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Mostapha Diss, Eric Kamwa, Muhammad Mahajne

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CRESE
30, avenue de l'Observatoire
25009 Besançon
France
http://crese.univ-fcomte.fr/

UNIVERSITÉ DE FRANCHE-COMTÉ

Borda rule as an almost first-order stochastic dominance rule

Mostapha Diss · **Eric Kamwa** · **Muhammad Mahajne**

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Abstract In single-winner elections and individuals expressing linear orderings, an alternative has first-order stochastic dominance if the cumulative standing for this alternative at each rank is higher than that of the other alternatives. It is well known that this criterion may fail in ranking the competing alternatives since the first-order stochastic dominance winner may not exist in some situations. Making an adaptation of a centrality measure from network theory, we introduce in this note a rule, called the almost first-order stochastic dominance rule, which selects the alternative having first-order stochastic dominance if such an alternative exists, otherwise it selects the alternative which is close to achieve first-order stochastic dominance. It turns out that this rule is equivalent to the well-studied Borda rule. This result highlights an unknown property of the Borda rule.

Keywords: Network, centrality, centrality measures, rankings, first-order stochastic dominance, scoring rules, Borda's rule.
JEL Classification Number: C71, D71, D72, D85.

1 Introduction

Voting theory is full of a multitude of voting rules that can allow groups of individuals to make collective decisions. In practice, the rules differ from one group to another and the choice of a rule often depends on criteria which may be specific to

M. Diss
CRESE EA3190, Univ. Bourgogne Franche-Comté, F-25000 Besançon, France. E-mail: mostapha.diss@univ-fcomte.fr

E. Kamwa
LC2S UMR CNRS 8053, Univ. des Antilles, F-97275 Schoelcher Cedex, France. E-mail: eric.kamwa@univ-antilles.fr

M. Mahajne
GATE UMR CNRS 5824, Univ Lyon, UJM Saint-Etienne, F-42023 Saint-Etienne, France. E-mail: mahajnah@gate.cnrs.fr

the group and to the objectives pursued. Most of the popular voting rules belong to the family of the so-called scoring rules. With scoring rules, points (weights) are given to the alternatives in the running according to the position they occupy in the preferences (rankings) of the individuals taking part in the collective decision-making process; the winner is the alternative with the highest total number of points.

It should nevertheless be noted that given a preference profile, that is a list of preferences of a group of individuals over a set of alternatives, varying the voting rule (the points) can also vary the outcome (Saari, 1992, 1999). According to Stein et al. (1999), it is important to investigate how changing the assignment of point values will affect the outcome of the aggregation process and one way to do this is to use stochastic dominance. This concept has been originally developed in the traditional expected utility framework, and is now widely used in many other fields including economics, finance, mathematics, insurance, etc.[1] Given a preference profile, if for each rank, the cumulative standings (frequencies) of an alternative a are greater or equal to those of alternative b, then a is said to have a *first-order stochastic dominance* over b. The cumulative standings of a given alternative represent the number of individuals that rank this alternative over a certain rank. According to Stein et al. (1999), stochastic dominance finds all its importance in situations in which we are not able to point out exactly which system of points is the right one to use; if an alternative a has first-order stochastic dominance over an alternative b, then a will get a higher score than b for any possible scoring rule. In other words, an alternative with first-order stochastic dominance over each of its opponents, when it exists for a preference profile, would win using any scoring rule. In such a case, we may say (with no ambiguity) that this profile exhibits a first-order stochastic dominance consensus since all the scoring rules agree; therefore, the choice of the points system does not matter in terms of "the right one to use".

Note that the existence of an alternative with first-order stochastic dominance over each of its opponents is not always guaranteed. Furthermore, for some preference profiles, it may happen that first-order stochastic dominance does not allow two alternatives to be ranked. Indeed, we can find situations in which most individuals rank an alternative a before an alternative b but there is no stochastic dominance between the two alternatives (Leshno and Levy, 2002). In such rather specific situations, Leshno and Levy (2002) argue that some relaxations may provide advantages over of stochastic dominance in several contexts. In this sense, they proposed, especially for research in finance, the *almost stochastic dominance* concept as a subsidiary approach to the stochastic dominance (see also Tzeng et al., 2013). According to Leshno and Levy (2002), with almost stochastic dominance rules, it is possible that the distribution of standings on two prospects under comparison do not obey any stochastic dominance but, with a small change made in the cumulative standings, stochastic dominance rules will reveal a preference. It should be noted that if an alternative stochastically dominates another, the almost stochastic domination is also established. In this note, our objective is to reclaim the idea of almost first-order stochastic dominance in the context of the aggrega-

[1] See for instance Levy (2006) for a review.

tion of individual preferences in order to propose a "relaxation" of the classical stochastic dominance as developed in this context.

So, in this spirit, in the absence of an alternative having the first-order stochastic dominance, we could construct a rule which would choose the alternative(s) requiring the least number of transformations in individual preferences in order to achieve first-order stochastic dominance. With such a rule, we will see that determining the least number of transformations in individual preferences requires the use of metrics. However, it is well-known that the use of metrics comes up against a computational problem whose complexity can increase with the number of alternatives in the running. We may well achieve the same result as the rule that we have just described while circumventing this complexity by resorting to the *Almost first-order Stochastic Dominance Rule* (ASDR) that we introduce in this note. ASDR is based on an adaptation, from network theory to voting theory, of a generic measure of centrality which sums the differences between each vertex's centrality score and that of the most central vertex. In network theory or graph theory, centrality measures the extent of inequality among vertex centrality scores and it usually describes the network positions of vertices. Without being exhaustive, the reader may refer to Freeman (1979), Golbeck (2015, 2013) and Marsden (2015) for an overview of the centrality measures. In this note, we show that our almost first-order stochastic dominance rule is simply equivalent to the Borda rule. This result thus highlights an unknown property of the Borda rule.

The rest of the paper is organized as follows: Section 2 is devoted to basic definitions; in Section 3, we introduce our almost first-order stochastic dominance rule and we provide our main result. Section 4 concludes.

2 Notation and definitions

Let $A = \{a, b, \ldots\}$ be a set of K alternatives and $N = \{1, 2, \ldots, n\}$ a finite set of n individuals ($K, n \geq 2$). Every individual $i \in N$ is assumed to have a linear order $\succ_i$ on A; this means that each individual ranks the alternatives from the most desirable one to the least desirable one. Let $\mathcal{L}(A)$ be the set of all linear orders on A and we refer to the elements of $\mathcal{L}(A)$ as preference relations. We denote by $\pi = (\succ_1, \ldots, \succ_n)$ a preference profile which gives the preference relations of all of the n individuals. We denote by $P = \mathcal{L}(A)^n$ the set all possible preference profiles. For any preference relation $\succ_i \in \mathcal{L}(A)$ and for any alternative $a \in A$, the rank of a in $\succ_i$ is defined by $rank_{\succ_i}(a) = K - |\{b \in A : a \succ_i b, b \neq a\}|$.

A social choice rule is a function that assigns a nonempty subset of alternatives in A to each preference profile $\pi \in P$. A special class of social choice rules consists of *scoring rules*. A scoring rule is characterized by a K-tuple $w = (w_1, w_2, \ldots, w_K)$ of non-negative scores (weights) with $w_1 \geq w_2 \geq \ldots \geq w_K$ and $w_1 > w_K$. Given a preference profile π, each individual i assigns w_k points to an alternative ranked k-th in her preference relation, for $k = 1, 2, \ldots, K$. That is, each individual assigns w_1 points to her most preferred alternative, w_2 points to the second best alternative and so on. The scoring rule associated with the scores in w chooses the alternative(s) with the maximum total score.

Given the set A and a preference profile π, it is well known that the scoring rule defined by w and the one defined by $\alpha w + \beta$ are equivalent, for any $\alpha > 0$ and $\beta \in \mathbb{R}$. Therefore, we will restrict our attention throughout the paper to the vector $w = (w_1, w_2, \dots, w_K)$ where $w_K = 0$.

For every preference profile $\pi \in P$, let us introduce the quantity $f_\pi^a(k)$ to define the number of individuals ranking alternative a at the position $k \in [1, K]$:

$$f_\pi^a(k) = |\{i \in N : rank(a) = k\}| \tag{1}$$

Using (1), the total score of an alternative a under the scoring rule w is given by

$$S_w(\pi, a) = \sum_{k=1}^{K-1} w_k f_\pi^a(k) \tag{2}$$

and the first-order cumulative standings (frequencies) for a is given by

$$\begin{aligned} F_\pi^a(k) &= |\{i \in N : rank(a) \leq k\}| \\ &= \sum_{j=1}^{k} f_\pi^a(j) \end{aligned} \tag{3}$$

Many well-known scoring rules have received a considerable amount of attention in the social choice literature due to their intuitive appeal and we can define these rules using the first-order cumulative standings. For instance, the *Plurality rule* (PR) is the voting rule associated with the scoring vector $(1, 0, \dots, 0)$; it easily comes that the score of alternative a under PR denoted (with no ambiguity) by $S_{\text{PR}}(\pi, a)$ is given by $S_{PR}(\pi, a) = F_\pi^a(1)$. The *Negative Plurality rule* (NPR) is associated with the vector $(1, 1, \dots, 1, 0)$ and $S_{NPR}(\pi, a) = F_\pi^a(K-1)$. The *Borda rule* (BR) is associated with the vector $(K-1, K-2, \dots, 1, 0)$ and $S_{BR}(\pi, a) = \sum_{k=1}^{K-1} F_\pi^a(k)$. Finally, for $1 \leq t \leq K-1$, the *t-approval voting rule* (t-AV) is associated with the vector $(\underbrace{1, \dots, 1}_{t}, 0, \dots, 0)$ and $S_{t-AV}(\pi, a) = F_\pi^a(t)$. Note that Saari (1999) showed that alternative a is chosen by all the scoring rules if and only if it is chosen by all the t-approval voting rules for $t = 1, 2, \dots, K-1$.

We now introduce the definition of first-order stochastic dominance as already given in Stein et al. (1999).[2]

Definition 1. For $a, b \in A$ and a given preference profile $\pi \in P$, alternative a has *first-order stochastic dominance* over alternative b, denoted by $a \succ_{\text{FOSD}} b$, if $F_\pi^a(k) \geq F_\pi^b(k)$ for all $1 \leq k \leq K-1$ with a strict inequality for at least one k.

We will then say that alternative a exhibits *first-order stochastic dominance* (or alternative a is the *first-order stochastic dominance winner*) if $a \succ_{\text{FOSD}} b$, for all $b \in A \setminus \{a\}$. Such an alternative should be unique if it exists.

[2] See also Kondratyev (2018).

It should be noted that for a given preference profile π, if $a \succ_{\text{FOSD}} b$, this implies $S_w(\pi, a) > S_w(\pi, b)$ for all w (Stein et al., 1999). More, if a exhibits first-order stochastic dominance, then it is the winner for all scoring vectors w. In other words, first-order stochastic dominance for a given alternative also means *scoring consensus* for this alternative since all the scoring rules will agree on it as the same winner.

3 From centrality to first-order stochastic dominance

Stochastic dominance seems to be very attractive from a theoretical viewpoint. However, some issues can be addressed for its practical application. Indeed, it is easy to find examples of preference profiles where stochastic dominance cannot rank the competing alternatives. In this sense, stochastic dominance might be too rigid in practice. Worse still, for some preference profiles, an alternative with first-order stochastic dominance may not exist. This reminds us of the concept of Condorcet winner, for instance, which is very popular in the social choice literature. A Condorcet winner is an alternative which is preferred to each of the other competing alternatives by a simple majority of individuals. Such an alternative does not always exist and it should be unique if it exists. A voting rule is called Condorcet consistent if it always chooses the Condorcet winner when one exists. Many voting rules have been introduced in the literature in order to implement the idea of selecting the closest alternative to the Condorcet winner when one does not exist.

In accordance with the spirit of Condorcet's analysis, one extension of the first-order stochastic dominance might be to define a voting rule which always selects the alternative having first-order stochastic dominance if such an alternative exists, otherwise it selects the alternative which needs the minimal amount of changes in the preference profile in order to achieve first-order stochastic dominance. More exactly, given a preference profile $\pi \in P$, we can define a rule (denoted by G) as follows:

$$G(\pi) = \arg\min_{a \in A} d(a)$$

With $d(a) = \min_{\pi' \in H(a)} d(\pi, \pi')$ and $H(a) = \{\pi' \in P : a \succ_{\text{FOSD}} b, \forall b \in A \setminus \{a\}\}$. The quantity $d(\pi, \pi')$ defines a metric and π' is a new preference profile (obtained from π) defined such that an alternative having first-order stochastic dominance exists. The two commonly used metrics are the following:[3]

– The *inversion metric* where $d(\pi, \pi')$ is the minimum number of pairwise adjacent transpositions needed to obtain π' from π. Alternatively, it is the number of pairs of alternatives in A that are ranked differently by π and π'. This metric is also known as the *Kemeny's metric* (Kemeny, 1959, Kemeny and Snell, 1962).

[3] A review of distance-based rules can be found in Truchon (2007).

– The *edge metric* where $d(\pi, \pi')$ is the minimum number of pairwise transpositions (not necessarily adjacent) needed to obtain π' from π.[4]

Note that in some preference profiles with a large K, it is well-known in the literature of social choice theory that it may be difficult to find the winner under these possibilities since the Kemeny and the edge metrics suffer from a high computational costs (Bartholdi et al., 1989, Brandes, 2001, Hemaspaandra et al., 2005). Therefore, we define a "substitute rule" which we call the *almost first-order stochastic dominance rule* in accordance with the spirit of this appellation in the literature.

Definition 2. Given a preference profile $\pi \in P$, the *Almost first-order Stochastic Dominance Rule* (ASDR) is defined as follows:

$$ASDR(\pi) = \arg\min_{a \in A} \tilde{d}(a)$$

where, $\tilde{d}(a) = \sum_{k=1}^{K-1} \left[\left(\max_{x \in A} F_\pi^x(k) \right) - F_\pi^a(k) \right]$.

The rational behind $\tilde{d}(a)$ is to sum the differences between the first-order cumulative standings of alternative a and that of the most first-order cumulative standing for every rank k. The winning alternative under ASDR is then the one minimizing this sum. As noticed before, $\tilde{d}(a)$ is a function adapted from a generic measure of centrality which sums the differences between each vertex's centrality score and that of the most central vertex (see for instance Freeman, 1979, Golbeck, 2015, 2013, Marsden, 2015). In network theory or graph theory, centrality measures the extent of inequality among vertex centrality scores and it usually describes the network positions of vertices. A measure of centrality is one of the core principles of network analysis and it measures how central or important a node is in the network, i.e., this is used as an estimate of its importance in the network. So, based on this generic measure, our ASDR is a relaxation of stochastic dominance, which allows small changes of the stochastic dominance principle to avoid situations where most individuals rank one alternative ahead another but stochastic dominance cannot rank them. With ASDR, we are now able to rank otherwise unrankable alternatives.

Note that ASDR also selects the alternative which has first-order stochastic dominance when it exists. This is simple to show: if a given alternative a has first-order stochastic dominance, we get $\max_{x \in A} F_\pi^x(k) = F_\pi^a(k)$ for all $1 \leq k \leq K-1$, then $\tilde{d}(a) = 0$. Let us consider an example in order to illustrate how ASDR operates.

Example 1. Consider the following profile of $n = 206$ individuals on $K = 4$ alternatives a, b, c, and d.

[4] Extending betweenness centrality for nodes, the edge metric relies on the computation of shortest paths in a network (Brandes, 2001). The betweenness centrality captures how much a given node is in-between others. It is measured with the number of shortest paths between any couple of nodes in the graphs that passes through the target node (Golbeck, 2015, 2013).

1	6	18	7	6	6	13	9	6	3	7	11	6	4	11	1	3	22	7	2	43	1	8	5
a	a	a	a	a	a	b	b	b	b	b	b	c	c	c	c	c	c	d	d	d	d	d	d
b	b	c	c	d	d	a	a	c	c	d	d	a	a	b	b	d	d	a	a	b	b	c	c
c	d	b	d	b	c	c	d	a	d	a	c	b	d	a	d	a	b	b	c	a	c	a	b
d	c	d	b	c	b	d	c	d	a	c	a	d	b	d	a	b	a	c	b	c	a	b	a

We report the values of the function $F_\pi^x(k)$ for each alternative as follows:

x	$F_\pi^x(1)$	$F_\pi^x(2)$	$F_\pi^x(3)$	$F_\pi^x(4)$
a	44	85	163	206
b	49	112	176	206
c	47	94	128	206
d	66	121	151	206
$\max F_\pi^x(k)$	66	121	176	206

Note first that $F_\pi^d(k)$ is maximal when $k = 1$ and $k = 2$, while it is not in the only case of $k = 3$, and naturally $F_\pi^d(k) = 206$ as for the other alternatives. However, we get $b \succ_{\text{FOSD}} a$, $b \succ_{\text{FOSD}} c$, $d \succ_{\text{FOSD}} c$ and there is no other domination relation in the sense of first-order stochastic dominance. It follows that there is no alternative which has first-order stochastic dominance. We then compute,

$$\begin{aligned}
\tilde{d}(a) &= (66 - 44) + (121 - 85) + (176 - 163) = 71 \\
\tilde{d}(b) &= (66 - 49) + (121 - 112) + (176 - 176) = 26 \\
\tilde{d}(c) &= (66 - 47) + (121 - 94) + (176 - 128) = 94 \\
\tilde{d}(d) &= (66 - 66) + (121 - 121) + (176 - 151) = 25
\end{aligned}$$

So, $ASDR(\pi) = \{d\}$.

The reader can easily check that $BR(\pi) = \{d\}$ since $S_{\text{BR}}(\pi, a) = 292$, $S_{\text{BR}}(\pi, b) = 337$, $S_{\text{BR}}(\pi, c) = 269$ and $S_{\text{BR}}(\pi, d) = 338$.

Our main result in this note is given in Theorem 1 and states that our almost first-order stochastic dominance rule is equivalent to the well-known Borda rule.

Theorem 1. *$ASDR(\pi) = BR(\pi)$.*

Proof Let us assume a given preference profile $\pi \in P$ in which $ASDR(\pi) = x$. This means that for all $y \in A \setminus \{x\}$, we get $\tilde{d}(x) - \tilde{d}(y) < 0$. Knowing that

$\max_{a\in A} F_\pi^a(K) = F_\pi^x(K) = F_\pi^y(K)$, let us compute $\tilde{d}(x) - \tilde{d}(y)$.

$$\begin{aligned}
\tilde{d}(x) - \tilde{d}(y) &= \left(\sum_{k=1}^{K-1} \left[\max_{a\in A} F_\pi^a(k) - F_\pi^x(k) \right] \right) - \left(\sum_{k=1}^{K-1} \left[\max_{a\in A} F_\pi^a(k) - F_\pi^y(k) \right] \right) \\
&= \left(\sum_{k=1}^{K-1} \max_{a\in A} F_\pi^a(k) \right) - \left(\sum_{k=1}^{K-1} F_\pi^x(k) \right) - \left(\sum_{k=1}^{K-1} \max_{a\in A} F_\pi^a(k) \right) - \left(\sum_{k=1}^{K-1} F_\pi^y(k) \right) \\
&= \left(\sum_{k=1}^{K-1} F_\pi^y(k) \right) - \left(\sum_{k=1}^{K-1} F_\pi^x(k) \right) \\
&= S_{\text{BR}}(\pi, y) - S_{\text{BR}}(\pi, x)
\end{aligned}$$

It follows that $\tilde{d}(x) - \tilde{d}(y) < 0 \iff S_{\text{BR}}(y) < S_{\text{BR}}(x)$. So, $ASDR(\pi) = BR(\pi)$. □

4 Conclusion

In this note, we have managed to highlight a new property of the Borda rule. As the stochastic dominance approach may fail as a ranking criterion in some situations, we have first showed that we may introduce a rule which always picks an alternative with first-order stochastic dominance when it exists and, otherwise, the alternative which needs a small number of changes in the individual preferences to reach first-order stochastic dominance. This rule may be difficult to handle due to the high computational costs of the metrics on which the rule is built. As a consequence and following one of the centrality measures, we have introduced an almost first-order stochastic dominance rule as a substitute and we have showed that it is exactly equivalent to the well-known Borda rule. To summarize, if the decision maker wants to follow the recommendations of the aforementioned measure of centrality to implement the principle of stochastic dominance in the context of preference aggregation, then she can use the Borda rule.

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References

Bartholdi III J., Tovey C.A., Trick M.A. (1989) Voting schemes for which it can be difficult to tell who won the election. Social Choice and Welfare 6:157-165.

Brandes U. (2001) A faster algorithm for betweenness centrality. Journal of Mathematical Sociology 25(2):163-177.

Freeman L.C. (1979) Centrality in social networks: conceptual clarification. Social Networks 1: 215-239.

Golbeck J. (2015) Analyzing networks. In *Introduction to Social Media Investigation*, Chapter 21, p. 221-235, Elsevier.

Golbeck J. (2013) Network Structure and Measures. In *Analyzing the Social Web*, Chapter 3, p. 25-44, Elsevier.

Hemaspaandra E., Spakowski H., Vogel J. (2005) The complexity of Kemeny elections. Theoretical Computer Science 349(3):382-391.

Kemeny J. (1959) Mathematics without numbers. Daedalus 88 (4): 577-591.

Kemeny J., Snell J. (1962) *Mathematical models in the social sciences*. Boston: Ginn.

Kondratyev A.Y. (2018) Positional voting methods satisfying the weak mutual majority and Condorcet loser principles. Automation and Remote Control 79(8):1489-1514.

Leshno M., Levy H. (2002) Preferred by "all" and preferred by "most" decision makers: almost stochastic dominance. Management Science 48 (8): 1074-1085.

Levy H. (2006) *Stochastic dominance: Investment decision making under uncertainty*. Springer US.

Marsden P.V. (2015) Measures of Network Centrality. In *International Encyclopedia of the Social & Behavioral Sciences* (2nd Edition), 2015, p. 532-539, Elsevier.

Saari D.G. (1992) Millions of election outcomes from a single profile. Social Choice and Welfare 9: 277-306.

Saari D.G. (1999) Explaining all three-alternative voting outcomes. Journal of Economic Theory 87(2):313-355.

Stein W.E., Mizzi P.J., Pfaffenberger R.C. (1994) A stochastic dominance analysis of ranked voting systems with scoring. European Journal of Operational Research 74(1): 78-85.

Truchon, M. (2007) Aggregation of rankings: A brief review of distance-based rules. CIRPEE Working Paper No. 05-34.

Tzeng L.Y., Huang R.J., Shih P.-T.(2013) Revisiting almost second-degree stochastic dominance. Management Science 59: 1250-1254.