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An axiomatization of the iterated h-index and applications to sport rankings☆

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Abstract

A variant of the h-index introduced in García-Pérez (2009), called the iterated h-index, is studied to evaluate the productivity of scholars. It consists of successive applications of the h-index so as to obtain a vector of h-indices. In particular, the iterated h-index fixes a drawback of the h-index since it allows for (lexicographic) comparisons of scholars with the same h-index. Two types or results are presented. Firstly, we provide an axiomatic characterization of the iterated h-index, which rests on a new axiom of consistency and extensions of axioms in the literature to a richer framework. Secondly, we apply the h-index and iterated h-index to offer alternative sport rankings in tennis, football and basketball. These applications clearly demonstrate that the iterated h-index is much more appropriate than the classical h-index.

Keywords: h-index, iterated h-index, axioms, sports ranking.
JEL code: D70, Z20

1. Introduction

The h-index (Hirsch, 2005) evaluates the individual performance of scholars based on the publications and their citations. It is equal to the integer h if h of his or her publications have at least h citations each, and his or her other publications have at most h citations each. Hirsch (2005) shows that the h-index is very suitable to measure the scientific production of theoretical

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physicists. Ever since, the h-index has been very popular and is nowadays widely used in numerous academic domains.

Nevertheless, the h-index suffers from some drawbacks inherent to its simplicity. For instance, a scholar with few fundamental publications possessing each a huge number of citations has a small h-index. Many variants of the h-index has been proposed to cope with these difficulties (see for instance Bornmanna et al., 2011, among others). Another drawback is that many scholars typically end up with equal small h-index, which means that the h-index cannot discriminate among them. This article considers a richer framework than the one usually considered in the literature and studies a variant of the h-index introduced in García-Pérez (2009), called the iterated h-index, to deal with this last problem. Our framework is richer in that an index assigns to each publication/citation vector a vector of integers with the following lexicographic interpretation. If a first index vector contains as least as many components as a second index vector, and if these components are at least as large as in the second index vector, then the scholar associated with the first index vector is considered as at least as productive as the scholar associated with the second index vector. We think that too much information is perhaps lost when computing one-dimensional indices. In this article, the (possibly) multi-dimensional indices can be seen as a trade-off between the original data (the publication vector) and a one-dimensional index. The iterated h-index belongs to this category: it contains possibly many components (dimensions), each of which resulting from the application of the classical h-index to a specific subset of the publication vector. In particular, the iterated h-index has at most as many components as the number of publication of the studied scholar. Our iterated h-index only slightly differs from the so-called multidimensional h-index in García-Pérez (2009) with respect to the treatment of non-cited publications. We obtain two types of results.

Firstly, we provide an axiomatic characterization of the iterated h-index by means of five axioms that are either new or adapted from axioms invoked in several characterizations of the h-index in the simpler classical framework. The recent and growing literature on the axiomatic characterizations of the h-index has been initiated in Woeginger (2008a). The first axiom imposes that the index has a unique component equal to one in the benchmark case where the scholar has a unique cited publication with a unique citation. The second axiom states that the index should be multiplied by an integer c if first, the number of citations of each publication is multiplied by c and second, the resulting publication vector is replicated c times (adapted from Quesada, 2011b). The third axiom requires that the index should not vary if the number of citations of only the "best" publications increases. In the classical framework, similar axioms are called Independence of irrelevant citations and Head-independence in Quesada (2011b) and Kongo (2014), respectively. The fourth axiom states that the first components of the index should not be affected if publications with a small number of citations are added. The fifth axiom imposes that if the "best" publications are removed, then the resulting index should be obtained from the original one by removing its "best" components. In other words, if two scholars a and b differ only with respect to the "best" publications in the sense that the publication vector of scholar a is obtained from the publication vector of b by deleting b's "best" publications, then a's index should be obtained by from b's index by deleting its "best" components. This axiom of consistency is new and is key to distinguish the iterated h-index from the h-index. Beyond the above-mentioned articles, other characterizations of the h-index are contained in Woeginger (2008b), Quesada (2010, 2011a), Hwang (2013), Miroiu (2013) and Bouyssou and Marchand (2014), where the latter article compares various indices from an axiomatic perspective. Other axiomatic approaches to construct index of scientific performance are developed in Palacios-Huerta and Volij (2004, 2014), Chambers and Miller (2014), Bouyssou and Marchand (2016) and Perry and Reny (2016), among others. For completeness, let us mention that García-Pérez (2009) does not provide axiomatic foundations of his multidimensional h-index. Beyond introducing the multi-dimensional h-index, García-Pérez (2009) presents some of its properties, and calibrates the productivity of professors of Methodology of the Behavioral Sciences in Spain.

Secondly, we apply both the h-index and the iterated h-index to sport rankings. More specifically, our approach is adapted to sports with duels such as tennis, football and basketball. For such a sport, the list of publications of a scholar is replaced by the list of matches won by a player or a team, while the number of citations of each publication is replaced by the number of match won by each player/team defeated by the studied player/team. Based on the 2106 European football leagues and NBA regular seasons, we clearly underline that the h-index has a limited ranking power in that too many players/teams end up with the same h-index, even if they have very different seasonal records. To the contrary, this is not much less the case with the iterated h-index. We also point out that the iterated h-index can be used as a good proxy for NBA ranking, and provides new insights for ATP tennis ranking. For the case of European football leagues, where typically several teams are close to each other in the ranking, the use of the ih-index can lead to substantial changes. As an example, in the 2015 French league, Rennes would move from position 9 to position 15, losing approximately 3 millions euros in the distribution of the TV rights associated to the current season's performance. We also discuss the impact of the competition structures of these sports on the iterated h-index.

The rest of the article is organized as follows. Section 2 provides definitions and notation. Section 3 introduces and motivates our axioms, and states and proves the axiomatic characterization of the iterated h-index. Section 4 presents the application to sport rankings. Section 5 concludes.

2. Index and iterated index

2.1. A richer class of indices

A scholar with some publications is formally described by a vector $x = (x_1, \ldots, x_{n_x})$ with nonnegative integer components $x_1 \geq x_2 \geq \cdots \geq x_{n_x}$; the kth component x_k of this vector states the total number of citations to this scholar's kth-most important publication. Let X denote the set of all finite vectors x, including the empty vector. For any $x \in X$, n_x^0 denotes the number of cited publications, i.e. $n_x^0 = \max\{k = 1, \ldots, n_x : x_k > 0\}$. We say that a vector $x = (x_1, \ldots, x_{n_x})$ is dominated by a vector $y = (y_1, \ldots, y_{n_y})$, if $n_x \leq n_y$ holds and if $x_k \leq y_k$ for $k = 1, \ldots, n_x$; we will write $x \leq y$ to denote this situation.

An (generalized) index is a function $f : X \longrightarrow X$ that assigns to each $x \in X$ a vector $f(x) = (f_1(x), \ldots, f_{q_x}(x))$ such that

- if $x = \varnothing$ or $x = (0, \ldots, 0)$, then $f(x) = \varnothing$;
- if $x \leq y$, then $f(x) \leq f(y)$.

The first item requires that the index is empty (i.e. has zero coordinate or equivalently $q_x = 0$) for each vector without any citation. The interpretation that we propose for the index is based on lexicographic comparisons. A scholar x is considered as at most as productive as a scholar y if $f(x)$ is lexicographically dominated by $f(y)$.[1] For the rest of the article, for any index f and any vector x, keep in mind that n_x and q_x stands for the number of components in x and $f(x)$, respectively.

For an index f on X, $x \in X$ and $c = 1, \ldots, q_x$, let $s(f, x, c) = \sum_{k=1}^{c} f_k(x)$, and set $s(f, x, 0) = 0$ by convention. Abusing notation, if $x = (x_1, \ldots, x_{n_x})$, we shall sometimes write $f(x_1, \ldots, x_{n_x})$ instead of $f((x_1, \ldots, x_{n_x}))$. Finally, let $X^1 \subseteq X$ be the (sub)class of vectors x such that $x_{n_x} \geq n_x$. In this generalized setup, we restate the h-index and introduce an iterated version of it.

The h-index assigns to each publication vector an integer h if h publications have at least h citations each, and if the other publications have at most h publications each. Below is the definition of the h-index adapted to our richer framework.

Formally, the **h-index** (Hirsch, 2005) is the index h on X which assigns to each $x \in X$ the vector $h(x) = (h_1(x))$ such that

$$h_1(x) = \max\{k = 1, \ldots, n_x : x_k \geq k\} \tag{1}$$

[1] Other interpretations are discussed in section 3.3.

if $x_1 > 0$ and $h_1(x) = \varnothing$ otherwise.

The iterated h-index consists of several successive applications of the h-index. More specifically, its first component is obtained by a first classical application of the h-index. If this h-index is equal to c, then the most c-th cited publications are removed, and the h-index is applied another time to the resulting smaller vector. This yields the second component of the iterated h-index. This step is repeated until all cited publications have been treated. As such, the iterated h-index permits to discriminate among scholars with the same h-index.

Formally, the **iterated h-index** is the index ih on X which assigns to each $x \in X$ the vector $ih(x) = (ih_1(x), \ldots, ih_{q_x}(x))$ such that for all $k = 1, \ldots, q_x$,

$$ih_k(x) = \max\{c = 1, \ldots, n_x - s(ih, x, k-1) : x_{s(ih,x,k-1)+c} \geq c\}$$

and $ih(x) = \varnothing$ if x is either empty or $x_1 = 0$.

By definition, $ih_1(x) = h_1(x)$, and $ih_1(x) \geq \cdots \geq ih_{q_x}(x)$. Note also that $h(x) = ih(x) = (n_x)$ for all $x \in X^1$. Furthermore, it is easy to check that $\sum_{k=1}^{q_x} ih_k(x) = n_x^0$, i.e. the sum of the components' value of the iterated h-index add up to the number of cited publications. The iterated h-index is the same as the multidimensional h-index in García-Pérez (2009), except that we associate with empty vectors or non-cited publications an empty component whereas García-Pérez (2009) uses a zero component.

As an example, pick $x = (9, 9, 7, 6, 6, 5, 4, 4, 2, 1, 1, 0)$. Then one has $h(x) = (5)$ and $ih(x) = (5, 3, 1, 1, 1)$. These computations are even easier to grasp by drawing the picture represented in Figure 1.

As mentioned in the introduction, our aim is to distinguish among scholars characterized by the same h-index. This is a reason why we use lexicographic comparisons. Hence, if $x = (9, 9, 7, 6, 6, 5, 4, 4, 2, 1, 1, 0)$ as before and if $y = (6, 6, 6, 6, 6, 6)$, so that $ih(y) = (6)$, then we consider that scholar x is less productive than scholar y.

2.2. Operations on X

For any vectors $x \in X$ and $y \in \mathbb{N}_*^{n_x}$, define the **addition of** x **and** y as the vector $(x + y)$ of dimension n_x such that $(x + y)_k = x_k + y_k$ for each $k = 1, \ldots, n_x$.

For any $x \in X$ and $c \in \mathbb{N}$, the **c-expansion of** x is the vector denoted by $(c \otimes x) \in X$ of dimension cn_x is defined, for all $k = 1, \ldots, cn_x$ as $(c \otimes x)_k = cx_{\lceil k/c \rceil}$, where for each real number $a \in \mathbb{R}_+$, $\lceil a \rceil$ is the smallest integer greater than or equal to a. In words, the number of each citation in x is multiplied by c and then, the resulting publications are copied c times. Also, for any $x \in X$ and $c \in \mathbb{N}$, the **c-multiplication of** x is the vector $cx \in X$ is given by $(cx_1, \ldots, cx_{n_x})$. As an example, if $x = (4, 4, 3, 1)$ and $c = 3$, then $(c \otimes x) = (12, 12, 12, 12, 12, 12, 9, 9, 9, 3, 3, 3)$ and $cx = (12, 12, 9, 3)$.

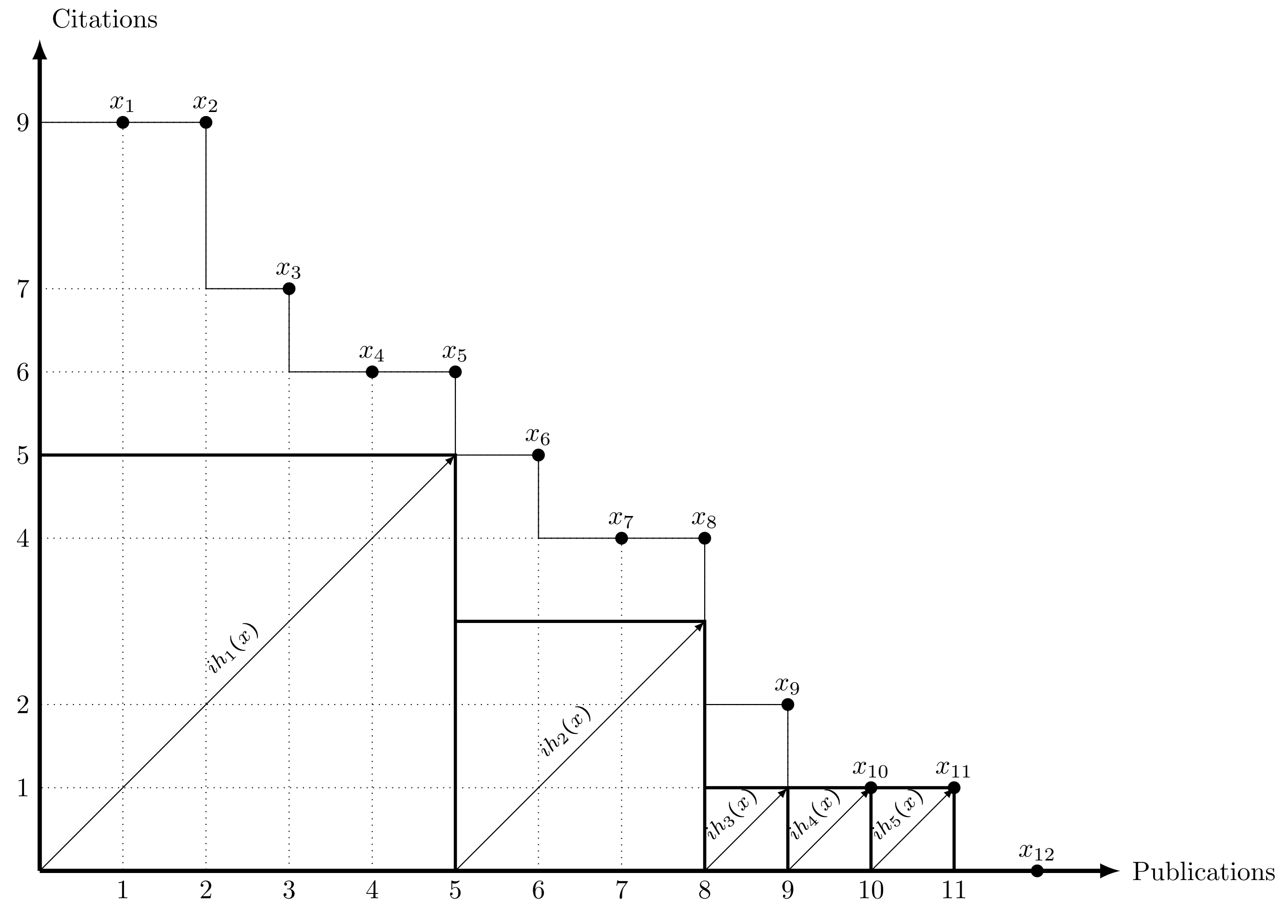

Figure 1: Graphical representation

For all $x \in X$ and $c \in \mathbb{N}$, define $d(x,c) = \arg\min_{k=1,\ldots,n_x}\{x_k : x_k < c\}$ if $c > x_{n_x}$ and $d(x,c) = n_x + 1$ if $c \leq x_{n_x}$ as the lowest position in x such that the associated publication as strictly less than c citations if such a position exists and $x_{n_x} + 1$ otherwise. Furthermore, for all $x \in X$ and $c \in \mathbb{N}$ define the **union of x with a c-cited publication** as the vector $x \cup (c)$ obtained from x by adding a publication with c citations in position $d(x,c)$ (each less-cited publication being moved from its original position to the immediately next one). Formally:

- $(x \cup (c))_k = x_k$ if $x_k \geq c$;
- $(x \cup (c))_k = x_{k-1}$ if $x_{k-1} < c$;
- $(x \cup (c))_{d(x,c)} = c$.

As an example, if $x = (6,5,5,4,3,1,1)$ and $c = 4$, then $d(x,c) = 5$ (since the publication with 3 citations is in position 5) and $x \cup (c) = (6,5,5,4,\mathbf{4},3,1,1)$ where the newly added publication is highlighted in bold.

For all $x \in X$ and $k = 1,\ldots,n_x$, define the **vector x without its k-th most cited publication** as $x\backslash(x_k) = (x_1,\ldots,x_{k-1},x_{k+1},\ldots,x_{n_x})$.

3. Axiomatic study

We begin this section by listing the axioms that we invoke. References to versions of the axiom already existing in the literature on the h-index are given in the introduction of the article and are not repeated here. Then we demonstrate the main characterization as well as an instructive preliminary result.

3.1. Axioms

This first axiom is new and provides a benchmark or normalization. If a researcher has a unique cited publication (and thus possibly many publications without any citation), and if this publication has received a unique citation, then the index has a unique component equal to 1.

One citation case (OC) If $\sum_{k=1}^{n_x} x_k = 1$, then $f(x) = (1)$.

The second axiom says that adding citations to the $f_1(x)$-th most cited publications has no impact on the index. The associated publications can be considered as the best of the studied scholar, and the axiom means that extra citations for these publications does not improve the scholar's productivity, ceteris paribus. In this sense, the added citations can be considered as superfluous.

Independence of superfluous citations (ISC) For all $x \in X$ and $y \in \mathbb{N}^{n_x}$ such that $y_k = 0$ whenever $k > f_1(x)$, if $(x + y) \in X$ then $f(x + y) = f(x)$.

The third axiom states that multiplying by c the number of citations and then by c the number of publications (as in a c-expansion) amounts to multiply by c the index. In other words, a change in scale of the scholar's production vector leads to the same change in scale for each of his/her index's component.

Homogeneity (H) For all $x \in X$ and all $c \in \mathbb{N}$, $f(c \otimes x) = cf(x)$.

The fourth axiom states that adding publications with at most $f_k(x)$ citations has no impact on the first k components of the index. In this sense, such ("weak") publications can be considered as irrelevant for these ("better") components. In order to state this axiom, for any index f on X and any $x \in X$, we adopt the convention $f_{q_x+1}(x) = 0$.

Independence of irrelevant publications (IIP) For all $x \in X$, all $k = 1, \dots, q_x + 1$ and all $c \in \mathbb{N}$ such that $c \leq f_k(x)$, it holds that $f_j(x) = f_j(x \cup (c))$ for each $j = 1, \dots, k$.

The last axiom involves the most $s(f, x, c)$-th cited publications. It states that these publications are removed (and are "rewarded" according to the first c components of the index in a sense), then the index of the new situation is the original index deprived of its first c components. In an other sense, the axiom means that deleting the best publications does not change the last components of the index.[2]

Consistency (C) For all $x \in X$ and $c = 1, \dots, q_x$, $f(x \backslash (x_1, \dots, x_{s(f,x,c)})) = f(x) \backslash (f_1(x), \dots, f_c(x))$.

3.2. Results

We start by proving a preliminary result on the class X^1, which states that axioms **OC**, **ISC** and **H** already characterize the classical h-index for the particular publication vectors in X^1. Since the ih-index coincides with the h-index on X^1, this result also characterizes the ih-index on this class.

Lemma 1. *An index f on X^1 satisfies **OC**, **ISC** and **H** if and only if $f = h$.*

[2] A more general version of this axiom can be stated by removing the publications "associated with" any set of components.

Before proving Lemma 1, note that **OC**, **ISC** and **H** are well-defined on X^1. More specifically, among the vectors of the form $x = (1, 0, \ldots, 0)$ that can be considered in **OC**, only $x = (1)$ belongs to X^1. Furthermore, for any $x \in X^1$ and $y \in \mathbb{N}^{n_x}$, note also that the vector $(x + y)$ is in X^1 if and only if $(x + y)$ is in X. Similarly, for any $x \in X^1$ and any $c \in \mathbb{N}$, $(c \otimes x) \in X^1$ as well.

Proof. It is clear that h satisfies the three axioms on X^1, and that $h(x) = (n_x)$ for all $x \in X^1$. Conversely, consider any index f on X^1 satisfying the three axioms. Pick any $x \in X^1$, so that it must be that $x_1 > 0$. Since $x_{n_x} \geq n_x$, x can be expressed as $x = (z + y)$, with $z \in X^1$ and $y \in \mathbb{N}^{n_x}$ such that $z = (n_x, \ldots, n_x)$ and $y = (x_1 - n_x, \ldots, x_{n_x} - n_x)$. It holds that $z = (n_x \otimes (1))$, so that **H** implies that $f(z) = f(n_x \otimes (1)) \overset{\mathbf{H}}{=} n_x f(1)$. Moreover, **OC** yields that $f(1) \overset{\mathbf{OC}}{=} (1)$. Thus, $f(z) = n_x(1) = (n_x)$. In particular, we have $f_1(z) = n_x$. Coming back to y, since y has n_x coordinates, **ISC** can be applied to z and y: $f(x) = f(z + y) \overset{\mathbf{ISC}}{=} f(z) = (n_x)$. Conclude that $f(x) = h(x)$. ∎

Proposition 1 below relies on Lemma 1 and add axioms **IIP** and **C** in order to characterize the ih-index on the full domain of publication vectors.

Proposition 1. *An index f on X satisfies **OC**, **ISC**, **H**, **IIP** and **C** if and only if $f = ih$.*

Proof. It is easy to check that ih satisfies the five axioms on X. Conversely, let f be any index satisfying the five axioms on X. To show that f is uniquely determined. So let $x \in X$. Since $f(x) = \varnothing$ if x is either empty or $x_1 = 0$ by definition of an index f, the shall only consider vectors x with some cited publications. For each $k = 1, \ldots, q_x$, denote by $x^{(k)}$ the sub-vector of x containing the publications in x those position is between $s(ih, x, k-1) + 1$ and $(s(ih, x, k)$, that is, $x^{(k)} = (x_{s(ih,x,k-1)+1}, \ldots, x_{s(ih,x,k)})$. So $x^{(1)}$ contains the $ih_1(x)$-th most cited publications, $x^{(2)}$ the next most $ih_2(x)$-th cited publications and so on until all cited publications have been taken into account. For each $k = 1, \ldots, q_x$, by definitions of ih and $x^{(k)}$, it holds that $x^{(k)} \in X^1$ since $x^{(k)}_{n_{x^{(k)}}} \geq ih_k(x) = s(ih, x, k) - s(ih, x, k-1) = n_{x^{(k)}}$. In particular, if $x \in X^1$, then $x = x^{(1)}$. Furthermore, it is easy to check that $ih_1(x^{(k)}) = ih_k(x)$ for each $k = 1, \ldots, q_x$. Thus, by Lemma 1, we have

$$f(x^{(k)}) = (f_1(x^{(k)})) = (ih_1(x^{(k)}) = (ih_k(x)) \tag{2}$$

for each $k = 1, \ldots, q_x$. For the rest of the proof, we demonstrate that $f_k(x)$ coincides with $ih_k(x)$ by induction on k.

Initialization. For $k = 1$, from the previous arguments $ih_1(x) = ih_1(x^{(1)}) = f_1(x^{(1)})$, and $ih_1(x) \geq s(ih, x, 1)$. Furthermore, for any $j \geq s(ih, x, 1)$, $x_j \in x^{(k)}$ for some $k = 2, \ldots, q_x$, and thus $x_j \leq s(ih, x, 1)$. This means that **IIP** can be used to obtain $f_1(x^{(1)}) \overset{\mathbf{IIP}}{=} f_1(x^{(1)} \cup (x_j))$. Thus, repeated applications of **IIP** yield

$$ih_1(x) = ih_1(x^{(1)}) = f_1(x^{(1)}) \overset{\textbf{IIP}}{=} f_1(x^{(1)} \cup \cdots \cup x^{(q_x)}) = f_1(x)$$

which means that $f_1(x) = ih_1(x)$ as desired.
Induction hypothesis. Assume that $f_k(x) = ih_k(x)$ for each $k < q$, $q = 2, \ldots, q_x$.
Induction step. Consider the component $f_q(x)$ of $f(x)$. Since each component $f_k(x)$, $k = 1, \ldots, q-1$, is known and coincides with $ih_k(x)$, $k = 1, \ldots, q-1$, by the induction hypothesis, an application of **C** yields that

$$f(x^{(q)} \cup \cdots \cup x^{(q_x)}) = f(x \backslash (x^{(1)} \cup \cdots \cup x^{(q-1)})) \overset{\textbf{C}}{=} f(x) \backslash (f_1(x), \ldots, f_{q-1}(x)) = (f_q(x), \ldots, f_{q_x}(x)).$$

In particular, this means that $f_q(x^{(q)} \cup \cdots \cup x^{(q_x)}) = f_q(x)$. Moreover, similarly as in the initialization, by **IIP**, we can write that

$$f_q(x^{(q)} \cup \cdots \cup x^{(q_x)}) \overset{\textbf{IIP}}{=} f_q(x^{(q)}) = ih_q(x),$$

where the last equality is the consequence of Lemma 1 pointed out in (2). Thus $f_q(x) = ih_q(x)$ for all $q = 1, \ldots, q_x$. This means that the most $s(ih(x, q_x) = n_x^0$-th cited publications have been treated. By **C**, we have

$$f(x \backslash (x_1, \ldots, x_{n_x^0}) \overset{\textbf{C}}{=} f(x) \backslash (f_1(x), \ldots, f_{q_x}(x))$$

Since $(x \backslash (x_1, \ldots, x_{n_x^0})$ is either empty or of the form $(0, \ldots, 0)$, The left-hand side is empty by definition of an index. As a consequence, the right-hand side is empty too, proving that f cannot have more nonempty coordinates. This completes the proof that $f = ih$. ■

It is worth noting that Proposition 1 provides an alternative formulation of the iterated h-index: for any $x \in X$, and $k = 1 \ldots, q_x$, it is given by $ih_k(x) = h(x^{(k)})$. The proof that the axioms in Proposition 1 are logically independent is made by exhibiting the following index on X:

- The h-index on X satisfies **OC**, **ISC**, **H**, **IIP** but violates **C**.
- The index f on X such that for each $x \in X$, $f(x) = \varnothing$ satisfies **ISC**, **H**, **IIP**, **C** but violates **OC**.
- The index f on X such that for each $x \in X$, $f(x) = (1)$ if $x_1 > 0$ and $f(x) = \varnothing$ otherwise satisfies **OC**, **ISC**, **IIP**, **C** but violates **H**.
- The index f on X such that for each $x \in X$, $f(x) = (n_x^0)$ if $n_x^0 \neq 0$ and $f(x) = \varnothing$ otherwise satisfies **OC**, **ISC**, **H**, **C** but violates **IIP**.

- The index f on X such that for each $x \in X$, $f(x) = (\min_{k=1,\ldots,n_x}\{n_k : x_k > 0\})$ if $x_1 > 0$ and $f(x) = \emptyset$ otherwise satisfies **OC**, **H**, **IIP**, **C** but violates **ISC**.

As a final remark, we can suggest a characterization of the h-index in our framework of possibly multidimensional indices. As pointed out in the preceding paragraph the h-index satisfies **OC**, **ISC**, **H** and **IIP**. The combination of these four axioms is not sufficient to characterize the h index by Proposition 1. A characterization can be obtained by strengthening axiom **IIP** as follows.

Strong independence of irrelevant publications (SIIP) For all $x \in X$, all $k = 1,\ldots,q_x + 1$ and all $c \in \mathbb{N}$ such that $c \leq f_1(x)$, it holds that $f(x) = f(x \cup (c))$.

This new axiom imposes that an index is invariant to adding a new publication with at most as citations as the first component of the index. It shares some similarities with axiom C14 – Square rightwards in Bouyssou and Marchand (2014). Combining **OC**, **ISC** and **H** with **SIIP** yields a characterization of the h-index given by (1). The proof is similar to those of Proposition 1 and is omitted.

3.3. Discussion

Alternative interpretations. Until now, we have adopted a lexicographic interpretation in order to compare scholars by means of their respective ih-index. Many other criteria are conceivable. As an example, scholars can also be compared via the Lorenz dominance (see Sen, 1973, for an introduction to this literature). Consider two scholars x and y, and their ih-index $ih(x) = (ih_1(x),\ldots,ih_{q_x}(x))$ and $ih(y) = (ih_1(y),\ldots,ih_{q_y}(y))$, respectively. According to the Lorenz dominance, scholar x is said to be as productive as scholar y if for all $k \in \{1,\ldots,\max\{q_x,q_y\}\}$, it holds that

$$\sum_{i=1}^{k} ih_i(x) \geq \sum_{i=1}^{k} ih_i(y).$$

Contrary to the lexicographic interpretation, it is obvious that the Lorenz domination does not yield a total order on the set of scholars, as pointed out in the next example. Suppose that $x = (12,9,9,7,7,6,3,2)$ and $y = (9,8,7,7,7,6,5,4)$, so that $ih(x) = (6,2)$ and $ih(y) = (5,4)$. Scholars x and y cannot be compared by using the Lorenz dominance since $ih_1(x) > ih_1(y)$ but $ih_1(x) + ih_2(x) < ih_1(y) + ih_2(y)$.

Another variant in the same spirit. The ih-index improves upon the h-index by processing the information contained in the tail of the publication record, i.e. by gratifying the citations of the least-cited publications. Another variant of the h-index can be constructed by considering more finely the head of the distribution instead of its tail. More specifically, the h-index potentially excludes some of the citations of the most-cited publications. Similarly to the ih-index, it is possible to further apply the h-index to these "remaining" citations, leading to a new multidimensional index, exactly as what the ih-index does for the "remaining" publications. In order to be clear, let us come back to the example depicted in Figure 1. The h-index singles out the five best publications. However, the first two publications have each 4 more citations than necessary to attain this result. Similarly, the three other concerned publications have 2, 1 and 1 more citations than necessary, respectively. Applying iteratively the above-mentioned principle, we obtain the multidimensional index $(5, 2, 2)$, where the first 2 indicates that within the five best publications, two have at least two "remaining" citations each, and the three other have at most two "remaining" citations each. The last 2 has a similar interpretation except that it deals only with the best two publications, and that the number of "remaining" citations is calculated after removing $5 + 2$ citations. García-Pérez (2012) even considers the possibility to combine extensions of the h-index in the tail and head areas simultaneously.

4. Alternative sport rankings based on the *h*-index and iterated *h*-index

In this section, we consider several sport competitions in which the ih-index can be calculated. ATP tennis, NBA basketball and European national football leagues are investigated. The first objective is to determine whether the ih-index provides relevant alternative rankings to the official rankings (ATP ranking for tennis, winning percentage for NBA, and total points by receiving three points for a win and one point for a draw for football leagues). The second objective is to discuss how the various competition formats influence the ih-index.

4.1. Tennis

For the computation of the ih-index, we have extracted data from the official website of the ATP (http://www.atpworldtour.com/). International tennis competition is mostly based on five types of events: the four grand slam, the ATP world tour, which includes the other most prestigious tournaments, the ATP challenger tour and the ITF circuit, composed of less prestigious tournaments, and the Davis cup, a team competition. In what follows, we only take into account grand slam and the ATP world tour matches, including qualification matches. Each professional player is associated with a vector in which each integer is the number of wins achieved a player he has defeated at grand slams and ATP world tour tournaments.

As an example, Mikhail Youzhny (ranked 127th at the 2015 ATP year-end ranking) is associated with vector $(41, 38, 23, 23, 22, 20, 19, 18, 18, 17, 14, 13, 10, 9, 0)$. This means that for the 2015 ATP season, Youzhny has won 15 grand slam and the ATP world tour matches, including qualifications. Among these wins, Gilles Simon is the defeated player with most wins (41), Viktor Troicki is the defeated player with the second most number of wins (38), and so on. The zero at the end of the vector corresponds to Youzhny's win against Yassine Idmbarek, a low-ranked player who had no win at grand slam and the ATP world tour level in 2015. The *ih*-index of Mikhail Youzhny is thus $(12, 2)$.

Table 1 summarizes, for the 2015 season, the *ih*-index (and the corresponding ranking), the ATP year-end ranking, the total number of ATP points, and the differences in these rankings for the top 50 players (according to the *ih*-index). If two players have the same total number of points or the same *ih*-index, ties shall be broken by using the the most total points from the grand slams as used in the official ATP ranking.

Table 1 reveals the following facts. The two ranking systems (ATP and *ih*-index) agree on the best 6 players. Moreover, 9 of the top 10 ATP players also belong to the top 10 *ih*-index players. Jo-Wilfried Tsonga (ATP–10, *ih*-index–17) is the only exception, because an injury prevented him from playing as many tournaments as the other top 10 players. Regarding the *ih*-index top 50, 47 players also belong to the ATP top 50, with some notable differences explained below. The major difference between the two ranking systems have four main sources. Firstly, as for Tsonga, some players have played less tournaments than the average, even if they enjoyed good performances. Beyond Tsonga, this is the case for Cilic, among others. Secondly, some players have played at the ATP challenger tour level, or even at the ITF future tour level. Among the best players, Benoit Paire is an example. He started the 2015 season with a low ranking, which forces him to play less prestigious tournament during the first tier of the season. Since we do not count such tournaments in our study, it is not surprising to observe that is *ih*-index ranking is lower than his ATP ranking. Similar explanations can be put forward for Leonardo Mayer. Thirdly, some players have been very successful in the less prestigious category of ATP world tour tournaments. Thus, they accumulated wins but not so many points. Examples of such players, having a better *ih*-index ranking than ATP ranking, are Dominic Thiem and Joao Sousa. Fourthly, we count qualification wins which provide only a small number of points. The ranking of some players is not good enough to enter main draws directly, so that they sometimes win many qualification matches. This is the case for Baghdatis and Bolelli, among others. They also achieve a better *ih*-index ranking than ATP ranking.

The tennis ranking provided by the *ih*-index is in line with other alternative rankings proposed in the literature, for instance by Dahl (2012). The *ih*-index is also useful to evaluate the strength of tennis players across years. In the past 2013 and 2014 ATP seasons, the players with the best *ih*-index were the two number one in the world: Rafael Nadal, $(36, 22, 15, 5)$ in 2013, and Novak Djokovic, $(35, 19, 9, 2)$ in 2014. Both players have a smaller *ih*-index than Novak Djokovic in 2015, which is among the best seasons ever achieved by a player on the ATP tour. In 2015, Djokovic's

Player	ih-index ranking	ih-index	ATP ranking	ATP points	Difference
Novak Djokovic	1	$(\mathbf{37}, 26, 14, 4, 1)$	1	16 585	=
Andy Murray	2	$(\mathbf{31}, 23, 6, 2, 1)$	2	8 945	=
Roger Federer	3	$(\mathbf{31}, 18, 11, 1)$	3	8 265	=
Stan Wawrinka	4	$(\mathbf{30}, 16, 6, 2)$	4	6 865	=
Rafael Nadal	5	$(\mathbf{28}, 19, 9, 3)$	5	5 230	=
Tomas Berdych	6	$(\mathbf{27}, 20, 7, 2, 1)$	6	4 620	=
Key Nishikori	7	$(\mathbf{26}, 16, 7, 2)$	8	4 235	▲1
John Isner	8	$(\mathbf{25}, 13, 5, 2)$	11	2 495	▲3
Richard Gasquet	9	$(\mathbf{25}, 13, 4, 1)$	9	2 850	=
David Ferrer	10	$(\mathbf{24}, 18, 9, 2)$	7	4 305	▼3
Gilles Simon	11	$(\mathbf{24}, 12, 4)$	15	2 145	▲4
Kevin Anderson	12	$(\mathbf{23}, 15, 4, 2, 2)$	12	2 475	=
Roberto Bautista Agut	13	$(\mathbf{22}, 12, 4, 2)$	25	1 480	▲12
Ivo Karlovic	14	$(\mathbf{22}, 12, 3)$	23	1 485	▲9
Dominic Thiem	15	$(\mathbf{22}, 12, 2)$	20	1 600	▲5
Gaël Monfils	16	$(\mathbf{21}, 10, 2)$	24	1 485	▲8
Jo-Wilfried Tsonga	17	$(\mathbf{21}, 9, 2)$	10	2 635	▼7
Milos Raonic	18	$(\mathbf{21}, 9, 2)$	14	2 170	▼4
Viktor Troicki	19	$(\mathbf{21}, 8, 4)$	22	1 487	▲3
Feliciano Lopez	20	$(\mathbf{21}, 8, 2, 1)$	17	1 690	▼3
Guillermo Garcia-Lopez	21	$(\mathbf{21}, 8, 2)$	27	1 430	▲6
Joao Sousa	22	$(\mathbf{21}, 8, 2)$	33	1 191	▲11
Bernard Tomic	23	$(\mathbf{20}, 11, 3, 2, 1)$	18	1 675	▼5
Steve Johnson	24	$(\mathbf{20}, 10, 6, 1)$	32	1 240	▲8
David Goffin	25	$(\mathbf{20}, 10, 4)$	16	1 880	▼9
Grigor Dimitrov	26	$(\mathbf{20}, 10, 2)$	28	1 360	▲2
Jack Sock	27	$(\mathbf{20}, 9, 4)$	26	1 465	▼1
Alexandr Dolgopolov	28	$(\mathbf{20}, 9, 2)$	36	1 135	▲8
Marin Cilic	29	$(\mathbf{19}, 11, 4)$	13	2 405	▼16
Simone Bolelli	30	$(\mathbf{19}, 10, 4)$	58	790	▲28
Philipp Kohlschreiber	31	$(\mathbf{19}, 7, 4)$	34	1 185	▲3
Fabio Fognini	32	$(\mathbf{19}, 7, 3)$	21	1 515	▼11
Marcos Baghdatis	33	$(\mathbf{19}, 5, 1)$	46	933	▲13
Gilles Muller	34	$(\mathbf{18}, 10, 1)$	38	1 105	▲4
Nick Kyrgios	35	$(\mathbf{18}, 7)$	30	1 260	▼5
Benoit Paire	36	$(\mathbf{18}, 6, 3, 1, 1)$	19	1 633	▼17
Borna Coric	37	$(\mathbf{18}, 5)$	44	941	▲7
Thomaz Belluci	38	$(\mathbf{17}, 8, 3, 2)$	37	1 105	▼1
Adrian Mannarino	39	$(\mathbf{17}, 7, 2)$	47	930	▲8
Jérémy Chardy	40	$(\mathbf{17}, 7, 1, 1)$	31	1 255	▼9
Pablo Cuevas	41	$(\mathbf{17}, 7, 1)$	40	1 065	▼1
Vasek Pospisil	42	$(\mathbf{16}, 7, 3, 1)$	39	1 075	▼2
Andreas Seppi	43	$(\mathbf{16}, 6, 3, 1)$	29	1 360	▼14
Donald Young	44	$(\mathbf{16}, 6, 1)$	48	907	▲4
Martin Klizan	45	$(\mathbf{16}, 5, 4, 1, 1)$	43	980	▼2
Jerzy Janowicz	46	$(\mathbf{16}, 5, 1)$	57	795	▲11
Lukas Rosol	47	$(\mathbf{16}, 4, 1)$	55	797	▲8
Fernando Verdasco	48	$(\mathbf{15}, 7, 2)$	49	900	▲1
Tommy Robredo	49	$(\mathbf{15}, 6, 1)$	42	1 000	▼7
Leonardo Mayer	50	$(\mathbf{15}, 5, 1, 1)$	35	1 150	▼15

Table 1: Tennis season 2015.

Team	*ih*-index ranking	*ih*-index	Conference ranking	Winning %	Difference
Toronto Raptors	1	$(\mathbf{35}, 17, 4)$	2	0.683	▲1
Cleveland Cavaliers	2	$(\mathbf{33}, 18, 6)$	1	0.695	▼1
Atlanta Hawks	3	$(\mathbf{33}, 12, 3)$	4	0.585	▲1
Miami Heats	4	$(\mathbf{33}, 12, 3)$	3	0.585	▼1
Boston Celtics	5	$(\mathbf{33}, 11, 4)$	5	0.585	=
Charlotte Hornets	6	$(\mathbf{32}, 12, 4)$	6	0.585	=
Detroit Pistons	7	$(\mathbf{32}, 10, 2)$	8	0.537	▲1
Indiana Pacers	8	$(\mathbf{31}, 11, 3)$	7	0.549	▼1
Chicago Bulls	9	$(\mathbf{31}, 10, 1)$	9	0.512	=
Washington Wizards	10	$(\mathbf{30}, 10, 1)$	10	0.500	=
Orlando Magic	11	$(\mathbf{27}, 8)$	11	0.427	=
Milwaukee Bucks	12	$(\mathbf{24}, 9)$	12	0.402	=
New York Knicks	13	$(\mathbf{23}, 9)$	13	0.390	=
Brooklyn Nets	14	$(\mathbf{19}, 2)$	14	0.256	=
Philadelphia 76ers	15	$(\mathbf{10})$	15	0.122	=

Table 2: NBA 2016 regular season – Eastern conference.

ih-index even surpasses Federer's *ih*-index $(37, 24, 13, 5, 3)$ in his great 2006 season. Similarly, Ruiz et al. (2013) rely of a data envelopment analysis to assess tennis players' performances. Finally, it would be nice to determine whether the *ih*-index is a better predictor for the outcome of tennis matches than the ATP official ranking, which is used by Clarke and Dyte (2000) and del Corrala and Prieto-Rodríguez (2010) to predict grand slam tournaments outcomes.

4.2. Basketball

Data come from http://www.basketball-reference.com/leagues/NBA_2016_games.html. In the 2016 NBA regular season, each of the 30 teams plays 82 matches against each other, and the ranking among them is calculated on the basis of the winning percentage. Teams are grouped into two conferences (Eastern and Western), and the 8 top teams in each conference are qualified from a playoff tournament which determines the NBA champion. In this section, we only study the regular season, and compare the official NBA ranking with those provided by the *ih*-index. Statistics are contained in tables 2 and 3.

These tables call up the following comments. Firstly, for the NBA regular season, a team has qualified for the playoff via the official NBA ranking if and only if it has also qualified by means of the *ih*-index ranking. In other words, the two rankings agree on the eight first teams in both conferences, but not on their orders. Secondly, it should be noted that many teams achieve the same winning percentage (for instance 4 teams in the eastern conference), which necessitates to use tie breaking rules. In the *ih*-index ranking, only two teams are in that case. Thirdly, even if the lists of qualified teams are the same under the two ranking systems, there are nevertheless small changes in the rankings that can have important consequences for the playoff phase. The reason is that the position in the bracket (and so the potential advantages going with a good position,

Team	*ih*-index ranking	*ih*-index	Conference ranking	Winning %	Difference
Golden State Warriors	1	$(\mathbf{41}, 23, 9)$	1	0.890	=
San Antonio Spurs	2	$(\mathbf{37}, 22, 8)$	2	0.817	=
Oklahoma City Thunder	3	$(\mathbf{33}, 17, 5)$	3	0.671	=
Los Angeles Clippers	4	$(\mathbf{33}, 17, 3)$	4	0.646	=
Portland Trails Blazzers	5	$(\mathbf{31}, 12, 1)$	5	0.537	=
Memphis Grizzlies	6	$(\mathbf{30}, 10, 2)$	7	0.512	▲1
Houston Rockets	7	$(\mathbf{30}, 10, 2)$	8	0.500	▲1
Dallas Mawericks	8	$(\mathbf{29}, 11, 2)$	6	0.512	▼2
Utah Jazz	9	$(\mathbf{29}, 10, 1)$	9	0.488	=
Denver Nuggets	10	$(\mathbf{26}, 7)$	11	0.402	▲1
Sacramento Kings	11	$(\mathbf{24}, 9)$	10	0.402	▼1
New Orleans Pelicans	12	$(\mathbf{24}, 6)$	12	0.366	=
Minnesota Timberwolves	13	$(\mathbf{23}, 6)$	13	0.354	=
Phoenix Suns	14	$(\mathbf{20}, 3)$	14	0.280	=
Los Angeles Lakers	15	$(\mathbf{16}, 1)$	15	0.207	=

Table 3: NBA 2016 regular season – Western conference.

such as playing a low-ranked team and the home-court advantage) depends on the rankings in the regular season. As an example, in the eastern conference, the ranking of the top 2 teams is inverted when the *ih*-index replaces the official winning percentage. The consequence is that Cleveland would have lost the home-court advantage in the conference final against Toronto. The *ih*-index and NBA winning percentage agree on the first four teams in each conference (but not in the same order in the Eastern conference), which means that the home-court advantage would be the same with the two ranking systems in the first round of playoffs. Fourthly, there is no change in ranking for the 14 teams that did not qualify for the playoff phase. Here too, these positions are important for the NBA draft, which is the annual event during which all NBA teams can draft promising players who are eligible and wish to join the league. The reason is that these 14 worst teams are assigned the first 14 choices by a lottery in which the probability to obtain the first choice is decreasing with the team ranking. Taylor and Trogdon (2002) and Price et al. (2010) point out that teams eliminated from playoffs can strategically lose games at the end of the season in order to increase their probability to get the first draft choice, while Lenten (2016) shows that a team's performances increase when this perverse incentive is eliminated. Motomura et al. (2016) prove that building a team through the draft is not the most successful strategy. Finally, we can point out a difference with the study on tennis. Each NBA team plays a fixed number of matches. In that sense, a NBA team cannot improve its *ih*-index by playing more games, contrary to a tennis player who can add extra tournaments to his calendar.

4.3. Football

The main European football leagues share the same ranking system. Each team plays twice against each other team, and add 3 points to its total in case of a win, and 1 point in case of

Team	ih-index ranking	ih-index	League ranking	Points	Difference	2015 TV rights
Paris	1	$(\mathbf{12}, 8, 4)$	1	83	=	15 714 696
Lyon	2	$(\mathbf{12}, 8, 2)$	2	75	=	13 663 055
Marseille	3	$(\mathbf{12}, 7, 2)$	4	69	▲1	10 323 682
Monaco	4	$(\mathbf{12}, 7, 1)$	3	71	▼1	11 873 326
Saint-Etienne	5	$(\mathbf{12}, 7)$	5	69	=	8 970 472
Bordeaux	6	$(\mathbf{12}, 5)$	6	63	=	7 802 783
Guingamp	7	$(\mathbf{12}, 3)$	10	49	▲3	4 452 497
Montpellier	8	$(\mathbf{11}, 5)$	7	56	▼1	6 787 876
Lille	9	$(\mathbf{11}, 5)$	8	56	▼1	5 893 011
Nice	10	$(\mathbf{11}, 2)$	11	48	▲1	3 874 109
Caen	11	$(\mathbf{11}, 1)$	13	46	▲2	2 924 680
Bastia	12	$(\mathbf{11}, 1)$	12	47	=	3 372 112
Reims	13	$(\mathbf{11}, 1)$	15	44	▲2	2 215 336
Toulouse	14	$(\mathbf{11}, 1)$	17	42	▲3	1 669 686
Rennes	15	$(\mathbf{10}, 3)$	9	50	▼6	5 129 102
Nantes	16	$(\mathbf{10}, 1)$	14	45	▼2	2 542 725
Lorient	17	$(\mathbf{9}, 3)$	16	43	▼1	1 920 685
Evian	18	$(\mathbf{7}, 4)$	18	37	=	0
Lens	19	$(\mathbf{7})$	20	29	▲1	0
Metz	20	$(\mathbf{7})$	19	30	▼1	0

Table 4: 2015 French league.

a draw. The only difference is the number of teams in the league, which is 20 for the French, Spanish, Italian and English leagues, and only 18 for the German league. Contrary to the NBA, there are no playoffs: the top-ranked team wins the championship. The league ranking determines which teams qualify for the UEFA champions league and the Europa league, and which teams are relegated to the second division league. On top of that, the ranking is also crucial for teams in order to obtain the best possible share in the TV (broadcasting) rights distribution. Table 4 provides an example based on the 2015 French league, where the last column indicates the share of the TV rights obtained by each teams for its current season's official ranking (the total is about 25% of the total TV rights for the French case). Data come from Wikipedia. As for tennis and basketball, we have chosen to use the same tie-breakers as for the official ranking.

Before discussing the particular case of the 2015 season, it should be noted that tie/draw results are not taken into account by the ih-index. As a consequence, the ih-index provides an incentive for teams to win that is similar to the rule giving three points for a win (instead of two) adopted by all national leagues for many years (see Guedes and Machado, 2002; Dilger and Geyer, 2009, for instance). The fairness of the three-point rule is sometimes disputed as underlined in Bring and Thuresson (2011), and we think that ih-index can be considered as a relevant consensual alternative.

Table 4 reveals some substantial differences between the ih-index and the official league ranking. Firstly, even if the five teams qualified for the European competitions are the same, the third spot for the champions league goes to Marseille instead of Monaco if the *ih*-index replaces the official ranking. Note also that the three relegated teams are the same with both rankings too. Secondly, the difference in rankings for some other teams is not negligible. For instance, Rennes falls from position 9 with the official ranking to position 15 with the *ih*-index ranking, which would translate into a loss of money of around 2.91 million euros. Furthermore, the *h*-index is obviously limited here since seven teams obtain an *h*-index of 12, while seven other teams obtain an *h*-index of 11. The evident explanation is that European football leagues feature a smaller number of matches per teams than the in a NBA regular season or than the number of annual matches for the best ATP players.

4.4. Discussion

The three applications to sport ranking considered so far clearly indicate that the (classical) *h*-index is perhaps not a good tool to rank teams and players, since many of them end up with the same *h*-index, even if they have very different season records. To the contrary, the *ih*-index has several components from which teams and players with the same *h*-index can be distinguished. Even in sports for which the regular season contains many games, the *h*-index could have a limited power. For instance, in the Major league baseball, teams play around 160 games during the season. The official ranking is the winning percentage as for the NBA basketball, but the difference in winning percentage between the best and worst teams is small. In the 2015 regular season, St. Louis Cardinals achieves the best winning percentage (0.617) while Philadelphia Phillies had the worse (0.389). The difference of 0.228 is much lower than for basketball (0.890 – 0.122 = 0.768 according to tables 2 and 3 for the 2016 season).

We believe that the *ih*-index provides a strong incentive system for players/teams since it potentially rewards more wins against high-ranked players/teams than against low-ranked players/teams. Bonus system exist or have existed in many sports rankings, and some of them are also based on the strength of the opponents. Between 1994 and 1999, the ATP ranking was including bonuses depending on the current ranking of the defeated players. For instance, a win against world number one was associated with a 50 points bonus (doubled at a grand slam event), which was a substantial amount. Another example is the Elo rating system, used in chess but also for calculating the FIFA Women's World Rankings, which incorporates bonuses according to the difference in ranking between two opponents.

5. Conclusion

The main purpose of this article was to introduce a new kind of index for measuring the productivity of scholars, by allowing multi-dimensions. Even though our study has been focused on the h-index, we think that extensions of other indices in the same vein would deserve interest. Another task which we leave for future works is to find other applications for these multidimensional indices. Sports ranking have provided an interesting example in this article. To the best of our knowledge, Hovden (2013) is the only other related work based on the h-index, which is used to evaluate the performance of video channels on YouTube.

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