary of the admissible set can make the winning-coalition constraints intersect even though their unrestricted hulls do not. Sections 4 and 5 answer these questions first for an arbitrary convex domain and then through the lower-dimensional faces of a polytope.

4. Coalitional Efficiency and Sequential Stability in Dimension k

The sequential core can be decomposed into coalition-specific constraints. This decomposition separates the unanimity problem within a coalition from the intersection problem across winning coalitions. Once deviations are restricted to the admissible domain, the first problem is a standard constrained multiobjective problem; the second remains a voting problem because it depends on which coalitions are declared winning by W . Recall that $C(v) = \bigcap_{S \in W} \text{Conv}(S)$, where $\text{Conv}(S) = \text{conv}\{q^i : i \in S\}$.

For every nonempty coalition $S \subseteq N$, define its admissible no-blocking locus by $\mathcal{L}_S = \{x \in D : \forall y \in D, \neg(y \succ_S x)\}$. Since parliamentary dominance is generated by winning coalitions, $C(G) = \bigcap_{S \in W} \mathcal{L}_S$. The locus $\mathcal{L}_S$ records how admissibility changes the blocking possibilities of coalition S , and it is exactly the weakly efficient set of a constrained Euclidean multiobjective location problem whose demand points are the ideal points of the members of S . This equivalence solves the within-coalition problem without introducing a new compromise criterion, so that the

parliamentary content is recovered once the projected efficiency loci are intersected according to the family of winning coalitions specified by the voting rule.

4.1. Coalition Efficiency and the Projection Formula

To distinguish the ordinary Pareto convention, define

$$\mathcal{E}_S = \left\{ x \in D : \begin{array}{l} \text{there is no } y \in D \text{ such that } \|y - q^i\| \leq \|x - q^i\| \text{ for all } i \in S, \\ \text{with a strict inequality for at least one } i \in S \end{array} \right\}.$$

Thus $\mathcal{L}_S$ excludes strict improvement for every member of S , whereas $\mathcal{E}_S$ excludes a weak improvement for all members with a strict gain for at least one.

Carrizosa et al. (1993, Theorem 2). Let $A \subset \mathbb{R}^k$ be a finite nonempty set of demand points and let $D \subseteq \mathbb{R}^k$ be nonempty, closed, and convex. For the Euclidean constrained location problem on D , the weakly efficient and efficient solution sets coincide, and both are equal to $\Pi_D(\text{conv}(A))$. We apply this characterization coalition by coalition, taking the demand points to be the ideal points of the members of the coalition.

Proposition 1. Let $G = (V, v)$ be a two-stage sequential spatial voting game with nonempty closed convex admissibility domain $D = \mathcal{D}(V)$. For every nonempty coalition $S \subseteq N$, the two coalition-specific efficiency notions coincide and satisfy

$$\mathcal{L}_S = \mathcal{E}_S = \Pi_D(\text{Conv}(S)).$$

Consequently, the sequential core is

$$C(G) = \bigcap_{S \in W} \Pi_D(\text{Conv}(S)).$$

Taking $A = \{q^i : i \in S\}$ in the result of Carrizosa et al. (1993) yields the coalition-specific identity. Proposition 1 is therefore the coalition-level representation imported from constrained location theory. The subsequent results use this representation to study the genuinely parliamentary problem: whether the loci associated with all winning coalitions can intersect at a common admissible policy.

A point in $\Pi_D(\text{Conv}(S))$ should accordingly be interpreted as an admissible policy at which coalition S has exhausted its feasible unanimous improvements, obtained as the projection of some point of the coalition hull rather than of a fixed averaging rule or of a particular member's ideal point: its preimage may be any point of $\text{Conv}(S)$. The projection formula therefore converts an unrestricted unanimity condition into the precise Pareto frontier relevant after the first-stage restriction.

The equivalence between weak and ordinary Euclidean efficiency has an immediate implication for parliamentary blocking. The strict-improvement convention commonly used for the spatial core and the Pareto convention based on weak improvement for all members lead to the same sequential solution under the maintained assumptions. The existence results below therefore do not depend on which of these two familiar conventions is used to define a successful coalitionary challenge.

Corollary 1. *Define*

$$\tilde{C}(G) = \left\{ x \in D : \begin{array}{l} \text{there are no } S \in W \text{ and } y \in D \text{ such that} \\ \|y - q^i\| \leq \|x - q^i\| \text{ for all } i \in S, \text{ with strict inequality for some } i \in S \end{array} \right\}.$$

Then $\tilde{C}(G) = C(G)$.

The projection formula also gives a coalition-specific support interpretation. For a nonempty $S \subseteq N$, define the coalition-weight simplex by $\Delta(S) = \{(\lambda_i)_{i \in S} \in \mathbb{R}_+^S : \sum_{i \in S} \lambda_i = 1\}$.

Corollary 2. *An alternative $x \in D$ belongs to $C(G)$ if and only if every winning coalition $S \in W$ admits weights $\lambda^S \in \Delta(S)$ such that*

$$x = \Pi_D \left(\sum_{i \in S} \lambda_i^S q^i \right).$$

Thus, every winning coalition that is unable to block a sequentially stable alternative admits a weighted barycenter of its members' ideal points whose closest admissible point is precisely that alternative. These coefficients are not voting weights and do not alter the rule W . They are coalition-specific supporting weights that certify the absence of a feasible unanimous improvement. Because different winning coalitions may require different supporting barycenters, sequential stability does not rely on a common interpersonal weighting scheme across the parliament.

Remark 1. Carrizosa et al. (1993) also characterize properly efficient locations. Applied to a coalition S , their result gives $PE_S = (\{q^i : i \in S\} \cap D) \cup \Pi_D(\text{ri Conv}(S))$, where ri denotes relative interior. Away from a coalition member's feasible ideal point, proper efficiency can therefore be supported by a barycenter lying in the relative interior of the coalition hull, equivalently by a representation in which the coalition members can all receive strictly positive weights. This refinement sharpens the support interpretation of Corollary 2: a properly efficient point outside the set of feasible individual ideals is backed by a barycenter that places positive weight on every coalition member, so proper efficiency distinguishes coalition-stable policies whose geometric support genuinely uses the whole coalition from policies whose support can be concentrated on a smaller subset of members.

Remark 2. The coalition projection argument extends beyond an unweighted Euclidean policy space. Carrizosa et al. (1993) show that the efficiency characterization holds for any ellipsoidal metric induced by a scalar product. If all parliamentary agents evaluate policy distance through the same positive-definite quadratic form, an invertible linear transformation reduces the problem to the Euclidean case while preserving convexity, polytopality, affine and face dimensions, and coalition intersections. Consequently, the projection representation, the preservation and boundary results of Section 4, and the effective-dimension and local-stabilization results of Section 5 carry over after replacing Euclidean projection by projection for that common scalar product.

4.2. Preservation, Projection, and Boundary Localization

The coalition projection formula also clarifies how the sequential solution relates to the unrestricted parliamentary core. The next results distinguish stability that is inherited from the unrestricted game from stability that is generated by the admissibility boundary of the policy set.

Corollary 3. *If $C(v) \neq \emptyset$, then*

$$\Pi_D(C(v)) \subseteq C(G).$$

In particular, $C(G) \neq \emptyset$, and $D \cap C(v) \subseteq C(G)$.

Hence an unrestricted core point that is already admissible remains stable, while an unrestricted core point outside the admissibility domain has a sequentially stable projection. This gives a strong preservation property: if the parliamentary game already possesses a stable policy, imposing admissibility cannot eliminate sequential stability altogether. The location of the stable policy may change, but the projection of every unrestricted core point continues to satisfy every projected winning-coalition constraint. The remaining question is where stability can arise when it is not inherited in this way.

Proposition 2. *For every two-stage sequential spatial voting game, sequential and unrestricted stability coincide in the interior of the admissible domain:*

$$C(G) \cap \text{Int}(D) = C(v) \cap \text{Int}(D).$$

Consequently, if $D \cap C(v) = \emptyset$, then $C(G) \subseteq \partial D$.

When D has nonempty interior, the proposition gives a substantive localization result: admissibility cannot create a new stable point strictly inside the feasible region. At an interior point, the local set of feasible directions is rich enough that projection does not alter the coalition-hull test, so new stability must arise where the feasibility restriction binds, namely on the boundary. If D is lower-dimensional in the ambient policy space, its Euclidean interior is empty and the boundary statement holds automatically in the ambient topology; the substantive content of the proposition is then specific to the full-dimensional case.

A winning coalition whose entire hull remains strictly inside the admissible domain prevents boundary-generated stability. For that coalition, admissibility leaves the whole unrestricted non-blocking set unchanged, so a candidate sequential core point would have to remain in the interior even though the interior contains no unrestricted core point.

Corollary 4. *Suppose that $D \cap C(v) = \emptyset$. If there exists $S \in W$ such that $\text{Conv}(S) \subseteq \text{Int}(D)$, then $C(G) = \emptyset$.*

When the unrestricted parliamentary core is empty, the same geometry can be written entirely in terms of boundary constraints.

Corollary 5. *If $C(v) = \emptyset$, then*

$$C(G) = \bigcap_{S \in W} [(\text{Conv}(S) \cap \partial D) \cup (\Pi_D(\text{Conv}(S)) \setminus \text{Conv}(S))].$$

The first component consists of coalition-hull points already located on the admissible boundary. The second consists of boundary points reached by projecting parts of the coalition hull that lie outside the admissible set. The representation separates two ways in which a winning coalition can accept a boundary policy: the policy may already satisfy the coalition's unrestricted hull condition,

or it may become efficient only because further movement is ruled out by admissibility. A nonempty sequential core requires one boundary point to satisfy this condition for every winning coalition.

5. Effective-Dimension Stabilization

The projection formula transforms sequential stability into an intersection problem for the family of coalition no-blocking loci $(\mathcal{L}_S)_{S \in W}$. The first result isolates the abstract requirement behind the dimensional reduction: convexity of these loci and the affine dimension of their union. Convexity is a genuine hypothesis at this level. Proposition 1 identifies $\mathcal{L}_S$ with the projection of a convex hull, but Euclidean projection onto a general convex set need not map a convex set to a convex set. The common-face and normal-region results below provide useful sufficient conditions under which the loci are convex and jointly lower-dimensional.

For an integer $h \geq 1$, call the family W h -intersecting if every collection of at most h winning coalitions has a common member. For $h = 2$, this is pairwise intersection. Under the maintained monotonicity of W , pairwise intersection is equivalent to properness of the associated simple game.

For later use, recall that a subset F of a convex set D is a *face* of D if, whenever a point of F lies in the relative interior of a segment contained in D , both endpoints of that segment belong to F . A face is *proper* when $F \neq D$. The affine dimension of F is the dimension of $\text{aff}(F)$. The affine-dimension theorem below applies to any nonempty closed convex domain; the normal-region results in Subsection 5.2 specialize the analysis to polytopes.

5.1. An Affine-Dimension Theorem

Theorem 1. *Let $G = (V, v)$ be a two-stage sequential spatial voting game with admissibility domain $D = \mathcal{D}(V) \subseteq \mathbb{R}^k$. Suppose that every winning-coalition locus $\mathcal{L}_S$ is convex, and define*

$$H = \text{aff}\left(\bigcup_{S \in W} \mathcal{L}_S\right), \quad r = \dim(H).$$

If W is $(r + 1)$ -intersecting, then $C(G) \neq \emptyset$.

The theorem separates coalition geometry from the parliamentary intersection rule. The number r is the effective affine dimension in which all no-blocking loci jointly live. If every collection of at most $r + 1$ winning coalitions share a representative, the projection of that representative's ideal point belongs to every corresponding locus. Helly's theorem (Danzer et al., 1963; Helly, 1923) then converts these local intersections into a common point for the entire family. The effective dimension can be strictly smaller than the ambient policy dimension k .

Remark 3. Let $\nu(W)$ denote the Nakamura number of the parliamentary simple game, that is, the minimum number of winning coalitions with empty intersection, with $\nu(W) = \infty$ when no such finite family exists. The family W is h -intersecting if and only if $\nu(W) > h$. Hence the coalition condition in Theorem 1 can equivalently be written $\nu(W) > r + 1$. The theorem therefore retains the classical Nakamura coalition-intersection logic while replacing the ambient policy dimension by the effective dimension of the projected no-blocking loci.

The face structure used below supplies a transparent way of verifying the convexity and dimensional requirements of Theorem 1.

Corollary 6. *Let $G = (V, v)$ be a two-stage sequential spatial voting game with admissibility domain $D = \mathcal{D}(V) \subseteq \mathbb{R}^k$. Suppose that D has a face F of affine dimension r and, for every $S \in W$,*

$$\Pi_D(\text{Conv}(S)) = \text{conv}\{\Pi_D(q^i) : i \in S\} \subseteq F.$$

If W is $(r + 1)$ -intersecting, then $C(G) \neq \emptyset$.

The equality in Corollary 6 makes each $\mathcal{L}_S$ a convex hull of projected ideal points and places the whole family in $\text{aff}(F)$. Its joint affine dimension may be smaller than r , in which case Theorem 1 applies with the smaller effective dimension. The face dimension therefore provides an immediately observable upper bound on the dimension relevant for stability.

Remark 4. Requiring only the individual projections $\Pi_D(q^i)$ to lie on a common face is not sufficient for Corollary 6. Euclidean projection onto a convex set is generally not affine, so the projection of a coalition hull need not coincide with the convex hull of the projected ideal points. For the same reason, verifying the equality only for minimal winning coalitions does not in general establish it for larger winning coalitions. The equality in the corollary is a sufficient geometric condition that guarantees convexity of all coalition loci and confines them to one affine space. Lemma 1 provides an operational criterion: on the local exterior regions identified there, projection is affine and the equality holds simultaneously for every coalition. Theorem 1 itself requires neither a face nor this equality once convexity and the joint affine dimension of the loci are known.

If F is a vertex, all projected constraints coincide there and the relevant affine dimension is zero. If F is an edge, pairwise intersection of winning coalitions is sufficient. If F has dimension two, triple intersection is sufficient. More generally, each dimension that remains after admissibility raises by one the order of common membership required among winning coalitions.

For symmetric quota rules, the institutional condition has an exact cardinal characterization.

Corollary 7. *Suppose the hypotheses of Corollary 6 hold and that parliamentary winning coalitions are generated by the symmetric quota ρ . If*

$$\rho > \frac{r}{r+1} n,$$

then the sequential core is nonempty. For a symmetric quota rule, this inequality is also equivalent to $(r + 1)$ -intersection of the winning-coalition family W_ρ .

The quota bound is therefore exact for the combinatorial premise used by the face result, although it is not asserted to be necessary for core existence in every particular geometry. Whenever the projected constraints are confined to a proper face, $r \leq k - 1$, and the sufficient parliamentary quota falls below the unrestricted bound recalled in Subsection 2.2. The gain comes from the lower-dimensional geometry on which decisive coalitions must agree, rather than from the size of the admissible set itself.

5.2. Normal Regions and Local Stabilization

For a polytope, the common-face condition can arise from the local geometry of Euclidean projection. For a proper face F of D , let $N_D(F)$ denote its outer normal cone. Away from the lower-dimensional boundaries separating normal regions, nearby exterior points project to the same face and the projection map is affine. A parliamentary profile contained in one such region therefore generates projected coalition hulls that inherit a common affine structure.

Lemma 1. *Let $D \subset \mathbb{R}^k$ be a polytope with nonempty interior. For Lebesgue-almost every exterior point $x \in \mathbb{R}^k \setminus D$, there exist a convex neighbourhood U of x and a proper face F_x of D such that*

$$0 \leq \dim(F_x) \leq k - 1, \quad U \subseteq \mathbb{R}^k \setminus D, \quad \Pi_D(U) \subseteq F_x.$$

Moreover, Π_D is affine on U . Hence, for every finite set $A \subset U$,

$$\Pi_D(\text{conv}(A)) = \text{conv}(\Pi_D(A)) \subseteq F_x.$$

The lemma makes the common-face specialization operational. It does not require the profile to be placed exactly on a normal ray or the representatives to share the same position: a whole neighbourhood of a generic exterior point is mapped affinely to one face. Once every parliamentary ideal point enters that neighbourhood, the projection of each coalition hull is the convex hull of the projected ideal points. The resulting coalition loci are convex and jointly confined to a proper face, so both the general affine-dimension theorem and Corollary 6 apply.

Theorem 2. *Let $D \subset \mathbb{R}^k$ be a polytope with nonempty interior, and let $x \in \mathbb{R}^k \setminus D$ be an exterior point satisfying Lemma 1. Denote by U its associated neighbourhood and by F_x the associated face. If every parliamentary ideal point belongs to U and W is $(\dim(F_x) + 1)$ -intersecting, then*

$$C(G) \neq \emptyset.$$

The required intersection order is governed by $\dim(F_x) \leq k - 1$ rather than by k . Sequential stability holds for every parliamentary profile confined to U , independently of how that profile arrived there; convergence toward x , considered next, is one way of producing such a profile but is not needed for the conclusion.

Corollary 8. *Let $D \subset \mathbb{R}^k$ be a polytope with nonempty interior and suppose that W is k -intersecting. Consider a sequence of parliamentary profiles $q_t = (q_t^i)_{i \in N}$ such that every representative's ideal point converges to the same exterior point $x \in \mathbb{R}^k \setminus D$. For Lebesgue-almost every such x , there exists a finite threshold t^* such that, for every $t \geq t^*$,*

$$C(G_t) \neq \emptyset, \quad G_t = (V, (N, W, q_t)).$$

Convergence of the parliamentary profile toward x guarantees, from some finite t^* onward, exactly the membership $q_t^i \in U$ required by Theorem 2, and k -intersection of W implies $(\dim(F_x) + 1)$ -intersection since $\dim(F_x) + 1 \leq k$; the corollary is proved in the Appendix by this direct substitution. The finite-threshold nature of the statement is worth isolating on its own terms: representatives do not have to reach the common limit point, and their ideal points need not coincide at any finite t .

Sequential stability appears as soon as the whole profile lies in one normal region and the projected coalition geometry is confined to a common lower-dimensional stabilizing face of the admissible polytope.

Corollary 9. *Under a symmetric parliamentary quota rule, the conclusion of Corollary 8 holds whenever $\rho > \frac{k-1}{k} n$.*

For symmetric quota rules, the difference between ambient and effective dimension becomes a difference between voting thresholds. A standard sufficient condition for a nonempty unrestricted core in $\mathbb{R}^k$ is $\rho > \frac{k}{k+1} n$, whereas generic exterior stabilization through a proper face requires only $\rho > \frac{k-1}{k} n$. The gap is substantive even in low dimension. In two dimensions the unrestricted sufficient bound is above two thirds, while the sequential face mechanism requires only a strict majority. In three dimensions the corresponding comparison is between a quota above three fourths and one above two thirds.

Corollary 10. *Let $D \subset \mathbb{R}^2$ be a polygon with nonempty interior and suppose the parliamentary simple game is proper. Consider any sequence of parliamentary profiles in which every representative's ideal point converges to the same exterior point x . For Lebesgue-almost every $x \in \mathbb{R}^2 \setminus D$, the sequential core is nonempty from some finite threshold onward. In particular, every symmetric strict-majority rule satisfies the required intersection condition.*

For symmetric quota rules, the last statement is the specialization of Corollary 9 to $k = 2$. The proper-game formulation is more general because it also covers nonsymmetric simple games.

5.3. Planar Implications and the Guiding Illustration

The planar case makes the argument particularly transparent and links it to a familiar property of simple games. A proper face of a polygon is either a vertex or an edge. If projected winning-coalition constraints fall on a common vertex, they share that point automatically. If they fall on a common edge, they become intervals. Properness guarantees that any two winning coalitions share a representative, so the corresponding intervals intersect; the interval form of Helly's theorem then gives a common point. The first-stage restriction has therefore converted a two-dimensional parliamentary problem into a one-dimensional intersection problem for which ordinary majority overlap can be sufficient.

Return to the setting of Subsection 3.3. If the unrestricted parliamentary core is nonempty, the three representative ideal points are aligned and the core reduces to the median parliamentary option q^* . If $q^* \in D$, Corollary 3 gives $q^* \in C(G)$. If $q^* \notin D$, the projected point $\Pi_D(q^*)$ belongs to $C(G)$.

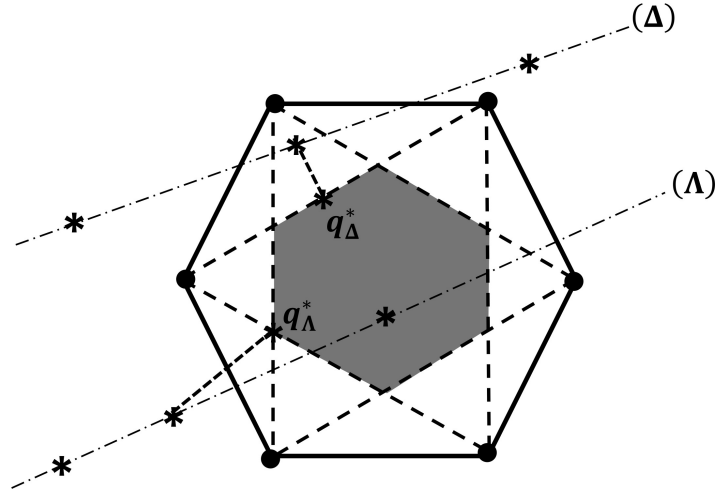


Figure 2: Median projection onto $D = C(V)$. Along (Λ) , $m_\Lambda = q_\Lambda^*$ is admissible; along (Δ) , $m_\Delta \notin D$ and $q_\Delta^* = \Pi_D(m_\Delta)$ is sequentially stable.

Figure 2 illustrates both possibilities. Along each parliamentary line, the aligned ideal points generate a median position. Under the core-based specification used in the illustration, $D = C(V)$; let m_Δ and m_Λ denote the two median positions, and write $q_\Delta^* = \Pi_D(m_\Delta)$ and $q_\Lambda^* = \Pi_D(m_\Lambda)$. When a median already belongs to D , projection leaves it unchanged; otherwise, its closest admissible counterpart is sequentially stable. In this second case, exclusion of the unrestricted equilibrium by the first stage does not prevent a stable outcome from existing at the second stage.

Suppose now that the unrestricted parliamentary core is empty. Figure 3 is drawn at the coalition level, with each triangle representing an illustrative winning-coalition hull. In configuration Γ_1 , projection produces the boundary point x^* . If this point belongs to every winning-coalition locus, Proposition 1 gives $x^* \in C(G)$. There is no unrestricted median to preserve or project; stability is generated by the common boundary geometry imposed on decisive coalitions by admissibility.

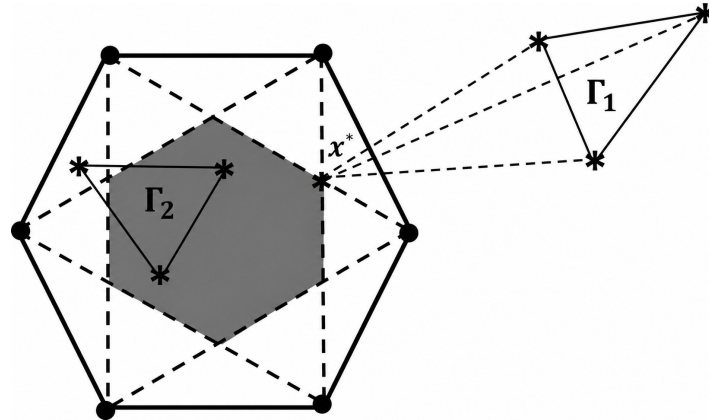


Figure 3: Projection-based stabilization. In Γ_1 , projected coalition hulls meet at $x^* \in C(G)$; in Γ_2 , a winning-coalition hull remains in $\text{Int}(D)$.

The convergence result has the same geometric source. Once the parliamentary ideal points enter a common normal region, their projected hulls lie on a common vertex or edge. With

strict majority, winning coalitions are pairwise intersecting, so Corollary 10 applies even though nearby nonaligned parliamentary profiles may continue to have an empty unrestricted core. The admissibility restriction is therefore doing more than selecting a point close to parliamentary preferences. It changes the geometry of every decisive coalition in a coordinated way, converting several two-dimensional constraints into a family of one-dimensional intervals whose common intersection contains a sequentially stable policy.

Configuration Γ_2 shows the obstruction identified in Corollary 4. A winning coalition whose entire convex hull is contained in $\text{Int}(D)$ retains an unchanged projected constraint in the interior. If $D \cap C(v) = \emptyset$, Proposition 2 excludes any sequentially stable point there, and the sequential core is empty. The examples therefore illustrate three of the four effects distinguished in the Introduction, namely preservation, relocation, and creation of stability, while configuration Γ_2 displays the obstruction to boundary-generated stability.

6. Conclusion

This paper studies parliamentary stability when collective choice is preceded by a stage that determines which policies remain politically admissible. Holding parliamentary preferences and winning coalitions fixed, the sequential core decomposes into two problems: a coalition-level constrained Euclidean efficiency problem, whose solution is the projected coalition hull identified by Carrizosa et al. (1993), and an intersection problem across winning coalitions governed by the voting rule. This decomposition yields invariance to the two standard blocking conventions, a barycentric support certificate for each winning coalition, and the preservation of unrestricted core points through projection. It also shows that, for a full-dimensional admissible set, stability created by the first stage is necessarily a boundary phenomenon. The effects of admissibility can accordingly be classified as preservation, relocation, boundary localization, and creation of stability.

The central result is that the dimension relevant for parliamentary agreement is the effective affine dimension of the projected no-blocking loci rather than the ambient policy dimension. When these loci are convex and span an affine space of dimension r , the Nakamura condition $v(W) > r + 1$ suffices for a nonempty sequential core; for symmetric quotas this becomes $\rho > \frac{r}{r+1}n$, which is exact for the underlying coalition-intersection property. Under the core-based rule, the popular stage generates a polytope, and around almost every exterior point Euclidean projection maps a whole neighbourhood affinely onto a proper face. Any parliamentary profile contained in such a neighbourhood therefore requires only k -fold intersection of winning coalitions. In the plane, properness, and hence every strict-majority quota, suffices, even though the unrestricted majority core may be empty.

These results also refine how the first stage should be interpreted. Its effect is governed not by the size of the admissible domain or by the amount of policy space removed, but by the position of the admissible boundary relative to parliamentary ideal points and the way this boundary transforms the hulls of decisive coalitions. Two admissible sets of similar size can therefore have very different implications for stability. Likewise, the same admissible set can interact differently with majority and supermajority rules because the order of coalition intersection required for a common point depends on the dimension of the face on which the projected constraints are jointly concentrated by the first stage.

The model is consequently suited to institutional settings in which one collective body or procedure delimits an acceptable policy range and another body makes the final decision. In empirical applications, the admissible domain could be constructed from observed constraints associated with a citizens' assembly, an electoral mandate, a constitutional range, or an earlier bargaining stage, while parliamentary ideal points could be estimated from roll-call votes, manifestos, or other spatial measures. The projected-hull representation then provides a direct diagnostic: it indicates whether observed stability is inherited from the unrestricted parliamentary game, obtained by projecting an existing core point, or supported by the lower-dimensional geometry of the admissible boundary.

The same framework suggests further theoretical questions without changing the distinction on which the analysis is built. One may endogenize the formation of the admissibility domain, characterize first-stage rules that systematically generate low-dimensional common faces, study how strategic actors anticipate the effect of admissibility on later parliamentary coalitions, or allow the policy metric to vary across groups. The results in this paper provide a benchmark for such extensions by identifying the exact coalition-level efficiency sets and the intersection conditions that turn them into parliamentary stability. The broader implication is that the dimension relevant for political agreement need not coincide with the dimension in which preferences are originally expressed; institutional sequencing can determine the lower-dimensional set on which decisive coalitions ultimately have to agree.

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Appendix: Proofs of Results

Proof of Proposition 1. Fix a nonempty coalition $S \subseteq N$, and write $Q_S = \{q^i : i \in S\}$. In the constrained Euclidean location problem with demand set Q_S and feasible set D , $\mathcal{L}_S$ is the weakly efficient set and $\mathcal{E}_S$ the efficient set. Theorem 2 of Carrizosa et al. (1993) therefore gives

$$\mathcal{L}_S = WE(Q_S, D) = E(Q_S, D) = \Pi_D(\text{conv}(Q_S)) = \Pi_D(\text{Conv}(S)) = \mathcal{E}_S.$$

This proves the coalition-specific identity. Restricting to winning coalitions and using the definition of parliamentary dominance gives $C(G) = \bigcap_{S \in W} \mathcal{L}_S = \bigcap_{S \in W} \Pi_D(\text{Conv}(S))$, as required. ■

Proof of Corollary 1. For every $S \in W$, Proposition 1 gives $\mathcal{L}_S = \mathcal{E}_S$. Hence $\tilde{C}(G) = \bigcap_{S \in W} \mathcal{E}_S = \bigcap_{S \in W} \mathcal{L}_S = C(G)$. ■

Proof of Corollary 2. By Proposition 1, $x \in C(G)$ if and only if $x \in \Pi_D(\text{Conv}(S))$ for every $S \in W$. For a fixed winning coalition, $\text{Conv}(S) = \{\sum_{i \in S} \lambda_i q^i : \lambda \in \Delta(S)\}$. Thus $x \in \Pi_D(\text{Conv}(S))$ if and only if some $\lambda^S \in \Delta(S)$ satisfies $x = \Pi_D(\sum_{i \in S} \lambda_i^S q^i)$. Applying this equivalence to every winning coalition proves the result. ■

Proof of Corollary 3. Take $z \in C(v)$. Since $z \in \text{Conv}(S)$ for every $S \in W$, we have $\Pi_D(z) \in \Pi_D(\text{Conv}(S))$ for every winning coalition. Proposition 1 then yields $\Pi_D(z) \in C(G)$. Hence

$\Pi_D(C(v)) \subseteq C(G)$, so $C(v) \neq \emptyset$ implies $C(G) \neq \emptyset$. If moreover $z \in D \cap C(v)$, then $\Pi_D(z) = z$, giving $D \cap C(v) \subseteq C(G)$. ■

Proof of Proposition 2. Fix $S \in W$. We first show that $\Pi_D(\text{Conv}(S)) \cap \text{Int}(D) = \text{Conv}(S) \cap \text{Int}(D)$. The inclusion from right to left is immediate because projection fixes every point of D . Conversely, let $x = \Pi_D(z) \in \text{Int}(D)$ for some $z \in \text{Conv}(S)$, and suppose $z \neq x$. Since x is interior, there exists $\varepsilon \in (0, 1)$ such that $w = x + \varepsilon(z - x) \in D$. But $\|z - w\| = (1 - \varepsilon)\|z - x\| < \|z - x\|$, contradicting $x = \Pi_D(z)$. Hence $z = x \in \text{Conv}(S)$, proving the coalition-specific identity.

Intersecting over W and using Proposition 1,

$$\begin{aligned} C(G) \cap \text{Int}(D) &= \left(\bigcap_{S \in W} \Pi_D(\text{Conv}(S)) \right) \cap \text{Int}(D) = \bigcap_{S \in W} (\text{Conv}(S) \cap \text{Int}(D)) \\ &= \left(\bigcap_{S \in W} \text{Conv}(S) \right) \cap \text{Int}(D) = C(v) \cap \text{Int}(D). \end{aligned}$$

If $D \cap C(v) = \emptyset$, this set is empty. Since $C(G) \subseteq D$ and D is closed, $C(G) \subseteq D \setminus \text{Int}(D) = \partial D$. ■

Proof of Corollary 4. Let $S \in W$ satisfy $\text{Conv}(S) \subseteq \text{Int}(D)$. Projection fixes this hull, so Proposition 1 gives $C(G) \subseteq \Pi_D(\text{Conv}(S)) = \text{Conv}(S) \subseteq \text{Int}(D)$. Yet $D \cap C(v) = \emptyset$ and Proposition 2 imply $C(G) \cap \text{Int}(D) = \emptyset$. Therefore $C(G) = \emptyset$. ■

Proof of Corollary 5. Assume $C(v) = \emptyset$. Proposition 2 gives $C(G) \subseteq \partial D$. Hence, by Proposition 1,

$$C(G) = \left(\bigcap_{S \in W} \Pi_D(\text{Conv}(S)) \right) \cap \partial D = \bigcap_{S \in W} (\Pi_D(\text{Conv}(S)) \cap \partial D).$$

Fix $S \in W$. We claim that $\Pi_D(\text{Conv}(S)) \cap \partial D = (\text{Conv}(S) \cap \partial D) \cup (\Pi_D(\text{Conv}(S)) \setminus \text{Conv}(S))$. The first set on the right is contained in the left because projection fixes D . If $x \in \Pi_D(\text{Conv}(S)) \setminus \text{Conv}(S)$, write $x = \Pi_D(z)$ with $z \in \text{Conv}(S)$. Then $z \notin D$, since otherwise $x = z \in \text{Conv}(S)$; therefore $x \in \partial D$. This proves the inclusion from right to left. The reverse inclusion follows by separating the cases $x \in \text{Conv}(S)$ and $x \notin \text{Conv}(S)$. Substitution into the preceding intersection yields the stated formula. ■

Proof of Theorem 1. Let $H = \text{aff}(\bigcup_{S \in W} \mathcal{L}_S)$ and $r = \dim(H)$. Each $\mathcal{L}_S$ is nonempty by Proposition 1 and convex by hypothesis.

Consider $S_1, \dots, S_j \in W$ with $1 \leq j \leq r + 1$. Since W is $(r + 1)$ -intersecting, choose $i^* \in \bigcap_{\ell=1}^j S_\ell$. Then $q^{i^*} \in \text{Conv}(S_\ell)$ for every ℓ , and Proposition 1 gives $\Pi_D(q^{i^*}) \in \mathcal{L}_{S_\ell}$ for every ℓ . Thus every subfamily of at most $r + 1$ loci has nonempty intersection.

Because N is finite, so is W . Helly's theorem applied in the r -dimensional affine space H yields $\bigcap_{S \in W} \mathcal{L}_S \neq \emptyset$. This intersection is $C(G)$, proving the theorem. ■

Proof of Corollary 6. For every $S \in W$, the hypothesis and Proposition 1 give $\mathcal{L}_S = \text{conv}\{\Pi_D(q^i) : i \in S\} \subseteq F$. Thus each locus is convex. If $r_{\text{eff}} = \dim \text{aff}(\bigcup_{S \in W} \mathcal{L}_S)$, then $r_{\text{eff}} \leq \dim(F) = r$. Hence $(r + 1)$ -intersection implies $(r_{\text{eff}} + 1)$ -intersection, and Theorem 1 gives $C(G) \neq \emptyset$. ■

Proof of Corollary 7. Assume first that $\rho > \frac{r}{r+1}n$. Take $S_1, \dots, S_j \in W_\rho$ with $1 \leq j \leq r+1$, and suppose their intersection is empty. Then $N = \bigcup_{\ell=1}^j (N \setminus S_\ell)$, so

$$n \leq \sum_{\ell=1}^j |N \setminus S_\ell| \leq j(n - \rho) \leq (r+1)(n - \rho),$$

which implies $\rho \leq \frac{r}{r+1}n$, a contradiction. Thus W_ρ is $(r+1)$ -intersecting, and Corollary 6 gives $C(G) \neq \emptyset$.

For the converse equivalence, suppose $\rho \leq \frac{r}{r+1}n$. Then $n - \rho > 0$ and $n/(n - \rho) \leq r+1$. Set $j = \lceil n/(n - \rho) \rceil \leq r+1$, and partition N into nonempty sets $B_1, \dots, B_j$, each of cardinality at most $n - \rho$. Define $S_\ell = N \setminus B_\ell$. Then $|S_\ell| \geq \rho$, so every S_ℓ is winning, while $\bigcap_{\ell=1}^j S_\ell = N \setminus \bigcup_{\ell=1}^j B_\ell = \emptyset$. Hence W_ρ is not $(r+1)$ -intersecting, establishing the equivalence. ■

Proof of Lemma 1. For a proper face F of D , let $\mathcal{R}(F) = \text{relint}(F) + \text{ri } N_D(F)$ be its normal cell. The normal-cone characterization of Euclidean projection implies that every $z \in \mathcal{R}(F)$ satisfies $\Pi_D(z) \in \text{relint}(F)$. On this cell, projection onto D coincides with orthogonal projection onto $\text{aff}(F)$ (Schneider, 2014, Section 1.2); hence Π_D is affine on $\mathcal{R}(F)$.

A polytope has finitely many faces. Its normal cells cover $\mathbb{R}^k \setminus D$ except for common boundaries contained in a finite union of polyhedral sets of dimension below k , hence of zero k -dimensional Lebesgue measure. Therefore, for Lebesgue-almost every exterior point x , there is a unique proper face F_x whose normal cell contains x in its relative interior.

Choose a convex neighbourhood U of such an x with $U \subset \mathcal{R}(F_x)$. Then $U \subseteq \mathbb{R}^k \setminus D$, $\Pi_D(U) \subseteq F_x$, and Π_D is affine on U . For any finite $A \subset U$, convexity gives $\text{conv}(A) \subset U$, and affinity gives $\Pi_D(\text{conv}(A)) = \text{conv}(\Pi_D(A)) \subseteq F_x$. This proves the lemma. ■

Proof of Theorem 2. Assume $q^i \in U$ for every $i \in N$, and for $S \subseteq N$ set $Q_S = \{q^i : i \in S\}$. Then $Q_S \subset U$. Lemma 1 gives, for every winning coalition,

$$\Pi_D(\text{Conv}(S)) = \Pi_D(\text{conv}(Q_S)) = \text{conv}(\Pi_D(Q_S)) = \text{conv}\{\Pi_D(q^i) : i \in S\} \subseteq F_x.$$

Thus Corollary 6 applies with $F = F_x$ and $r = \dim(F_x)$. Since W is $(\dim(F_x) + 1)$ -intersecting, $C(G) \neq \emptyset$. ■

Proof of Corollary 8. Fix an exterior point x for which Lemma 1 holds, with associated neighbourhood U and face F_x . Let $r_x = \dim(F_x)$, so $r_x + 1 \leq k$.

For every $i \in N$, $q_t^i \rightarrow x$ implies $q_t^i \in U$ for all sufficiently large t . Since N is finite, there is a common t^* such that $q_t^i \in U$ for every $i \in N$ and every $t \geq t^*$. Moreover, k -intersection of W implies $(r_x + 1)$ -intersection. Theorem 2 therefore gives $C(G_t) \neq \emptyset$ for every $t \geq t^*$. Since the lemma holds for Lebesgue-almost every exterior x , the result follows. ■

Proof of Corollary 9. Take $S_1, \dots, S_j \in W_\rho$ with $1 \leq j \leq k$, and suppose $\bigcap_{\ell=1}^j S_\ell = \emptyset$. The complement argument gives

$$n \leq \sum_{\ell=1}^j |N \setminus S_\ell| \leq j(n - \rho) \leq k(n - \rho),$$

so $\rho \leq \frac{k-1}{k}n$, contradicting the hypothesis. Thus W_ρ is k -intersecting, and Corollary 8 applies. ■

Proof of Corollary 10. For a monotonic simple game, properness is equivalent to pairwise intersection of winning coalitions. When $k = 2$, this is exactly the 2-intersection condition required by Corollary 8. Under a symmetric strict-majority rule, $\rho > n/2$, so $|S_1| + |S_2| \geq 2\rho > n$ for any two winning coalitions S_1, S_2 ; hence $S_1 \cap S_2 \neq \emptyset$. The rule is therefore proper, and the same conclusion follows. ■

Declaration of Competing Interests

The authors declare that they have no competing interests.