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Re examining confidence intervals for ratios of parameters

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Abstract

We consider the problem of constructing a confidence intervals (CIs) for nonlinear functions of the parameters. The classical approaches for constructing CIs are the Delta method and the Fieller method. These methods can be implemented in any context in econometrics and statistics.

There are two main reasons for the failure of these two methods. The first is the bias of the parameters estimator. In many econometric and statistical applications, the estimator of the nonlinear functions of the parameters is biased. The second problem is that the estimated parameters have non-normal and asymmetric distributions.

We extended the Delta method to obtain a better approximation by using the Edgeworth expansion. We then proposed a new interval by correcting the skewness in the Edgeworth expansion. Such bias-corrected confidence intervals are easy to compute and the coverage probability converges to the nominal level at a rate of $O(n^{-1/2})$ where n is the sample.size

We also define the bias of the nonlinear functions of the parameters and we propose a bias-corrected estimator that is identical to the almost unbiased ratio estimator proposed by Tin (1965). We then correct the CIs according to the Delta method and the Edgeworth expansion. Thus we develop new methods for constructing of confidence intervals that take into account both the bias and the skewness of the distribution of the nonlinear functions of the parameters.

We conduct a simulation study to compare the confidence intervals of our new methods with the two classical methods. The methods evaluated include Fieller's interval, Delta interval; Delta with the bias correction interval; Edgeworth expansion interval, and Edgeworth expansion with the bias correction interval. The results show that our new methods, generally, give good coverage probability and the confidence width . When data are from skewed distributions, the Edgeworth expansion and the Edgeworth expansion with the bias correction should be recommended for constructing confidence intervals for nonlinear functions of estimated parameters.

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JEL Classification, C12; C13

1 Introduction

Many econometric and statistical applications are interested in tests of the nonlinear functions of the parameters, which can be expressed as the ratio of two unknown parameters including the ratio of regression coefficients, the ratio of the two linear functions such as the ratio of affine transformations of random variables and generally the ratio of the two nonlinear functions.

A non-exhaustive list of examples of econometric models where inferences for the ratio of parameters are used as follows: the long-run elasticities and flexibilities in dynamic models, (Li and Maddala 1999; Dorfman et al. 1990; Bernard et al. 2007; Hirschberg et al. 2008); the willingness to pay value, i.e, the maximum price an agent would pay to obtain an improvement in a particular attribute of a desired good or service, (Lye and Hirschberg 2018); the turning point in a quadratic specification model where the estimated relationship is either a U-shaped or an inverted U-shaped curve for example Kuznet and Beveridge curves, in applications to dynamic panel data, (Bernard et al. 2019; Lye and Hirschberg 2018); the determination of the non-accelerating inflation rate of unemployment (NAIRU), for example a Phillips curve, (Staiger et al. 1997,Hirschberg and Lye 2010a; Lye, and Hirschberg, 2018); the structural parameter in an exactly identified system of equations as estimated by the two-stage least squares method (Hirschberg and Lye 2017, Lye, and Hirschberg, 2018, Andews et al..2019); the notion of weak instruments in econometric models (Woglom 2001); inequality indices, (Dufour et al. 2024; Dufour et al. 2018); structural impulse responses, (Olea et al. 2021). Lye and Hirschberg (2018) give some other examples of econometric models.

Other examples of statistical applications, include cost-effectiveness analysis (Briggs and Fenn (1998); and the comparison of health outcomes across spatial domains (Beyene and Moineddin 2005); bioequivalence assessment, dose-response analysis, (Sitter and Wu 1993; Faraggi, et al. 2003, Wang et al. 2015). For other statistical applications, see Franz (2007) .

However, the statistical properties of the ratio of parameters can be problematic because the analytical expressions of the moments are generally not available, e.g. the ratio of asymptotically normally distributed random variables is a non-central Cauchy distribution. Moreover, if the denominator of the ratio is not significantly different from zero, the probability distribution of the ratio shows unusual behaviour, and the confidence intervals are unbounded. Another problem worth highlighting is the bias of the estimator in a finite sample when studying a nonlinear function of parameters.

To test the null hypothesis of the nonlinear functions of parameters, we use confidence intervals (CIs). The two widely used approaches for constructing CIs are the Fieller method and the Delta method. The advantage of these methods is that they can be implemented in any context and are easy to compute, they do not require the use of intensive calculation and sampling strategies as would be needed when using a Bootstrap or Bayesian method, (Hirschberg and Lye, 2010, 2017; Lye and Hirschberg 2018).

Fieller (1954) proposed a method to derive the confidence interval (CI) of the ratio of two random variables. In Fieller's method, it assumes that both the numerator and the denominator of the ratio follow normal distribution. The method is based on the inversion of the pivotal $t-$statistic, it gives an exact CI for achieving the required coverage probability. The Fieller's CI is asymmetric around the ratio estimate, which is a good property, as it can be reflects the skewness of the small sample distribution of the ratio. However, if the denominator of the ratio is not significantly different from zero, Fieller's CI will be unbounded, being either the entire real line or the union of two disconnected infinite intervals. It has a positive probability to produce CI with infinite length. Furthermore, Fieller's interval requires finding roots of a quadratic equation and these can be imaginary. In addition, if this quadratic equation has one root, the confidence interval will be half-open.

The Delta method is based on the first-order Taylor expansion by considering nonlinear functions of parameters. By assuming asymptotic normality in large samples, this method produces a symmetric and bounded CI, unlike the Fieller method. However, the Delta method often has an inaccurate coverage probability (Dufour 1997) and unbalanced tail errors even at moderate sample sizes (Hirschberg and Lye 2010). A geometric interpretation of the Fieller and Delta methods can be found in von Luxbur and Franz (2009); Hirschberg and Lye,(2010). According to Hirschberg and Lye,(2010), if the true value of the ratio has the same sign as the correlation coefficient between the numerator and the denominator then the Delta and Fieller intervals may be very similar even if the denominator has a high variance. However, if the signs are opposite and the precision of the denominator is low, then the Delta method has poorer performance.

Moreover, there are two potential problems with these Fieller and Delta methods; first, the estimator of the parameters is biased in the nonlinear functions of parameters. Second, the estimated parameters have non-normal and asymmetric distribution. Thus the variance of the estimated parameters is not useful in constructing confidence intervals, Dorfman et al. (1990); Li and Madalla (1999).

In order to overcome the disadvantage of the previous methods, some numerical procedures have been proposed in the literature such as the parametric bootstrap method and the nonparametric bootstrap method (bootstrap standard, bootstrap $t-$statistic; bootstrap percentile, bootstrap bias-corrected, bootstrap bias-corrected and accelerated) see (Krinsky and Robb (1986); Dorfman et al. (1990), Li and Madalla (1999), among others. The CIs obtained from these iterative procedures are bounded and are more computationally intensive.

Dorfman et al. (1990) compared the Delta and Fieller methods and three types of the single bootstrap and found that the bootstrap did not achieve nominal coverage and that all methods performed reasonably well.

The bootstrap percentile-t and the Delta methods confidence intervals are very close to each other in many cases in terms of the length of the confidence intervals, Li and Madalla (1999).

It should be noted that all the previous methods do not take into account the bias of the estimator which should be a prerequisite for constructing a reliable confidence interval.

In this regard, the paper has five main contributions. *First*, we propose a novel analytical approach that modifies the Delta method to reduce the effect of skewness. The method is based on the Edgeworth expansion (Hall 1992a). We then propose a new easy to compute confidence interval for the ratio of parameters and the interval has the coverage probability converging to the nominal level at a rate of $O(n^{-1/2})$ where n is the sample size. *Second*, the source of

potential bias is due to the nonlinearity of the ratio $\widehat{\theta} = \widehat{\theta}_1 / \widehat{\theta}_2$ in terms of $\widehat{\theta}_1$ and $\widehat{\theta}_2$. It is well known that even when exact unbiased estimators of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are available, the estimator $\widehat{\theta}$ could still be badly biased in finite samples. This is because the expected value of a ratio of random variables is not equal to the ratio of the expected values and the use of the ratio of the expected values leads to a biased estimate of the true measure although it is a consistent estimator. We consider a second-order term in the Taylor series expansion to determine the bias and we propose a bias-corrected estimator which is identical to the almost unbiased ratio estimator proposed by Tin (1965). *Third,* we investigate the problem of approximating the variance of a nonlinear function of parameters based on a second-degree Taylor series expansion. Unfortunately, when calculating the variance of the second-degree Taylor expansion, most authors (Hayya et al. 1975, Wang et al. 2015) did not take into account the possible covariances between the random variables which is indispensable because it provides a better approximation. This variance is none other than the variance of the bias-corrected estimator (or the variance of the almost unbiased ratio estimator of Tin (1965)). *Fourth*, we define a modified version of the Delta method, correct the estimator of the bias, and calculate the corresponding variance. This can be helpful in terms of more accurate coverage probabilities for the CIs. *Fifth*, we propose a novel analytical approach to construct the CI for the ratio estimate. Our method, Edgeworth expansion with bias-corrected estimator uses the Edgeworth expansion but adopts an estimator corrected for the bias and its variance. The method always produces a bounded CI. Simulation results show that it generally outperforms the Edgeworth expansion in terms of controlling the coverage probabilities and the average width and is particularly useful when the data are skewed.

The rest of this paper is organised as follows. Section 2 presents some highlights. Section 3 studies the different methods for constructing CIs, the Fieller and Delta methods and we will develop the Edgeworth expansions for the Delta method. Section 4 provides an analytical form of the bias that can be used to construct the bias-corrected estimator and to calculate the variance of the bias-corrected estimator. Section 5 presents the confidence intervals with the bias-corrected estimator. Section 6 presents some econometric applications. The simulation study and the results are presented in Section 7 and Section 8 concludes the paper.

2 Some highlights

2.1 Definitions, notation

Let X and Y be two random variables, we assume that the moments exist then the expected value of X is denote by $E(X)$, the variance of X by $V(X)$, and the coefficient of variation of X is defined $CV(X) = \sqrt{V(X)/X^2} =.\frac{\sqrt{V(X)}}{X}$. A similar notation will be used for the random variable Y. The covariance of X and Y is defined by $Cov(X,Y) = E(XY) - E(X)E(Y)$, the correlation coefficient between X and Y is defined by $\rho = \frac{Cov(X,Y)}{\left(\sqrt{V(X)}\right)\left(\sqrt{V(Y)}\right)}$ so it satisfies $-1 \leq \rho \leq 1$ and $Cov(X,Y) = \rho\sqrt{V(X)}\sqrt{V(Y)}$. The coefficient of co-variation of X and Y is defined by $CV(X,Y) = \frac{Cov(X,Y)}{XY}$ which can be expressed as the produit of the correlation coefficient and the coefficients of variation of X and Y respectively: $CV(X,Y) = \rho\frac{\sqrt{V(X)}}{X}\frac{\sqrt{V(Y)}}{Y} = \rho CV(X)CV(Y)$. We use the notation $[a \pm b]$ for the interval $[a-b,\ a+b]$ $(b \succ 0)$.

2.2 The ratio estimator is biased.

Let $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are consistent estimators of θ_1 and θ_2,respectively, $E(\widehat{\theta}_1) = \theta_1$ and $E(\widehat{\theta}_2) = \theta_2$ and $\widehat{\theta} = \widehat{\theta}_1/\widehat{\theta}_2$ is a consistent estimator of the ratio $\theta = \theta_1/\theta_2$. It is well known that the ratio of two unbiased estimators is not, in general, itself an unbiased estimator, i.e $E(\widehat{\theta}_1/\widehat{\theta}_2) \neq E(\widehat{\theta}_1)/E(\widehat{\theta}_2) = \theta_1/\theta_2$

The expected value of the ratio between $\widehat{\theta}_1$ and $\widehat{\theta}_2$, provided that all moments exist, is given by

$$\begin{aligned} E(\widehat{\theta}_1/\widehat{\theta}_2) &= E(\widehat{\theta}_1 \times 1/\widehat{\theta}_2) \\ &= E(\widehat{\theta}_1) \times E(1/\widehat{\theta}_2) + Cov(\widehat{\theta}_1, 1/\widehat{\theta}_2) \end{aligned}$$

If $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are independent or if $\widehat{\theta}_1$ and $1/\widehat{\theta}_2$ are uncorrelated, then

$$E(\widehat{\theta}_1 \times 1/\widehat{\theta}_2) = E(\widehat{\theta}_1) \times E(1/\widehat{\theta}_2).$$

It is well known that $E(1/\widehat{\theta}_2) \neq 1/E(\ \widehat{\theta}_2)$, Jensen's inequality implies that $E(1/\widehat{\theta}_2) \geq 1/E(\ \widehat{\theta}_2)$ because the function $1/z$ is convex for $z \succ 0$ or $z \prec 0$, then we have

$$E(\widehat{\theta}_1/\widehat{\theta}_2) = E(\widehat{\theta}_1) \times E(1/\widehat{\theta}_2) \succeq E(\widehat{\theta}_1)/E(\widehat{\theta}_2)$$

and using that $E(\widehat{\theta}_1) = \theta_1$ and $E(\widehat{\theta}_2) = \theta_2$ we have

$$E(\widehat{\theta}_1/\widehat{\theta}_2) \succeq \theta_1/\theta_2$$

This result shows that the estimator of the ratio of two unbiased estimators is, in general, biased,

We will now consider a more general framework which can be precise the bias of $\widehat{\theta}$

Note that the covariance of $\widehat{\theta}_1/\widehat{\theta}_2$ and $\widehat{\theta}_2$ is

$$\begin{aligned} Cov(\widehat{\theta}_2, \widehat{\theta}_1/\widehat{\theta}_2) &= E(\widehat{\theta}_2 \times \widehat{\theta}_1/\widehat{\theta}_2) - E(\widehat{\theta}_2) \times E(\widehat{\theta}_1/\widehat{\theta}_2) \\ &= E(\widehat{\theta}_1) - E(\widehat{\theta}_2) \times E(\widehat{\theta}_1/\widehat{\theta}_2) \end{aligned}$$

Then, by rearranging these terms, provided that $E(\widehat{\theta}_2) \neq 0$, we obtain the expected value of the ratio between $\widehat{\theta}_1$ and $\widehat{\theta}_2$

$$E(\widehat{\theta}_1/\widehat{\theta}_2) = E(\widehat{\theta}_1)/E(\widehat{\theta}_2) - 1/E(\widehat{\theta}_2) \times Cov(\widehat{\theta}_2, \widehat{\theta}_1/\widehat{\theta}_2)$$

and using that $E(\widehat{\theta}_1) = \theta_1$ and $E(\widehat{\theta}_2) = \theta_2$ we get

$$E(\widehat{\theta}_1/\widehat{\theta}_2) = \theta_1/\theta_2 - 1/\theta_2 \times Cov(\widehat{\theta}_2, \widehat{\theta}_1/\widehat{\theta}_2)$$

which can be written as

$$E(\widehat{\theta}) = \theta - 1/\theta_2 \times Cov(\widehat{\theta}_2, \widehat{\theta}_1/\widehat{\theta}_2)$$

and the bias of $\widehat{\theta}$ is

$$Bias(\widehat{\theta}) = E(\widehat{\theta}) - \theta = -1/\theta_2 \times Cov(\widehat{\theta}_2, \widehat{\theta}_1/\widehat{\theta}_2)$$

The size of the bias of $\widehat{\theta}$ depending on both θ_2 and the covariance between $\widehat{\theta}_2$ and the ratio $\widehat{\theta}_1\ /\widehat{\theta}_2$.

Consequently, the absolute value of the bias is

$$\left|Bias(\widehat{\theta})\right| = \frac{\left|\rho\sqrt{V(\widehat{\theta})}\sqrt{V(\widehat{\theta}_2)}\right|}{\theta_2} \preceq \frac{\sqrt{V(\widehat{\theta})}\sqrt{V(\widehat{\theta}_2)}}{\theta_2}$$

where $\sqrt{V(\widehat{\theta})}$ and $\sqrt{V(\widehat{\theta}_2)}$ are the standard errors of $\widehat{\theta}$ and $\widehat{\theta}_2$ respectively. Thus an upper bound to the ratio of the absolute value of the bias to its standard error is given by

$$\frac{\left|Bias(\widehat{\theta})\right|}{\sqrt{V(\widehat{\theta})}} \preceq \frac{\sqrt{V(\widehat{\theta}_2)}}{\theta_2} = CV(\widehat{\theta}_2)$$

where $CV(\widehat{\theta}_2)$ is the coefficient of variation of $\widehat{\theta}_2$.

Remark 1 *If $\widehat{\theta}_2$ is distributed $N(\theta_2,\ V(\widehat{\theta}_2))$ then the coefficient of variation $CV(\widehat{\theta}_2)$ is simply the inverse of the $t-$ statistic of $\widehat{\theta}_2$. This result implies that an upper bound to the relative absolute bias of the ratio $\widehat{\theta}$ is the inverse of the $t-$statistic of the denominator.*

For a large sample size, the bias in the ratio estimator $\widehat{\theta}$ is negligible as compared to its standard error. It is well known that the variance of estimator $V(\widehat{\theta}_2)$ is of $O(n^{-1})$ then also the bias in $(\widehat{\theta})$ is also $O(n^{-1})$. Cochran (1977) has shown that if the coefficient of variation of $\widehat{\theta}_2$ is less than 0.1, then the bias is small relative to the standard error. Furthermore, it is difficult to obtain an analytical expression of the bias, as we will see later using an approximation of the ratio of the parameters gives an analytical form of the bias.

3 Methods

3.1 The Delta method (or the Taylor's series expansion)

The Delta method (often referred to as the Taylor's series expansion) estimates the variance of a nonlinear function of two or more random variables is given by taking a first-order Taylor expansion around the mean value of the variables and calculating the variance for this expression. In the case of the ratio of parameterss $\widehat{\theta} = g(\widehat{\theta}_1, \widehat{\theta}_2) = \widehat{\theta}_1/\widehat{\theta}_2$, the variance of $\widehat{\theta}$ is

$$V(\widehat{\theta}) = G'\Sigma G$$

where G is a Jacobian vector containing all the first-order partial derivatives and Σ is the variance-covariance matrix of $\widehat{\theta}_1$ and $\widehat{\theta}_2$, defined as follows

$$\Sigma = \begin{bmatrix} V(\widehat{\theta}_1) & Cov(\widehat{\theta}_1\widehat{\theta}_2) \\ Cov(\widehat{\theta}_2\widehat{\theta}_1) & V(\widehat{\theta}_2) \end{bmatrix}$$

We have

$$V(\widehat{\theta}) = \frac{1}{\widehat{\theta}_2^2}\left[V(\widehat{\theta}_1) - 2\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)Cov(\widehat{\theta}_1, \widehat{\theta}_2) + \left(\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\right)V(\widehat{\theta}_2)\right]$$

(Full derivation details can be see in Appendix).

To construct a confidence interval for the ratio $\theta = \theta_1/\theta_2$, we assume that $n^{1/2}(\widehat{\theta} - \theta)$ is asymptotically normal distributed with zero mean and variance $V(\widehat{\theta})$.

Let $\widehat{V}(\widehat{\theta})$ be a consistent estimator of $V(\widehat{\theta})$, the Delta method $100(1-\alpha)\%$ confidence limits for the ratio θ_1/θ_2 is given by:

$$CI_D : \frac{\widehat{\theta}_1}{\widehat{\theta}_2} \pm z_{\alpha/2} Q_D$$

where $Q_D = \sqrt{\widehat{V}(\widehat{\theta})} = \frac{1}{\widehat{\theta}_2}\left[\widehat{V}(\widehat{\theta}_1) - 2\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)\widehat{Cov}(\widehat{\theta}_1, \widehat{\theta}_2) + \left(\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\right)\widehat{V}(\widehat{\theta}_2)\right]^{1/2}$, the estimated standard error of $\widehat{\theta}$ and $z_{\alpha/2}$ is the $(\alpha/2)\ th$ quantile for standard normal distribution.

It is useful to express the variance of $\widehat{\theta}$ in terms of coefficients of variation and coefficients of co-variation to give a more concrete interpretation of the results in empirical applications; then, the variance of $\widehat{\theta}$ in terms of the coefficient of variation and the coefficient of co-variation of $\widehat{\theta}_1$ and $\widehat{\theta}_2$. is given by

$$\widehat{V}(\widehat{\theta})^* = \frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left[\widehat{CV}(\widehat{\theta}_1)^2 - 2\widehat{CV}(\widehat{\theta}_1, \widehat{\theta}_2) + \widehat{CV}(\widehat{\theta}_2)^2\right]$$

where $\widehat{CV}(\widehat{\theta}_i)\ i = 1, 2$ is the estimate of coefficient of variation for a random variable $\widehat{\theta}_i$ and $\widehat{CV}(\widehat{\theta}_1, \widehat{\theta}_2) = \widehat{\rho}\widehat{CV}(\widehat{\theta}_1)\widehat{CV}(\widehat{\theta}_2)$ is the estimate of coefficient of co-variation for the two random variables $\widehat{\theta}_1$ and $\widehat{\theta}_2$

The $100(1-\alpha)\%$ confidence intervals becomes

$$CI_D : \frac{\widehat{\theta}_1}{\widehat{\theta}_2} \pm z_{\alpha/2} Q_D^*$$

where $Q_D^* = \sqrt{\widehat{V}(\widehat{\theta})^*} = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left[\widehat{CV}(\widehat{\theta}_1)^2 - 2\widehat{CV}(\widehat{\theta}_1, \widehat{\theta}_2) + \widehat{CV}(\widehat{\theta}_2)^2\right]^{1/2}$ is the estimated standard error of $\widehat{\theta}$ in terms of the coefficient of variation and the correlation coefficient.

This method assumes that $\widehat{\theta}$ is normally distributed and thus symmetrical around its mean. However, the assumption of normality is clearly strong as there is no guarantee that $\widehat{\theta}$ is normally distributed.

However, for large sample sizes (or rather small coefficients of variation) the distribution of a ratio may be close to normal.

The assumption of a normal distribution may be justified in the case of large samples, but it is unlikely that the distribution of a ratio will generally follow a well-behaved distribution. Furthermore, the assumption of a normal distribution may be quite inaccurate if the data have a skewed distribution.

3.2 The Fieller method

Fieller (1954) proposed a general procedure for constructing confidence limits for the ratio of the means of two normal distributions. In Fieller's method, the ratio variable is transformed into a linear function. The confidence interval of the ratio variable can be obtained by solving out the quadratic roots of the linear function.

For testing the null hypothesis $H_0 : \frac{\theta_1}{\theta_2} = \gamma$ equivalently it is written as on a linear combination of the parameters $H_0' : \theta_1 - \gamma\theta_2 = 0$, the method assumes that $\widehat{\theta}_1$and $\widehat{\theta}_2$ follow a joint normal distribution function such that $\widehat{\theta}_1 - \gamma\widehat{\theta}_2$ is normally distributed. Hence, the pivotal statistic for this test is:

$$T = \frac{\widehat{\theta}_1 - \gamma\widehat{\theta}_2}{\sqrt{\widehat{V}(\widehat{\theta}_1) - 2\gamma\widehat{Cov}(\widehat{\theta}_1\widehat{\theta}_2) + \gamma^2\widehat{V}(\widehat{\theta}_2)}}$$

which is $t-$distribution with *df* degrees of freedom under the null hypothesis.

Let $t_{\alpha/2,df}$ denotes the $100(1-\alpha/2)$th percentile of the $t-$distribution with *df* degrees of freedom, we have

$$P\left[T^2 \leq t^2_{\alpha/2,df}\right] = 1 - \alpha$$

By replacing the expression of square T and rearranging gives a quadratic equation in γ.

$$a\gamma^2 + b\gamma + c \preceq 0$$

where $a = 1 - t^2_{\alpha/2,df}\frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}$, $b = -2\frac{\theta_1}{\theta_2}\left(1 - t^2_{\alpha/2,df}\frac{\widehat{Cov}(\widehat{\theta}_1\widehat{\theta}_2)}{\widehat{\theta}_1\widehat{\theta}_2}\right)$, and $c = \left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)^2\left(1 - t^2_{\alpha/2,df}\frac{\widehat{V}(\widehat{\theta}_1)}{\widehat{\theta}_1^2}\right)$. The parameters, a, b and c can be expressed in terms of coefficient variation and coefficient co-variation $a = 1 - t^2_{\alpha/2,df}\widehat{CV}(\widehat{\theta}_2)^2$, $b = -2\frac{\theta_1}{\theta_2}\left(1 - t^2_{\alpha/2,df}\widehat{CV}(\widehat{\theta}_1, \widehat{\theta}_2)\right)$, and $c = \left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)^2\left(1 - t^2_{\alpha/2,df}CV(\widehat{\theta}_1)^2\right)$. Finding an explicit form for the confidence intervals for γ requires solving the quadratic equation. The solution of this inequality depends on the sign of a

and $d = b^2 - 4ac$, the discriminant of the quadratic equation. We can expressed d as follows

$d = 4\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)^2 t^2_{\alpha/2,df}\left\{\left[\widehat{CV}(\widehat{\theta}_2) - \widehat{\rho}\widehat{CV}(\widehat{\theta}_1)\right]^2 + a\widehat{CV}(\widehat{\theta}_1)^2(1-\widehat{\rho}^2)\right\}$ Hence, $a \succ 0$ also implies $d \succ 0$

If $d \succ 0$, let γ_L and γ_U $(\gamma_L \prec \gamma_U)$ be the two real valued solutions to the quadratic equation in γ by changing the inequality into an equality. This gives the bounds of the Fieller interval in the case $a \succ 0$. These two roots are the lower and upper limits of the $(1-\alpha)$ confidence interval. The bounds of the interval are given by

$$CI_F : [\gamma_L,\ \gamma_U] = \frac{1}{1-g}\left\{\frac{\widehat{\theta}_1}{\widehat{\theta}_2} - g\frac{\widehat{Cov}(\widehat{\theta}_1\widehat{\theta}_2)}{\widehat{V}(\widehat{\theta}_2)} \pm t_{\alpha/2,df}Q_F\right\}$$

where $Q_F = \frac{1}{\widehat{\theta}_2}\left[\widehat{V}(\widehat{\theta}_1) - 2\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)\widehat{Cov}(\widehat{\theta}_1,\widehat{\theta}_2) + \left(\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\right)\widehat{V}(\widehat{\theta}_2) - g\left(\widehat{V}(\widehat{\theta}_1) - \frac{\widehat{Cov}(\widehat{\theta}_1\widehat{\theta}_2)^2}{\widehat{V}(\widehat{\theta}_2)}\right)\right]^{1/2}$
and $g = t^2_{\alpha/2,df}\frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}$

An equivalent form of the Fieller confidence interval (CI) in terms of the coefficient of variation and the coefficient of co-variation of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ is given by

$$CI_F : [\gamma_L,\ \gamma_U] = \frac{1}{1-h}\left\{\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left(1 - h\widehat{\rho}\frac{\widehat{CV}(\widehat{\theta}_1)}{\widehat{CV}(\widehat{\theta}_2)}\right) \pm t_{\alpha/2,df}Q_F^*\right\}$$

where $Q_F^* = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left[\widehat{CV}(\widehat{\theta}_1)^2 - 2\widehat{CV}(\widehat{\theta}_1,\widehat{\theta}_2) + \widehat{CV}(\widehat{\theta}_2)^2 - h\widehat{CV}(\widehat{\theta}_1)^2(1-\widehat{\rho}^2)\right]^{1/2}$
and $h = t^2_{\alpha/2,df}\widehat{CV}(\widehat{\theta}_2)^2$

However, if $a \prec 0$ the Fieller CI will be unbounded. Hence, if $d \succ 0$ the Fieller CI will be the complement of a finite interval $(-\infty, \gamma_U) \cup (\gamma_L, \infty)$ and if $d \prec 0$ the Fieller CI will be the whole real line $(-\infty, +\infty)$.

Other intervals may be considered when $a = 0$, the Fieller CI will be $\left]-\infty,\ \frac{-c}{b}\right]$ if $b \succ 0$ otherwise, it will be $\left[\frac{-c}{b},\ \infty\right[$ if $b \prec 0$.

Remark 2 *1) In the case of finite interval, the condition $a \succ 0$ is equivalent to $\left|\frac{\widehat{\theta}_2}{\sqrt{\widehat{V}(\widehat{\theta}_2)}}\right| \succ t_{\alpha/2,df}$ which means rejecting the null hypothesis $H_0\ : \theta_2 = 0$, i.e. θ_2 is significantly different from zero. The test of this null hypothesis is the first step of Scheffé's procedure, (Scheffé, 1970).*

2) The $t-$statistic $\left|\frac{\widehat{\theta}_2}{\sqrt{\widehat{V}(\widehat{\theta}_2)}}\right|$ is equal to $\left|\frac{1}{\widehat{CV}(\widehat{\theta}_2)}\right|$ the absolute inverse of the coefficient of variation for $\widehat{\theta}_2$, so the null hypothesis is rejected if the coefficient of variation for $\widehat{\theta}_2$ is negligible. (A high coefficient of variation for $\widehat{\theta}_2$ means a low statistical value).

It should be noted that the null hypothesis $H_0' : \theta_1 - \gamma\theta_2 = 0$, was obtained from the non-linear relationship $\frac{\theta_1}{\theta_2} = \gamma$ only when $\theta_2 \neq 0$. However, Fieller's method does not take this information into account. Therefore, the Fieller CI

has the potential to overestimate the confidence length.

The advantage of Fieller's method over the Delta method is that it takes into account the potential skewness of the sampling distribution of the ratio estimator and therefore may not be symmetric around the point estimate.

Fieller's method provides an exact solution subject to the joint normality assumption. However, it has been argued that the assumption of joint normality may be difficult to justify, particularly when sample sizes are small. In particular, the random variable follows a skewed distribution, which may cause problems for the normality assumption.

The normal approximation is a rather rough approximation, especially when sample sizes are not large; it does not take into account the skewness of the underlying distribution which is often the main source of error of the normal approximation. To remove the effect of the skewness, we develop the Edgeworth expansion.

3.3 Edgeworth expansion

The Delta method-based confidence interval is not very robust and can be quite inaccurate in practice for non-normal data. It produces intervals that are symmetric around the point estimate, so it does not take skewness into account. The correction for skewness used in our confidence intervals is based on the Edgeworth expansion.

We propose a method based on the Edgeworth expansion to modify the Delta intervals to remove the effect of skewness. The expansion provides a way to correct for the skewness in the data and to derive new confidence intervals for the ratio parameters. Thus we consider two aspects: first an Edgeworth expansion is derived for the Delta method for a ratio of parameters on a normal random variable and second by using the inverse of the Edgeworth expansions which are the quantiles of the distribution that is the Cornish-Fisher expansion, we construct an approximate confidence interval which contains a $n^{-1/2}$ order correction for the effect of skewness.

The Delta method can be easily extended for a better approximation by using Edgeworth expansion .

Let $U = \widehat{V}(\widehat{\theta})^{-1/2}\sqrt{n}(\widehat{\theta} - \theta)$ where $\widehat{\theta} = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}$, $\theta = \frac{\theta_1}{\theta_2}$ and $\widehat{V}(\widehat{\theta})$ the estimate of $V(\widehat{\theta})$ in Delta method, we assume that the distriubtion of a random variable U has the Edgeworth expansion (Hwang, 2019, Hall, 1992b)

$$F(x) = P(U \leq x) = \Phi(x) - n^{-1/2}\kappa\frac{1}{6}(x^2 - 1)\phi(x) + O(n^{-1/2})$$

where $\Phi(x)$ and $\phi(x)$ are the standard normal distribution and density functions respectively, κ is the skewness, and n is the sample size. This expansion can be interpreted as the sum of the normal distribution $\Phi(x)$, and an error due to the skewness of the distribution. When the error (the $n^{-1/2}$ skewness correction) in absolute value is small, U can be accurately approximated by a normal distribution. Conversely, when the error in absolute value is large, the second term in the formulation cannot be ignored and therefore the normal approximation would not be as accurate. The $n^{-1/2}$ skewness correction is an even function of x which means that it changes the distribution function symmetrically about zero. Thus, the skewness of the distribution F has a significant effect, especially when the sample size n is small.

To construct asymptotic confidence intervals, we should invert the Edgeworth expansions to obtain expansions of distribution quantiles. Such expansions are known as Cornish-Fisher expansions.

For any $0 \prec \alpha \prec 1$, let ξ_α be the $\alpha - th$ quantile of distribution $F(.)$, which is the solution to $F(\xi_\alpha) = \alpha$. This quantile of distribution $\xi_\alpha = F^{-1}(\alpha)$ admits a Cornish-Fisher expansion of the form (Hwang, 2019).

$$\xi_\alpha = z_\alpha + n^{-1/2}\widehat{\kappa}\frac{1}{6}(z_\alpha^2 - 1) + O(n^{-1/2})$$

where $\widehat{\kappa}$ is the estimate of κ and z_α is the $\alpha - th$ quantile of the standard normal distribution.

The $100(1-\alpha)\%$ Edgeworth expansion confidence interval for the ratio $\frac{\theta_1}{\theta_2}$ is given by

$$CI_E : \left[\frac{\widehat{\theta}_1}{\widehat{\theta}_2} - \xi_{1-\alpha/2}Q_D, \quad \frac{\widehat{\theta}_1}{\widehat{\theta}_2} - \xi_{\alpha/2}Q_D\right]$$

where $Q_D = \frac{1}{\widehat{\theta}_2}\left[\widehat{V}(\widehat{\theta}_1) - 2\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)\widehat{Cov}(\widehat{\theta}_1, \widehat{\theta}_2) + \left(\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\right)\widehat{V}(\widehat{\theta}_2)\right]^{1/2}$ and $\xi_{\alpha/2}$ and $\xi_{1-\alpha/2}$ are the $(\alpha/2)th$ and $(1-\alpha/2)th$ quantiles of distribution $F(.)$.

Or in terms of the coefficient of variation and the coefficient of co-variation

$$CI_E : \left[\frac{\widehat{\theta}_1}{\widehat{\theta}_2} - \xi_{1-\alpha/2}Q_D^*, \quad \frac{\widehat{\theta}_1}{\widehat{\theta}_2} - \xi_{\alpha/2}Q_D^*\right]$$

where $Q_D^* = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left[\widehat{CV}(\widehat{\theta}_1)^2 - 2\widehat{CV}(\widehat{\theta}_1, \widehat{\theta}_2) + \widehat{CV}(\widehat{\theta}_2)^2\right]^{1/2}$ and $\widehat{CV}(\widehat{\theta}_1, \widehat{\theta}_2) = \widehat{\rho}\widehat{CV}(\widehat{\theta}_1)\widehat{CV}(\widehat{\theta}_2)$ $\xi_{\alpha/2}$ and $\xi_{1-\alpha/2}$ are the $(\alpha/2)th$ and $(1-\alpha/2)th$ quantiles of distribution $F(.)$.

For positively skewed data, the true $1-\alpha/2$ quantile $\xi_{1-\alpha/2}$ is larger than the associated standard normal quantiles $z_{\alpha/2}$ and similarly the true lower quantile $\xi_{\alpha/2}$ is larger than $-z_{\alpha/2}$.

From the Cornish-Fisher expansion, we can state the asymptotic coverage probability of the proposed intervals

The coverage probability of confidence intervals is given by

$$P(\theta = \frac{\theta_1}{\theta_2} \in CI_E) = 1 - \alpha + O(n^{-1/2}).$$

4 Bias-correction analysis

4.1 Bias of estimator

In Section 1, we showed that the parameter θ is a biased estimator of the ratio parameters. It's essential to determine the expected direction and magnitude of this bias.

It is well known in the literature that the ratio of the parameters uses only first-order expansions to approximate asymptotic sampling distributions. However, calculating higher-order expansions can also be useful given that they can be used to estimate the bias of the ratio of the parameters and the analytical form of the bias obtained can be used to construct the bias-corrected estimator.

We consider a second-order term in the Taylor series expansion. This additional second-order term can be helpful, in the sense of more accurate coverage probabilities for the CIs.

Let θ is $g(\theta_1, \theta_2) = \theta_1/\theta_2$. then from a second-order Taylor's series expansion,

$$g(\widehat{\theta}_1, \widehat{\theta}_2) = g(\theta_1, \theta_2) + G' \begin{pmatrix} \widehat{\theta}_1 - \theta_1 \\ \widehat{\theta}_2 - \theta_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \widehat{\theta}_1 - \theta_1 \\ \widehat{\theta}_2 - \theta_2 \end{pmatrix}' H \begin{pmatrix} \widehat{\theta}_1 - \theta_1 \\ \widehat{\theta}_2 - \theta_2 \end{pmatrix}$$

where G is a Jacobian vector containing all the first-order partial derivatives and H is a Hessian matrix containing all the second partial derivative for the nonlinear function $g(\widehat{\theta}_1, \widehat{\theta}_2)$ evaluated at θ_1 and θ_2

In this section, a delta approximation of the bias, based on a second-order Taylor series expansion, is used to estimate the bias of the ratio of parameters. Then we construct the bias-corrected estimator and derive its variance.

Proposition 3 *For a ratio of parameters $\theta = \frac{\theta_1}{\theta_2}$, a second-order Taylor's series expansion gives the approximation of bias*

$$Bias(\widehat{\theta}) = E(\widehat{\theta}) - \theta = \frac{1}{2}(vecH)'vec(\Sigma)$$

where $vec(.)$ denotes the vectorisation operator which stacks the columns of the matrix and H is a Hessian matrix of second order partial derivatives and Σ is the variance-covariance matrix of $\widehat{\theta}_1$ and $\widehat{\theta}_2$.

Proof. (see Appendix) ■

Proposition 4 *Let $\widehat{H}$ and $\widehat{\Sigma}$ be the estimates of H and Σ respectively, the estimate of bias is given by*

$$\begin{aligned}\widehat{Bias}(\widehat{\theta}) &= \frac{1}{2}(vec\widehat{H})' vec\widehat{\Sigma} \\ \widehat{Bias}(\widehat{\theta}) &= -\frac{1}{\widehat{\theta}_2^2}\widehat{Cov}(\widehat{\theta}_1, \widehat{\theta}_2) + \frac{\widehat{\theta}_1}{\widehat{\theta}_2^3}\widehat{V}(\widehat{\theta}_2)\end{aligned}$$

which can also be written as

$$\widehat{Bias}(\widehat{\theta}) = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left[\frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2} - \frac{\widehat{Cov}(\widehat{\theta}_1, \widehat{\theta}_2)}{\widehat{\theta}_2\widehat{\theta}_1}\right]$$

where $\frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2} - \frac{\widehat{Cov}(\widehat{\theta}_1, \widehat{\theta}_2)}{\widehat{\theta}_2\widehat{\theta}_1}$ can be viewed as a correction factor to the estimated ratio estimator.

This bias is identical to Tin's bias, Tin (1965). It uses the same information as the correction factor formed by subtracting $\frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}$ from $\frac{\widehat{Cov}(\widehat{\theta}_2, \widehat{\theta}_1)}{\widehat{\theta}_2\widehat{\theta}_1}$. This bias is $O(n^{-1})$ Tin (1965). Our bias is derived by a different method. Tin (1965) and David and Sukhatme (1974) used an asymptotic series expansion of the ratio estimator under certain conditions. The high-order of Tin's bias formulation was given by David and Sukhatme (1974)

To obtain the sign of the bias, we express the bias as a function of the coefficient of variation and the coefficient of co-variation

The bias can be expessed in terms of the coefficient of variation and the coefficient of co-variation of $\widehat{\theta}_1$ and $\widehat{\theta}_2$

$$\widehat{Bias}(\widehat{\theta})^* = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left[\widehat{CV}(\widehat{\theta}_2)^2 - \widehat{CV}(\widehat{\theta}_1, \widehat{\theta}_2)\right]$$

An another alternative form of the bias is

$$\widehat{Bias}(\widehat{\theta})^* = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}\widehat{CV}(\widehat{\theta}_1)\widehat{CV}(\widehat{\theta}_2)\left[\frac{\widehat{CV}((\widehat{\theta}_2)}{\widehat{CV}(\widehat{\theta}_1)} - \widehat{\rho}\right]$$

where $\widehat{\rho}$ is the estimate of the correlation coefficient between θ_1 and θ_2.

Following this latter formula, if the coefficient of variation of $\widehat{\theta}_2$ is close to zero, then the bias may be negligible relative to the variation in $\widehat{\theta}$. Furthermore, if the coefficient of variation of $\widehat{\theta}_2$: $\widehat{CV}(\widehat{\theta}_2)$ is greater than the coefficient of variation of $\widehat{\theta}_1$: $\widehat{CV}(\widehat{\theta}_1)$, the absolute value of the bias increases if the correlation between $\widehat{\theta}_1$ and $\widehat{\theta}_2$ becomes zero or negative. Similarly, if $\widehat{CV}(\widehat{\theta}_1) \succ \widehat{CV}(\widehat{\theta}_2)$, the bias is negative for a high positive correlation coefficient. Furthermore, if $\widehat{\rho} = \frac{\widehat{CV}((\widehat{\theta}_2)}{\widehat{CV}(\widehat{\theta}_1)}$, then the ratio estimator is unbiased.

4.2 The bias-corrected estimator

The bias given in the previous propositions can be used to construct bias-corrected estimators of θ

Proposition 5 *The bias-corrected estimator for θ is given by*

$$\widehat{\theta}_{BC} = \widehat{\theta} - \widehat{Bias}(\widehat{\theta}) = \frac{\widehat{\theta}_1}{\widehat{\theta}_2} - \frac{1}{2}(vec\widehat{H})' vec\widehat{\Sigma}$$

$$\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC} = \frac{\widehat{\theta}_1}{\widehat{\theta}_2} + \frac{1}{\widehat{\theta}_2^2}\widehat{Cov}(\widehat{\theta}_1, \widehat{\theta}_2) - \frac{\widehat{\theta}_1}{\widehat{\theta}_2^3}\widehat{V}(\widehat{\theta}_2)$$

which can also be written as

$$\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC} = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left\{1 + \left[\frac{\widehat{Cov}(\widehat{\theta}_1, \widehat{\theta}_2)}{\widehat{\theta}_1\widehat{\theta}_2} - \frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\right]\right\}$$

where $1 + \left[\frac{\widehat{Cov}(\widehat{\theta}_1, \widehat{\theta}_2)}{\widehat{\theta}_1\widehat{\theta}_2} - \frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\right]$ can be considered as a correction factor to the estimated ratio estimator.

Proof. (see Appendix). ■

This bias-corrected estimator for θ has the same structure as Tin's (1965) almost unbiased ratio estimator in the sense that its bias is of $O(n^{-2})$, i.e. the bias of $\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC}$ converges to zero at a fast rate than that of $\frac{\widehat{\theta}_1}{\widehat{\theta}_2}$. Tin called it a "modified ratio estimator". He has shown that his estimator is better than other competing estimators of population mean, up to the second order of approximation and it is equivalent to the Beale (1962) estimator up to the first order of approximation. Tin's estimator has been studied theoretically and via simulation by, Dalabehera and Sahoo (1995), Swain and Dash (2020) and they found Tin's estimator generally to be less biased and more efficient compared with other proposed ratio estimators.

The bias-corrected estimator $\widehat{\theta}_{BC}$ in terms of coefficient of variation and the coefficient of co-variation of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ is

$$\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)^*_{BC} = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left\{1 + \left[\widehat{CV}(\widehat{\theta}_1,\widehat{\theta}_2) - \widehat{CV}(\widehat{\theta}_2)^2\right]\right\}$$

An other equivalent form to

$$\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)^*_{BC} = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left\{1 + \widehat{CV}(\widehat{\theta}_1)\widehat{CV}(\widehat{\theta}_2)\left[\widehat{\rho} - \frac{\widehat{CV}(\widehat{\theta}_2)}{\widehat{CV}(\widehat{\theta}_1)}\right]\right\}$$

where $1+\left[\widehat{CV}(\widehat{\theta}_1,\widehat{\theta}_2)-\widehat{CV}(\widehat{\theta}_2)^2\right]$ or $1+\widehat{CV}(\widehat{\theta}_1)\widehat{CV}(\widehat{\theta}_2)\left[\widehat{\rho}-\frac{\widehat{CV}(\widehat{\theta}_2)}{\widehat{CV}(\widehat{\theta}_1)}\right]$ can be considered as a correction factor to the estimated ratio estimator.

In the next, we examine the case where the numerator and denominator of a ratio are independent. In this case, we will specify the bias and the bias-corrected estimator in the following proposition:

Proposition 6 *If $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are independent, we have*

(1a) The estimate of the bias is $\widehat{Bias}(\widehat{\theta})=\frac{\widehat{\theta}_1}{\widehat{\theta}_2^3}\widehat{V}(\widehat{\theta}_2)$

(1b) Equivalently, the estimate of the bias is $\widehat{Bias}(\widehat{\theta})=\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\widehat{CV}(\widehat{\theta}_2)^2=\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\frac{1}{t(\widehat{\theta}_2)^2}$

where $t(\widehat{\theta}_2)^2$ denotes the square of $t-$statistic for $\widehat{\theta}_2$ and $\frac{1}{t(\widehat{\theta}_2)^2}$ can be considered as a correction factor to the estimated ratio estimator.

The bias of the estimator of the ratio is the estimator of the ratio weighted by the square of the coefficient of variation of $\widehat{\theta}_2$ (the inverse of the square of $t-$statistic for $\widehat{\theta}_2$).

(2a) The bias-corrected estimator for θ is $\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC}=\frac{\widehat{\theta}_1}{\widehat{\theta}_2}-\frac{\widehat{\theta}_1}{\widehat{\theta}_2^3}\widehat{V}(\widehat{\theta}_2)$

(2b) Equivalently, the bias-corrected estimator for θ is $\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC}=\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left[1-\widehat{CV}(\widehat{\theta}_2)^2\right]=\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left[1-\frac{1}{t(\widehat{\theta}_2)^2}\right]$

The bias-corrected estimator of the ratio parameter is the estimator of the ratio weighted by the simple statistic $\left[1-\frac{1}{t(\widehat{\theta}_2)^2}\right]$, this weight will be less than one because $\widehat{CV}(\widehat{\theta}_2)^2$ is positive. Thus, the bias-corrected estimator of the ratio is smaller than the estimator of the ratio.

4.3 The variance of the bias-corrected estimator

As we have shown, the bias-corrected estimator $\widehat{\theta}_{BC}$ corresponds to the Tin (1965) almost unbiased ratio estimator, also known as the modified ratio estimator. The approximation of the variance of $\widehat{\theta}$ with a second-order term expressed in terms of the coefficient of variation and the coefficient of co-variation of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ is identical to the variance of the almost unbiased ratio estimator. We therefore use this variance as the variance of the the bias-corrected estimator.

Proposition 7 *The variance of the bias-corrected estimator for θ*

$$\widehat{V}\left[(\widehat{\theta}_{BC})\right]==\underbrace{G'\widehat{\Sigma}G}_{\text{first-order part}}+\underbrace{\frac{1}{2}(vec\widehat{H})'(\widehat{\Sigma}\otimes\widehat{\Sigma})vec\widehat{H}}_{\text{second-oder part}}$$

where the first order part $G'\widehat{\Sigma}G$ corresponds to the asymptotic (first-order) variance of the estimator and the second order part permit to take into account the correlation between the random variables and $\otimes$ denotes the Kronecker product.

$$\widehat{V}\left[(\widehat{\theta}_{BC})\right] = \underbrace{\frac{1}{\widehat{\theta}_2^2}\left[\widehat{V}(\widehat{\theta}_1) - 2\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)\widehat{Cov}(\widehat{\theta}_1,\widehat{\theta}_2) + \left(\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\right)\widehat{V}(\widehat{\theta}_2)\right]}_{\text{first-order approximation}}$$
$$\underbrace{+\frac{1}{\widehat{\theta}_2^4}\widehat{V}(\widehat{\theta}_2)\left[\widehat{V}(\widehat{\theta}_1) - 4\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)\widehat{Cov}(\widehat{\theta}_1,\widehat{\theta}_2) + 2\left(\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\right)\widehat{V}(\widehat{\theta}_2)\right] + \frac{1}{\widehat{\theta}_2^4}\widehat{Cov}(\widehat{\theta}_1,\widehat{\theta}_2)^2}_{\text{additional part from second-order approximation}}$$

which can also be written by

$$\widehat{V}\left[(\widehat{\theta}_{BC})\right] = \frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left\{\begin{array}{c}\underbrace{\left[\frac{\widehat{V}(\widehat{\theta}_1)}{\widehat{\theta}_1^2} - 2\frac{\widehat{Cov}(\widehat{\theta}_1,\widehat{\theta}_2)}{\widehat{\theta}_1\widehat{\theta}_2} + \frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\right]}_{\text{first-order approximation}} \\ \underbrace{+\frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\left[\frac{\widehat{V}(\widehat{\theta}_1)}{\widehat{\theta}_1^2} - 4\frac{\widehat{Cov}(\widehat{\theta}_1,\widehat{\theta}_2)}{\widehat{\theta}_1\widehat{\theta}_2} + 2\frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\right] + \frac{\widehat{Cov}(\widehat{\theta}_1,\widehat{\theta}_2)^2}{\widehat{\theta}_1^2\widehat{\theta}_2^2}}_{\text{additional part from second-order approximation}}\end{array}\right\}$$

Thus, the variance $V\left[(\widehat{\theta}_{BC})\right]$ can be express in terms of coefficient variation of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ by

$$\widehat{V}\left[(\widehat{\theta}_{BC})\right]^* = \frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left\{\begin{array}{c}\underbrace{\left[\widehat{CV}(\widehat{\theta}_1)^2 - 2\widehat{\rho}\widehat{CV}(\widehat{\theta}_1)\widehat{CV}(\widehat{\theta}_2) + \widehat{CV}(\widehat{\theta}_2)^2\right]}_{\text{first-order approximation}} \\ \underbrace{+\widehat{CV}(\widehat{\theta}_2)^2\left[\widehat{CV}(\widehat{\theta}_1)^2 - 4\widehat{\rho}\widehat{CV}(\widehat{\theta}_1)\widehat{CV}(\widehat{\theta}_2) + \widehat{\rho}^2\widehat{CV}(\widehat{\theta}_1)^2 + 2\widehat{CV}(\widehat{\theta}_2)^2\right]}_{\text{additional part from second-order approximation}}\end{array}\right\}$$

where $\widehat{\rho}$ is the estimate of the correlation coefficient between θ_1 and θ_2. (Full derivation details can be see in Appendix.)

This variance is identical to the variance of the almost unbiased ratio estimator (or the variance of the modified ratio estimator) of Tin (1965), see also David and Shutkame (1975).

Proposition 8 *If $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are independent, we have*

(1) The variance of the bias-corrected estimator for θ

$$\begin{aligned}
\widehat{V}\left[(\widehat{\theta}_{BC})\right] &= \underbrace{\frac{1}{\widehat{\theta}_2^2}\left[\widehat{V}(\widehat{\theta}_1)+\left(\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\right)\widehat{V}(\widehat{\theta}_2)\right]}_{\text{first-order approximation}} + \underbrace{\frac{1}{\widehat{\theta}_2^4}\widehat{V}(\widehat{\theta}_2)\left[\widehat{V}(\widehat{\theta}_1)+2\left(\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\right)\widehat{V}(\widehat{\theta}_2)\right]}_{\text{additional part from second-order approximation}} \\
&= \frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left\{\underbrace{\left[\frac{\widehat{V}(\widehat{\theta}_1)}{\widehat{\theta}_1^2}+\frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\right]}_{\text{first-order approximation}} + \underbrace{\frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\left[\frac{\widehat{V}(\widehat{\theta}_1)}{\widehat{\theta}_1^2}+2\frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\right]}_{\text{additional part from second-order approximation}}\right\}
\end{aligned}$$

(2) The variance $\widehat{V}\left[(\widehat{\theta}_{BC})\right]$ can be express in terms of coefficient variation of $\widehat{\theta}_1$ and $\widehat{\theta}_2$

$$\widehat{V}\left[(\widehat{\theta}_{BC})\right]^* = \frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left\{\underbrace{\left[\widehat{CV}(\widehat{\theta}_1)^2+\widehat{CV}(\widehat{\theta}_2)^2\right]}_{\text{first-order approximation}} + \underbrace{\widehat{CV}(\widehat{\theta}_2)^2\left[\widehat{CV}(\widehat{\theta}_1)^2+2\widehat{CV}(\widehat{\theta}_2)^2\right]}_{\text{additional part from second-order approximation}}\right\}$$

5 Confidence intervals with bias-corrected estimator

In this section, we would construct new confidence intervals that take into account the bias of the estimator for the Delta method, and both the bias of the estimator and the asymmetry of the distribution for the Edgeworth expansion method.

5.1 Delta method based confidence interval with bias-corrected estimator

Let $\widehat{V}\left[(\widehat{\theta}_{BC})\right]$ be a consistent estimator of $V\left[(\widehat{\theta}_{BC})\right]$, the variance of the bias-corrected estimator for θ then the standard error of $\widehat{\theta}_{BC}$ is

$$Q_{BC} = \sqrt{\widehat{V}\left[(\widehat{\theta}_{BC})\right]}$$

or in terms of coeffficient of variation and coeffficient of co-variation

$$Q_{BC}^* = \sqrt{\widehat{V}\left[(\widehat{\theta}_{BC})\right]^*}$$

And the bias-corrrected estimator is

$$\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC} = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left\{1+\left[\frac{\widehat{Cov}(\widehat{\theta}_1,\widehat{\theta}_2)}{\widehat{\theta}_1\widehat{\theta}_2} - \frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\right]\right\}$$

or in terms of coeffficient of variation and coeffficient of co-variation

$$\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)^*_{BC} = \frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left\{1+\left[\widehat{CV}(\widehat{\theta}_1,\widehat{\theta}_2) - \widehat{CV}(\widehat{\theta}_2)^2\right]\right\}$$

where $\widehat{CV}(\widehat{\theta}_1,\widehat{\theta}_2) = \widehat{\rho}\widehat{CV}(\widehat{\theta}_1)\widehat{CV}(\widehat{\theta}_2)$.

The $100(1-\alpha)\%$ confidence limits of the Delta method bias-corrrected for the ratio θ_1/θ_2 is given by:

$$CI_{Dbc} : \left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC} \pm z_{\alpha/2} Q_{BC}$$

where $z_{\alpha/2}$ is the $(\alpha/2)$ th quantile for standard normal distribution..
Or in terms of coeffficient of variation and coefficient of co-variation

$$CI_{Dbc} : \left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC} \pm z_{\alpha/2} Q^*_{BC}$$

5.2 Edgeworth expansion based confidence interval with bias-corrected estimator

For the Edgeworth expansion based confidence interval, we use the same correct term for the estimator of the ratio parameters, then the $100(1-\alpha)\%$ confidence interval for the ratio $\frac{\theta_1}{\theta_2}$ based Edgeworth expansion becomes

$$CI_{Ebc} : \left[\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC} - \xi_{1-\alpha/2} Q_{BC},\quad \left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC} - \xi_{\alpha/2} Q_{BC}\right]$$

where $\xi_{\alpha/2}$ and $\xi_{1-\alpha/2}$ are the $(\alpha/2)th$ and $(1-\alpha/2)th$ quantiles of distribution.
Or in terms of coeffficient of variation and coefficient of co-variation

$$CI_{Ebc} : \left[\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)^*_{BC} - \xi_{1-\alpha/2} Q^*_{BC},\quad \left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)^*_{BC} - \xi_{\alpha/2} Q^*_{BC}\right]$$

6 Some econometric applications

6.1 The ratio of two linear combinations of parameters

Many of the nonlinear functions studied in economic applications are expressed in the functional form of a ratio of two linear combinations of parameters. In this section, we consider the test of one such nonlinear function.

We will specify the bias of the estimator, the bias-corrected estimator, and its variance. Note that the formulations of the confidence intervals are given in the previous section. We will see that the calculations are quite simple and do not require intensive computation.

Consider the general linear model

$$Y = X\beta + \varepsilon$$

where Y is an $n \times 1$ vector of observations, X is a $n \times k$ full-rank design matrix, β is a $k \times 1$ vector of unknown parameters, and ε is an $n \times 1$ vector of normal random errors with zero mean and variance $\sigma^2 I : \varepsilon \sim N(0, \sigma^2 I)$. The OLS estimators of unknown parameters are $\widehat{\beta} = (X'X)^{-1}X'Y$ and $\widehat{\sigma}^2 = \widehat{\varepsilon}'\widehat{\varepsilon}/n - k$ where $\widehat{\varepsilon}$ are the OLS residuals

Consider a null hypothesis for the ratio of two linear combinations of parameters

$H_0 : \quad \theta = \frac{K'\beta}{L'\beta}$

where K and L are $k \times 1$ vectors of known constants.

We have the following different terms:

$\theta_1 = K'\beta,\ \theta_1^2 = (K'\beta)^2, \qquad \widehat{V}(\widehat{\theta}_1) = K'\widehat{V}(\widehat{\beta})K = \widehat{\sigma}^2 K'(X'X)^{-1}K$

$\theta_2 = L'\beta,\ \theta_2^2 = (L'\beta)^2\ ,\ \theta_2^3 = (L'\beta)^3,\ \widehat{V}(\widehat{\theta}_2) = L'\widehat{V}(\widehat{\beta})L = \widehat{\sigma}^2 L'(X'X)^{-1}L$

$\theta_1\theta_2 = (K'\beta)(L'\beta),\ \theta_1^2\theta_2^2 = (K'\beta)^2(L'\beta)^2,\ \widehat{Cov}(\widehat{\theta}_1, \widehat{\theta}_2) = \widehat{Cov}(K'\widehat{\beta}, L'\widehat{\beta}) = \widehat{\sigma}^2 K'(X'X)^{-1}L$

By replacing all these terms in the formulation of the bias for $\widehat{\theta}$, the bias-corrected estimator $\widehat{\theta}_{bc}$, and the variance of the bias-corrected estimator $\widehat{V}(\widehat{\theta}_{bc})$, we have the following proposition

Proposition 9 *(i) The bias for $\widehat{\theta}$ is*

$$\widehat{Bias}(\widehat{\theta}) = -\frac{1}{(L'\beta)^2}\widehat{\sigma}^2 K'(X'X)^{-1}L + \frac{K'\widehat{\beta}}{(L'\widehat{\beta})^3}\widehat{\sigma}^2 L'(X'X)^{-1}L$$

which can also be written by

$$\widehat{Bias}(\widehat{\theta}) = \frac{K'\widehat{\beta}}{L'\widehat{\beta}}\left[\frac{\widehat{\sigma}^2 L'(X'X)^{-1}L}{(L'\widehat{\beta})^2} - \frac{\widehat{\sigma}^2 K'(X'X)^{-1}L}{(K'\widehat{\beta})(L'\widehat{\beta})}\right]$$

where $\left[\frac{\widehat{\sigma}^2 L'(X'X)^{-1}L}{(L'\widehat{\beta})^2} - \frac{\widehat{\sigma}^2 K'(X'X)^{-1}L}{(K'\widehat{\beta})(L'\widehat{\beta})}\right]$ *can be considered as a corrrection factor to the estimated ratio estimator..*

(ii) The bias-corrected estimator for θ *is given by*

$$\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC} = \frac{K'\widehat{\beta}}{L'\widehat{\beta}} + -\frac{1}{(L'\beta)^2}\widehat{\sigma}^2 K'(X'X)^{-1}L - \frac{K'\widehat{\beta}}{(L'\widehat{\beta})^3}\widehat{\sigma}^2 L'(X'X)^{-1}L$$

which can be written by

$$\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC} = \frac{K'\widehat{\beta}}{L'\widehat{\beta}}\left\{1 + \widehat{\sigma}^2\left[\frac{K'(X'X)^{-1}L}{(K'\widehat{\beta})(L'\widehat{\beta})} - \frac{L'(X'X)^{-1}L}{(L'\widehat{\beta})^2}\right]\right\}$$

where $1 + \widehat{\sigma}^2\left[\frac{K'(X'X)^{-1}L}{(K'\widehat{\beta})(L'\widehat{\beta})} - \frac{L'(X'X)^{-1}L}{(L'\widehat{\beta})^2}\right]$ *can be considered as a correction factor for the estimated ratio estimator.*

(iii) The variance of the bias-corrected estimator for θ

$$\widehat{V}\left[(\widehat{\theta}_{BC})\right] = \frac{(K'\widehat{\beta})^2}{(L'\widehat{\beta})^2}(A_1 + A_2)$$

where A_1 is the asymptotic (first-order) variance of estimator

$$A_1 = \widehat{\sigma}^2\left[\frac{K'(X'X)^{-1}K}{(K'\widehat{\beta})^2} - 2\frac{K'(X'X)^{-1}L}{(K'\beta)(L'\beta)} + \frac{L'(X'X)^{-1}L}{(L'\widehat{\beta})^2}\right]$$

and A_2 is the additional part from second-order approximation

$$A_2 = \frac{\widehat{\sigma}^2 L'(X'X)^{-1}L}{(L'\widehat{\beta})^2}\widehat{\sigma}^4\left[\frac{K'(X'X)^{-1}K}{(K'\widehat{\beta})^2} - 4\frac{K'(X'X)^{-1}L}{(K'\beta)(L'\beta)} + 2\frac{L'(X'X)^{-1}L}{(L'\widehat{\beta})^2}\right] + \frac{\widehat{\sigma}^4 (K'(X'X)^{-1}L)^2}{(K'\widehat{\beta})^2(L'\widehat{\beta})^2}$$

Next, we consider the case where the numerator and the denominator of the ratio are independent.

Proposition 10 *(i) If* $\widehat{\theta}_1$ *and* $\widehat{\theta}_2$ *are independent, then the bias for* $\widehat{\theta}$ *become*

$$\widehat{Bias}(\widehat{\theta}) = \frac{K'\widehat{\beta}}{(L'\widehat{\beta})^3}\widehat{\sigma}^2 L'(X'X)^{-1}L$$

which can be written by

$$\widehat{Bias}(\widehat{\theta}) = \frac{K'\widehat{\beta}}{L'\widehat{\beta}}\left[\frac{\widehat{\sigma}^2 L'(X'X)^{-1}L}{(L'\widehat{\beta})^2}\right]$$

where $\left[\frac{\widehat{\sigma}^2 L'(X'X)^{-1}L}{(L'\widehat{\beta})^2}\right]$ *can be considered as a correction factor for the estimated ratio estimator.*

(ii) The bias-corrected estimator for θ *is given by*

$$\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC} = \frac{K'\widehat{\beta}}{L'\widehat{\beta}} - \frac{K'\widehat{\beta}}{(L'\widehat{\beta})^3}\widehat{\sigma}^2 L'(X'X)^{-1}L$$

which can be written by

$$\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right)_{BC} = \frac{K'\widehat{\beta}}{L'\widehat{\beta}}\left[1 - \frac{\widehat{\sigma}^2 L'(X'X)^{-1}L}{(L'\widehat{\beta})^2}\right]$$

where $1 - \frac{\widehat{\sigma}^2 L'(X'X)^{-1}L}{(L'\widehat{\beta})^2}$ *can be considered as a correction factor for the estimated ratio estimator.*

(iii) The variance of the bias-corrected estimator for θ

$$\widehat{V}\left[(\widehat{\theta}_{BC})\right] = \frac{(K'\widehat{\beta})^2}{(L'\widehat{\beta})^2}\widehat{\sigma}^2 \left\{ \begin{array}{c} \underbrace{\left[\frac{K'(X'X)^{-1}K}{(K'\widehat{\beta})^2} + \frac{L'(X'X)^{-1}L}{(L'\widehat{\beta})^2}\right]}_{\textit{first-order approximation}} \\ \underbrace{+\frac{L'(X'X)^{-1}L}{(L'\widehat{\beta})^2}\widehat{\sigma}^2\left[\frac{K'(X'X)^{-1}K}{(K'\widehat{\beta})^2} + 2\frac{L'(X'X)^{-1}L}{(L'\widehat{\beta})^2}\right]}_{\textit{additional part from second-order approximation}} \end{array} \right\}$$

We will illustrate this result with an econometric application to show the simplicity of calculation for our method. Let's take the case of the turning point, which has been the subject of numerous economic applications.

6.2 The turning point.

Consider a classical linear model described by the quadratic regression model

$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + \varepsilon$$

where y is the dependent variable and x the independent variable and ε is an unobserved random error term with $E(\varepsilon) = 0$ and $V(\varepsilon) = \sigma^2$. A common example of such model is the Kuznets (1955) curve that proposes the relationship between income inequality and income, can be represented by an inverted U shaped curve. The turning point is given by

$$\theta = \frac{\theta_1}{\theta_2} = -\frac{\beta_1}{2\beta_2}$$

In this case $K = (0, -1, 0)'$ and $L = (0, 0, 2)'$
$\theta_1 = -\beta_1,\qquad \theta_1^2 = \beta_1^2,\qquad\qquad V(\theta_1) = V(\beta_1) = \sigma^2_{\beta_1}$
$\theta_2 = 2\beta_2,\qquad \widehat{\theta}_2^2 = 4\beta_2^2\ ,\qquad \widehat{\theta}_2^3 = 8\beta_2^3\ ,\qquad V(\theta_2) = 4V(\beta_2) = 4\sigma^2_{\beta_2}$
$\theta_1\theta_2 = -2\beta_1\beta_2,\qquad \theta_1^2\theta_2^2 = 4\beta_1^2\beta_2^2\ ,\qquad \widehat{Cov}(\widehat{\theta}_1, \widehat{\theta}_2) = 2\widehat{Cov}(\widehat{\beta}_1, \widehat{\beta}_2) = -2\widehat{\sigma}_{\widehat{\beta}_1\widehat{\beta}_2}$

In the formulation of the bias for $\widehat{\theta}$, the bias-corrected estimator for θ and its variance, by replacing all these terms, we have the following proposition:

Proposition 11 *(i) The bias for $\widehat{\theta}$ is*

$$\widehat{Bias}(\widehat{\theta}) = \frac{1}{2}\left[\frac{1}{\widehat{\beta}_2^2}\widehat{\sigma}_{\widehat{\beta}_1\widehat{\beta}_2} - \frac{\widehat{\beta}_1}{\widehat{\beta}_2^3}\widehat{\sigma}^2_{\widehat{\beta}_2}\right]$$

which can be written as

$$\widehat{Bias}(\widehat{\theta}) = -\frac{1}{2}\frac{\widehat{\beta}_1}{\widehat{\beta}_2}\left(\frac{\widehat{\sigma}^2_{\widehat{\beta}_2}}{\widehat{\beta}_2^2} - \frac{\widehat{\sigma}_{\widehat{\beta}_1\widehat{\beta}_2}}{\widehat{\beta}_1\widehat{\beta}_2}\right)$$

(ii) The bias can be express in terms of the coefficients of variation and the coefficient of co-variation of $\widehat{\beta}_1$ and $\widehat{\beta}_2$

$$\begin{aligned}\widehat{Bias}(\widehat{\theta})^* &= -\frac{1}{2}\frac{\widehat{\beta}_1}{\widehat{\beta}_2}\left[\widehat{CV}((\widehat{\beta}_2)^2 - \widehat{\rho}\widehat{CV}(\widehat{\beta}_1)\widehat{CV}(\widehat{\beta}_2)\right] \\ &= -\frac{1}{2}\frac{\widehat{\beta}_1}{\widehat{\beta}_2}\left[\left(\frac{1}{t(\widehat{\beta}_2)}\right)^2 - \widehat{\rho}\left(\frac{1}{t(\widehat{\beta}_1)}\right)\left(\frac{1}{t(\widehat{\beta}_2)}\right)\right]\end{aligned}$$

where $t(\widehat{\beta}_i)$ denotes the $t-statistic$ for $\widehat{\beta}_i$ for $i = 1, 2,$
An another alternative form of the bias is

$$\widehat{Bias}(\widehat{\theta})^* = -\frac{1}{2}\frac{\widehat{\beta}_1}{\widehat{\beta}_2}\left(\frac{1}{t(\widehat{\beta}_1)}\right)\left(\frac{1}{t(\widehat{\beta}_2)}\right)\left[\frac{t(\widehat{\beta}_1)}{t(\widehat{\beta}_2)} - \widehat{\rho}\right]$$

(iii) The bias-corrected estimator for θ

$$\widehat{\theta}_{BC} = -\frac{1}{2}\frac{\widehat{\beta}_1}{\widehat{\beta}_2} - \frac{1}{2}\left[\frac{1}{\widehat{\beta}_2^2}\widehat{\sigma}_{\widehat{\beta}_1\widehat{\beta}_2} - \frac{\widehat{\beta}_1}{\widehat{\beta}_2^3}\widehat{\sigma}^2_{\widehat{\beta}_2}\right]$$

which can be written as

$$\widehat{\theta}_{BC} = -\frac{1}{2}\frac{\widehat{\beta}_1}{\widehat{\beta}_2}\left[1 + \left(\frac{\widehat{\sigma}_{\widehat{\beta}_1\widehat{\beta}_2}}{\widehat{\beta}_1\widehat{\beta}_2} - \frac{\widehat{\sigma}^2_{\widehat{\beta}_2}}{\widehat{\beta}_2^2}\right)\right]$$

(iv) The bias-corrected estimator of $\widehat{\theta}$ in terms of the coefficient of variation and the coefficient of co-variation of $\widehat{\beta}_1$ and $\widehat{\beta}_2$ is

$$\begin{aligned}
\widehat{\theta}_{BC} &= -\frac{1}{2}\frac{\widehat{\beta}_1}{\widehat{\beta}_2}\left[1+\left(\widehat{\rho}\widehat{CV}(\widehat{\beta}_1)\widehat{CV}(\widehat{\beta}_2)-\widehat{CV}((\widehat{\theta}_2)^2)\right)\right] \\
&= -\frac{1}{2}\frac{\widehat{\beta}_1}{\widehat{\beta}_2}\left[1+\left(\widehat{\rho}\left(\frac{1}{t(\widehat{\beta}_1)}\right)\left(\frac{1}{t(\widehat{\beta}_2)}\right)-\frac{1}{t(\widehat{\beta}_2)^2}\right)\right]
\end{aligned}$$

An another alternative form is

$$\widehat{\theta}_{BC} = -\frac{1}{2}\frac{\widehat{\beta}_1}{\widehat{\beta}_2}\left[1+\left(\frac{1}{t(\widehat{\beta}_1)}\right)\left(\frac{1}{t(\widehat{\beta}_2)}\right)\left(\widehat{\rho}-\frac{t(\widehat{\beta}_1)}{t(\widehat{\beta}_2)}\right)\right]$$

Proposition 12 *(v)The variance of the bias-corrected estimator for* θ

$$\widehat{V}\left[(\widehat{\theta}_{BC})\right] = \frac{1}{4}\frac{\widehat{\beta}_1^2}{\widehat{\beta}_2^2}\left\{\underbrace{\left[\frac{\widehat{\sigma}^2_{\widehat{\beta}_1}}{\widehat{\beta}_1^2}-2\frac{\widehat{\sigma}_{\widehat{\beta}_1\widehat{\beta}_2}}{\widehat{\beta}_1\widehat{\beta}_2}+\frac{\widehat{\sigma}^2_{\widehat{\beta}_2}}{\widehat{\beta}_2^2}\right]}_{\textit{first-order approximation}} + \underbrace{\frac{\widehat{\sigma}^2_{\widehat{\beta}_2}}{\widehat{\beta}_2^2}\left[\frac{\widehat{\sigma}^2_{\widehat{\beta}_1}}{\widehat{\beta}_1^2}-4\frac{\widehat{\sigma}_{\widehat{\beta}_1\widehat{\beta}_2}}{\widehat{\beta}_1\widehat{\beta}_2}+2\frac{\widehat{\sigma}^2_{\widehat{\beta}_2}}{\widehat{\beta}_2^2}\right]+\frac{(\widehat{\sigma}_{\widehat{\beta}_1\widehat{\beta}_2})^2}{\widehat{\beta}_1^2\widehat{\beta}_2^2}}_{\textit{additional part from second-order approximation}}\right\}$$

(vi)Thus the variance $V\left[(\widehat{\theta}_{BC})\right]$ *can be express in terms of coefficient variation of* $\widehat{\beta}_1$ *and* $\widehat{\beta}_2$ *by*

$$\widehat{V}\left[(\widehat{\theta}_{BC})\right]^* = \frac{1}{4}\frac{\widehat{\beta}_1^2}{\widehat{\beta}_2^2}\left\{\begin{array}{c}\underbrace{\left[\widehat{CV}(\widehat{\beta}_1)^2-2\widehat{\rho}\widehat{CV}(\widehat{\beta}_1)\widehat{CV}(\widehat{\beta}_2)+\widehat{CV}(\widehat{\beta}_2)^2\right]}_{\textit{first-order approximation}} \\ \underbrace{+\widehat{CV}(\widehat{\beta}_2)^2\left[\widehat{CV}(\widehat{\beta}_1)^2-4\widehat{\rho}\widehat{CV}(\widehat{\beta}_1)\widehat{CV}(\widehat{\beta}_2)+\widehat{\rho}^2\widehat{CV}(\widehat{\beta}_1)^2+2\widehat{CV}(\widehat{\beta}_2)^2\right]}_{\textit{additional part from second-order approximation}}\end{array}\right\}$$

This variance is easily calculated using $t-statistics$ for $\widehat{\beta}_i$ for $i=1,2$.

$$\widehat{V}\left[(\widehat{\theta}_{BC})\right]^* = \frac{1}{4}\frac{\widehat{\beta}_1^2}{\widehat{\beta}_2^2}\left\{\begin{array}{c}\underbrace{\left[\frac{1}{t(\widehat{\beta}_1)^2}-2\widehat{\rho}\left(\frac{1}{t(\widehat{\beta}_1)}\right)\left(\frac{1}{t(\widehat{\beta}_2)}\right)+\frac{1}{t(\widehat{\beta}_2)^2}\right]}_{\text{first-order approximation}} \\ \underbrace{+\frac{1}{t(\widehat{\beta}_2)^2}\left[\frac{1}{t(\widehat{\beta}_1)^2}-4\widehat{\rho}\left(\frac{1}{t(\widehat{\beta}_1)}\right)\left(\frac{1}{t(\widehat{\beta}_2)}\right)+\widehat{\rho}^2\frac{1}{t(\widehat{\beta}_1)^2}+2\frac{1}{t(\widehat{\beta}_2)^2}\right]}_{\text{additional part from second-order approximation}}\end{array}\right\}$$

7 Simulation study

7.1 Overview

In this section, we carry out a simulation study to assess the coverage probabilities of the methods presented in the previous section. We also examine, the average length of the confidence intervals. We evaluate the performance of the Fieller interval, the Delta method interval without and with bias correction and the Edgeworth interval without and with bias correction. Let $X_1, ..., X_n$ be i.i.d. observations from some distributions F with mean μ_X and variance σ_X^2 , $Y_1, ..., Y_n$ be i.i.d. observations from some distributions G with mean μ_Y and variance σ_Y^2 and $\rho\sigma_X\sigma_Y$ the covariance between $X_i's$ and $Y_j's$ where ρ is the correlation coefficient. Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ and their ratio $\widehat{\theta} = \frac{\overline{X}}{\overline{Y}}$ is a consistent estimator of $\theta = \frac{\mu_X}{\mu_Y}$

We generate data from three bivariate distributions: a bivariate normal distribution, and two positively skewed family of distributions. The two families that we consider are the bivariate lognormal distribution and the bivariate mixture ($X_i's$ are lognormal and $Y_j's$ are normal) distribution. We choose three correlation coefficients between X_i and Y_j (-0,8, 0,1, 0,8) and four sample sizes (25, 50, 100, 1000). We use 10 000 data sets. The data are generated as follows:

(a) Bivariate Normal Distribution

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim_{i.i.d} N_2\left(\begin{pmatrix} \mu_X = 7 \\ \mu_Y = 5 \end{pmatrix}, \begin{pmatrix} \sigma_X^2 = 2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 = 1 \end{pmatrix}\right)$$

(b) Bivariate Mixture Distribution

$(X_i) = e^{\widetilde{X}_i}$

$$\begin{pmatrix} \widetilde{X}_i \\ Y_i \end{pmatrix} \sim_{i.i.d} N_2\left(\begin{pmatrix} \mu_{\widetilde{X}} = 5 \\ \mu_Y = 4 \end{pmatrix}, \begin{pmatrix} \sigma_{\widetilde{X}}^2 = 0,2 & \rho\sigma_{\widetilde{X}}\sigma_Y \\ \rho\sigma_{\widetilde{X}}\sigma_Y & \sigma_Y^2 = 0,5 \end{pmatrix}\right)$$

(c) Bivariate Lognormal Distribution

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim_{i.i.d} \exp\left\{N_2\left(\begin{pmatrix} \mu_X = 5 \\ \mu_Y = 4 \end{pmatrix}, \begin{pmatrix} \sigma_X^2 = 0,2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 = 0,5 \end{pmatrix}\right)\right\}$$

7.2 Results

The results of our simulation are presented in Table 1. The values presented in the table are confidence intervals based on the Fieller method, the Delta method, the Delta method with the bias correction (denoted by Dbc), the Edgeworth method, and the Edgeworth method with the bias correction (denoted by Ebc). The values of the average width (denoted by Width) are the average lengths of the corresponding intervals. For data generated from normal distribution, all intervals give good performance. That is, all coverage probabilities are closer to the nominal level. Average interval lengths (Width) are also comparable for all methods. The Fieller and the Delta confidence intervals are in many cases very close to each other in terms of the coverage probabilities and we can also observe that the average interval lengths for Delta method with the bias correction (Dbc) are less wide than for the Delta method without the bias correction which means that the estimator is more accurate. We also observe that the average interval lengths for the Edgeworth method with the bias correction (Ebc) are narrower than for the Edgeworth method without the bias correction. However, for data generated from bivariate mixture and bivariate lognormal distributions, Delta methods confidence intervals are obviously inadequate, the coverage probabilities are lower than the nominal level. Fieller's intervals are also insufficient in terms of coverage probabilities. All the other methods give coverage probabilities lower than the nominal level. The Dbc intervals outperform Delta intervals. The Dbc intervals give better coverage probabilities than Delta intervals. They are comparable and sometimes better than the Fieller intervals. Note that the Delta interval has the longest average width whereas the Dbc interval has the shortest average width. The same applies to the Ebc compared to the Edgeworth expansion. We also observe that the Ebc interval performs much better than the Edgeworth interval. This can be explained by the fact that the estimated ratio is biased. Overall, the Edgeworth and the Edgeworth bias corrected appear to be best in terms of coverage probabilities and average width (width). To explore how the correlation coefficients affect the coverage probabilities we performed simulations for different values (-0.8, 0.1, 0.8) from Table 1. The simulation results showed that the correlation coefficients have an impact on the coverage probabilities. The sample sizes have a substantial impact on the coverage probabilities for almost all methods. Among all the methods, the Edgeworth bias-corrected (Ebc) method seems to give a narrower average than the others. The important conclusion from our simulation is that one should use the Edgeworth bias corrected, rather than the Edgeworth expansion. We also consider other sample sizes and other correlation structures. The results are similar and are not reported here.

In summary, the Edgeworth without and with the bias correction have good performance in terms of coverage probability and average width and should be recommended for constructing confidence intervals when data are from skewed distributions.

Table 1. *Coverage probability and average width (Width) of 95% confidence intervals.*

ρ	Fieller	Width	Delta	Width	Dbc	Width	Edgeworth	Width	Ebc	Width
(a) Bivariate Normal Distribution										
$n = 25$										
0.8	0.9505	1.9625	0.9491	1.9455	0.9495	1.9364	0.9531	2.0523	0.9532	1.1935
0.1	0.9463	2.0572	0.9452	2.0443	0.9458	2.0115	0.9512	2.0365	0.9511	2.0136
-0.8	0.9485	2.4198	0.9482	2.4041	0.9473	2.1464	0.9528	2.0523	0.9529	2.0310
$n = 50$										
0.8	0.9505	1.9625	0.9506	1.9485	0.9504	1.9275	0.9526	2.0497	0.9522	1.9210
0.1	0.9489	2.0577	0.9480	2.0443	0.9464	2.0324	0.9510	2.0342	0.9515	2.0387
-0.8	0.9476	2.4187	0.9469	2.4036	0.9478	2.1685	0.9521	2.0415	0.9519	2.0450
$n = 100$										
0.8	0.9504	1.9753	0.9501	1.9753	0.9503	1.9212	0.9524	2.0520	0.9521	1.9215
0.1	0.9477	2.0678	0.9489	2.0621	0.9463	2.0218	0.9503	2.0365	0.9506	2.0240
-0.8	0.9504	2.3953	0.9468	2.3975	0.9475	2.2358	0.9506	2.0522	0.9505	2.0486
$n = 1000$										
0.8	0.9501	1.9780	0.9501	1.9658	0.9500	1.9245	0.9520	2.0568	0.9520	1.9146
0.1	0.9476	2.0749	0.9469	2.0581	0.9460	2.0510	0.9504	2.0412	0.9500	2.0168
-0.8	0.9500	2.3763	0.9470	2.3860	0.9477	2.2045	0.9515	2.0495	0.9514	2.0475
(b) Bivariate Mixture Distribution										
$n = 25$										
0.8	0.8286	90.13	0.8214	86.48	0.8297	86.42	0.8674	85.86	0.8815	84.53
0.1	0.8713	107.30	0.8474	103.57	0.8512	103.45	0.8671	102.93	0.8705	102.49
-0.8	0.8970	139.34	0.8570	133.28	0.8980	132.51	0.9013	131.14	0.9051	130.57
$n = 50$										
0.8	0.8485	91.70	0.8329	88.26	0.8496	87.57	0.8816	87.51	0.8898	86.76
0.1	0.8707	107.18	0.8430	103.78	0.8514	102.12	0.8904	101.45	0.9009	101.14
-0.8	0.8945	138.37	0.8553	132.36	0.8598	132.17	0.9002	130.78	0.9121	129.41
$n = 100$										
0.8	0.8623	90.87	0.8610	89.24	0.8726	85.21	0.9002	86.45	0.9132	87.10
0.1	0.8798	106.87	0.8725	102.53	0.8798	101.21	0.9045	101.24	0.9187	102.25
-0.8	0.9015	137.21	0.9104	130.87	0.8805	130.54	0.9068	130.36	0.9208	129.21
$n = 1000$										
0.8	0.8674	91.10	0.8735	90.13	0.8765	84.25	0.9165	88.12	0.9218	84.59
0.1	0.8723	105.34	0.8806	102.14	0.8725	102.22	0.9046	101.14	0.9284	101.21
-0.8	0.9001	136.21	0.9312	131.51	0.9422	130.57	0.9185	130.03	0.9298	129.25
(c) Bivariate Lognormal Distribution										
$n = 25$										
0.8	0.8119	1.5936	0.8076	1.4075	0.8121	1.2761	0.8618	1.4063	0.8715	1.2326
0.1	0.9027	2.6677	08546	2.4743	0.9037	2.4106	0.8934	2.4530	0.8963	2.2078
-0.8	0.9232	3.4301	0.8688	3.1422	0.8721	3.1256	0.9066	3.1047	0.9158	3.0985
$n = 50$										
0.8	0.8472	2.8002	0.8351	1.4512	0.8486	1.4150	0.8845	1.4526	0.9005	1.4328
0.1	0.9055	2.6200	0.8610	2.4462	0.9065	2.3812	0.8981	2.4175	0.9002	2.4076
-0.8	0.9169	3.0737	0.8688	3.1422	0.8765	3.1027	0.9058	3.1107	0.9084	2.9615
$n = 100$										
0.8	0.8417	1.1407	0.8427	1.0821	0.8612	1.1835	0.8766	1.0665	0.9106	1.1078
0.1	0.9130	1.8217	0.8819	1.7727	0.9139	1.6941	0.9078	1.7516	0.9178	1.6851
-0.8	0.9244	2.2228	0.8869	2.1603	0.8981	2.2844	0.9134	2.1281	0.9223	2.0675
$n = 1000$										
0.8	0.8626	1.1691	0.8573	1.1126	0.8621	1.1076	0.8938	1.1010	0.9115	1.0981
0.1	0.9088	1.8157	0.8823	1.7375	0.9054	1.6975	0.9057	1.7434	0.9182	1.6896
-0.8	0.9248	2.2106	0.8965	2.2186	0.9045	2.2081	0.9146	2.1126	0.9268	2.0198

Note: Dbc: Delta method with the bias correction; Ebc: Edgeworth method with the bias correction; Width: average confidence interval lenghts; ρ : correlation coefficients.

8 Conclusion

We have developed new methods for constructing confidence intervals for the nonlinear functions of parameters. In many practical applications, the distribution of the data is not symmetric, in particular when the sample size is small. We propose that the Edgeworth expansion to the statistics makes it possible to remedy this inconvenience. Then the Delta method can be extended to obtain a better approximation using the Edgeworth expansion. Furthermore, we have shown that the nonlinear functions of the parameters are biased and we have given an analytical form of the bias of the ratio of the parameters. This has allowed us to define bias-corrected estimators and, more particularly, to calculate the variance associated with these bias-corrected estimators. We have therefore proposed two other new methods: the Delta method with bias correction and the Edgeworth expansion with bias correction.

The results of the simulation study showed that our methods generally have better coverage probabilities and confidence width and are narrower than the Delta method and Fieller's method. In the case of bivariate normality, the Delta with bias correction intervals gives better coverage probabilities than the Delta intervals. They are comparable and sometimes better than Fieller's intervals. When the data have been generated from a skewed distribution, the Edgeworth without and with the bias correction have good performance in terms of controlling the coverage probabilities and average length intervals. Therefore, in this situation, we recommend using the Edgeworth without and with bias correction to construct a reliable confidence interval for nonlinear functions of the estimated parameters.

9 APPENDIX

The Delta method is useful to approximate the moments of the nonlinear functions of parameters by using Taylor's series expansion. In the literature, only first-order expansions are used to approximate asymptotic sampling distributions. The Delta method provides a compromise to approximate the asymptotic sampling distribution of the ratio parameters $\theta = \theta_1/\theta_2$ where θ_1 and θ_2 are unknwon parameters. However, higher-order expansions are also useful because they can be used to estimate the bias of the ratio parameters and the analytical form of the bias obtained can be used to construct the bias-corrected estimator. We begin with how the variance of the ratio of the parameters in the main text can be approximated with the Delta method. We then extend this approach to obtain the higher-order terms necessary to estimate the bias and derive a bias-corrected estimator.

The variance of a first order Taylor's series expansion,

Let θ is $g(\theta_1, \theta_2) = \theta_1/\theta_2$. On the basis of Taylor's series expansion, the Delta method approximates the variance of a function of estimators of parameters $g(\widehat{\theta}_1, \widehat{\theta}_2)$ which estimates $g(\theta_1, \theta_2)$. Since $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are unbiased estimators of θ_1 and θ_2 respectively i.e $E(\widehat{\theta}_i) = \theta_i$ for $i = 1, 2$, the variance of $\widehat{\theta}$ is

$$V(\widehat{\theta}) = V(g(\widehat{\theta}_1, \widehat{\theta}_2) = G'\Sigma G \qquad \text{A1}$$

where G is a Jacobian vector containing all the first-order partial derivatives of $g(\widehat{\theta}_1, \widehat{\theta}_2)$ evaluated at θ_i for $i = 1, 2$.

$$G' = \left[\frac{\partial g(\widehat{\theta}_1, \widehat{\theta}_2)}{\partial \widehat{\theta}_1}, \frac{\partial g(\widehat{\theta}_1, \widehat{\theta}_2)}{\partial \widehat{\theta}_2}\right] = \left[\frac{1}{\theta_2}, \frac{-\theta_1}{\theta_2^2}\right]$$

and Σ is the variance-covariance matrix of $\widehat{\theta}_1$ and $\widehat{\theta}_2$ defined as follows

$$\Sigma = \begin{bmatrix} V(\widehat{\theta}_1) & Cov(\widehat{\theta}_1\widehat{\theta}_2) \\ Cov(\widehat{\theta}_2\widehat{\theta}_1) & V(\widehat{\theta}_2) \end{bmatrix}$$

Solving Eq.A.1 and using the estimators $\widehat{\theta}_1$ and $\widehat{\theta}_2$ to substitute for unknown parameters θ_1 and θ_2 respectively we get the variance of $\widehat{\theta}$

$$V(\widehat{\theta}) = \frac{1}{\widehat{\theta}_2^2}\left[V(\widehat{\theta}_1) - 2\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right) Cov(\widehat{\theta}_1, \widehat{\theta}_2) + \left(\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\right) V(\widehat{\theta}_2)\right]$$

which can be written by

$$V(\widehat{\theta}) = \frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left[\frac{V(\widehat{\theta}_1)}{\widehat{\theta}_1^2} - 2\frac{Cov(\widehat{\theta}_1, \widehat{\theta}_2)}{\widehat{\theta}_1\widehat{\theta}_2} + \frac{V(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\right]$$

Thus, the variance $V(\widehat{\theta})$ can be express in terms of the coefficient of variation and the coefficient of co-variation of $\widehat{\theta}_1$ and $\widehat{\theta}_2$

$$\begin{aligned} V(\widehat{\theta})^* &= \frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left[CV(\widehat{\theta}_1)^2 - 2CV(\widehat{\theta}_1, \widehat{\theta}_2) + CV(\widehat{\theta}_2)^2\right] \\ &= \frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left[CV(\widehat{\theta}_1)^2 - 2\rho CV(\widehat{\theta}_1)CV(\widehat{\theta}_2) + CV(\widehat{\theta}_2)^2\right] \end{aligned}$$

where ρ is the correlation coefficient between θ_1 and θ_2

Bias of estimator

The first-order Taylor's series approximations may not be accurate in some applications because of bias from truncation of the Taylor's series or small-sample bias in the asymptotic regression parameter variances used in the Taylor's series formulas. A second order Taylor's series expansios of $g(\widehat{\theta}_1, \widehat{\theta}_2)$ is

$$g(\widehat{\theta}_1, \widehat{\theta}_2) = g(\theta_1, \theta_2) + G' \begin{pmatrix} \widehat{\theta}_1 - \theta_1 \\ \widehat{\theta}_2 - \theta_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \widehat{\theta}_1 - \theta_1 \\ \widehat{\theta}_2 - \theta_2 \end{pmatrix}' H \begin{pmatrix} \widehat{\theta}_1 - \theta_1 \\ \widehat{\theta}_2 - \theta_2 \end{pmatrix} \qquad \text{A2}$$

where H is a Hessian matrix containing all the second partial derivatives of $g(\widehat{\theta}_1, \widehat{\theta}_2)$ evaluated at θ_i $i = 1, 2$.

$$H = \begin{bmatrix} \frac{\partial^2 g(\widehat{\theta}_1, \widehat{\theta}_2)}{\partial \widehat{\theta}_1^2}, \frac{\partial^2 g(\widehat{\theta}_1, \widehat{\theta}_2)}{\partial \widehat{\theta}_1 \partial \widehat{\theta}_2} \\ \frac{\partial^2 g(\widehat{\theta}_1, \widehat{\theta}_2)}{\partial \widehat{\theta}_2 \partial \widehat{\theta}_1}, \frac{\partial^2 g(\widehat{\theta}_1, \widehat{\theta}_1)}{\partial \widehat{\theta}_2^2} \end{bmatrix} = \begin{bmatrix} 0, -\frac{1}{\theta_2^2} \\ -\frac{1}{\theta_2^2}, \frac{2\theta_1}{\theta_2^3} \end{bmatrix}$$

By taking expectation of Eq. A2 and since $E(\widehat{\theta}_i - \theta_i) = 0$ for $i = 1, 2$, we obtain

$$\begin{aligned} E\left[g(\widehat{\theta}_1, \widehat{\theta}_2)\right] &= g(\theta_1, \theta_2) + \frac{1}{2} tr\{H\Sigma\} \\ E(\widehat{\theta}) &= \theta + \frac{1}{2} tr\{H\Sigma\} \end{aligned}$$

where $tr(.)$ denotes the trace of matrix, then the bias for $\widehat{\theta}$ is defined by

$$Bias(\widehat{\theta}) = E(\widehat{\theta}) - \theta = \frac{1}{2} tr\{H\Sigma\} = \frac{1}{2}(vecH)' vec\Sigma$$

where $vec(.)$ denotes the vectorisation operator which stacks the columns of the matrix and the matrix H is symmetric so that $vecH' = vecH$

Since H and Σ are unknown, we estimate bias as

$$\begin{aligned} \widehat{Bias}(\widehat{\theta}) &= \frac{1}{2} tr\left\{\widehat{H}\widehat{\Sigma}\right\} = \frac{1}{2}(vec\widehat{H})' vec\widehat{\Sigma} \\ \widehat{Bias}(\widehat{\theta}) &= -\frac{1}{\widehat{\theta}_2^2} \widehat{Cov}(\widehat{\theta}_1, \widehat{\theta}_2) + \frac{\widehat{\theta}_1}{\widehat{\theta}_2^3} \widehat{V}(\widehat{\theta}_2) \end{aligned}$$

which can be written as

$$\widehat{Bias}(\widehat{\theta})=\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left[\frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}-\frac{\widehat{Cov}(\widehat{\theta}_1,\widehat{\theta}_2)}{\widehat{\theta}_1\widehat{\theta}_2}\right]$$

The bias-corrected estimator

We have obtained an analytic form of the bias and the estimate bias of the ratio parameters can be used to correct the estimator, the bias-corrected estimator for θ is given by

$$\begin{aligned}\widehat{\theta}_{BC} &= \widehat{\theta}-\widehat{Bias}(\widehat{\theta})=\widehat{\theta}-\frac{1}{2}(vec\widehat{H})'vec\widehat{\Sigma}\\ \widehat{\theta}_{BC} &= \frac{\widehat{\theta}_1}{\widehat{\theta}_2}+\frac{1}{\widehat{\theta}_2^2}\widehat{Cov}(\widehat{\theta}_1,\widehat{\theta}_2)-\frac{\widehat{\theta}_1}{\widehat{\theta}_2^3}\widehat{V}(\widehat{\theta}_2)\end{aligned}$$

which can be written as

$$\widehat{\theta}_{BC}=\frac{\widehat{\theta}_1}{\widehat{\theta}_2}+\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\left[\frac{\widehat{Cov}(\widehat{\theta}_1,\widehat{\theta}_2)}{\widehat{\theta}_1\widehat{\theta}_2}-\frac{\widehat{V}(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\right]$$

The calculation of the variance of the second order Taylor series reveals the covariances between the random variables.and gives a better approximation.

The approximation of the variance of $\widehat{\theta}$ with a second-order term

To facilitate notation, let us define the random vector $z=\left(\widehat{\theta}_1-\theta_1\ \ ,\widehat{\theta}_2-\theta_2\right)'$ with $E(z)=0$, $E(zz')=\Sigma$ and z is a normal random variable $.z\sim N(0,\Sigma)$

We can rewrite the second order of Taylor's expansion as follows

$$g(\widehat{\theta}_1,\widehat{\theta}_2)=g(\theta_1,\theta_2)+G'z+\frac{1}{2}z'Hz$$

and its variance is

$$\begin{aligned}V(\ g(\widehat{\theta}_1,\widehat{\theta}_2)) &= V\left\{g(\theta_1,\theta_2)+G'z+\frac{1}{2}z'Hz\right\}\\ &= V(G'z)+\frac{1}{4}V\left(z'Hz\right)+Cov\left(G'z,z'Hz\right)\end{aligned}$$

To obtain the variance $V(\ g(\widehat{\theta}_1,\widehat{\theta}_2))$ we need to calculate the three terms

(i) $V(G'z)=G'\Sigma G$

(ii) $\frac{1}{4}V\left(z'Hz\right)=\frac{1}{4}\left\{E\left[z'Hz\right]^2\ -\ \left[E(z'Hz)\right]^2\right\}$

$=\frac{1}{4}\left\{\left[tr(H\Sigma)\right]^2+2tr(H\Sigma)^2-\left[tr(H\Sigma)\right]^2\right\}$

$=\frac{1}{2}tr(H\Sigma)^2$

(iii) $Cov\left(G'z,z'Hz\right)=G'E\left[zz'Hz\right]$

$=G'E\left[z\otimes zz'\right]'vecH$

$$= 0$$

since odd moments of z are zero. Thus the linear form $G'z$ and the quadratic form $z'Hz$ are uncorrelated.

By combining these three results, we obtain the following result

$$\begin{aligned} V\left[g(\widehat{\theta}_1, \widehat{\theta}_2)\right] &= \underbrace{G'\Sigma G}_{\text{first-order part}} + \underbrace{\frac{1}{2} tr\left[(H\Sigma)^2\right]}_{\text{second-order part}} \\ &= \underbrace{G'\Sigma G}_{\text{first-order part}} + \underbrace{\frac{1}{2}(vecH)'(\Sigma \otimes \Sigma) vecH}_{\text{second-oder part}} \end{aligned}$$

where $\otimes$ denotes the Kronecker product, the first order part $G'\Sigma G$ is the variance of $\widehat{\theta}$ corresponding to a first order approximation and the second order part permit to take into account the correlation between the random variables.

It which yields

$$\begin{aligned} V\left[g(\widehat{\theta}_1, \widehat{\theta}_2)\right] &= \underbrace{\frac{1}{\widehat{\theta}_2^2}\left[V(\widehat{\theta}_1) - 2\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right) Cov(\widehat{\theta}_1, \widehat{\theta}_2) + \left(\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\right) V(\widehat{\theta}_2)\right]}_{\text{first-order part}} \\ &\quad \underbrace{+\frac{1}{\widehat{\theta}_2^4} V(\widehat{\theta}_2)\left[V(\widehat{\theta}_1) - 4\left(\frac{\widehat{\theta}_1}{\widehat{\theta}_2}\right) Cov(\widehat{\theta}_1, \widehat{\theta}_2) + 2\left(\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\right) V(\widehat{\theta}_2)\right] + \frac{1}{\widehat{\theta}_2^4} Cov(\widehat{\theta}_1, \widehat{\theta}_2)^2}_{\text{second-oder part}} \end{aligned}$$

which can be written as

$$\begin{aligned} V\left[g(\widehat{\theta}_1, \widehat{\theta}_2)\right] &= \underbrace{\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left[\frac{V(\widehat{\theta}_1)}{\widehat{\theta}_1^2} - 2\frac{Cov(\widehat{\theta}_1, \widehat{\theta}_2)}{\widehat{\theta}_1\widehat{\theta}_2} + \frac{V(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\right]}_{\text{first-order part}} \\ &\quad \underbrace{+\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left\{\frac{V(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\left[\frac{V(\widehat{\theta}_1)}{\widehat{\theta}_1^2} - 4\frac{Cov(\widehat{\theta}_1, \widehat{\theta}_2)}{\widehat{\theta}_1\widehat{\theta}_2} + 2\frac{V(\widehat{\theta}_2)}{\widehat{\theta}_2^2}\right] + \frac{Cov(\widehat{\theta}_1, \widehat{\theta}_2)^2}{\widehat{\theta}_1^2\widehat{\theta}_2^2}\right\}}_{\text{second-order part}} \end{aligned}$$

Thus the variance $V\left[g(\widehat{\theta}_1, \widehat{\theta}_2)\right]$ can be express in terms of the coefficient of variation and the coefficient of co-variation of $\widehat{\theta}_1$ and $\widehat{\theta}_2$.

$$
\begin{aligned}
V\left[g(\widehat{\theta}_1,\widehat{\theta}_2)\right] &= \underbrace{\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left[CV(\widehat{\theta}_1)^2 - 2CV(\widehat{\theta}_1,\widehat{\theta}_2) + CV(\widehat{\theta}_2)^2\right]}_{\text{first-order part}} \\
&\quad \underbrace{+\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left\{CV(\widehat{\theta}_2)^2\left[CV(\widehat{\theta}_1)^2 - 4CV(\widehat{\theta}_1,\widehat{\theta}_2) + 2CV(\widehat{\theta}_2)^2\right] + CV(\widehat{\theta}_1,\widehat{\theta}_2)^2\right\}}_{\text{second-order part}} \\
&= \underbrace{\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left[CV(\widehat{\theta}_1)^2 - 2\rho CV(\widehat{\theta}_1)CV(\widehat{\theta}_2) + CV(\widehat{\theta}_2)^2\right]}_{\text{first-order part}} \\
&\quad \underbrace{+\frac{\widehat{\theta}_1^2}{\widehat{\theta}_2^2}\left\{CV(\widehat{\theta}_2)^2\left[CV(\widehat{\theta}_1)^2 - 4\rho CV(\widehat{\theta}_1)CV(\widehat{\theta}_2) + \rho^2 CV(\widehat{\theta}_1)^2 + 2CV(\widehat{\theta}_2)^2\right]\right\}}_{\text{second-order part}}
\end{aligned}
$$

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