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Abstract

We study the relationship between two cooperative games which arise from very different situations. On the one hand, the labeled network game which is defined to study how to allocate a certain flow in a network among agents that control different parts of the network. On the other hand, the museum pass game which is defined to analyze how to distribute the profit obtained from the use of passes which provide visitors unlimited access to the collaborating museums. We establish that both problems are related in the sense that a museum pass game can be written as a labeled network game and some labeled network games can be written as museum pass games.

Keywords: Game theory, labeled network games, museum pass games, Shapley value

1 Introduction

Many articles in the literature analyze different network models from a game theoretical point of view (see, for example, Jackson and Zenou, 2014). Some network games related to labeled network games are the so-called flow games. These games were first introduced by Kalai and Zemel (1982). In flow games the directed arcs of a network with a source node and a sink node are controlled by different agents and the worth of a coalition is the total flow that can be sent from the source to the sink throughout the arcs controlled by the members of the coalition. These games coincide with the class of non negative totally balanced games (Shapley and Shubik, 1969). Derks and Tijs (1985, 1986) extended flow games to the case of multi-commodity flow situations. Curiel et al. (1989) studied flow games with committee control and Reijnierse et al. (1996) analyzed simple flow games. All these networks games are cooperative games. In Guha et al. (2018) non cooperative multi-player flow games are studied.

In this note, we consider labeled networks as those used in Algaba et al. (2019b), in which there is one perfectly divisible unit of flow or traffic to allocate between different nodes of the network. Multiple arcs connecting two nodes are allowed, for this reason they are labeled in order to distinguish each other. The part of the unit of flow between two nodes can go throughout different routes and this is known due to different reasons, for example, the real flow is observed during a time window. In the network there are agents that control different sets of labels. Thus, the set of arcs of the network is partitioned among the agents. The worth

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obtained by a coalition of agents is the part of the unit of flow that they can obtain by using only their arcs. This network model is more general than the one introduced by Algaba et al. (2019a) to study the profit allocation problem in horizontal cooperation in public transport systems. Note also that the flow is fixed as exogenous.

The museum pass problem was originally posed by Ginsburgh and Zang (2003) and developed by Béal and Solal (2010), Casas-Méndez et al. (2011), Wang (2011) and Bergantiños and Moreno-Ternero (2015 and 2016), among others. The museum pass game models how to share the profit that a coalition of museums can obtain when they offer a limited time subscription or access pass allowing unlimited usage of their museums. Museum games are convex (Shapley, 1971) and their Shapley values (Shapley, 1953) have a simple expression.

The main goal is to prove that labeled network games and museum pass games are closely related. Therefore, labeled network games are convex and their Shapley values can be easily calculated.

The rest of the article is organized as follows. In Section 2, we recall some basic elements of cooperative game theory and introduce the labeled network game and the museum pass game. Section 3 is devoted to establish the relation between the labeled network game and the museum pass game.

2 Preliminaries

Cooperative games In this section, we recall some basic elements of cooperative game theory. A *cooperative game with transferable utility* or TU-game, is a pair (N, v), where N denotes the finite set of *players* and $v : 2^N \to \mathbb{R}$ the *characteristic function*, with $v(\varnothing) = 0$. A group of players $S \subseteq N$ is called a *coalition* and $v(S)$ is the *worth* obtained by a coalition, i.e., what the agents in S may obtain by themselves.

A cooperative game (N, v) is said to be convex if for all $S \subset T \subseteq N \setminus \{i\}$, it holds that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

The convexity property reflects a snowball effect, i.e., the larger is the coalition the higher is the contribution. Therefore, for this kind of games the collaboration is very profitable because the players can increase significantly the total profit.

Given a TU-game (N, v), an *allocation* or *payoff vector* is a vector $(x_i)_{i \in N} \in \mathbb{R}^n$ assigning to player $i \in N$ the amount x_i. An allocation $(x_i)_{i \in N} \in \mathbb{R}^n$ is *efficient* if $\sum_{i \in N} x_i = v(N)$, and it is *coalitionally rational* if $\sum_{i \in S} x_i = x(S) \geq v(S)$, for each $S \subseteq N$. The set of all efficient allocations that satisfy coalitional rationality is called the *core* (see Gillies, 1953 and 1959, Shapley, 1955 and Zhao, 2018) and is denoted by $core(N, v)$.

A *solution* for TU-games is a function ψ that assigns a payoff vector $\psi(N, v)$ to every TU-game. One of the most used solutions is the Shapley value (Shapley, 1953) that assigns to each player the average of all her marginal contributions with respect to all possible coalitions of players. Formally, for each $i \in N$,

$$\phi_i(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} \left(v(S \cup \{i\}) - v(S) \right),$$

where $s = |S|$ and $n = |N|$.

Labeled network allocation problems The *labeled network allocation problem* (see, Algaba et al., 2019b) is given by the 3-tuple $\texttt{LN} = (\mathcal{N}, N, \mathcal{L})$ where $\mathcal{N} = (\mathcal{G}, \mathcal{R}, f)$ is a network with $\mathcal{G} = (V, L, A)$ such that V is the set of nodes, L is the set of labels and $A \in N \times N \times L$ is a set of labeled arcs (consisting of an arc together with a label); $\mathcal{R}$ is a set of labeled routes[5] between nodes of V, and $f : \mathcal{R} \to [0, 1]$ is the distribution of one unit of flow among all labeled routes in $\mathcal{R}$ such that $\sum_{r\in\mathcal{R}} f(r) = 1$. The finite set N of cardinality n contains the agents controlling the different labels of the problem as indicated by the partition of the labels $\mathcal{L} = \{L_1, L_2, ..., L_n\}$. We simply assume that $L_i \neq \varnothing$ for each $i \in N$.

With each labeled network allocation problem, we may associate a *labeled network game* $(N, v^{\texttt{LN}})$ defined as follows

$$v^{\texttt{LN}}(S) = \sum_{r\in\mathcal{R}:L(r)\subseteq\bigcup_{i\in S} L_i} f(r), \quad \forall\, S \subseteq N, \tag{1}$$

where $L(r)$ is the set of different labels in labeled route r.

We conclude this paragraph with the following remarks. First, note the following with regard to the structure of the characteristic function. On the one hand, the characteristic function measures the part of the unit of flow using the feasible labeled routes provided exclusively by agents in coalition S or alternatively, the probability that the flow goes along a labeled route provided exclusively by agents in coalition S. On the other hand, $v^{LN}(N) = 1$ because we sum up all parts of the unit of flow or all probabilities of an exhaustive set of disjoint independent events.

Museum pass problems The *museum pass game* was introduced by Ginsburgh and Zang (2003). The museum pass problem is given by the 3-tuple $\texttt{M} = (N, M, K)$, where N is the set of museums participating in the pass program, M is the set of customers having bought a pass (i.e. the set of pass holders) and $K : M \to N$ specifies the set of museums $K(j) \subseteq N$ visited by customer $j \in M$.

With each museum pass problem, we may associate a *museum pass game* $(N, v^{\texttt{M}})$ defined as follows

$$v^{\texttt{M}}(S) = \sum_{j\in M:K(j)\subseteq S} 1, \quad S \subseteq N. \tag{2}$$

Note that $v^{\texttt{M}}(S)$ counts the number of pass holders who only visited some or all museums in coalition S. Assuming that the price of a pass is normalized to unity, this quantity is equivalent to the total revenue generated by a pass program allowing visits of museums in S only.

3 Relation between the labeled network game and the museum pass game

In this section, we study the relation between the labeled network game and the museum pass game. We establish certain relationships between both classes of games. These relationships allow us to conclude that labeled network games are convex and to obtain a simple expression for the Shapley value of these games.

[5] A *labeled route* connecting two nodes $i, j \in V$ in a labeled graph (V, L, A) is a sequence of labeled arcs $\{(i, i_1, l_1), (i_1, i_2, l_2), ..., (i_{k-1}, j, l_k)\} \subseteq A$.

To begin with, for each museum pass problem, we show that it is possible to construct an associated labeled network allocation problem with the same player set in such a way that the resulting museum pass game and labeled network game are proportional, i.e. they are multiplicatively connected to a constant.

Lemma 1. *Each museum pass problem is TU-proportional to a labeled network allocation problem: for each* $\mathtt{M}$*, there is* $\mathtt{LN(M)}$ *such that* $v^{\mathtt{LN(M)}} = cv^{\mathtt{M}}$ *for some constant* $c \in \mathbb{R}_{++}$*.*

Proof. Let $\mathtt{M} = (N, M, K)$ be any museum pass problem. Let us construct a labeled network allocation problem $\mathtt{LN(M)}$ as follows. As a start, the set of players N is the same in $\mathtt{M}$ and $\mathtt{LN(M)}$. Next, we consider the labeled network $(\mathcal{N}, \mathcal{R}, f)$ where $\mathcal{G} = (V, L, A)$ is such that

- $V = N \cup \{n+1\}$,
- $L = N$,
- $A = \{(i, j, i)) : i < j; i, j \in N\} \cup \{(n, n+1, n)\}$.

Let us also define $\mathcal{R} = \{\{(i_1, i_2, i_1)(i_2, i_3, i_2), \dots, (i_{p-1}, i_p, i_{p-1})(i_p, i_{p+1}, i_p)\}, i_1 < i_2 < \dots < i_p < n+1, p = 1, 2, \dots, n\}$ and f as:

$$f(r) = \frac{|\{j \in M : K(j) = L(r)\}|}{|M|}, \quad \forall r \in \mathcal{R}. \tag{3}$$

Furthermore, we take the partition of the labels as $\mathcal{L}$ with $L_i = \{i\}$ for all $i \in N$, i.e., this partition contains all singletons. Since there are $|M|$ customers, it is clear that $\sum_{r \in \mathcal{R}} f(r) = 1$ and $f(r) = 0$ if $r \notin \mathcal{R}$. This implies that $\mathtt{LN(M)}$ is a labeled network allocation problem.

It remains to prove that $v^{\mathtt{M}} = cv^{\mathtt{LN(M)}}$ for some constant $c \in \mathbb{R}_{++}$. So pick any coalition $S \subseteq N$. By (1), (3) and the fact that $L_i = \{i\}$ for each $i \in N$, we have

$$\begin{aligned} v^{\mathtt{LN(M)}}(S) &= \sum_{r \in \mathcal{R} : L(r) \subseteq \cup_{i \in S} L_i} f(r) \\ &= \sum_{r \in \mathcal{R} : L(r) \subseteq S} \frac{|\{j \in M : K(j) = L(r)\}|}{|M|} \\ &= \frac{1}{|M|} \sum_{j \in M : K(j) \subseteq S} |\{j \in M : K(j) = L(r)\}| \\ &= \frac{1}{|M|} v^{\mathtt{M}}(S), \end{aligned}$$

where the third equality comes from the condition $K(j) = L(r)$ in (3). Hence, the proof is complete by setting $c = 1/|M|$. □

Similarly, for a generic class of labeled network allocation problems, we show that it is possible to construct an associated museum pass problem with the same player set in such a way that the resulting labeled network game and museum pass game are proportional. We only need to impose the following minor restriction on a labeled network problem. A labeled network problem $(\mathcal{N}, N, \mathcal{L})$ with $\mathcal{N} = (\mathcal{G}, \mathcal{R}, f)$ is called *rational* if $f(r) \in \mathbb{Q}$ for each labeled route $r \in \mathcal{R}$.

Lemma 2. *Each rational labeled network problem is TU-proportional to a museum pass problem: for each* $\mathtt{LN}$*, there is* $\mathtt{M(LN)}$ *such that* $v^{\mathtt{M(LN)}} = cv^{\mathtt{LN}}$ *for some constant* $c \in \mathbb{R}_{++}$*.*

Proof. Let $\mathtt{LN} = (\mathcal{N}, N, \mathcal{L})$ with $\mathcal{N} = (\mathcal{G}, \mathcal{R}, f)$ and $\mathcal{G} = (V, L, A)$ be any rational labeled network problem. Since $f(r) \in \mathbb{Q}$ for each $r \in \mathcal{R}$, there is a function F such that $F(r) \in \mathbb{N}$ for each $r \in \mathcal{R}$ and

$$\frac{F(r')}{\sum_{r \in \mathcal{R}} F(r)} = f(r'), \quad \forall r' \in \mathcal{R}. \tag{4}$$

From $\mathtt{LN}$, we construct the museum pass problem $\mathtt{M(LN)} = (N, M, K)$ as follows. The set of players N is the same in $\mathtt{LN}$ and $\mathtt{M(LN)}$. The set of pass holders contains $\sum_{r \in \mathcal{R}} F(r)$ customers and can be partitioned into $|\mathcal{R}|$ types of customers, one for each feasible labeled route in $\mathcal{R}$. For each type of customers r, there are exactly $|F(r)|$ customers. Therefore, each customer j in M has an associated labeled route r_j and each labeled route r has $|F(r)|$ customers. Finally, for each customer $j \in M$ define

$$K(j) = \{i \in N : L_i \cap L(r_j) \neq \emptyset\}. \tag{5}$$

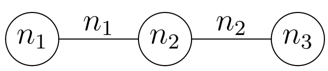
We shall prove that $v^{\mathtt{LN}} = cv^{\mathtt{M(LN)}}$ for some constant $c \in \mathbb{R}_{++}$. Choose any coalition $S \subseteq N$. By (2), the fact that there are $F(r)$ identical customers for each $r \in \mathcal{R}$, (5) and (4), we have

$$\begin{aligned} v^{\mathtt{M(LN)}}(S) &= \sum_{j \in M : K(j) \subseteq S} 1 \\ &= \sum_{r \in R : \{i \in N : L_i \cap L(r) \neq \emptyset\} \subseteq S} F(r) \\ &= \sum_{r \in R : L(r) \subseteq \cup_{i \in S} L_i} F(r) \\ &= \sum_{r' \in R} F(r') \sum_{r \in R : L(r) \subseteq \cup_{i \in S} L_i} f(r) \\ &= \sum_{r' \in R} F(r') v^{\mathtt{LN}}(S). \end{aligned}$$

Thus, setting $c = \sum_{r' \in R} F(r')$ completes the proof. □

Note that if we apply the procedure used in Lemma 1 to the museum pass problem defined in Lemma 2, we do not obtain, in general, the original graph $\mathcal{G} = (V, L, A)$ of the initial rational labeled network, but a labeled network problem with the same feasible labeled routes. However, if we apply the procedure used in Lemma 2 to the labeled network defined in Lemma 1, which is easy to check that it is rational, we obtain the original museum pass problem up to a constant for the number of visitors, as in the following example.

Example 1. *Consider the following museum pass problem:* $N = \{n_1, n_2\}$, $M = \{m_1, m_2, m_3\}$, $K(m_1) = \{n_1\}$, $K(m_2) = \{n_2\}$, $K(m_3) = \{n_1, n_2\}$. *The associated network is:*



The labeled routes are: $r_1 = \{(n_1, n_2, n_1)\}, r_2 = \{(n_2, n_3, n_2)\}, r_3 = \{(n_1, n_2, n_1)(n_2, n_3, n_2)\}$ *and* $f(r_1) = f(r_2) = f(r_3) = \frac{1}{3}$

Going from the network to the museum pass problem, we may define $F(r_1) = F(r_2) = F(r_3) = k$ *for some* $k \in \mathbb{N}$ *in order to obtain a museum pass problem with* $3k$ *customers, which coincides with the original one only if* $k = 1$.

The immediate consequence of Lemmas 1 and 2 is the following result.

Theorem 1. *The classes of rational labeled network games and museum pass games are the same except for the product by a positive constant.*

We conclude this article with several remarks.

First, from Theorem 1, we can conclude that rational labeled network games are convex. They are even totally positive (Vasil'ev, 1975), i.e. all Harsanyi dividends (Harsanyi, 1959) are non-negative. Obviously, not all totally positive games can be generated by rational labeled network problems. If the museum pass problem is augmented by the price $p \in \mathbb{R}_{++}$ of a pass, then the class of museum pass games coincides with the class of totally positive games.

Second, the Shapley value of a rational labeled network game $(N, v^{\mathtt{LN}})$ can be written for each $i \in N$ as follows:

$$\phi_i(N, v^{\mathtt{LN}}) = \sum_{r \in \mathcal{R}} \frac{f(r)}{|L(r)|} \delta_i(L(r)),$$

where $\delta_i(L(r)) = 1$, if $L_i \cap L(r) \neq \emptyset$, and $\delta_i(L(r)) = 0$, otherwise. More generally, the link between museum pass games and rational labeled network games established in Theorem 1 also extends to solutions for TU-games ψ that are homogeneous, that is $\psi(N, cv) = c\psi(N, v)$ for each TU-game (N, v) and any real number $c \in \mathbb{R}$. The Shapley value is only one among numerous homogeneous solutions for TU-games.

Third, the Shapley value of a rational labeled network game lies in its core (Shapley, 1971).

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