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Dynamic equilibrium in games with randomly arriving players

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Abstract

There are real strategic situations where nobody knows *ex ante* how many players there will be in the game at each step. Assuming that entry and exit could be modelized by random processes whose probability laws are common knowledge, we use dynamic programming and piecewise deterministic Markov decision processes to investigate such games. We study the dynamic equilibrium in games with randomly arriving players in discrete and continuous time for both finite and infinite horizon. Existence of dynamic equilibrium in discrete time is proved and we develop explicit algorithms for both discrete and continuous time linear quadratic problems. In both cases we offer a resolution for a Cournot oligopoly with sticky prices.

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JEL Classification: C72, C61, L13
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1 Introduction

1.1 Two examples

In 1921 Luigi Pirandello published his play *Six Characters in Search of an Author*. Beyond the content of the play, we refer to the original stylistic method used: the interweaving of a play within a play. Indeed this is the story of the rehearsal of a play involving a troop leader and its actors, disrupted by the arrival of a family of six looking for a writer to write their history, who end up playing their own roles in front of the troop.

Using the same type of stylistic process one could imagine a fiction where an author is confronted with an individual who tells him that he has been perfectly cloned and asks him to write a play where every minute one of its clones can (or not) show up and stay, and neither he nor the author nor the clones know exactly how many they will be at every minute of the play.

Beyond the fictional aspect we believe that this "fable" [Rubinstein, 2006] or "analogy" [Gilboa et al., 2014] can be used to model some economic situations where nobody knows *ex ante* how many players will enter the game. One may of course think of the arrival of competitors on a given market, as we will develop later in Cournot oligopoly. But this is not the only possible application. To illustrate this fact, consider for example the following two situations:

1.1.1 Exemple 1: Selling personnal data

In 1996, [Laudon, 1996] proposed a mechanism where the private data of individuals are aggregated into bundles and leased on an open market to all potential operators. Each individual who wishes to participate contacts an infomediary and offers whatever part of his personal data he determines as advantageous. The latter aggregates these informations with those of other individuals with similar characteristics and sells to operators (firms or governments) the possibility of making an offer to the group for a limited time. Individuals may vary at each time step the amount of personal data they wish to release, and receive a remuneration function of that amount, and of the value of the bundle they are in. They can also leave the group (opt out) if they feel that the inconvenience caused is not offset by the perceived income. It is tacitly assumed that the value of a bundle of private data increases with, and at a faster rate than, the number of agents therein. In such a case nobody knows *ex ante* how many people will be in a group and how much data they will choose to reveal. And if the gain per period is equally divided between individuals, each one wonders how much he will receive according to the fact that the larger is the group the bigger his income.

1.1.2 Exemple 2: Solidarity fund

In international law, the protection of foreign property rests with local authorities. So no public funds in France will compensate the French owners which suffered a loss abroad by a natural disaster or a serious political crisis. But few states fulfill their responsibilities in such cases. To solve this problem a bill was introduced in the French Senate on July 28, 2016. It suggests the creation of a solidarity fund with: gifts and bequests, a fraction of 10% of the proceeds of the establishment of passports, and a 10% levy on dormant inheritances (that is to say in the absence of heirs and will in favor of a third party, or renunciation of the heirs, or the heirs out of time in their claims).

The management of such a fund would then be subject to a double randomness. First, given its constitution, it is difficult to precisely predict at the beginning of each year what the amount of this fund will be. However it seems possible to make an estimate. According to us, the main problem lies in the second source of randomness, that is the number of relevant nationals who might be eligible for such a compensation.

The manager of the fund faces the problem of deciding how to share it among the beneficiaries. The solution that the first nationals concerned be fully repaid and the following ones be to exhaustion of the fund may seem unfair. Therefore the fund manager might consider the following alternative: each year, he offers each claimant the choice of either being paid a given percentage of its remaining claim as a final settlement, or getting a smaller amount obtained by dividing the available funds equally among those who choose that second solution, and staying in the pool of claimants. This raises the issue of how long those will wait to be reimbursed for their loss. It should be noted that, unlike in the previous example, here the larger the set of beneficiaries the lesser the compensation they will receive at each period. We will see, however, why this example is actually somewhat beyond the scope of our theory.

1.2 Random number of players in the literature

Classical game theory models belong to the "fixed-n" paradigm and can not be applied to this kind of problems. To the best of our knowledge there exists currently in game theory three ways to manage uncertainty on the number of players in the game. First it could be modelized by assuming that there is a common knowledge number of potential players and a stochastic process chooses wich ones will be active players (see [Levin and Ozdenoren, 2004] for example). A second way to model such games is to use population games, as Poisson games, where the number of players really in the game is supposed to be drawn from a random variable, whose probability distribution is commonly known (as in [Myerson, 1998b] and [Myerson, 1998a]). Finally, a third way is to use games with a large number of players modeled as games with infinitely many players, see [Khan and Sun, 2002] for a survey, or mean field games [Lasry and Lions, 2007, Caines, 2014] for instance. (Evolutionary games, which also involve an infinite number of players, are not really games where an equilibrium is sought.)

Unfortunately, as we discuss in [Bernhard and Deschamps, 2016b], the kind of games we mentionned in our examples could not be analysed with such tools. Indeeed our diagnostic is that where the number of players is random, there is no time involved, and therefore no concept of entry. Typical examples are auction theory, see [Levin and Ozdenoren, 2004] or Poisson games, see [Myerson, 1998b, De Sinopoli et al., 2014] . Where there is a time structure in the game, the number of players is fixed, such as in stochastic games, see [Neyman and Sorin, 2003], or in generalized secretary problems, see [Ferguson, 2005]. And in the literature on entry equilibrium, such as [Samuelson, 1985, Breton et al., 2010], the players are the would-be entrants, the number of which is known.

One notable exception is the article [Kordonis and Papavassilopoulos, 2015] which explicitly deals with a dynamic game with random entry. In this article, the authors describe a problem more complicated than ours on at least three counts: 1/ There are two types of players: a major one, an incumbent, who has an infinite horizon, and identical minor ones that enter at random and leave after a fixed time T (although the authors mention that they can also deal with the case where T is random), 2/ Each player has its own state and dynamics. Yet, the criteria of the players only depend on a mean value of these states, simplifying the analysis, and opening the way for a simplified analysis in terms of mean field in the large number of minor players case, and 3/ All the dynamics are noisy. (See however, our paragraph 2.1.3). It is simpler than ours in that it does not attempt to foray away from the discrete time, linear dynamics, quadratic payoff (L.Q.) case. Admittedly, our results in the nonlinear case are rather theoretical and remain difficult to use beyond the L.Q. case. But we do deal with the continuous time case also. Due to the added complexity, the solution proposed is much less explicit than what we offer in the linear quadratic problem. Typically, the authors solve the two maximization problems with opponents' strategies fixed and state that if the set of strategies is "consistent", i.e. solves the fixed point problem inherent in a Nash equilibrium, then it is the required equilibrium. The algorithm proposed to solve the fixed point problem is the natural Picard iteration. A convergence proof is only available in a very restrictive case.

1.3 Overview

Contrary to our first article [Bernhard and Deschamps, 2016b], here we seek a dynamic equilibrium, using the tools of dynamic programming (discrete tile) and piecewise deterministic Markov decision processes (or piecewise deterministic control systems (continuous time), see [Sworder, 1969, Rishel, 1975, Vermes, 1985, Haurie and Moresino, 2000]).

In the discrete time case (section 2), the resulting discrete Isaacs equation obtained is rather involved. As usual, it yields an explicit algorithm in the finite horizon, linear-quadratic case via a kind of discrete Riccati equation. The infinite horizon problem is briefly considered. It seems to be manageable only if one limits the number of players present in the game. In that case, the linear quadratic problem seems solvable via essentially the same algorithm, although we have no convergence proof, but only very convincing numerical evidence.

We then consider the continuous time case (section 3), with a Poisson arrival process. While the general Isaacs equation obtained is as involved as in the discrete time case, the linear quadratic case is simpler, and, provided again that we bound the maximum number of players allowed in the game, it yields an explicit algorithm. It takes a sign hypothesis not very realistic for an economic application to get in addition a convergence proof to the solution of the infinite horizon case.

In both discrete and continuous time, we briefly examine the case where players may leave the game. We also offer an example of a L.Q. game as one of Cournot oligopoly with sticky prices.

The paper concludes with a summary of findings and limitations.

2 Discrete time

2.1 The problem

2.1.1 Players, dynamics and payoff

Time t is an integer. An horizon $T \in \mathbb{N}$ is given, and we will write $\{1, 2, \dots, T\} = \mathbb{T}$, thus $t \in \mathbb{T}$. A state space X is given. A dynamic system in X may be controlled by an arbitrary number of agents. The number m of agents varies with time. We let $m(t)$ be that number at time t. The agents arrive as a Bernoulli process with variable probability; i.e. at each time step there may arrive only one player, and this happens with a probability p^m when m players are present, independently of previous arrivals. We call t_n the arrival time of the n-th player, $s_n \in \mathsf{S}$ its decision (or control).

We distinguish the *finite case* where X and S are finite sets, from the *infinite* *case* where they are infinite. In that case, they are supposed to be topologic spaces, S compact.

Note concerning the notation *We use lower indices to denote players, and upper indices to denote quantities pertaining to that number of agents in the game. An exception is* S^m *which is the cartesian power set* $\mathsf{S} \times \mathsf{S} \times \cdots \times \mathsf{S}$ m *times.* We use the notation:

$$s^m = (s_1^m, s_2^m, \dots, s_m^m) \in \mathsf{S}^m\,, \quad v^{\times m} = (\overbrace{v, v, \dots, v}^{m\,\text{times}})\,,$$
$$s^{m \backslash n} = (s_1, \dots, s_{n-1}, s_{n+1}, \dots, s_m)\,,$$
$$\{s^{m \backslash n}, s\} = \{s, s^{m \backslash n}\} = (s_1, \dots, s_{n-1}, s, s_{n+1}, \dots, s_m)$$

The dynamics are ruled by the state equation in X:

$$x(t+1) = f^{m(t)}(t, x(t), s^{m(t)}(t))\,, \quad x(0) = x_0\,. \tag{1}$$

A double family of stepwise payoffs, for $n \le m \in \mathbb{T}$ is given: $L_n^m : \mathbb{T} \times \mathsf{X} \times \mathsf{S}^m \to \mathbb{R} : (t, x, s^m) \mapsto L_n^m(t, x, s^m)$, as well as a discount factor $r \le 1$. The overall payoff of player n, which it seeks to maximize, is

$$\Pi_n^e(t_n, x(t_n), \{s^m\}_{m \ge n}) = \mathbb{E} \sum_{t=t_n}^{T} r^{t-t_n} L_n^{m(t)}(t, x(t), s^{m(t)}(t))\,. \tag{2}$$

Moreover, all players are assumed to be identical. Specifically, we assume that

1. The functions f^m are invariant by a permutation of the s_n,
2. the functions L_n^m enjoy the properties of a game with identical players as described in Appendix A. That is: a permutation of the s_n produces an identical permutation of the L_n^m.

Finally, in the infinite case, the functions f^m and L^m are all assumed continuous.

2.1.2 Pure strategies and equilibria

We have assumed that the current number of players in the game at each step is common knowledge. We therefore need to introduce $m(t)$-dependent controls: denote by $S_n \in \mathsf{S}_n = \mathsf{S}^{T-n+1}$ a complete n-th player's decision, i.e. an application $\{n, \dots, T\} \to \mathsf{S} : m \mapsto s_n^m$. We recall the notation for a strategy profile: $s^m = (s_1^m, s_2^m, \dots, s_m^m) \in \mathsf{S}^m$. We also denote by S^m a decision profile: $S^m = (S_1, S_2, \dots, S_m)$. It can also be seen as a family $S^m = (s^1, s^2, \dots, s^m)$. The set of elementary controls in S^t is best represented by Table 1 where $s_n^m(t)$ is the control used by player n at time t if there are m players in the game at that time.

	Player				
$m(t)$	1	2	$\cdots$	t	
1	$s_1^1(t)$				$s^1(t)$
2	$s_1^2(t)$	$s_2^2(t)$			$s^2(t)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$		$\vdots$
t	$s_1^t(t)$	$s_2^t(t)$	$\dots$	$s_t^t(t)$	$s^t(t)$
	$S_1(t)$	$S_2(t)$	$\dots$	$S_t(t)$	

Table 1: Representation of $S^t(t)$, the section at time t of an open-loop profile of strategies $S(\cdot)$. In the rightmost column: the names of the lines, in the last line: the names of the columns.

A partial strategy profile $(S_1, \dots, S_{n-1}, S_{n+1}, \dots, S_m)$ where S_n is missing, will be denoted $S^{m \setminus n}$. An open-loop profile of strategies is characterized by a sequence $S(\cdot) : \mathbb{T} \ni t \mapsto S^t(t)$. A partial open-loop strategy profile where $S_n(\cdot)$ is missing will be denoted $S^{\setminus n}(\cdot)$.

The payoff $\Pi_n^e(t_n, x(t_n), S(\cdot))$ is a mathematical expectation conditioned on the pair $(t_n, x(t_n))$, which is a random variable independent from $S_n(\cdot)$.

Definition 2.1 *An open loop dynamic pure Nash equilibrium is a family history* $\widehat{S}(\cdot)$ *such that*

$$\forall n \in \mathbb{T}, \forall (t_n, x(t_n)) \in \mathbb{T} \times \mathsf{X}, \forall S_n(\cdot) \in \mathsf{S}_n, \; \Pi_n^e(\{\widehat{S}^{\setminus n}(\cdot), S_n(\cdot)\}) \leq \Pi_n^e(\widehat{S}(\cdot)) . \tag{3}$$

Definition 2.2 *A Nash equilibrium will be called* uniform *if at all times, all players present in the game use the same decision, i.e., with our notations, if, for all t, for all m, $\hat{s}^m(t) = \hat{s}(t)^{\times m}$ for some sequence $\hat{s}(\cdot)$.*

Remark 2.1 *A game with identical players may have non uniform pure equilibria, and even have pure equilibria but none uniform. However, if it has a unique equilibrium, it is a uniform equilibrium (see appendix A).*

However, we will be interested in *closed loop* strategies, and more specifically *state feedback* strategies; i.e. we assume that each player is allowed to base its control at each time step t on the current time, the current state $x(t)$ and the current number $m(t)$ of players in the game. We therefore allow families of state feedbacks indexed by the number m of players:

$$\varphi^m = (\varphi_1^m, \varphi_2^m, \dots, \varphi_m^m) : \mathbb{T} \times \mathsf{X} \to \mathsf{S}^m$$

and typically let

$$s_n^m(t) = \varphi_n^m(t, x(t))\,.$$

We denote by $\Phi_n \in \mathcal{F}_n$ a whole family $(\varphi_n^m(\cdot,\cdot)\,, m \in \{n, \ldots, T\})$ (the complete strategy choice of a player n), Φ a complete strategy profile, $\Phi^{\setminus n}$ a partial strategy profile specifying their strategy Φ_ℓ for all players except player n. A closed loop strategy profile Φ generates through the dynamics and the entry process a random open-loop strategy profile $S(\cdot) = \Gamma(\Phi)$. With a transparent abuse of notation, we write $\Pi_n^e(\Phi)$ for $\Pi_n^e(\Gamma(\Phi))$.

Definition 2.3 *A closed loop dynamic pure Nash equilibrium is a profile* $\widehat{\Phi}$ *such that*

$$\forall n \in \mathbb{T}\,, \forall (t_n, x(t_n)) \in \mathbb{T} \times \mathsf{X}\,, \forall \Phi_n \in \mathcal{F}_n\,, \quad \Pi_n^e(\{\widehat{\Phi}^{\setminus n}, \Phi_n\}) \le \Pi_n^e(\widehat{\Phi})\,. \tag{4}$$

It will be called uniform *if it holds that* $\hat{\varphi}^m = \hat{\varphi}^{\times m}$.

We further notice that using state feedback strategies (and dynamic programming) will naturally yield time consistent and subgame perfect strategies.

2.1.3 Mixed strategies and disturbances

For the sake of simplicity, we will emphasize pure strategies hereafter. But of course, a pure Nash equilibrium may not exist. In the discrete time case investigated here, we can derive existence results if we allow mixed strategies.

Let $\mathcal{S}$ be the set of probability distributions over S. Replacing S by $\mathcal{S}$ in the definitions of open-loop and closed-loop strategies above yields equivalent open-loop and closed-loop behavioral mixed strategies. By behavioral, we mean that we use sequences of random choices of controls and not random choices of sequences of controls. See [Bernhard, 1992] for a more detailed analysis of the relationship between various concepts of mixed strategies for dynamic games.

In case the strategies are interpreted as mixed strategies, $s^{m(t)}(t)$ in equations (1) and (2) are random variables, and the pair $(m(\cdot), x(\cdot))$ is a (controlled) markov chain. But since anyhow, $m(\cdot)$ is already a markov chain even with pure strategies, the rest of the analysis is unchanged.

We might go one step further and introduce disturbances in the dynamics and the payoff. Let $\{w(\cdot)\}$ be a sequence of independent random variables in $\mathbb{R}^\ell$, and add the argument $w(t)$ in both f^m and L_n^m. All results hereafter in the discrete time problem remain unchanged (except for formula (9) where one term must be added). We keep with the undisturbed case for the sake of simplicity of notation, and because in the continuous time case, to be seen later, it spares us the Ito terms in the equations.

2.2 Isaacs' equation

2.2.1 Finite horizon

We use dynamic programming, and therefore Isaacs' equation in terms of a family of Value functions $V_n^m : \mathbb{T} \times \mathsf{X} \to \mathbb{R}$. It will be convenient to associate to any such family the family W_n^m defined as

$$W_n^m(t,x) = (1 - p^m)V_n^m(t,x) + p^m V_n^{m+1}(t,x)\,, \tag{5}$$

and the Hamiltonian functions

$$H_n^m(t,x,u^m) := L_n^m(t,x,s^m) + rW_n^m(t+1, f_n^m(t,x,s^m))\,. \tag{6}$$

We write Isaacs' equation for the general case of a non uniform equilibrium, but the uniform case will be of particular interest to us.

Theorem 2.1 *: An subgame perfect equilibrium $\widehat{\Phi} = \{\hat\varphi_n^m\}$ exists, if and only if there is a family of functions V_n^m satisfying the following Isaacs equation, which makes use of the notation* (5), (6)*:*

$$\forall n \le m \in \mathbb{T}\,,\ \forall (t,x) \in \{0,\dots,T\} \times \mathsf{X}\,,\ \forall s \in \mathsf{S}\,,$$
$$V_n^m(t,x) = H_n^m(t,x,\hat\varphi^{\times m}(t,x)) \ge H_n^m(t,x,\{\hat\varphi^{m\setminus n}(t,x), s\})\,,$$
$$\forall m \in \mathbb{T}\,,\ \forall x \in \mathsf{X}\,,\ V_n^m(T+1,x) = 0\,.$$

And then, the equilibrium payoff of player n joining the game at time t_n at state x_n is $V_n^n(t_n,x_n)$. If the equilibrium is uniform, i.e. for all $n \le m$, $\hat\varphi_n^m = \hat\varphi_1^m$, then $V_n^m = V_1^m$ for all m,n (and we may call it V^m).

Proof This is a classical dynamic programming argument. We notice first that the above system can be written in terms of conditional expectations given (m,x) as

$$\forall n \le m \in \mathbb{T}\,,\ \forall (t,x) \in \{0,\dots,T\} \times \mathsf{X}\,,\ \forall s \in \mathsf{S}\,,$$
$$\begin{aligned} V_n^m(t,x) &= \mathbb{E}^{m,x}\Big[L_n^m(t,x,\hat\varphi^m(t,x)) \\ &\quad + rV_n^{m(t+1)}\big(t+1, f^m(t,x,\hat\varphi^m(t,x))\big)\Big] \\ &\ge \mathbb{E}^{m,x}\Big[L_n^m(t,x,\{\hat\varphi^{m\setminus n}(t,x),s\}) \\ &\quad + rV_n^{m(t+1)}\big(t+1, f^m(t,x,\{\hat\varphi^{m\setminus n}(t,x),s\})\big)\Big] \end{aligned}$$
$$\forall m \in \mathbb{T}\,,\ \forall x \in \mathsf{X}\,,\ V_n^m(T+1,x) = 0\,.$$

Assume first that all players use the strategy $\hat{\varphi}$. Fix an initial time t_n (which may or may not be the arrival time of the n-th player) an state x_n and an initial m. Assume all players use their control $\hat{\varphi}_n(t, x(t))$, and consider the random process $(m(t), x(t))$ thus generated. For brevity, write $\hat{s}^m(t) := \hat{\varphi}^m(t, x(t))$. Write the equality in theorem 2.1 at all steps of the stochastic process $(m(t), x(t), \hat{s}^{m(t)}(t))$:

$$V^m(t, x(t)) = \mathbb{E}^{m(t),x(t)} \left[L_n^{m(t)}\big(t, x(t), \hat{s}^{m(t)}(t)\big) + rV_n^{m(t+1)}\big(t+1, x(t+1)\big) \right].$$

Multiply by r^{t-t_n}, take the a priori expectation of both sides and use the theorem of embedded conditional expectations, to obtain

$$\mathbb{E}\Big[-r^{t-t_n} V_n^{m(t)}(t, x(t)) + r^{t-t_n} L_n^{m(t)}\big(t, x(t), \hat{s}^{m(t)}(t)\big) + r^{t+1-t_n} V_n^{m(t+1)}(t+1, x(t+1)) \Big] = 0\,.$$

Sum these equalities from t_n to T and use $V_n^m(T+1, x) = 0$ to obtain

$$-V_n^m(t_n, x_n) + \mathbb{E}\left[\sum_{t=t_n}^{T} r^{t-t_n} L_n^{m(t)}(t, x(t), \hat{s}^{m(t)}) \right] = 0\,,$$

hence the claim that the payoff of all players from (t_n, x_n, m) is just $V_n^m(t_n, x_n)$, and in particular the payoff of player n as in the theorem.

Assume now that player n deviates from $\hat{\varphi}_n$ according to any sequence $s_n(\cdot)$. Exactly the same reasoning, but using the inequality in the theorem, will lead to $V_n(t_n, x_n) \geq \Pi_n^e$. We have therefore shown that the conditions of the theorem are sufficient for the existence of a subgame perfect equilibrium.

Finally, assume that the subgame perfect equilibrium exists. Let $V_n^m(t, x)$ be defined as the payoff to player n in the subgame starting with m players at (t, x). The equality in the theorem directly derives from the linearity (here, additivity) of the mathematical expectation. And if at one (m, t, x) the inequality were violated, for the subgame starting from that situation, a control $s_n(t) = s$ would yield a higher expectation for player n, which is in contradiction with the fact that $\widehat{\Phi}$ generates an equilibrium for all subgames.

Concerning a uniform equilibrium, observe first that (for all equilibria), for all m, n, for all $x \in \mathsf{X}$, $V_n^m(T+1, x) = 0$. Assume that $V_n^m(t+1, x) = V_1^m(t+1, x)$. Observe that then, in the right hand side of Isaacs' equation, only L_n^m depends on n. let π be a permutation that exchanges n and 1. By hypothesis, $L_n^m(t, x, \hat{\varphi}^{\pi[m]}(t, x)) = L_1^m(t, x, \hat{\varphi}^m)$. But for a uniform equilibrium, it also holds that $\hat{\varphi}^{\pi[m]}(t, x) = \hat{\varphi}^m(t, x)$. Hence $V_n^m(t, x) = V_1^m(t, x)$. ∎

Isaacs' equation in the theorem involves a sequence of Nash equilibria of the Hamiltonian. In general, stringent conditions are necessary to ensure existence of a pure equilibrium. However, our hypotheses ensure existence of a mixed equilibrium (see, e.g. [Ekeland, 1974] and [Bernhard, 1992]). And since the equation is constructive via backward induction, we infer

Corollary 2.1 *A dynamic subgame perfect Nash equilibrium in behavioural strategies exists in the finite horizon discrete time game..*

A natural approach to using the theorem is via Dynamic Programming (backward induction). Assume that we have discretized the set of reachable states in N_t points at each time t. (Or $x \in \mathsf{X}$, a finite set) The theorem brings the determination of a subgame perfect equilibrium set of strategies to the computation of $\sum_t t \times N_t$ Nash equilibria (one for each value of m at each (t,x)). A daunting task in general. However, the search for a *uniform* equilibrium may be much simpler. On the one hand, there is now a one-parameter family of functions $V^m(t,x)$, and, in the infinite case, if all functions are differentiable (concerning W_n^m this is *not* guaranteed by regularity hypotheses on f^m and L_n^m) and if the equilibrium is interior, the search for each static Nash equilibrium is brought back to solving an equation of the form (36):

$$\partial_{s_1} L_1^m(t,x,s^{\times m}) + r\partial_x W^m(t+1, f^m(t,x,s^{\times m}))\,\partial_{s_1} f^m(t,x,s^{\times m}) = 0\,.$$

We will see that in the linear quadratic case that we will consider, this can be done.

2.2.2 Infinite horizon

We consider the same problem as above, with both f^m and L_n^m independent from time t. We assume that the L_n^m are uniformly bounded by some number L, and we let the payoff of the n-th player in a (sub)game starting with n players at time t_n and state $x(t_n) = x_n$ be

$$\Pi_n^e(t_n, x_n; S(\cdot)) = \mathbb{E}\sum_{t=t_n}^{\infty} r^{t-t_n} L_n^{m(t)}(x(t); s^{m(t)}(t))\,. \tag{7}$$

We look for a subgame perfect equilibrium set of strategies $\hat\varphi_n^m(x)$. Isaacs equation becomes an implicit equation for a bounded infinite family of functions $V_n^m(x)$. Using the time invariant form of equations (5) and (6), we get:

Theorem 2.2 *Let $r < 1$. Then, a subgame perfect equilibrium $\widehat\Phi$ of the infinite horizon game exists if and only if there is a two-parameter infinite family of uniformly bounded functions $V_n^m(x)$ satisfying the following Isaacs equation:*

$$\forall n \le m \in \mathbb{N}\,,\ \forall x \in \mathsf{X}\,,\ \forall s \in \mathsf{S}\,,$$

$$V_n^m(x) = H_n^m(x, \hat\varphi^m(x)) \ge H_n^m(x, \{\hat\varphi^{m\setminus n}(x), s\})\,.$$

Then, the equilibrium payoff of player n joining the game at state x_n is $V_n^n(x_n)$. If the equilibrium is uniform, $V_n^m = V_1^m$ for all n, m.

Proof The proof proceeds along the same lines as in the finite horizon case. In the summation of the sufficiency proof, there remains a term $r^{T-t_n}V^m(x(T))$ that goes to zero as T goes to infinity, because the functions V^m have been assumed to be bounded. And this is indeed necessary since the bound assumed on the L_n^m implies that the Value functions are bounded by $L/(1-r)$. ∎

We restrict our attention to uniform equilibria, so that we have a one-parameter family of Value functions V^m. But it is infinite. To get a feasible algorithm, we make the following assumption:

Hypothesis 2.1 *There is a finite $M \in \mathbb{N}$ such that $p^M = 0$.*

Thanks to that hypothesis, there is a finite number M of Value functions to consider. There remains to find an algorithm to solve for the fixed points bearing on the family $\{V^m(x)\}_m$ for all $x \in \mathsf{X}$. We offer the *conjecture* that the mapping from the family $\{V^m(t+1,\cdot)\}_m$ to the family $\{V^m(t,\cdot\}_m$ in the finite horizon Isaacs equation is a contraction in an appropriate distance. If so, then it provides an algorithm of "iteration on the Value" to compute the $V^m(x)$ of the infinite horizon problem. (We will offer a different conjecture in the linear quadratic case.)

Remark 2.2 *Hypothesis 2.1 is natural in case the payoff is decreasing with the number of players and there is a fixed entry cost. Otherwise, it may seem artificial and somewhat unfortunate. Yet, we may notice that for any numerical implementation, we are obliged to consider only a bounded (since finite) set of x. We are accustomed to doing so, relying upon the assumption that very large values of x will be reached very seldom, and play essentially no role in the computation. In a similar fashion, we may think that very large values of $m(t)$ will be reached for very large t, which, due to the discount factor, will play a negligible role in the numerical results. This is an unavoidable feature of numerical computations, not really worse in our problem than in classical dynamic programming.*

2.3 Entering and leaving

2.3.1 Methodology

It would be desirable to extend the theory to a framework where players may also leave the game at random. However, we must notice that although our players are identical, the game is not anonymous. As a matter of fact, players are labelled by their rank of arrival, and their payoffs depend on that rank. We must therefore propose exit mechanisms able to take into account *who* leaves the game. Before doing so, we agree on the fact that once a player has left the game, it does not re-enter. (Or if it does, this new participation is considered as that of another player.) Let T_n be the exit time of the player of rank n, a random variable. We now have

$$\Pi_n^e(t_n, x(t_n), S(\cdot)) = \mathbb{E} \sum_{t=t_n}^{T_n} r^{t-t_n} L_n^{m(t)}(t, x(t), s^{m(t)}(t)) \,.$$

In defining the controls of the players, we may no longer have $n \leq m \leq t$ as previously, and Table 1 must be modified accordingly. Let $N(m)$ be the maximum possible rank of players present when there are m of them, and $M(n)$ the maximum possible number of players present when player n is. Then $s^m(t) = \{s_n^m\}_{n \leq N(m)}$ and $S_n(t) = \{s_n^m(t)\}_{m \leq M(n)}$. And of course, a choice of $s_n^m(t)$ means the decision that player of rank n chooses at time t if there are m players present at that time, *including himself*.

We also insist that the probabilities of entry (or exit) are functions such as p^m of the current number of players present, and not of the rank of entry.

When a player leaves the game, from the next time step on it will not get any payoff. Thus, we may just consider that for it, the Value functions $V_n^m(t+1, x)$ are null. To take this into account we determine the probabilities $\mathbb{P}^{m,k}$ that there be k players at the next time step *and that the focal player has not left*, knowing that there are m players present at the current step. And then, Theorem 2.1 above and its proof remain unchanged upon substituting

$$W_n^m = \sum_k \mathbb{P}^{m,k} V_n^k$$

to equation (5). (In the Bernoulli entry-only version of the problem, we may set $\mathbb{P}^{m,m+1} = p$ and $\mathbb{P}^{m,m} = (1-p)$.)

We propose several entry and exit mechanisms as examples.

2.3.2 A joint scheme

In this scheme, there is a probability q^m that one player leaves the game at the end of a step where there are m players present. (And of course, $q^0 = 0$.) Moreover, we add the dictum that should one player actually leave, which one leaves is chosen at random with uniform probability among the players present. As a consequence, each player present has a probability q^m/m to leave the game at (the end of) each time step. Let $m(t) = m$, then the probabilities that a given player among the m present at step t be still present at time $t+1$ and that $m(t+1)$ take different values is given by the following table:

$m(t+1)$	probability
$m+1$	$\mathbb{P}^{m,m+1} = p^m(1-q^m)\,,$
m	$\mathbb{P}^{m,m} = p^m q^m \dfrac{m-1}{m} + (1-p^m)(1-q^m)$
$m-1$	$\mathbb{P}^{m,m-1} = (1-p^m)q^m \dfrac{m-1}{m}\,.$

2.3.3 Individual schemes

The previous scheme is consistent with our entry scheme. But it might not be the most realistic. We propose two other schemes.

In the first, each player, once it has joined the game, has a probability q of leaving the game at each time step, independently of the other players and of the past and current arrivals sequence. We need powers of p and q. So, to keep the sequel readable, we take them constant, and upper indices in the table below are powers. It is only a matter of notation to take them dependent on m. In computing the probability that a given number of players has left, we must remember that those must be chosen among the other $m-1$ players, and that the focal player must have remained. The corresponding table of probabilities is now

$m(t+1)$	probability
$m+1$	$\mathbb{P}^{m,m+1} = p(1-q)^m\,,$
$1 < k \le m$	$\mathbb{P}^{m,k} = \frac{(m-1)!}{(m-k)!(k-2)!} q^{m-k}(1-q)^{k-1}\left[\dfrac{(1-p)(1-q)}{k-1} + \dfrac{pq}{m-k+1}\right],$
1	$\mathbb{P}^{m,1} = (1-p)(1-q)q^{m-1}\,.$

A more coherent scheme, but that drives us away from the main stream of this article, is one where there is a finite pool of M agents who are eligible to enter the game. At each time step, each of them has a probability p of actually entering. Once into the game, each has a probability q of leaving at each time step, and if so, it re-enters the pool. In that case, we set

$$\mathcal{L}^{m,k} = \{\ell \in \mathbb{N} | \ell \ge 0\,,\ \ell \ge m-k\,,\ \ell \le m-1\,,\ \ell \le M-k\}$$

and we have, for all m, k less or equal to M:

$$\mathbb{P}^{m,k} = \sum_{\ell \in \mathcal{L}^{m,k}} \begin{pmatrix} m-1 \\ \ell \end{pmatrix} \begin{pmatrix} M-m \\ k-m+\ell \end{pmatrix} p^{k-m+\ell}(1-p)^{M-k+\ell} q^{\ell}(1-q)^{m-\ell}\,.$$

2.3.4 Beyond the Bernoulli process

At this stage, it is not difficult to generalize our model to one where several players may join the game at each instant of time, provided that it remains a finite Markov chain. Introduce probabilities p_ℓ^m that ℓ players join the game when m players are already there. In a similar fashion, in the so called "joint scheme" above, we might have probabilities q_ℓ^m that ℓ players leave at the same time.

Set $p_j^m = 0$ for any $j < 0$. We then have

$$\mathbb{P}^{m,k} = \sum_{\ell=0}^{m-1} \frac{m-\ell}{m} q_\ell^m p_{k-m-\ell}^m \,. \tag{8}$$

2.4 Linear quadratic problem

2.4.1 The problem

We consider an academic example as follows: the state space in $\mathsf{X} = \mathbb{R}^d$, the control set $\mathsf{S} = \mathbb{R}^a$. the dynamics are defined by a sequence of square $d \times d$ matrices $A(t)$ and a sequence of $d \times a$ matrices $B(t)$ and

$$x(t+1) = A(t)x(t) + B(t) \sum_{n=1}^{m(t)} s_n(t) \,.$$

The payoff of player n is given in terms of two sequences of square matrices $Q^m(t)$ and $R(t)$, the first nonnegative definite, the second positive definite, as [1]

$$\Pi_n^e = \mathbb{E} \sum_{t=t_n}^{T} r^{t-t_n} \left[\|x(t)\|^2_{Q^{m(t)}(t)} - \|s_n(t)\|^2_{R(t)} \right] \,.$$

The idea is that the players, through their controls s_n, collectively produce some good x, worth $\|x(t)\|^2_{Q(t)}$ at time t. Either it is a public good, and they all benefit equally, then $Q_m = Q$, or it is a company earnings, and they share equally the dividends, and then $Q^m = (1/m)Q$. But each of them pays its effort at each time step t an amount $\|s_n(t)\|^2_{R(t)}$. (It is only for notational convenience that we do not let R depend on m.)

This is not quite the classical linear quadratic problem, because both terms in the payoff have different signs. As a consequence, to avoid an indefiniteness in the

[1] We use prime for transposed and $\|x\|^2_Q = x'Qx$, $\|s\|^2_R = s'Rs$.

payoff, we must add the dictum that all players are constrained to using decision sequences $s_n(\cdot)$ of finite weighted norm, (or "energy") in the sense that

$$\sum_{t=t_n}^{T} r^{t-t_n}\|s_n(t)\|^2_{R(t)} < \infty\,.$$

2.4.2 Solution via the Riccati equation

As usual, we seek a solution with a quadratic Value function. We look for a uniform equilibrium, and a one-parameter family of Value functions of the form

$$V^m(t,x) = \|x\|^2_{P^m(t)}\,. \tag{9}$$

Notice first that, according to the terminal condition in theorem 2.1, for all $m \le T$, $P^m(T+1) = 0$. Assume, as a recursion hypothesis, that $V^m(t+1,x)$ is, for all m, a quadratic form in x, i.e. that there exist symmetric matrices $P^m(t+1)$ such that

$$V^m(t+1,x) = \|x\|^2_{P^m(t+1)}\,.$$

Since for any controls of the others, each player may always use $s_n = 0$ and that way ensure itself a nonnegative payoff, it follows that $P^m(t+1)$ is nonnegative. Isaacs equation is now

$$\begin{aligned} V^m(t,x) = \max_s \Big\{ & \|x\|^2_{Q^m(t)} - \|s\|^2_{R(t)} \\ & + r\Big[(1-p^m)\|A(t)x + (m-1)B(t)\hat{s} + B(t)s\|^2_{P^m(t+1)} \\ & \quad + p^m\|A(t)x + (m-1)B(t)\hat{s} + B(t)s\|^2_{P^{m+1}(t+1)}\Big]\Big\}, \end{aligned}$$

the maximum in s being reached at $s = \hat{s}$. Let [2]

$$W^m(t+1) = r\left[(1-p^m)P^m(t+1) + p^m P^{m+1}(t+1)\right]. \tag{10}$$

These are symmetric non-negative definite matrices, and $W^m(T+1) = 0$. Isaacs' equation can be written

$$V^m(t,x) = \max_s \Big\{ \|x\|^2_{Q^m(t)} - \|s\|^2_{R(t)} + \|A(t)x + (m-1)B(t)\hat{s} + B(t)s\|^2_{W^m(t+1)} \Big\}.$$

The right hand side is a (non homogeneous) quadratic form in s, with a quadratic term coefficient

$$K^m(t) = B'(t)W^m(t+1)B(t) - R(t)\,. \tag{11}$$

[2]Notice that contrary to whet we did in the nonlinear case, we include the factor r in W^m, to simplify further expressions

At time T, this is the negative definite matrix $-R(T)$. The optimum $s(T)$ is clearly zero, leading to $P^m(T) = Q^m(T)$, and therefore

$$K^m(T-1) = rB'(T-1)[(1-p^m)Q^m(T)+p^m Q^{m+1}(T)]B(T-1) - R(T-1)\,.$$

If this matrix is negative definite, there is a finite maximum in s. In the case where one $K^m(t)$ would fail to be at least nonpositive definite, s could be chosen so as to make the form arbitrarily large positive, the subgame starting from this node (m, t, x) could not have an equilibrium. In the case where one $K^m(t)$ would be nonpositive definite but singular, the quadratic form would have a finite maximum only if, for all x

$$s'K^m(t)^m s = 0 \Rightarrow s'B'(t)W^m(t+1)[A(t)x + mB(t)s] = 0.$$

Now, $s'K^m(t)s = 0$ implies that $s'B'W^m(t+1)Bs = s'Rs$ and therefore is not identically zero. But the requirement would be that for those s in the kernel of K, $s'B'W^m(t+1)Ax = ms'Rs$ for all x. The left hand side of this last inequality is linear in x and could be constant only if it were zero. In conclusion, the quadratic form cannot have a finite maximum for all x, and therefore the game no subgame perfect equilibrium.

Assume therefore that all $K^m(t)$ for $t, m \le T-1$ are negative definite. Equating the derivative with respect to s to zero, and equating all controls, yields

$$[B'(t)W^m(t+1)B(t) - R(t)]\hat{s} + B'(t)W^m(t+1)[A(t)x + (m-1)B(t)\hat{s}] = 0\,,$$

or, rearranging

$$\left[mB'(t)W^m(t+1)B(t) - R(t)\right]\hat{s} = B'(t)W^m(t+1)A(t)x\,. \tag{12}$$

If this equation has no solution for some (m, t), a uniform subgame perfect equilibrium cannot exist. If it has a solution, there exists a matrix $F^m(t)$ such that

$$\hat{s} = -F^m(t)x =: \hat{\varphi}_1^m(t, x). \tag{13}$$

We write it

$$F^m(t) = \left[mB'(t)W^m(t+1)B(t) - R(t)\right]^{-1} B'(t)W^m(t+1)A(t)\,, \tag{14}$$

knowing that indeed, the inverse might have to be replaced by a pseudo inverse. Finally, placing this value of s in the right hand side, we find that $V^m(t, x)$ is indeed a quadratic form in x. Thus we have proven that (9) holds, with

$$\begin{aligned} P^m(t) = {} & Q^m(t) - F^{m\prime}(t)R^m(t)F^m(t) \\ & + [A'(t) - mF^{m\prime}(t)B'(t)]W^m(t+1)[A(t) - mB(t)F^m(t)]\,, \end{aligned}$$

and after substituting $F^m(t)$ and reordering:

$$\begin{aligned} P^m(t) =& Q^m(t) + A'(t)W^m(t+1)A(t) - \\ & A'(t)W^m(t+1)B(t)[mB'(t)W^m(t+1)B(t) - R(t)]^{-1} \\ & [m^2B'(t)W^m(t+1)B(t) - (2m-1)R(t)] \\ & [mB'(t)W^m(t+1)B(t) - R(t)]^{-1}B'(t)W^m(t+1)A(t)\,, \end{aligned} \tag{15}$$

$$\forall m \in \mathbb{T}\,, \quad P^m(T) = Q^m(T)\,. \tag{16}$$

Recall that each matrix W^m involves P^{m+1}. But there cannot be more than T players at any time in the game (and T of them only at $T-1$, the final decision time.) Therefore, starting with $P^T(T) = Q^T$ and computing the $P^m(t)$ backward, this is a constructive algorithm. We therefore end up with the following:

Theorem 2.3 *The finite horizon, linear quadratic problem admits a uniform subgame perfect equilibrium if and only if equations* (15) *have a solution over* $[0,T]$, *—i.e. all equations* (12) *have a solution—, leading to negative definite matrices* $K^m(t)$ *as defined by equation* (11). *If these conditions are satisfied, the unique uniform equilibrium is given by equations (9,10,13,14,15,16).*

Entering and leaving It is now easy to get the solution of the same problem with one of our extended entry and exit mechanisms: according to equation (8), it suffices to replace the definition (10) of W^m by

$$W^m(t+1) = r\sum_{k=1}^{m+1} \mathbb{P}^{m,k}P^k(t+1)$$

with the relevant set of probabilities $\mathbb{P}^{m,k}$.

Infinite horizon We might want to consider the infinite horizon game with all system matrices constant. Notice first that the problem has a meaning only if the matrix A is stable (or at least has a spectral radius strictly less than $r^{-1/2}$). Otherwise, all players might just play $s_n(t) = 0$ and get an infinitely large payoff. But even so, and contrary to the case where Q would be nonpositive definite, we do not know the asymptotic behavior of the Riccati equations, let alone whether it has the necessary stabilizing properties to lead to a Nash equilibrium. (See [Mageirou, 1976].)

2.5 Example: Cournot oligopoly with sticky prices

2.5.1 The model

We provide a slightly different example of a linear quadratic dynamic game, as an intertemporal equilibrium in a Cournot oligopoly with random arrivals at a constant probability p and no exit. A dynamic effect may materialize only if the players' actions at one step affect the future market situation. A typical example is an intertemporal production constraint, such as in [van den Berg et al., 2012]. Here we consider a situation where prices adjust from one time period to the next one with some stickiness. This is also a concern in a large segment of the literature. See e.g. [Gordon, 1990, Blinder et al., 1998, Goldberg and Hellerstein, 2007, Anderson and Simester, 2010] and the references therein. For the sake of simplicity, we assume that the demand is fixed. We could make it vary randomly from time step to time step.

In our model, we have a classic linear inverse demand law linking prices P with the total production Q of all players:

$$P = a - bQ\,,$$

but we assume that prices do not adjust instantly to the new production level of each period. To the contrary, at the begining of each period, prices are those of the previous period, until they adjust to the new production level. Due to this stickiness, a share θ of the production of each period is sold at the price of the previous period, and a share $1-\theta$ at the new price determined by the inverse demand law.[3] For convenience, we call $\underline{P}(t)$ the closing price of period $t-1$, which is the price at the begining of period t. We have, if m players are present:

$$\underline{P}(t+1) = a - b\sum_{j=1}^{m} q_j(t) \tag{17}$$

Finally, let $r \leq 1$ be a discount factor. We assume that the production costs have been normalized to zero (or included in a), and that p, a, b, θ, and r are common knowledge. We therefore have for the expected profit Π_i^e of player i:

$$\Pi_i^e = \mathbb{E}\sum_{t=1}^{T} r^t \left[\theta \underline{P}(t) + (1-\theta)\left(a - b\sum_{j=1}^{m(t)} q_j(t)\right)\right] q_i(t)\,. \tag{18}$$

[3] This could result from a constant production rate and a linearly increasing price from the previous one to the one corresponding to the new production rate, which would be reached no later than the end of the current period, and where it would stick until the end of the period. In that case, $\theta \leq 1/2$, $\theta/(1-\theta) \leq 1$.

Observe that equations (17) and (18) define a linear quadratic problem essentially of the form considered in the previous subsection, except that there are cross terms in $P_{-}q$ and non homogeneous terms both in the dynamics and in the payoff. We therefore expect to find a Value function as a second degree polynomial in P_{-}:

$$V^m(t, P_{-}) = K^m(t)P_{-}^2 + L^m(t)P_{-} + M^m(t)\,. \tag{19}$$

2.5.2 Solution

Let the *degree of stickiness* be defined as

$$\delta = \frac{\theta}{1-\theta}\,,$$

the ratio of the weights of the "inherited" price to the "market closure" price. Hence,

$$\Pi_i^e = (1-\theta)\mathbb{E}\sum_{t=1}^{T} r^t \left(\delta P_{-}(t) + a - b\sum_{j=1}^{m(t)} q_j(t)\right) q_i(t).$$

We also introduce the following notation:

$$\begin{aligned}
\alpha(t) &= \frac{rb}{1-\theta}[(1-p)K^m(t+1) + pK^{m+1}(t+1)]\,,\\
\beta(t) &= \frac{rb}{1-\theta}[(1-p)L^m(t+1) + pL^{m+1}(t+1)]\,,\\
\gamma(t) &= \frac{rb}{1-\theta}[(1-p)M^m(t+1) + pM^{m+1}(t+1)]\,.
\end{aligned}$$

With these, assuming the form (19) for $V_i^\ell(t+1)$ and for $\ell = m, m+1$, Isaacs' equation reads:

$$\begin{aligned}
\frac{V_i^m(t,P_{-})}{1-\theta} = \max_{q_i\in\mathbb{R}_+}\Bigg\{ &\left(\delta P_{-} + a - b\sum_{j=1}^{m} q_j\right) q_i + \\
&\frac{1}{a}\left[\alpha(t)\left(a - b\sum_{j=1}^{m} q_j\right)^2 + \beta(t)\left(a - b\sum_{j=1}^{m} q_j\right) + \gamma(t)\right]\Bigg\}.
\end{aligned}$$

It is a tedious but straightforward calculation to see that, on the one hand, the maximum in q_i exists if and only if $\alpha(t) < 1$, and on the other hand, that then, this yields

$$q_i^\star = \frac{\delta P_{-} + (1-2\alpha(t))a - \beta(t)}{b(-2m\alpha(t) + m + 1)} \tag{20}$$

and V_i^m of the form (19) with

$$K^m(t) = \frac{\theta^2}{1-\theta}\,\frac{m(m-2)\alpha(t)+1}{b(-2m\alpha(t)+m+1)^2}\,,$$
$$L^m(t) = 2\theta\,\frac{2ma\alpha^2(t)+[m\beta(t)-(2m+1)a]\alpha(t)-\frac{1}{2}(m^2+1)\beta(t)+a}{b(-2m\alpha(t)+m+1)^2}\,,$$
$$M^m(t) = (1-\theta)\left[\frac{(m\beta(t)+a)^2(1-\alpha(t))}{b(-2m\alpha(t)+m+1)^2}+\frac{\gamma(t)}{b}\right],$$

to be initialized at $\alpha(T)=\beta(T)=\gamma(T)=0$.[4]

2.5.3 Simplest example

As a very simple example, we consider a monopolist who has been such for a long time (e.g. the historical incumbent in a newly deregulated market). Then it enters a two period game where at the first period, it remains alone, but is warned that a competitor might show up in the second and last period, with a probability p. Cost production are normalized to zero. Due to the long monopoly before the true game starts, $\underline{P}(1) = a/2$, the monopoly price.

We get for the second period Value function:

$$\text{monopoly} \to \quad K^1(2) = \tfrac{\theta^2}{4b(1-\theta)}\,,\quad L^1(2) = \tfrac{2\theta a}{4b}\,,\quad M^1(2) = (1-\theta)\tfrac{a^2}{4b}\,,$$
$$\text{Cournot duopoly} \to \quad K^2(2) = \tfrac{\theta^2}{9b(1-\theta)}\,,\quad L^2(2) = \tfrac{2\theta a}{9b}\,,\quad M^2(2) = (1-\theta)\tfrac{a^2}{9b}\,,$$

hence, not surprisingly,

$$V^1(2) = \frac{[\theta\underline{P}+(1-\theta)a]^2}{4b(1-\theta)} \quad\text{and}\quad V^2(2) = \frac{[\theta\underline{P}+(1-\theta)a]^2}{9b(1-\theta)}.$$

Let

$$\varpi = \frac{1-p}{4}+\frac{p}{9}\,,$$

which decreases from 1/4 to 1/9 as p increases from 0 to 1. We obtain

$$\alpha(1) = \varpi r\delta^2\,,\qquad \beta(1) = 2\varpi ra\delta\,,\qquad \gamma(1) = \varpi ra^2\,.$$

Notice therefore that if we accept the hypothesis that $\theta < 1/2$, i.e. $\delta < 1$, then the existence condition $\alpha < 1$ is met for all p. Then we get

$$q_1^\star(1) = \frac{1+\frac{\delta}{2}-2\varpi r\delta(\delta+1)}{1-\varpi r\delta^2}\,\frac{a}{2b} = \frac{(1-\theta)(1-\frac{\theta}{2})+2\varpi r\theta}{(1-\theta)^2-\varpi r\theta^2}\,\frac{a}{2b}.$$

[4]The equality $\alpha(T)=\beta(T)=\gamma(T)$ is somewhat absurd ... since these three parameters do not have the same physical dimension: α is dimensionless, β has the dimension of a price and γ of the square of a price.

Several remarks are in order:

1. As expected, for $\theta = 0$, we recover $q^\star(1) = a/2b$, the monopoly production.

2. If θ is very small (less than $2(1-r)/(2-r)$, slightly less than $2(1-r)$), for any p, $q^\star(1)$ is larger than the monopoly level. The first player takes advantage of the small effect of the previous price to get a slightly larger overall profit.

3. For larger θ, the first period production is larger than the equilibrium production if and only if $p > 1.8[1-(1-\theta)/(r(1-\theta/2))]$, a threshold increasing with θ, up to slightly less than 60% as θ reaches $1/2$.

4. For an arrival probability of 60% or more, and still assuming that θ is no more than $1/2$, $q^\star(1)$ is always higher than the monopoly production. This means that then, the prices are lowered by the mere risk of competition, even if it does eventually not materialize.

   The explanation of this "ambiguous effect" of the stickiness is that for a sizable stickiness, if the probability of arrival of a competitor is small enough, the oligopolist has better lower its initial production to let the price go up, increasing by stickiness the average price of the second period where it can benefit from this higher price. But if the probability of arrival of a competitor is large, the expected benefit of the second period is too much decreased, and we are back in the previous scenario where the risk of competition suffices to induce a lowering of the prices.

3 Continuous time

3.1 The problem

3.1.1 Players, dynamics and payoff

We consider a game with randomly arriving (or arriving and leaving) players as in the previous section, but in continuous-time. The players arrive as a Poisson process of variable intensity: The interval lengths $t_{m+1} - t_m$ between successive arrivals are independent random variables obeying exponential laws with intensity λ_m:

$$\mathbb{P}(t_{m+1} - t_m > \tau) = \mathrm{e}^{-\lambda^m \tau}$$

for a given sequence of positive λ^m. An added difficulty, as compared to the discrete time case, is that the number of possible arrivals is unbounded, even for the finite horizon problem. For that reason, the sequence λ^m is a priori infinite. But we assume that the λ^m are bounded by a fixed Λ. As a matter of fact, for any practical use of the theory, we will have to assume that the λ^m are all zero for m larger than a given integer M, thus limiting the number of players to M. Alternatively, for a finite horizon T, we may notice that for any M, the probability $\mathbb{P}(m(t) > M)$ is less than $(\Lambda T)^M/M!$ and therefore goes to zero as $M \to \infty$, and take argument to neglect very large m's.

The dynamic system is also in continuous time. The state space X is now the Euclidean space $\mathbb{R}^d$, or a subset of it, and the dynamics

$$\dot{x} = f^{m(t)}(t, x, s^{m(t)}), \quad x(0) = x_0 .$$

Standard regularity and growth hypotheses hold on the functions f^m to insure existence of a unique solution in X over $[0, T]$ to the dynamics for every m-tuple of measurable functions $s^m(\cdot) : [0, T] \to \mathsf{S}^m$.

A positive discount factor ρ is given, and the performance indices are given via

$$\mathcal{L}_n(t_n, x(t_n), \{s^m(\cdot)\}_{m\in\mathbb{N}}) = \int_{t_n}^{T} \mathrm{e}^{-\rho(t-t_n)} L_n^{m(t)}(t, x(t), s^{m(t)}(t))\,\mathrm{d}t$$

as

$$\Pi_n^e(t_n, x(t_n), \{u^m(\cdot)\}_{m\in\mathbb{N}}) = \mathbb{E}\mathcal{L}_n(t_n, x(t_n), \{s^m(\cdot)\}_{m\in\mathbb{N}}) . \tag{21}$$

The functions L_n^m are assumed to be continuous and uniformly bounded.

As in the discrete time case, we consider identical players, i.e. the functions f^m are invariant by a permutation of the s_n, and the functions L_n^m enjoy the properties of a game with identical players as detailed in the appendix A.

3.1.2 Strategies and equilibrium

We seek a state feedback equilibrium. Let $\mathcal{A}^m$ be the set of *admissible feedbacks* when m players are present. A control law $\varphi : [0, T] \times \mathsf{X} \to \mathsf{S}$ will be in $\mathcal{A}^m$ if, on the one hand, the differential equation

$$\dot{x} = f^m(t, x, \varphi(t, x)^{\times m})$$

has a unique solution for any initial data $(t_n, x_n) \in [0, T] \times \mathsf{X}$, and on the other hand, for every measurable $s(\cdot) : [0, T] \to \mathsf{S}$, the differential equation

$$\dot{x} = f^m(t, x(t), \{s(t), \varphi(t, x(t))^{\times m\setminus 1}\})$$

has a unique solution over $[0, T]$ for any initial data $(t_n, x_n) \in [0, T] \times \mathsf{X}$.

We define a state feedback pure equilibrium as in the previous section, namely via definition 2.3. Moreover, we shall be concerned only with uniform such equilibrium strategies.

3.1.3 Mixed strategies and disturbances

We have rather avoid the complexity of mixed strategies in continuous time (see, however, [Elliot and Kalton, 1972]), as experience teaches us that they are often unnecessary.

Adding disturbances to the dynamics and payoff as in the discrete time problem is not difficult. But the notation need to be changed to that of diffusions, and we would get extra second order terms in Isaacs equation, due to Ito calculus. All results carry over with the necessary adaptations. We keep with the deterministic set up for the sake of simplicity.

3.2 Isaacs equation

3.2.1 Finite horizon

The Isaacs equation naturally associated with a uniform equilibrium in this problem is as follows, where $\hat{s}$ stands for the argument of the maximum (we write V_t and V_x for the partial derivatives of V):

$$\begin{aligned}
&\forall (t,x) \in [0,T] \times \mathsf{X}, \quad (\rho + \lambda^m)V^m(t,x) - \lambda^m V^{m+1}(t,x) - V_t^m(t,x) - \\
&\max_{s \in \mathsf{S}} \Big[V_x^m(t,x) f^m(t,x,\{s, \hat{s}^{\times m \setminus 1}\}) + L_1^m(t,x,\{s,\hat{s}^{\times m \setminus 1}\}) \Big] = 0\,, \qquad (22) \\
&\forall x \in \mathsf{X}, \quad V^m(T,x) = 0\,.
\end{aligned}$$

As already mentioned, even for a finite horizon, the number of players that may join the game is unbounded. Therefore, equation (22) is an infinite system of partial differential equations for an infinite family of functions $V^m(t,x)$. We will therefore make use of the hypothesis similar to 2.1:

Hypothesis 3.1 *There exists an integer M such that $\lambda^M = 0$.*

As hypothesis 2.1 of the discrete time case, this is a natural hypothesis in case of a decreasing payoff and fixed finite entry cost, and akin to classical approximations of the Isaacs equation in dynamic programming algorithms.

Under that hypothesis, using the tools of piecewise deterministic Markov decision Processes, we have the following easy extension of [Fleming and Soner, 1993]:

Theorem 3.1 *A uniform subgame perfect equilibrium exists if and only if there exists a family of admissible feedbacks $\varphi^m \in \mathcal{A}^m$ and a family of bounded uniformly continuous functions $V^m(t,x)$ that are, for all $m \leq M$, viscosity solutions of the partial differential equation* (22)*. Then, $s_n(t) = \hat{\varphi}^{m(t)}(t,x(t))$ is a uniform subgame perfect equilibrium, and the equilibrium payoff of player n joining the game at time t_n and state x_n is $V^n(t_n,x_n)$.*

A sketch of the proof is given in appendix B.1.

The question naturally arises of what can be said of the problem without the hypothesis 3.1. To investigate this problem, we consider an "original problem" defined by its infinite sequence $\{\lambda^m\}_{m\in\mathbb{N}}$, assumed bounded :

$$\exists\Lambda > 0 : \quad \forall m \in \mathbb{N}\,,\ \lambda^m \le \Lambda\,.$$

and a family of "modified problems" depending on an integer M, where we modify the sequence $\{\lambda^m\}$ at λ^M that we set equal to zero. (And therefore all λ^m for $m > M$ are irrelevant: there will never be more than M players.) The theorem above holds for all modified problems, whatever the M chosen. We call $V^{m|M}$ (a finite family) the solution of the corresponding equation (22). They yield the equilibrium value of the payoff Π^{eM} in the modified problems.

We propose in appendix B.2 arguments in favor of the following

Conjecture 3.1 *As M goes to infinity, the equilibrium state feedbacks φ^M of the modified problems converge, in L^1 (possibly weighted by a weight $\exp(-\alpha\|x\|)$) toward an equilibrium feedback $\varphi^\star$ of the original problem, and the functions $V^{m|M}$ converge in C^1 toward the equilibrium value V^m. Consequently, theorem 3.1 holds for the original, unmodified problem.*

3.2.2 Infinite horizon

We assume here that the functions f^m and L^m_n are time invariant, and $\rho > 0$. We set

$$\Pi^e_n(t_n, x(t_n), \{s^m(\cdot)\}_{m\in\mathbb{N}}) = \mathbb{E}\int_{t_n}^{\infty} \mathrm{e}^{-\rho(t-t_n)} L_n^{m(t)}(t, x(t), s^{m(t)})\,\mathrm{d}t\,.$$

As expected, we get

Theorem 3.2 *Under hypothesis 3.1, a uniform subgame perfect equilibrium in infinite horizon exists if and only if there exists a family of admissible feedbacks $\hat\varphi^m \in \mathcal{A}^m$ and a family of bounded uniformly continuous functions $V^m(x)$ that are, for all m, viscosity solutions of the following partial differential equation, where $\hat s$ stands for $\hat\varphi^m(t,x)$ and the minimum is reached precisely at $s = \hat s$:*

$$\forall x \in \mathsf{X}\,, \qquad 0 = (\rho + \lambda^m)V^m(x) - \lambda^m V^{m+1}(x) - \tag{23}$$

$$\max_{s\in\mathsf{S}}\Big[V^m_x(x)f^m(x, \{s, \hat u^{\times m\backslash 1}\}) + L^m_1(x, \{s, \hat s^{\times m\backslash 1}\})\Big] \tag{24}$$

Then, $s_n(t) = \hat\varphi^{m(t)}(x(t))$ is a uniform subgame perfect equilibrium, and the equilibrium payoff of player n joining the game at state x_n is $V^n(x_n)$.

The proof involves extending equation (22) to the infinite horizon case, a sketch of which is provided in appendix B.1, relying on the boundedness of the functions V^m to ensure that $\exp(-\rho T)V^m(x(T))$ goes to zero as T increases to infinity. The rest is exactly as in the previous subsubsection.

The original problem without the bounding hypothesis 3.1 requires a different approach from that of the previous subsection, because in infinite horizon, it is no longer true that $\mathbb{P}(\Omega_M)$ is small. Indeed it is equal to one if the hypothesis does not hold and the λ^m have a lower bound.

3.3 Entering and leaving

As in the discrete time case, we may extend the theory to the case where the players mays also leave the game. We consider that once a player has left, it does not re-enter. We let T_n be the exit time of player n. In the joint exit mechanism, the process that one of the m players present may leave is a Poisson process with intensity μ^m, and if one does, it is one of the players present with equal probability. In the individual scheme, each of the m players present has a Poisson exit process with probability μ^m.

We leave it to the reader to check that Isaacs' equation now reads

$$(\rho + \mathbb{P}^{m,m})V^m(t,x) - \mathbb{P}^{m,m+1}V^{m+1}(t,x) - \mathbb{P}^{m,m-1}V^{m-1}(t,x) - V^m_t(t,x)$$
$$- \max_{s\in\mathsf{S}}\Big[V^m_x(t,x)f^m(t,x,\{s,\hat{s}^{m\backslash 1}\}) + L^m_1(t,x,\{s,\hat{s}^{m\backslash 1}\})\Big] = 0\,,$$

where the coefficients $\mathbb{P}^{m,\ell}$ are given by the following table:

scheme	$\mathbb{P}^{m,m-1}$	$\mathbb{P}^{m,m}$	$\mathbb{P}^{m,m+1}$
joint	$\frac{m-1}{m}\mu^m$	$\lambda^m + \mu^m$	λ^m
individual	μ^m	$\lambda^m + m\mu^m$	λ^m

(25)

3.4 Linear quadratic problem

3.4.1 Finite horizon

We turn to the non standard linear quadratic case, where the dynamics are given by piecewise continuous (or even measurable) time dependent matrices $A(t)$ and $B(t)$ of dimensions, respectively $d \times d$ and $d \times a$ (both could be m-dependent)

$$\dot{x} = A(t)x + B(t)\sum_{n=1}^{m} s_n\,,$$

but where the payoff has two terms of opposite signs, given by a discount factor ρ, and two families of piecewise continuous symmetric matrices: nonnegative $d \times d$ matrices $Q^m(t)$ and positive definite $a \times a$ matrices $R(t)$, as

$$\Pi_n^e = \mathbb{E}\left[\int_{t_n}^T \mathrm{e}^{-\rho(t-t_n)}\left(\|x(t)\|^2_{Q^{m(t)}(t)} - \|s_n(t)\|^2_{R(t)}\right)\mathrm{d}t\right].$$

The economic interpretation is as in the discrete time case, sharing a resource $\|x(t)\|^2_{Q(t)}$ jointly produced at an individual cost $\|u_n(t)\|^2_{R(t)}$. It is for pure notational convenience that we do not let $R(t)$ depend on m, as we shall need its inverse $R^{-1}(t)$.

As in the discrete time case, we must restrict all players to finite weighted norm decisions:

$$\int_{t_n}^T \mathrm{e}^{-\rho(t-t_n)}\|s_n(t)\|^2_{R(t)}\,\mathrm{d}t < \infty. \tag{26}$$

We again seek a uniform solution with Value functions

$$V^m(t,x) = \|x\|^2_{P^m(t)}\,. \tag{27}$$

Isaacs equation now reads

$$\rho\|x\|^2_{P^m(t)} = \max_{s\in\mathsf{S}}\Big[\|x\|^2_{\dot{P}^m(t)} + 2x'P^m(t)(A(t)x + B(t)s + (m-1)B(t)\hat{s}) \\ + \|x\|^2_{Q^m(t)} - \|s\|^2_{R(t)}\Big] + \lambda^m\big(\|x\|^2_{P^{(m+1)}(t)} - \|x\|^2_{P^m(t)}\big).$$

We drop explicit time dependences of the system matrices for legibility. We obtain

$$\hat{s} = R^{-1}B'P^m(t)x \tag{28}$$

and

$$\dot{P}^m - (\rho+\lambda^m)P^m + P^mA + A'P^m + (2m-1)P^mBR^{-1}B'P^m + Q^m + \lambda^mP^{m+1} = 0 \tag{29}$$

with the terminal condition

$$P^m(T) = 0\,. \tag{30}$$

As a corollary of theorem 3.1, we have proved the following:

Corollary 3.1 *If the Riccati equations* (29) (30) *all have a solution over* $[0,T]$*, the finite horizon linear quadratic problem has a unique uniform equilibrium given by equations (27,28,29,30).*

Note that in equation (29), the right hand side is locally Lipshitz continuous in P^m. Therefore, if it is finite dimensional, i.e. under hypothesis 3.1, it is locally integrable backwards. We infer:

Corollary 3.2 *Under hypothesis 3.1, there exists a positive $T^\star$ such that for all $T < T^\star$, the finite horizon linear quadratic problem has a unique uniform equilibrium.*

Entering and leaving We may of course deal with the case where players may leave the game in the same way as before, replacing in the Riccati equation (29) the term

$$\lambda^m\big(P^{m+1}(t) - P^m(t)\big)$$

by

$$\mathbb{P}^{m,m-1}P^{m-1}(t) - \mathbb{P}^{m,m}P^m(t) + \mathbb{P}^{m,m+1}P^{m+1}(t)\,.$$

the $\mathbb{P}^{m,k}$ being given by the table (25). The Riccati equations can no longer be integrated in sequence from $m = M$ down to $m = 1$. But they can still be integrated backward jointly, as a finite dimensional ordinary differential equation. As long as all $P^m(t)$ exist, the interpretation as Value functions still guarantees their nonnegativity.

Several remarks are in order :

Remark 3.1

1. *If the maximum number of players that may enter the game is bounded, i.e. under hypothesis 3.1, this is an explicit algorithm to determine whether a uniform subgame perfect equilibrium can be thus computed and to actually compute it.*

2. *We only stated a sufficiency theorem. What happens if some of the Riccati equations diverge before $t = 0$ is a complicated matter, supposedly more complicated than for a simple zero-sum two-player differential game. And even in that case, we know that, under some non generic condition, a saddle point (a Nash equilibrium) may "survive" such a conjugate point. See [Bernhard, 1979, Bernhard, 1980].*

3. *In the case where the $Q^m(t)$ would be nonpositive definite, and under hypothesis 3.1, we can prove that the Riccati equations do have a solution over $[0, T]$, proving the existence of the uniform subgame perfect equilibrium. (See [Bernhard and Deschamps, 2016a]).*

3.4.2 Infinite horizon

We consider now the case where the system matrices A, B, Q^m, and R are constant, and the problem with payoff

$$\Pi_n^e = \mathbb{E} \int_{t_n}^{\infty} \mathrm{e}^{-\rho(t-t_n)} [\|x(t)\|^2_{Q^{m(t)}} - \|s\|^2_R] \, \mathrm{d}t \, .$$

We still assume (3.1) and constrain all players to finite weighted norm decisions as in (26) with $T = \infty$. Furthermore, we assume that the matrix A has all its eigenvalues with real parts strictly smaller than ρ, to rule out the trivial case where all players could just play $s(t) = 0$ and get an infinitely large payoff, and that $Q^M > 0$. We may state the following:

Theorem 3.3 *If the M Riccati equations* (29) *when integrated backward from $P^m(0) = 0$, have a limit $\bar{P}^M$ a $t \to -\infty$, for a large enough ρ it holds that*

$$\bar{P}^M B R^{-1} B' \bar{P}^M - Q^M < 0 \, , \tag{31}$$

and then, the strategy profile $\hat{s}_n(t) = R^{-1} B' \bar{P}^m(t) x(t)$ is a uniform subgame perfect pure Nash equilibrium.

Proof We aim to apply a modification of theorem 3.2. Notice first that if the limits exist, then the $\bar{P}^m$ solve the algebraic Riccati equations

$$-(\rho + \lambda^m) P^m + P^m A + A' P^m + (2m-1) P^m B R^{-1} B' P^m + Q^m + \lambda^m P^{m+1} = 0 \, . \tag{32}$$

Therefore, the value functions $\|x\|^2_{\bar{P}^m}$ satisfy the stationary Isaacs equation of Theorem 3.2.

Since the (variable) $P^m(t)$ are all positive definite, the $\bar{P}^m$ are nonnegative definite. But they are even positive definite. As a matter of fact, (32) shows that $x' \bar{P}^M x = 0$, which implies $\bar{P}^M x = 0$, is impossible since $\lambda^M = 0$ and $Q^M > 0$. Then, recursively, the same holds for all $m < M$. We also notice that using a standard comparison theorem for ordinary differential equations for $x' P^M x$ with a constant x and equation (29), we see that the $P^m(t)$ are decreasing in ρ, and using (32) divided through by ρ, we see that their limits as $\rho \to \infty$ are zero. ($\P^m(t)$ being decreasing with ρ is bounded.)

To apply the reasoning of 3.2, because the integrand in the payoff is not uniformly bounded, we need first to verify that $\exp(-\rho t) \|x(t)\|^2_{\bar{P}^m}$ goes to zero when all players use their strategies $\hat{s}_n$. Since in infinite time, the number of players will almost surely reach M, the asymptotic behavior of the system is ruled by the control with $\bar{P}^M$. A direct calculation using (32) shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-\rho t}\|x(t)\|^2_{\bar{P}^M}\right] = x'[\bar{P}^M B R^{-1} B' \bar{P}^M - Q^M]x.$$

Apply a standard Lyapunov theory for linear system to conclude that under the condition (31), the Value functions indeed all go to zero as $t \to \infty$.

We need also to check that this limit also holds when all players but one use the strategies $\hat{s}_n$, while the other one plays any admissible control $s(t)$. This will hold if the system where $M-1$ players play according to $\hat{s}$ and the last one zero is stable. Apply again a Lyapunov theory. We find that under this condition,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-\rho t}\|x(t)\|^2_{\bar{P}^M}\right] = -x'[\bar{P}^M B R^{-1} B' \bar{P}^M + Q^M]x.$$

Therefore, the system is indeed stabilized. ▮

Remark 3.2 *In the case where $Q^m \leq 0$, we can prove that indeed the Riccati equations have a limit as $t \to -\infty$. The stability condition is the same, but corresponds to the stability of the system with $M-1$ players using their Nash control.*

3.5 Example: Cournot oligopoly with sticky prices

3.5.1 The model

We propose a continuous time equivalent of our example of subsection 2.5. We assume that the fixed inverse demand curve is now driven by the production rates $\dot{q}_i$ of the players and their sum $\dot{Q}$:

$$P_+ = a - b\dot{Q}.$$

Sticky prices will now be described as the output of a low-pass filter excited by the market clearing price P_+. Let $\delta \in (0, \infty)$ be the degree of stickiness:

$$\delta \dot{P} = P_+ - P.$$

Hence, if $\delta = 0$, $P = P_+$, no stickiness, while in the limit as $\delta = \infty$, P stays forever at its initial value. To avoid technical complications with so called "cheap control" problems, we assume here a production cost with a quadratic term $c\dot{q}^2$ with $c > 0$. As a consequence, the payoff is

$$\Pi^e_n = \mathbb{E}\int_{t_n}^{T} \mathrm{e}^{-\rho(t-t_n)}\left(P\dot{q}_n - c\dot{q}_n^2\right)\,\mathrm{d}t\,.$$

3.5.2 Solution

We expect a Value function of the form

$$V^m(t,P) = K^m(t)P^2 + L^m(t)P + M^m(t)\,.$$

The Isaacs equation reads, for $\delta \neq 0$,

$$\begin{aligned}
&\dot K^m P^2 + \dot L^m P + \dot L^m - (\rho+\lambda)(K^m P^2 + L^m P + M^m)\\
&+ \lambda(K^{m+1}P^2 + L^{m+1}P + M^{m+1})\\
&+ \max_{\dot q_i}\left\{\frac{1}{\delta}(2K^m P + L^m)[a-(m-1)b(\dot q^m)^\star - b\dot q_i] + P\dot q_i - c\dot q_i^2\right\} = 0\,.
\end{aligned}$$

it follows that

$$(\dot q^m)^\star = \frac{1}{2c}\left[\left(1-\frac{2b}{\delta}K^m\right)P - \frac{b}{\delta}L^m\right]\,,$$

and identifying terms in P^2, P and without P, we obtain (for $\delta \neq 0$)

$$\begin{aligned}
&\dot K^m - \left[\rho+\lambda+\frac{1}{\delta}\left(2+\frac{mb}{c}\right)\right]K^m + \frac{(2m-1)b^2}{c\delta^2}K^{m2} + \frac{1}{4c} + \lambda K^{m+1} = 0\,,\\
&\dot L^m - \left[\rho+\lambda+\frac{1}{\delta}\left(1+\frac{mb}{2c}\right) - \frac{(2m-1)b^2}{c\delta^2}K^m\right]L^m + \frac{2a}{\delta}K^m + \lambda L^{m+1} = 0,\\
&\dot M^m - (\rho+\lambda)M^m + \frac{a}{\delta}L^m + \frac{(m-\frac{1}{2})b^2}{2c\delta^2}L^{m2} + \lambda M^{m+1} = 0\,,
\end{aligned}$$

initialized at $K^m(T) = L^m(T) = M^m(T) = 0$.

If we accept hypothesis 3.1 limiting the maximum number of players to M (not to be mistaken for $M^m(t)$ above), this becomes a high dimensional ordinary dfferential equation, which can be integrated by groups of three variables (K^m, L^m, M^m) from $m = M$ with $\lambda = 0$, down to $m = 1$.

A careful analysis of the above differential equations shows that, when integrated backward from zero, all the (K^m, L^m, M^m) converge to some asymptotic values as $T - t \to \infty$. It remains to check whether the equilibrium production rate $(\dot q^m)^\star$ remains positive as is desirable.

A more complete analysis of the dependance of this solution on the degree of stickiness requires a detailed analysis of the above dufferential equations and is beyond the scope of this article. We only want to stress that this problem is within the scope of the linear quadratic version of our theory.

4 Conclusion

In our previous article [Bernhard and Deschamps, 2016b] we investigate dynamic games with randomly arriving players and propose a way to find a sequence of static equilibria for games in discrete time in finite and infinite horizon. With this one we resolve several limitations of the model therein since here we have: a true dynamic equilibrium, variable entry probability (or density), possibility of group entry (if there is a finite number of players) and some exit mechanisms, all done in discrete and continuous time.

Here the tools of piecewise deterministic Markov decision processes have been extended to games with random player arrivals. We have chosen some specific problems within this wide class, namely identical players (there might be several classes of players as in, e.g. [Tembine, 2010]). We have emphasized Bernoulli arrival process in the discrete time case, Poisson in the continuous time case, with no exit. Yet, we have given a few examples of other schemes, with exit at random time also.

We have also considered a restricted class of linear quadratic problems as illustration. All these might be extended to other cases. The present article shows clearly how to proceed. The question is to find which other cases are both interesting and, if possible, amenable to feasible computational algorithms.

In that respect, the unbounded number of players in the infinite horizon discrete time problem, and in all cases in continuous time, poses a problem, mainly computational in the former case, also mathematical in the later, because of the difficulty of dealing with an infinite set of partial differential equations. The computational problem, however, is nothing very different from that of discretizing an infinite state space.

Finally, we may point out the main weakness of our theory: our agents have no idiosyncratic state, wich means, for example, that this theory does not apply as such to our example 1.1.2, since the amount of their remaining claim is such a private state of the players. Nor could we deal with classes of agents, as *e.g.* in [Tembine, 2010], or exit linked with the agent state, typically an exit after a fixed time horizon as in [Kordonis and Papavassilopoulos, 2015].

Nevertheless we consider that our model can probably be used to better understand some real life economic problems and if not at least slightly extend economic modelling.

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A Games with identical players

A.1 Model and properties

By assumption, in the game considered here, all players are identical. To reflect this fact in the mathematical model, we need to consider permutations $\pi^m \in \Pi^m$ of the elements of $\{1, 2, \ldots, m\}$. We also recall the notation

$$s^{m\backslash n} := (s_1, \ldots, s_{n-1}, s_{n+1}, \ldots, s_m)\,,$$
$$\{s^{m\backslash n}, s\} := (s_1, \ldots, s_{n-1}, s, s_{n+1}, \ldots, s_m)$$

Furthermore, we denote

$$s^\pi = s^{\pi[m]} := \left(s_{\pi(1)}, s_{\pi(2)}, \ldots, s_{\pi(m)}\right)\,,$$
$$s^{\pi[m]\backslash\pi(n)} := \left(s_{\pi(1)}, \ldots, s_{\pi(n-1)}, s_{\pi(n+1)}, \ldots, s_{\pi(m)}\right)\,,$$
$$\{s^{\pi[m]\backslash\pi(n)}, s\} := \left(s_{\pi(1)}, \ldots, s_{\pi(n-1)}, s, s_{\pi(n+1)}, \ldots, s_{\pi(m)}\right)\,,$$
$$s^{\times m} := (s, s, \ldots, s) \in \mathsf{S}^m\,.$$

Definition A.1 *A m-person game $\{J_n : \mathsf{S}^m \to \mathbb{R}\}$, $n = 1, \ldots, m$ will be called a* game with identical players *if, for any permutation π of the set $\{1, \ldots, m\}$, it holds that*

$$\forall n \leq m\,, \quad J_n(s_{\pi(1)}, \ldots, s_{\pi(m)}) = J_{\pi(n)}(s_1, \ldots, s_m)\,. \tag{33}$$

We shall write this equation as $J_n(s^{\pi[m]}) = J_{\pi(n)}(s^m)$.

An alternate definition of a game with identical players is given by the following:

Lemma A.1 *A game with identical player is defined by a function $G : \mathsf{S} \times \mathsf{S}^{m-1} \to \mathbb{R}$ invariant by a permutation of the elements of its second argument, i.e. such that,*

$$\forall s \in \mathsf{S}\,, \forall v^{m-1} \in \mathsf{S}^{m-1}\,, \forall \pi \in \Pi^{m-1}\,, \quad G(s, v^{m-1}) = G(s, v^{\pi[m-1]})\,. \tag{34}$$

And the J_n are defined by

$$J_n(s^m) = G(s_n, s^{m\backslash n}) \tag{35}$$

Proof It is clear that if the J_n are defined by (35) with G satisfying (34), they satisfy (33). Indeed, then

$$J_n(s^{\pi[m]}) = G(s_{\pi(n)}, s^{\pi[m]\setminus\pi(n)}) = G(s_{\pi(n)}, s^{m\setminus\pi(n)}) = J_{\pi(n)}(s^m)\,.$$

Conversely, assume that the J_n satisfy (33). Define

$$G(s_1, s^{m\setminus 1}) = J_1(s^m)\,.$$

Let $\pi_1 \in \Pi^{m-1}$, and π defined by $\pi(1) = 1$, and for all $j \geq 2$, $\pi(j) = \pi_1(j-1)$. (i.e. π is any permutation of Π^m that leaves 1 invariant.) It follows from (33) that

$$G(s_1, s^{m\setminus 1}) = J_1(s^m) = J_{\pi(1)}(s^m) = J_1(\{s_1, s^{\pi_1[m\setminus 1]} = G(s_1, s^{\pi_1[m\setminus 1]})\,.$$

Therefore G is invariant by a permutation of the elements of its second argument. Let now π be a permutation such that $\pi(1) = n$. We have

$$J_n(s^m) = J_{\pi(1)}(s^m) = J_1(s^\pi) = G(s_{\pi(1)}, s^{\pi\setminus\pi(1)}) = G(s_n, s^{m\setminus n})\,,$$

which is equation (35). And this proves the lemma. ∎

The main fact is that the set of pure Nash equilibria is invariant by a permutation of the decisions:

Theorem A.1 *Let $\{J_n : \mathsf{S}^m \to \mathbb{R}\}$, $n = 1, \ldots, m$ be a game with identical players. Then if $\hat{s}^m$ is a Nash equilibrium, so is $\hat{s}^{\pi[m]}$.*

Proof Consider $J_n(\hat{s}^{\pi[m]})$, and then substitute some s to $\hat{s}_{\pi(n)}$ in the argument. Because $J_n(s^{\pi[m]}) = J_{\pi(n)}(s^m)$, it follows that

$$J_n(\{\hat{s}^{\pi[m]\setminus\pi(n)}, s\}) = J_{\pi(n)}(\{\hat{s}^{m\setminus\pi(n)}, s\}) \leq J_{\pi(n)}(\hat{s}^m) = J_n(\hat{s}^{\pi[m]})\,.$$

And this is true for all $n \leq m$, which proves the theorem. ∎

Example An example of the above reasoning is as follows. Let $m = 2$ and by hypothesis, $\forall(s_1, s_2)$, $J_1(s_2, s_1) = J_2(s_1, s_2)$. Let $(\hat{s}_1, \hat{s}_2)$ be a Nash equilibrium. Let us show that $(\hat{s}_2, \hat{s}_1)$ is also an equilibrium:

$$\forall s\,, \quad J_1(s, \hat{s}_1) = J_2(\hat{s}_1, s) \leq J_2(\hat{s}_1, \hat{s}_2) = J_1(\hat{s}_2, \hat{s}_1).$$

Corollary A.1 *A pure Nash equilibrium of a game with identical players can be unique only if it is* uniform*, i.e. with all players using the same control:*

$$\exists \hat{s} \in \mathsf{S} : \forall n \leq m\,, \quad \hat{s}^m_n = \hat{s}\,.$$

Existence of such a Nash equilibrium is not guaranteed, and even if it exists, it might not be the only one. However there is a simple way to look for one. Let us first assert the following fact:

Theorem A.2 *Let* $\{J_n : \mathsf{S}^m \to \mathbb{R}\}, n = 1, \dots, m$ *be a game with identical players. If the function* $s_1 \mapsto J_1(\{s^{m\setminus 1}, s_1\})$ *is concave, so are all the functions* $s_n \mapsto J_n(\{u^{m\setminus n}, s_n\})$.

Proof Let $\tilde{s}^m = (s_n, s_2, \dots, s_{n-1}, s_1, s_{n+1}, \dots, s_m)$, and let $\pi^{1,n}$ be the permutation that exchanges 1 and n. Then, $s^m = \tilde{s}^{\pi^{1,n}}$. Thus,

$$J_n(s^m) = J_n(\tilde{s}^{\pi^{1,n}}) = J_1(\tilde{s}^m) = J_1(s_n, \dots)\,.$$

Now, J_1 is by hypothesis concave in its first argument, here u_n. Therefore J_n is concave in s_n. ∎

Finally, we shall use the corollary of the following theorem[5]:

Theorem A.3 *Let* $\{J_n : \mathsf{S}^m \to \mathbb{R}\}, n = 1, \dots, m$ *be a game with identical players. Let* $s \in \mathsf{S}$ *and* $s^{\times m} = (s, s, \dots, s) \in \mathsf{S}^m$. *Then*

$$\forall n \le m\,, \quad \mathrm{D}_n J_n(s^{\times m}) = \mathrm{D}_1 J_1(s^{\times m})\,.$$

Proof Observe first that obviously,

$$\forall n \le m\,, \quad J_n(s^{\times m}) = J_1(s^{\times m}).$$

Let now $\tilde{s}^m = (s + \delta s, s, \dots, s)$, and as previously $\pi^{1,n}$ be the permutation that exchanges 1 and n. Let perturb the n-th control in $J_n(s^{\times m})$ by δs. We get

$$J_n(s, \dots, s, s + \delta s, s, \dots, s) = J_n(\tilde{s}^{\pi^{1,n}}) = J_1(\tilde{s}^m).$$

Therefore, the differential quotients involved in $\mathrm{D}_n J_n(s^{\times m})$ and $D_1 J_1(s^{\times m})$ are equal, hence the result. ∎

Corollary A.2 *If* $s_1 \mapsto J_1(s^m)$ *is concave, an interior solution* $\hat{s} \in \mathsf{S}$ *of the equation*

$$D_1 J_1(s^{\times m}) = 0 \tag{36}$$

yields a uniform Nash equilibrium $\hat{s}^{\times m}$.

[5]Where we use Dieudoné's notation $\mathrm{D}_k J$ for the partial derivative of J with respect to its k-th variable

A.2 Examples of games with identical players

The best known example of game with identical players is Cournot's duopoly. This, incidentally, is an *aggregative game* according to the definition of [Kukushkin, 1994], which are a sub-class of games with identical players. We propose three 2-player games with identical players with different structures of equilibria. Let $i \in \{1, 2\}$:

$$\Pi_i(s_1, s_2) = (s_1 - s_2)^2 - as_i^2 \,. \tag{37}$$

A.2.1 Example 1

In this example, $\mathsf{S} = \mathbb{R}$ and $1 < a < 2$. Then $s_1 \mapsto \Pi_1(s_1, s_2)$ is concave for all s_2. Moreover

$$\mathrm{D}_1\Pi_1(s_1, s_2) = 2(1 - a)s_1 - 2s_2$$

so that the unique maximum in s_1 is reached at $s_1 = -s_2/(1 - a)$. Therefore a Nash equilibrium requires that

$$\begin{aligned} (a-1)s_1 + s_2 &= 0\,, \\ s_1 + (a-1)s_2 &= 0\,. \end{aligned}$$

The determinant of the matrix of this system is $a(a - 2) < 0$. Therefore, the matrix is invertible, the only solution is $s_1 = s_2 = 0$. There is a single (pure) Nash equilibrium, which is uniform.

The question of whether there can exist a mixed Nash equilibrium is investigated as follows: let s_2 be a random variable (a mixed strategy). Clearly, then

$$\mathbb{E}\Pi_1(s_1, s_2) = (1 - a)s_1^2 - 2\mathbb{E}(s_2)s_1 + \mathbb{E}(s_2^2)\,.$$

This has a unique maximum at $s_1 = -\mathbb{E}(s_2)/(a-1)$. Therefore Player 1's strategy is necessarily pure, but then Player 2's strategy also.

A.2.2 Example 2

We use the same example as above, but with $a = 2$. Then any pair $(s_1, s_2) = (s, -s)$ solves the Nash necessary condition, and in view of the concavity of the payoffs, is a (pure) Nash equilibrium. Indeed $\Pi_1(-v, v) - \Pi_1(s, v) = (s+v)^2 \geq 0$, and symmetrically $\Pi_2(s, -s) - \Pi_2(s, v) = (s+v)^2 \geq 0$. The set of Nash equilibria is, as predicted, invariant by a permutation $s_1 \leftrightarrow s_2$. (However, $\Pi_i(s, -s) = -2s^2$, so that both players prefer the equilibrium $(0, 0)$.)

No mixed equilibrium is possible for the same reason as above.

A.2.3 example 3

We use now $\mathsf{S} = [0,1]$ and $a < 1$. Now $s_1 \mapsto \Pi_1(s_1, s_2)$ is convex for all s_2. Therefore a maximum in s_1 can only be reached at $s_1 = 0$ or $s_1 = 1$. Observe that

$$\Pi_1(1, s_2) - \Pi_1(0, s_2) = 1 - a - 2s_2 \,.$$

Therefore, for $s_2 < (1-a)/2$, the maximum of Π_1 is reached at $s_1 = 1$, while for $s_2 > (1-a)/2$, it is reached at $s_1 = 0$. We therefore find two pure Nash equilibria: $(1,0)$ and $(0,1)$.

Indeed, once it is established that pure Nash equilibria can only be found at $s_i \in \{0, 1\}$, we can investigate the matrix game

$s_1 \backslash s_2$	0		1	
		0		$1-a$
0	0		1	
		1		$-a$
1	$1-a$		$-a$	

The two pure Nash equilibria appear naturally. We can also look for a mixed equilibrium, obtained for $\mathbb{P}(s_i = 0) = (1+a)/2$, $\mathbb{P}(s_i = 1) = (1-a)/2$. In that case $\mathbb{E}(s_2) = (1-a)/2$. This an mixed equilibrium of the matrix game, *and also an equilibrium of the game over the unit square*, since the maxima can only be attained at 0 or 1. (The fact that also $\Pi_1(0, (1-a)/2) = \Pi_1(1, (1-a)/2)$ is a coincidence, due to the fact that $\Pi_1(1, s_2) - \Pi_1(0, s_2)$ is affine in s_2.)

B Continuous Isaacs equation

B.1 Modified, bounded m, problem

We first evaluate the following mathematical expectation, given t_m:

$$\mathcal{S}^m = \mathbb{E}\left[\int_{t_m}^{t_{m+1}} \mathrm{e}^{-\rho t} L^m(t, x(t), u^m(t))\,\mathrm{d}t + \mathrm{e}^{-\rho t_{m+1}} V^{m+1}(t_{m+1}, x(t_{m+1}))\right] .$$

given that both $L^m(t)$ and $V^{m+1}(t)$ are taken equal to zero if $t > T$. We have

$$\mathcal{S}^m = \mathrm{e}^{-\lambda^m(T-t_m)} \int_{t_m}^{T} \mathrm{e}^{-\rho t} L^m(t, x(t), u^m(t))\,\mathrm{d}t +$$
$$\int_{t_m}^{T} \lambda^m \mathrm{e}^{-\lambda^m(\tau - t_m)} \left[\int_{t_m}^{\tau} \mathrm{e}^{-\rho t} L^m(t, x(t), u^m(t))\,\mathrm{d}t + \mathrm{e}^{-\rho\tau} V^{m+1}(\tau, x(\tau))\right] d\tau \,.$$

Exchanging the order of summations in the double integral, changing the name of the integration variable in the second, it comes, after cancellation of the first term with one of those coming from the double integral:

$$\mathcal{S}^m = \int_{t_m}^{T} \mathrm{e}^{-\lambda_m(t-t_m)-\rho t}\big(L^m(t,x(t),u^m(t)) + \lambda^m V^{m+1}(t,x(t))\big)\,\mathrm{d}t\,. \tag{38}$$

We turn to the Isaacs equation (22), and deal with it as if the Value functions V^m were of class C^1. Multiply both sides of the equation by $\exp(-\lambda(t-t_m)-\rho t)$ and rewrite it as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\lambda^m(t-t_m)-\rho t}V^m(t,x(t)\right) + \mathrm{e}^{-\lambda^m(t-t_m)-\rho t}L^m(t,x(t),u^m(t))$$
$$+\,\lambda^m\mathrm{e}^{-\lambda^m(t-t_m)-\rho t}V^{m+1}(t,x(t)) \le 0\,,$$

being understood that the lagrangian derivative and L^m are evaluated at $u^m(t) = \{u(t),\hat{u}^{(m\backslash 1)}(t)\}$, and that the inequality becomes an equality for $u(t) = \hat{u}(t)$. Integrating from t_m to T, we recognize $\mathcal{S}^m$ and write

$$\mathrm{e}^{-\rho t_m}V^m(t_m,x(t_m)) \ge \mathrm{e}^{-(\lambda^m+\rho)T+\lambda^m t_m}V^m(T,x(T))$$
$$+\,\mathbb{E}\left[\int_{t_m}^{t_{m+1}} L^m(t,x(t),u^m(t)) + \mathrm{e}^{-\rho t_{m+1}}V^{m+1}(t_{m+1},x(t_{m+1}))\right].$$

In the finite horizon version, we have $V^m(T,x) = 0$, so that the first term in the right hand side cancels, and we are left with

$$\mathrm{e}^{-\rho t_m}V^m(t_m,x(t_m)) \ge$$
$$+\,\mathbb{E}\left[\int_{t_m}^{t_{m+1}} L^m(t,x(t),u^m(t)) + \mathrm{e}^{-\rho t_{m+1}}V^{m+1}(t_{m+1},x(t_{m+1}))\right]$$

if player one, say, deviates alone from $\hat{u}^m(t)$, and equality if $u^m(t) = \hat{u}^{(m)}(t)$. In the infinite horizon case, use the fact that V^m is bounded to see that the same first term of the r.h.s. cancels in the limit as T goes to infinity.

With this last inequality, we proceed as in discrete dynamic programing: take the a priori expectation of both sides, sum for all $m \le M$, cancel the terms that appear on both sides of the sum and use $t_1 = 0$ (the first player starts at time 0) to get

$$V^1(0,x_0) \ge \mathbb{E}\int_0^T \mathrm{e}^{-\rho t}L^{m(t)}(t,x(t),u^m(t))\,\mathrm{d}t = \Pi_1^e(0,x_0,u^m)\,,$$

for $u^m(t) = \{u(t),u^{(m\backslash 1)}(t)\}$, and equality if $u^m(t) = \hat{u}^{(m)}(t)$.

Having restricted our search to state feedback strategies and to a uniform equilibrium of identical players, and ignoring the intrinsic fixed point problem that for each (m,t,x) the maximizing control be precisely $\hat{\phi}^m(t,x)$ used by all other players, the inequality in definition 2.3 defines a unique maximization problem. As a consequence, in the case where the functions V^m are not globally C^1, both the necessary and the sufficiency characters with viscosity solutions are derived from this calculation in the same way as for one-player control problems. But a major difference with that case is that here, existence is far from granted. On the one hand, the fixed point for each (m,t,x) may not exist, and on the other hand, if it always does, it might not define an admissible strategy as characterized in paragraph 3.1.2. The situation is more complex for many player games than for two player games, where one can dispense with state feedback strategies. For these difficult technical matters, see [Evans and Souganidis, 1984, Friedman, 1994, Quincampoix, 2009, Laraki and Sorin, 2015].

B.2 Unmodified unbounded m problem

We aim to extend theorem 3.1 to the unmodified problem where the number of players who may join the game before the time T is unbounded, and therefore equation (22) involves an infinite number of functions V^m. We simplify the notations as follows. Given two admissible state feedbacks ϕ and ψ, let

$$G(\phi,\psi) = \Pi_1^e(\{\phi, \psi^{\times m(t)\setminus 1}\})$$

and the same with upper index M (respectively N) be the corresponding quantity in the modified problem where $\lambda^M = 0$ (resp. $\lambda^N = 0$).

We make the following hypotheses which would need to be converted into hypotheses bearing on the data f^m and L^m of the problem, probably via the hamiltonian

$$H^m(t,x,p,u,v) = \langle p, f(t,x,\{u, v^{\times m\setminus 1}\})\rangle + L(t,x,\{u, v^{\times m\setminus 1}\})\,.$$

We endow the set of state feedbacks with the topology of L^1 and assume:

Hypothesis B.1

1. *The function $\phi \mapsto G(\phi,\psi)$ is, for all ψ quasi concave with a unique maximum and differentiable.*

2. *there exists a positive number β such that,*

$$\begin{aligned}
&\forall M \in \mathbb{N}\,,\ \forall \phi, \chi, \psi \in \mathcal{A}\,,\ \forall \mu \in [0,1] \\
&G^M\big((1-\mu)\phi + \mu\chi, \psi)\big) \leq \\
&(1-\mu)G^M(\phi,\psi) + \mu G^M(\chi,\psi) + \frac{\beta}{2}\mu(1-\mu)\|u-v\|^2\,.
\end{aligned}$$

If $\phi \mapsto G^M(\phi,\psi)$ is of class C^2, this is equivalent to

$$\forall \phi, \chi, \psi \in \mathcal{A}\,, \quad |\langle \mathrm{D}_{11}G(\phi,\psi)\chi,\chi\rangle| \leq \beta\|\chi\|^2\,.$$

3. *For all M and ψ, the map $\phi \mapsto D_1G^M(\phi,\phi)$ is locally invertible in a neighborhood of zero with an inverse locally uniformly Lipshitz of modulus γ. If $(\phi,\psi) \mapsto G^M(\phi,\psi)$ is of class C^2, it suffices that the operator $\mathrm{D}_{11}G(\phi,\psi) + \mathrm{D}_{12}G(\phi,\psi)$ be onto, with an inverse uniformly bounded by a positive number γ.*

With this set of hypotheses, too abstract at this stage, we can prove the conjecture 3.1. We first prove a simple lemma.

Let $\mathbb{P}$ be the probability structure induced by the entry process in the original problem, $\mathbb{E}$ the mathematical expectation in that probability law, $\mathbb{P}^M$ the probability law induced by the modified problem with $\lambda^M = 0$, and $\mathbb{E}^M$ the mathematical expectation in that law. We prove the following lemma.

Lemma B.1 *Let $X(\omega)$ be a bounded random variable measurable on the sigma-field generated by the entry process. $\mathbb{E}^M X$ converges to $\mathbb{E}X$ as M goes to infinity.*

Proof In the original problem, let Ω^M be the set of events for which $m(T) < M$ and Ω_M the complement: events such that $m(T) \geq M$. These sets belong to the sigma-field generated by the entry process. We have

$$\mathbb{E}(X) = \int_{\Omega^M} X(\omega)\,\mathrm{d}\mathbb{P}(\omega) + \int_{\Omega_M} X(\omega)\,\mathrm{d}\mathbb{P}(\omega)$$

and similarly for $\mathbb{E}^M X$. Now, both laws coincide over Ω^M. Therefore

$$\begin{aligned}
|\mathbb{E}X - \mathbb{E}^M X| &= \left|\int_{\Omega_M} X(\omega)\,\mathrm{d}\big(\mathbb{P}(\omega) - \mathbb{P}^M(\omega)\big)\right| \\
&\leq \sup_{\omega\in\Omega_M} |X(\omega)|\,\big(\mathbb{P}(\Omega_M) + \mathbb{P}^M(\Omega_M)\big)\ .
\end{aligned}$$

Notice finally that $\mathbb{P}(\Omega^M) = \mathbb{P}^M(\Omega^M)$, and therefore for their complements:

$$\mathbb{P}(\Omega_M) = \mathbb{P}^M(\Omega_M) = \mathbb{P}(m(T) \geq M) < \frac{(\Lambda T)^M}{M!}$$

which goes to zero with M. As a consequence, $\mathbb{E}^M X$ converges to $\mathbb{E}X$ as M goes to infinity. ∎

Let $M < N$ be two integers. Let φ^M and φ^N be the equilibrium feedbacks of the modified problems G^M and G^N respectively. Using the lemma, we see that given a positive number ε, there exists an integer K such that for any M and N lager than K, and any φ,

$$|G^M(\varphi, \varphi^N) - G^N(\varphi, \varphi^N)| \leq \varepsilon\,.$$

It follows that

$$\forall \varphi\,, G^M(\varphi, \varphi^N) \leq G^N(\varphi, \varphi^N) + \varepsilon \leq G^N(\varphi^N, \varphi^N) + \varepsilon \leq G^M(\varphi^N, \varphi^N) + 2\varepsilon\,.$$

From the fact that $G^M(\varphi^N, \varphi^N)$ is close to the maximum in ϕ of $G^M(\phi, \varphi^N)$ and hypothesis 2, we may derive that

$$\|\mathrm{D}_1 G^M(\varphi^N, \varphi^N)\| \leq 2\sqrt{\beta\varepsilon}.$$

On the other hand, $\mathrm{D}_1 G^M(\varphi^M, \varphi^M) = 0$. From hypothesis 3 we conclude that

$$\|\varphi^N - \varphi^M\| \leq 2\gamma\sqrt{\beta\varepsilon}\,.$$

Hence the sequence $\{\varphi^M\}$ is Cauchy, and thus converges to some $\varphi^\star$. Hence the $V^{m|M}$ converge, and because all satisfy the P.D.E. they converge in C^1.

The hypotheses B.1 can be made mode concrete with the following approach, only sketched here. We consider again the system of partial differential

$$\begin{aligned}
\rho W^m(t,x) &= W^m_t(t,x) + H^m(t,x,W^m_x(t,x),u,v) \\
&\qquad + \lambda^m[W^{m+1}(t,x) - W^m(t,x)]\,, \\
W^m(T,x) &= 0\,.
\end{aligned}$$

When we set furthermore, for all $(t,x) \in [0,T] \times \mathbb{R}^d$, $W^{M+1}(t,x) = 0$ in the system above, this uncouples the equation for W^M from the other ones, and allows one to consider the system in decreasing order of m as a finite sequence of P.D.E's. We denote with $W^{m|M}$ such a family of solutions with $m \leq M$.

Choose a pair of admissible state feedbacks ϕ and ψ. Consider the above system with, for all (t,x), $u = \phi^{m(t)}(t,x)$ and $v = \psi^{m(t)}(t,x)$. It follows from the analysis of the previous subsection that a viscosity solution exists and that, if $x(t_n) = x_n$

$$W^{m|M}(t_n, x_n) = G_n^M(\phi, \psi)\,.$$

And the $V^{m|M}(t,x)$ are the equilibrium values for a uniform equilibrium, obtained for $\phi(t,x) = \psi(t,x) = \varphi^M(t,x)$. We will use the shorthand notation u^M (respectively u^N) in the equations. As a consequence, for instance

$$V^{m|M}(t_m, x(t_m)) = G_m^M(\varphi^M, \varphi^M) = \min_{\phi} G_m^M(\phi, \varphi^M)\,.$$

It follows that each $\varphi^{m|M}$ maximizes the criterion $\mathcal{S}^m$ (38), and that, for M and N large enough, $|W^{m|M} - W^{m|N}|$ is smaller than an arbitrarily chosen ε. Let

$$H^m(t,x,p,u,v) = L^m(t,x,\{u, v^{\times m\setminus 1}\}) + \langle p, f^m(t,x,\{u, v^{\times m\setminus 1}\})\rangle\,.$$

The second derivative version of hypothesis B.1.2 derives from the standard second variation theory and the hypothesis that the solution $y(t)$ of the linear differential equations

$$\dot{y} = \mathrm{D}_2 f^m(t,x,(\varphi^M(t,x))^{\times m})y + \mathrm{D}_{u_1} f^m(t,x,(\varphi^M(t,x))^{\times m})w(t)$$

and the second derivative

$$\begin{pmatrix} \mathrm{D}_{22}H^m & \mathrm{D}_{24}H^m \\ \mathrm{D}_{42}H^m & \mathrm{D}_{44}H^m \end{pmatrix}$$

are uniformly bounded. Then we conclude that the L^∞ norm

$$\|\mathrm{D}_4 H^m(t,x,W_x^{m|M}(t,x), u^N, u^N)\|_\infty$$

is less than $2\sqrt{\beta\varepsilon}$, and that thus the equation

$$\mathrm{D}_4 H^m(t, W_x^{m|M}(t,x), u, u) = 0$$

has a solution u^M close to u^N, leading to the conclusion that the sequence $\{\varphi^M\}$ is Cauchy and thus convergent.