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Abstract

This paper introduces the q-fixed majority property for committee selection rules, which extends the traditional fixed majority principle to a flexible framework. We examine conditions under which the committee scoring rules satisfy the q-fixed majority property. Focusing on (weakly) separable rules, we find that the Bloc rule is the only which satisfies it for all $q > 1/2$. In addition, the q-bottom majority property is introduced, highlighting conditions under which committees can be excluded based on voter consensus.

Keywords: Voting, multiwinner elections, committee scoring rules, q-fixed majority.
JEL classification: D71, D72.

1 Introduction

The study of voting systems, particularly in multi-winner elections, is a crucial area of research in social choice theory. This paper delves into the efficiency of committee scoring rules under the *q-fixed majority property*, a concept that extends the classical majority principle from single-winner elections to the multi-winner context. Multi-winner voting rules, also known as committee selection rules, are designed to select a subset of candidates (a committee) based on the preferences of a group of voters. These rules are fundamental in various applications, such as parliamentary elections, board member selections, and other decision-making processes that require collective representation.

The majority principle is a cornerstone of democratic decision-making. In single-winner elections, this principle asserts that if a candidate is most preferred by a strict majority (more than 50%) of voters, that candidate should be declared the winner. Debord [7] extended this principle to multi-winner elections with the *fixed majority property*. A committee selection rule satisfies the fixed majority property if, whenever a strict majority of voters ranks all members of a committee of size k in the top k positions (k being the targeted size of the committee to be selected), that committee should be the unique winner.

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This paper introduces the *q-fixed majority property*, a generalization of the fixed majority property. The q-fixed majority property applies to situations where a proportion q ($\frac{1}{2} < q \leq 1$) of voters supports a given committee. If this proportion of voters ranks all members of a committee in the top k positions in their individual rankings, the voting rule should select that committee as the winner. We also introduce the *q-bottom majority property*, which requires that if a proportion q of voters ($\frac{1}{2} < q \leq 1$) ranks all members of a given committee in the bottom k positions of their individual rankings, that committee should not be selected as the winner.

The q-fixed majority property and the q-bottom majority property capture a broader range of majority support scenarios, reflecting the varying degrees of consensus required in different electoral contexts. Similarly, [5, 6] generalized the Condorcet winner concept to the q-Condorcet winner, underscoring the relevance of such generalizations in voting theory. The q-fixed majority property provides a more flexible framework for evaluating the performance of committee selection rules under different majority requirements.

In single-winner elections, a scoring rule assigns scores to candidates based on their positions in voters' preferences, and the candidate with the highest total score wins. Committee scoring rules were introduced as an extension of scoring rules to multi-winner elections, where the goal is to select a committee of a fixed size k. Committee scoring rules assign a score to each possible committee based on the voters' preferences and select the committee with the highest score ([9]). These rules can be either separable, when the score of a committee is calculated as the sum of the scores of its individual members, or non-separable, when the score is determined by the overall composition of the committee. Well-known examples of separable committee scoring rules include the k-Plurality rule, k-Borda rule, and Bloc rule.

The core of this paper is to explore how committee scoring rules perform when subjected to the q-fixed majority property and to the q-bottom majority property. Many committee scoring rules do not satisfy the fixed majority property. Faliszewski et al. [11] characterized the committee scoring rules that do, identifying the Bloc rule as a notable example. The primary contribution of this paper is the analysis of the q-fixed majority property and q-bottom majority property for committee scoring rules. We provide theoretical insights into the conditions under which various committee scoring rules satisfy these properties. By introducing a threshold value for q, we determine the scenarios where the q-fixed majority and the q-bottom majority property are guaranteed.

The paper is structured as follows: Section 2 provides preliminary definitions and sets up the theoretical framework for committee scoring rules. Section 3 first introduces the q-fixed majority property, presenting key propositions and proofs. It then delves into the implications of this property for various types of committee scoring rules. Section 4 presents our findings about the q-bottom majority property and Section 5 concludes.

2 Preliminary definitions

2.1 Setup

Consider a non-empty set A of m alternatives (or candidates) and a non-empty set N of n voters (or individuals) with $m \geq 3$ and $n \geq 2$. Alternatives are denoted by small letters a, b, c, $\ldots$, or

a_1, a_2, a_3, etc. Voters are denoted by positive integers 1, 2, 3, etc. We denote by $\mathbb{N}$ the set of all non-negative integers and by $\mathbb{N}^*$ the set of all positive integers. Throughout the paper, we simply write $[r]$ to denote the set $\{1, \ldots, r\}$ for any positive integer $r \in \mathbb{N}^*$.

We assume that each voter $i \in N$ is endowed with a preference p_i which is a linear order on the set of candidates; that is, the preference of a voter is then a complete, anti-symmetric and transitive binary relation on A. For any two candidates a and b, we will write $a \succ_i b$ if voter i prefers a to b and the ranking $p_i = a_1 \succ_i a_2 \succ_i a_3 \succ_i \cdots$ of voter i will be sometimes written $a_1 a_2 a_3 \cdots$. The set of all linear orders on A is denoted $\mathcal{P}(A)$. A preference profile (or simply a profile) is a collection $p = (p_i)_{i \in N}$ specifying the preferences of all voters. The set of profiles with n voters is denoted by $\mathcal{P}^n$ the set of all possible profile is then $\cup_{n=2}^{\infty} \mathcal{P}(A)^n$. Given the set of candidates and the set of voters, we consider in this paper the setting where the goal is to select a fixed-size subset of candidates, called a *committee*, by aggregating the preferences of all voters. For any integer $k \in [m-1]$, a committee of size k is defined as any k-element subset of A. The set containing all possible committees of size k for the set A is denoted by 2^A_k. We focus on committee sizes k such that $k \in [m-1]$, as the case $k = m$ is straightforward. In this framework, a committee selection Rule or multi-winner voting rule is defined as any mapping $\mathcal{R}$ that assigns, to any profile p and any committee size $k \in [m-1]$, the set $\mathcal{R}(p, k)$ comprising the winning committee(s). This set is referred to as the social outcome of the pair (p, k) under the CSR $\mathcal{R}$.

2.2 Committee scoring rules

The study of this paper focuses on the family of committee scoring rules introduced by [9], which assign to each committee a score according to a (committee) scoring function, with respect to the profile, and select the committee(s) with the maximum score. This family of committee scoring rules has been proposed as an extension of the well-known family of scoring rules for single-winner elections defined and characterized by [17]; [16] have provided a characterization of committee scoring, analogue to that of [17]. Recall that a scoring rule for single-winner elections is defined via a scoring vector, which is a vector of decreasing real numbers $\alpha = (\alpha_1, \cdots, \alpha_m)$ satisfying $\alpha_1 > \alpha_m$, such that each voter gives α_1 points to her most favoured candidate, α_2 points to her second-ranked candidate, and so forth until α_m points to her bottom-ranked candidate. The score gained by any candidate $a \in A$ across a preference profile $p \in \mathcal{P}^n$ with respect to the scoring vector α is given by

$$S_\alpha(p, a) = \sum_{i=1}^{n} \alpha_{r(p_i, a)}$$

where $r(p_i, a)$ is the rank of a in the ranking p_i.

As the scoring rules for single-winner are based on the candidates' ranks in voters preferences, the first step for providing an extension of these rules has been to define the rank of any committee of size $k \in [m-1]$ within a voter preference. Given a committee $W \in 2^A_k$, the rank $r(p_i, W)$ of the committee W with regards to the linear order p_i of a given voter i is the increasing sequence $(i_1, \ldots, i_k)$ obtained by ordering the set $\big\{r(p_i, a) : a \in W\big\}$. For instance, assume that the set of candidates is $A = \{a, b, c, d, e\}$, the preference of voter i is $p_i = bcade$, and the committee size is $k = 3$. Then, the rank of the committee $W = \{a, c, e\}$ in voter's i

preference relation is $r(p_i, W) = (2, 3, 5)$. We denote by $[m]_k$ the set of all possible increasing sequences of k elements from $[m]$. In other words, $[m]_k$ stands for the set of all possible ranks of a given size-k committee in a given linear ranking. Given two committee ranks $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$, we say that I *dominates* J, which is denoted by $I \succeq J$, if $i_t \leq j_t$ for all $t \in [k]$. In particular, $I_0 = (1, \ldots, k)$ dominates any other rank and $J_0 = (m - k + 1, \ldots, m)$ is dominated by any other rank. We can immediately deduce that for any committee $W \in 2^A_k$ and any voter $i \in N$, we have

$$f_k(J_0) \leq f_k\big(r(p_i, W)\big) \leq f_k(I_0).$$

Definition 1 *A committee scoring function is a function $f_k : [m]_k \to \mathbb{R}_+$ such that for all $I, J \in [m]_k$, $I \succeq J$ implies $f_k(I) \geq f_k(J)$. Given a committee scoring function f_k, the score of a committee $W \in 2^A_k$ with respect to the committee scoring function f_k and a preference profile p is defined by*

$$S_{f_k}(p, W) = \sum_{i \in N} f_k\Big(r(p_i, W)\Big).$$

Definition 2 *A committee selection rule $\mathcal{R}$ is a committee scoring rule if there is a family of scoring functions $f = (f_k)_{k \leq m-1}$ such that for any size $k \leq m - 1$ and any preference profile p, the set of winning committees with respect to (p, k) is the set of all committees of size k with the highest score across scoring function f_k. We will denote such a rule by $\mathcal{R}_f$ and we have,*

$$\mathcal{R}_f(p, k) = \Big\{W \in 2^A_k : S_{f_k}(p, W) \geq S_{f_k}(p, W') \text{ for all } W' \in 2^A_k\Big\}.$$

It is clear that the committee scoring rules constitute a very large family of multi-winner rules and the most studied committee scoring rules are undoubtedly the (weakly) separable committee scoring rules that rate the candidates separately according to a single-winner scoring vector and selects the k candidates with the highest scores. The accuracy *weakly* is needed if the underlying scoring vector depends on the committee size; otherwise, the rule is simply said to be separable, without the "weakly" accuracy. Thus, every (weakly) separable committee scoring rule defined through a scoring vector α is then a committee scoring rule associated with the family of scoring functions $(f_k)_{k \leq m-1}$ defined by

$$f_k(i_1, \ldots, i_k) = \sum_{t=1}^{k} \alpha_{i_t}.$$

Note that the subclass of separable committee scoring rules can be seen as the intersection between the committee scoring rules and the *candidate-wise* procedures already defined and studied in [14, 15]. A procedure is described as a candidate-wise procedure if the score of a given committee is the sum of the scores of all the candidates belonging to that committee, each of them considered as a 1-size committee.

The well-known rules in the class of (weakly) separable committee scoring rules are k-Plurality rule (also called *Single Non Transferable Voting rule*, SNTV), k-Borda rule, k-Antiplurality rule, and the Bloc rule, defined by the scoring vectors $\alpha^P = (1, 0, 0, \ldots, 0)$, $\alpha^B = (m-1, m-2, \ldots, 2, 1, 0)$, $\alpha^{AP} = (1, 1, \ldots, 1, 0)$, and $\alpha^{k,Bl} = (\underbrace{1, \ldots, 1}_{k \text{ times}}, \underbrace{0, \ldots, 0}_{m-k \text{ times}})$ respectively. We can then remark the Bloc rule is weakly separable, since its associated scoring vector depends on the committee size.

Note that the wide family of committee scoring rules contains other interesting rules which are not (weakly) separable. We can think about the β-Chamberlin-Courant rule (β-CC) and the α_k-Chamberlin-rule (α_k-CC) defined with the scoring functions

$$f_k^{\beta-CC}(i_1, \ldots, i_k) = \alpha_{i_1}^B = m - i_1$$

$$f_k^{\alpha_k-CC}(i_1, \ldots, i_k) = \alpha_k(i_1)$$

where $\alpha_k : [m] \to \mathbb{R}_+$ is defined by $\alpha_k(t) = 1$ if $t \leq k$ and $\alpha_k(t) = 0$ otherwise. We refer the reader to [12] and [11] for further discussion about committee scoring rules.

2.3 The fixed majority property

As in the single-winner setting, several properties have been proposed in the literature in order to evaluate the performances of different multi-winner voting rules. While some properties of multi-winner rules are intrinsic to this setting and are not readily applicable to a single-winner context, it is nevertheless worthwhile to note that the majority of these properties represent an extension of the principles governing single-winner voting rules within the multi-winner framework (see for instance 2, 3, 4, 9, and 13).

In the single-winner setting, the *absolute majority principle* requires that if a candidate is ranked at the top position by a strict majority of voters, then this candidate should be the unique winner of the election. The Plurality rule has been identified as the only scoring rule satisfying the absolute majority principle. This property has been extended to the multi-winner framework by [7] and it has been called *fixed majority*. A committee scoring rule (CSR) $\mathcal{R}$ satisfies the fixed majority property if, for every profile p and every committee size k, if there exists a strict majority of voters who rank all members of a committee $W \in 2^A_k$ within the top k positions (in any order), then $\mathcal{R}(p, k) = \{W\}$. It has been shown that many committee scoring rules fail to satisfy the fixed majority property. By the way, [11] have characterized all the committee scoring rules satisfying this property, allowing for the identification of those that are equivalent to the Plurality rule in the multi-winner context, including the Bloc rule. In the next section, we will provide further proofs that the Bloc rule is the only (weakly) separable committee scoring rule satisfying the generalized fixed majority property.

3 The q-fixed majority property

In this section, we argue that the size of the majority of voters ranking the committee members within the top k positions plays a crucial role in satisfying the fixed majority property. We extend this idea to the *q-fixed majority* property, where $q \in \left(\frac{1}{2}, 1\right]$ represents the proportion of voters who rank all the committee members in the top k positions. A similar approach was taken by [5, 6], who introduced the q-Condorcet winner to generalize the concept of the Condorcet winner in the single-winner setting.

Definition 3 *Let $q \in (\frac{1}{2}; 1]$ be a real number and $\mathcal{R}$ be a CSR. We say that $\mathcal{R}$ satisfies the q-fixed majority property if for any committee W of size $k \in [m-1]$ and any profile p, if there is a majority of qn voters who rank the members of W in the top k positions (in any order), then $\mathcal{R}(p, k) = \{W\}$.*

Definition 3 can be seen as a generalization of the fixed majority property because, as the quota q approaches $\frac{1}{2}$, it aligns with the definition of fixed majority provided by [7]. Moreover, if a CSR satisfies the fixed majority property, then it satisfies the q-fixed majority property for all $q > 1/2$.

Based on Definition 3, we conjecture that for a given committee size and committee scoring rule, there exists a specific value q_0 of the quota such that the q-fixed majority is guaranteed. Proposition 1 below substantiates this conjecture. Before presenting the proposition, we first introduce additional notations.

Given k the committee size and f_k a committee scoring function, we denote by $\Gamma(f_k) = f_k(I_0) - f_k(J_0)$ the maximum difference of scores assigned by f_k to any two committee ranks, and $\gamma(f_k) = f(I_0) - f_k(1, \cdots, k-1, k+1)$ where $(1, \cdots, k-1, k+1)$ is the committee rank obtained from I_0 by replacing k by $k+1$. So, $\gamma(f_k)$ is the difference of scores assigned by f_k to I_0 and $(1, \cdots, k-1, k+1)$.

Proposition 1 *Let $\mathcal{R}_f$ be the committee scoring rule defined through the family $f = \big(f_k\big)_{k \leq m-1}$ and let k be the given committee size. If $q > \frac{\Gamma(f_k)}{\Gamma(f_k)+\gamma(f_k)}$, then $\mathcal{R}_f$, satisfies the q-fixed majority property.*

Proof. Let $\mathcal{R}_f$ a committee scoring rule and let k be the committee size. Let $W \in 2^A_k$ be a committee and p a profile such that qn voters in p rank the members of W on the top k positions, with $q \in (\frac{1}{2}, 1]$. Assume that $\mathcal{R}_f(p, k) \neq \{W\}$. Then, there is a committee $T \in \mathcal{R}_f(p, k)$ with $T \neq W$, which means that there is at least one candidate from T that does not belong to W, and conversely. Let N_1 be the set of voters who rank W on the top k positions and $N_2 = N \setminus N_1$. Since $T \in \mathcal{R}_f(p, k)$, it holds that

$$S_{f_k}(p, T) - S_{f_k}(p, W) \geq 0. \tag{1}$$

On the other hand, we have

$$\begin{aligned} S_{f_k}(p, T) - S_{f_k}(p, W) &= \sum_{i \in N_1} \Big[f_k\big(r(p_i, T)\big) - f_k\big(r(p_i, W)\big)\Big] + \sum_{i \in N_2} \Big[f_k\big(r(p_i, T)\big) - f_k\big(r(p_i, W)\big)\Big] \\ &\leq |N_1|\Big[f_k(1, \cdots, k-1, k+1) - f_k(I_0)\Big] + |N_2|\Gamma(f_k) \\ &= (1-q)n\Gamma(f_k) - qn\gamma(f_k) \;\; (\text{ since } |N_1| = qn) \end{aligned} \tag{2}$$

Equations (1) and (2) together imply that

$$q \leq \frac{\Gamma(f_k)}{\Gamma(f_k) + \gamma(f_k)}.$$

This proves that if $q > \frac{\Gamma(f_k)}{\Gamma(f_k)+\gamma(f_k)}$, the q-fixed majority property is satisfied by $\mathcal{R}_f$. ∎

We can observe that Proposition 1 gives a sufficient condition (on the value q) so that a committee scoring rule satisfies the q-fixed majority property. However, we are not sure that the quota $q_0(f_k) = \frac{\Gamma(f_k)}{\Gamma(f_k)+\gamma(f_k)}$ given in Proposition 1 is a threshold, since we cannot definitely prove that the q-fixed majority is failed when q is less than or equal to that quota. This challenging problem is mainly due to the fact that the family of committee scoring functions encompasses a large number of functions that are difficult to handle. Nevertheless, for the class of (weakly) separable committee scoring rules, Theorem 1 below provides a threshold of the value q, which is obviously lower than the quota provided in Proposition 1.

For any scoring vector $\alpha = (\alpha_1, \cdots, \alpha_m)$, let $\Gamma(\alpha) = \alpha_1 - \alpha_m$ and $\gamma^k(\alpha) = \alpha_k - \alpha_{k+1}$.

Theorem 1 *Let $\mathcal{R}_f = \mathcal{R}_\alpha$ be a (weakly) separable committee scoring rule, k be the committee size, and $q \in (\frac{1}{2}, 1]$. The rule $\mathcal{R}_\alpha$ satisfies the q-fixed majority property if and only if $q > \frac{\Gamma(\alpha)}{\Gamma(\alpha)+\gamma^k(\alpha)}$.*

Proof. Let $\mathcal{R}_\alpha$ be a (weakly) separable scoring rule defined with the scoring vector $\alpha = (\alpha_1, \cdots, \alpha_m)$ and let $k \in [m-1]$ be the given committee size. Let $W \in 2^A_k$ be a committee and p be a preference profile such that qn voters rank the members of W on the top k positions with $q \in (\frac{1}{2}; 1]$; denote by N_1 the set of all those voters, and $N_2 = N \setminus N_1$. Assume that $\mathcal{R}_\alpha(p, k) \neq \{W\}$. Then, there exists $T \in 2^A_k \setminus \{W\}$ such that $T \in \mathcal{R}_\alpha(p, k)$ and, the sets $T \setminus W$ and $W \setminus T$ are non-empty and they have the same cardinality (because W and T have the same cardinality). Let $a \in \underset{x \in T\setminus W}{\arg\max} S_\alpha(p, x)$ be the candidate from $T \setminus W$ with the maximum individual score, and $b \in \underset{x \in W\setminus T}{\arg\min} S_\alpha(p, x)$ be the candidate from $W \setminus T$ with the minimum individual score. It follows that

$$\begin{aligned}
S_\alpha(p, T) - S_\alpha(p, W) &= S_\alpha(p, T \setminus W) - S_\alpha(p, W \setminus T) \\
&= \sum_{x \in T\setminus W} S_\alpha(p, x) - \sum_{x \in W\setminus T} S_\alpha(p, x) \\
&\leq |T \setminus W| \Big[S_\alpha(p, a) - S_\alpha(p, b) \Big] \\
&= |T \setminus W| \Big[\sum_{i \in N_1} \Big(\alpha_{r(p_i,a)} - \alpha_{r(p_i,b)} \Big) + \sum_{i \in N_2} \Big(\alpha_{r(p_i,a)} - \alpha_{r(p_i,b)} \Big) \Big] \\
&\leq |T \setminus W| \Big[|N_1|(\alpha_{k+1} - \alpha_k) + |N_2|(\alpha_1 - \alpha_m) \Big] \\
&= |T \setminus W| \Big[(1-q)n\Gamma(\alpha) - qn\gamma^k(\alpha) \Big]. \qquad (3)
\end{aligned}$$

On the other hand, since $T \in \mathcal{R}_\alpha(p, k)$, it holds that

$$S_{f_k}(p, T) - S_{f_k}(p, W) \geq 0. \qquad (4)$$

Equations (3) and (4) together lead to $(1-q)n\Gamma(\alpha) - qn\gamma^k(\alpha) \geq 0$, which implies that

$$q \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha) + \gamma^k(\alpha)}.$$

Thus, $\mathcal{R}_\alpha$ satisfies the q-fixed majority property if $q > \frac{\Gamma(\alpha)}{\Gamma(\alpha)+\gamma^k(\alpha)}$.

Conversely, assume that $q \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha)+\gamma^k(\alpha)}$. Let $W \in 2^A_k$ such that $W = \{a_1, \cdots, a_k\}$, pick a candidate $x \in A \setminus W$ (such a candidate exists since $k \leq m-1$), and set $T = \big(W \setminus \{a_k\}\big) \cup \{x\} = \{a_1, \cdots, a_{k-1}, x\}$. Consider any preference profile on the following form:

$$qn \text{ voters: } a_1 a_2 \cdots a_k x \cdots$$

$$(1-q)n \text{ voters: } x \cdots a_1 a_2 \cdots a_k.$$

For such a profile, we have

$$\begin{aligned}
S_\alpha(p,T) - S_\alpha(p,W) &= S_\alpha(p,x) - S_\alpha(p,a_k) \\
&= qn(\alpha_{k+1} - \alpha_k) + (1-q)n(\alpha_1 - \alpha_m) \\
&= n\Gamma(\alpha) - qn\big(\Gamma(\alpha) + \gamma^k(\alpha)\big) \\
&\geq n\Gamma(\alpha) - n\Gamma(\alpha) = 0 \text{ Since } q \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha) + \gamma^k(\alpha)}
\end{aligned}$$

Hence, we have $S_\alpha(p,T) - S_\alpha(p,W) \geq 0$ which implies that $\mathcal{R}_\alpha(p,k) \neq \{W\}$ and therefore, $\mathcal{R}_\alpha$ fails to satisfy the q-fixed majority property. ∎

From Theorem 1, we derive Corollary 1 regarding the four well-known (weakly) separable committee scoring rules presented in Section 2.2.

Corollary 1 *The following results hold:*

- *The k-Plurality rule satisfies the q-fixed majority property only for $k = 1$ and in this case, it meets the classical fixed majority since $\frac{\Gamma(\alpha^P)}{\Gamma(\alpha^P)+\gamma^k(\alpha^P)} = 1/2$.*
- *The k-Anti-Plurality rule satisfies the q-fixed majority property only for $k = m-1$ and in this case, it fulfills the classical fixed majority since for $k = m-1$, we have $\frac{\Gamma(\alpha^{AP})}{\Gamma(\alpha^{AP})+\gamma^k(\alpha^{AP})} = 1/2$*
- *The k-Borda rule satisfies the q-fixed majority property for any value of k if and only if $q > \frac{m-1}{m}$. This is consistent with the well-known result on the majority efficiency of the Borda rule for single-winner elections (see [1] and [8]).*
- *The Bloc rule satisfies the q-fixed majority property for all $q > 1/2$ and for any committee size, since $\frac{\Gamma(\alpha^{Bl})}{\Gamma(\alpha^{Bl})+\gamma^k(\alpha^{Bl})} = 1/2$ for all $k \in [m-1]$. Therefore, The Bloc rule satisfies the fixed majority property for any committee size.*

Corollary 1 shows that the Bloc rule is the only one among four rules mentioned above that always satisfies the fixed majority property regardless of the committee size. Indeed, [11] characterized the Bloc rule as the only committee scoring rule satisfying both the fixed majority property and the non-crossing monotonicity property (see [9] for further details on the latter property). However, [10] showed that (weakly) separable committee scoring rules are the only committee scoring rules satisfying the non-crossing monotonicity property. Therefore, we can deduce that the Bloc rule is the only (weakly) separable committee scoring rule satisfying the fixed majority property. We can also deduce this result directly from Theorem 1 above.

Corollary 2 *The Bloc rule is the only (weakly) separable committee scoring rule satisfying the fixed majority property.*

Proof. Let $\mathcal{R}_\alpha$ be a (weakly) separable committee scoring rule. It has been proved in Theorem 1 that for $q \in (\frac{1}{2}; 1]$, $\mathcal{R}_\alpha$ satisfies the q-fixed majority if and only if $q > \frac{\Gamma(\alpha)}{\Gamma(\alpha)+\gamma^k(\alpha)}$. Hence, $\mathcal{R}_\alpha$ satisfies the fixed majority property if and only if $\frac{\Gamma(\alpha)}{\Gamma(\alpha)+\gamma^k(\alpha)} = 1/2$, which implies that $\Gamma(\alpha) = \gamma^k(\alpha)$. This equation means that $0 \leq \alpha_1 - \alpha_k = \alpha_m - \alpha_{k+1} \leq 0$, which implies that $\alpha_1 - \alpha_k = \alpha_{k+1} - \alpha_m = 0$. Therefore, we deduce that $\alpha_1 = \alpha_k$ and $\alpha_{k+1} = \alpha_m$. Thus, $\mathcal{R}_\alpha$ is the Bloc rule. ■

4 The q-bottom majority property

In Section 3, we have defined a q-fixed majority committee as a committee whose members are ranked in the top-k positions by a proportion $q \in (\frac{1}{2}, 1]$ of voters and the q-fixed majority property requires that such a committee should be the unique winning committee according to the considered preference profile. In this section, we present a dual property that we call *q-bottom majority* property, which requires that any committee whose members are ranked in the bottom-k positions in a profile by a proportion $q \in (\frac{1}{2}, 1]$ of voters, cannot be a winning committee for this profile.

Definition 4 *Let $q \in (\frac{1}{2}; 1]$ be a real number and $\mathcal{R}$ be a CSR. We say that $\mathcal{R}$ satisfies the q-bottom majority property if for any committee W of size $k \in [m-1]$ and any profile p, if there is a majority of qn voters who rank the members of W in the bottom-k positions (no matter the order), then $W \notin \mathcal{R}(p, k)$.*

Note that if we focus only on a strict majority of voters, we obtain the *bottom majority* property similar to the fixed majority property and, if a CSR satisfies the bottom majority property, then it satisfies the q-bottom majority property for all $q > 1/2$. It is not hard to check that, not all the committee scoring rules satisfy the bottom majority property.

Example 1 *Consider the set of five candidates $A = \{a, b, c, d, e\}$ and the following preference profile with five voters, where each column represents the ranking of a voter.*

$$p = \begin{bmatrix} a & b & c & d & e \\ b & a & d & e & c \\ c & e & e & c & d \\ d & d & a & b & a \\ e & c & b & a & b \end{bmatrix}$$

Assume that the goal is to select a committee of size 2 using the 2-Plurality rule. Then, it can be checked that $\{a, b\}$ a winning committee while its members are ranked in the bottom-2 positions by a strict majority of voters. Hence, the 2-Plurality fails to satisfy the bottom majority property.

Similar to Proposition 1, Proposition 2 below gives a condition on the proportion q so that a committee scoring rule $\mathcal{R}_f$ satisfies the q-bottom majority property for a given committee size. First, let k be the committee size and f_k be a scoring function. Denote by $\theta(f_k) = f_k(m-k, m-k+2, \cdots, m) - f_k(J_0)$ the difference of scores assigned by f_k to the committee ranks $(m-k, m-k+2, \cdots, m)$ and $J_0 = (m-k+1, \cdots, m)$. Recall that J_0 is the committee rank that consists of the k bottom candidate positions (that is, the worst possible rank of any committee in any linear order), and $(m-k, m-k+2, \cdots, m)$ is the committee rank obtained from J_0 by substituting the position $m-k+1$ by $m-k$.

Proposition 2 *Let $\mathcal{R}_f$ be the committee scoring rule defined through the family $f = \big(f_k\big)_{k \leq m-1}$ and let k be the given committee size. If $q > \frac{\Gamma(f_k)}{\Gamma(f_k)+\theta(f_k)}$, then $\mathcal{R}_f$ satisfies the q-bottom majority property.*

Proof. Let $\mathcal{R}_f$ a committee scoring rule and let k be the committee size. Let $W \in 2^A_k$ be a committee and p a profile such that qn voters in p rank the members of W in the bottom-k positions, with $q \in (\frac{1}{2}, 1]$. Assume that $W \in \mathcal{R}_f(p, k)$. Then, for every committee $T \in 2^A_k \setminus \{W\}$, we have

$$S_{f_k}(p, W) - S_{f_k}(p, T) \geq 0 \tag{5}$$

On the other hand, let N_1 be the set of voters who rank the members of W in the q-bottom positions and $N_2 = N \setminus N_1$. For any committee $T \in 2^A_k \setminus \{W\}$, the difference of scores between W and T is

$$\begin{aligned} S_{f_k}(p, W) - S_{f_k}(p, T) &= \sum_{i \in N_1} \Big[f_k\big(r(p_i, W)\big) - f_k\big(r(p_i, T)\big)\Big] + \sum_{i \in N_2} \Big[f_k\big(r(p_i, W)\big) - f_k\big(r(p_i, T)\big)\Big] \\ &\leq |N_1|\Big[f_k(J_0) - f_k(m-k, m-k+1, \cdots, m)\Big] + |N_2|\Big[f_k(I_0) - f_k(J_0)\Big] \\ &= (1-q)n\Gamma(f_k) - qn\theta(f_k). \end{aligned} \tag{6}$$

Equations (5) and (6) together imply that

$$q \leq \frac{\Gamma(f_k)}{\Gamma(f_k) + \theta(f_k)}.$$

This proves that if $q > \frac{\Gamma(f_k)}{\Gamma(f_k)+\theta(f_k)}$, then $\mathcal{R}_f$ have the q-bottom majority property. ■

The proportion $q_0(f_k) = \frac{\Gamma(f_k)}{\Gamma(f_k)+\theta(f_k)}$ given in the above proposition holds for the whole class of committee scoring rules. However, for the class of (weakly) separable committee scoring rules, this quota can be significantly lower.

Proposition 3 *Let $\mathcal{R}_f = \mathcal{R}_\alpha$ be a (weakly) separable committee scoring rule, k be the committee size, and $q \in (\frac{1}{2}, 1]$. Let $\Gamma(\alpha) = \alpha_1 - \alpha_m$ and $\theta^k(\alpha) = \alpha_{m-k} - \alpha_{m-k+1}$. If $q > \frac{\Gamma(\alpha)}{\Gamma(\alpha)+\theta^k(\alpha)}$, then $\mathcal{R}_\alpha$ satisfies the q-bottom majority property.*

Proof. Let $\mathcal{R}_\alpha$ be a (weakly) separable scoring rule defined with the scoring vector $\alpha = (\alpha_1, \cdots, \alpha_m)$ and let $k \in [m-1]$ be the given committee size. Let $W \in 2^A_k$ be a committee and p be a preference profile such that qn voters rank the members of W on the bottom-k positions with $q \in (\frac{1}{2}; 1]$. Assume that $W \in \mathcal{R}_\alpha(p, k)$. Literally, it means that the individual score of any W member is greater than or equal to the individual score of any candidate that does not belong to W; that is, $S_\alpha(p,a) \geq S_\alpha(p,b)$ for all $a \in W$ and $b \in A \setminus W$. However, for any such two candidates, we have

$$S_\alpha(p,a) \leq qn\alpha_{m-k+1} + (1-q)n\alpha_1 \text{ and } S_\alpha(p,b) \geq qn\alpha_{m-k} + (1-q)n\alpha_m.$$

Therefore, we deduce that $qn\alpha_{m-k+1} + (1-q)n\alpha_1 \geq qn\alpha_{m-k} + (1-q)n\alpha_m$, which implies that $q \leq \frac{\Gamma(\alpha)}{\Gamma(\alpha)+\theta^k(\alpha)}$. Thus, if $q > \frac{\Gamma(\alpha)}{\Gamma(\alpha)+\theta^k(\alpha)}$, then $\mathcal{R}_\alpha$ has the q-bottom majority property. ■

It can be checked that the quota provided for (weakly) separable rules is better (lower) than the one provided for the whole class of committee scoring rules. Unfortunately, we cannot obtain a similar result as in Theorem 1 in the sense that we cannot really show that the sufficient condition given in Proposition 3 is also necessary; this is a pending challenge left in this paper.

From Proposition 1 and Proposition 2, we can deduce the following Corollary.

Corollary 3 *Let $\mathcal{R}_f$ be a committee scoring rule defined through the family $f = (f_k)_{1\leq k\leq m-1}$, and k be the fixed committee size. Let $\phi(f_k) = \min\left\{\gamma(f_k), \theta(f_k)\right\}$ and $q \in (\frac{1}{2}, 1]$. If $q > \frac{\Gamma(f_k)}{\Gamma(f_k)+\phi(f_k)}$, then $\mathcal{R}_f$ satisfies both the q-fixed majority and the q-bottom majority properties.*

A similar result can be obtained from Theorem 1 and Proposition 3 for the (weakly) separable scoring rules, by providing a lower quota of the proportion q, which is sufficient to guarantee both properties. Moreover, for (weakly) separable scoring rules, Corollary 4 below provides a straightforward relation between the two properties for a specific value of the committee size.

Corollary 4 *For the committee size $k = m/2$, if a (weakly) separable committee scoring rule satisfies the q-fixed majority property, then it satisfies the q-bottom majority property.*

The proof of Corollary 4 is straightforward because when $k = m/2$, we have $\gamma^k(\alpha) = \theta^k(\alpha)$ (since $m-k=k$). Therefore, satisfying the q-fixed majority implies that $q > \frac{\Gamma(\alpha)}{\Gamma(\alpha)+\gamma^k(\alpha)} = \frac{\Gamma(\alpha)}{\Gamma(\alpha)+\theta^k(\alpha)}$ and then we also have the q-bottom majority.

5 Conclusion

This paper has introduced and examined the q-fixed majority property in the context of committee scoring rules, extending the classical fixed majority principle to accommodate varying degrees of majority support. Our exploration reveals that while many traditional committee scoring rules do not satisfy the fixed majority property, the q-fixed majority provides a more flexible framework that aligns with a broader range of electoral scenarios.

Furthermore, the discussion of the q-bottom majority property complements our understanding of majority rule in committee selection, highlighting scenarios where certain committees should be excluded based on broader voter consensus. Our findings contribute to the ongoing discourse on the efficiency and fairness of multi-winner election systems, offering new theoretical insights that could inform both academic research and practical applications in social choice theory.

Future research could investigate similar studies that highlight the consensus among a large group of voters in approval-based committee selection, where each voter submits a list of approved candidates rather than ranking all candidates.

References

[1] E. Baharad and S. Nitzan. The borda rule, condorcet consistency and condorcet stability. *Economic Theory*, 22(3):685–688, 2003.

[2] S. Barberà and D. Coelho. How to choose a non-controversial list with k names. *Social Choice and Welfare*, 31(1):79–96, 2008.

[3] D. Bubboloni, M. Diss, and M. Gori. Extensions of the simpson voting rule to the committee selection setting. *Public Choice*, 183:151–185, 2020.

[4] D. Coelho. *Understanding, evaluating and selecting voting rules through games and axioms.* Universitat Autònoma de Barcelona,, 2005.

[5] S. Courtin, M. Martin, and I. Moyouwou. The q-condorcet efficiency of positional rules. *Theory and Decision*, 79(1):31–49, 2015.

[6] S. Courtin, M. Martin, and B. Tchantcho. Positional rules and q-condorcet consistency. *Review of Economic Design*, 19:229–245, 2015.

[7] B. Debord. Prudent k-choice functions: Properties and algorithms. *Mathematical Social Sciences*, 26(1):63–77, 1993.

[8] M. Diss and M. Gori. Majority properties of positional social preference correspondences. *Theory and Decision*, 92(2):319–347, 2022.

[9] E. Elkind, P. Faliszewski, P. Skowron, and A. Slinko. Properties of multiwinner voting rules. *Social Choice and Welfare*, 48(3):599–632, 2017.

[10] P. Faliszewski, P. Skowron, A. Slinko, and N. Talmon. Committee scoring rules: Axiomatic classification and hierarchy. In *IJCAI*, pages 250–256, 2016.

[11] P. Faliszewski, P. Skowron, A. Slinko, and N. Talmon. Multiwinner analogues of the plurality rule: axiomatic and algorithmic perspectives. *Social Choice and Welfare*, 51(3):513–550, 2018.

[12] P. Faliszewski, P. Skowron, A. Slinko, and N. Talmon. Committee scoring rules: Axiomatic characterization and hierarchy. *ACM Transactions on Economics and Computation (TEAC)*, 7(1):1–39, 2019.

[13] E. Kamwa. On stable rules for selecting committees. *Journal of Mathematical Economics*, 70:36–44, 2017.

[14] D. M. Kilgour. Approval balloting for multi-winner elections. In *Handbook on approval voting*, pages 105–124. Springer, 2010.

[15] D. M. Kilgour and E. Marshall. Approval balloting for fixed-size committees. In *Electoral Systems*, pages 305–326. Springer, 2012.

[16] P. Skowron, P. Faliszewski, and A. Slinko. Axiomatic characterization of committee scoring rules. *Journal of Economic Theory*, 180:244–273, 2019.

[17] H. P. Young. Social choice scoring functions. *SIAM Journal on Applied Mathematics*, 28(4):824–838, 1975.