CENTRE DE RECHERCHE
SUR LES STRATÉGIES ÉCONOMIQUES

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January 2016

Working paper No. 2016–1

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Abstract

We provide characterizations of the equal division values and their convex mixtures, using a new axiom on a fixed player set based on player nullification which requires that if a player becomes null, then any two other players are equally affected. Meanwhile, we also present a global and useful framework which enables differential calculus within the space of values, allowing to *smoothly* implement other axioms.

Keywords: Player nullification, smoothness, equal division, equal surplus division.

JEL code: C71
AMS subject classification: 91A12

1. Introduction

Reconciling individual and social interests is a common theme in many economics fields. In that matter, cooperative game theory is a useful tool. This article deals with cooperative games with transferable utility (denoted as TU-games). A TU-game is defined by a characteristic function which associates to any subset of the player set the worth created by the cooperation of its members. For any TU-game, each player is given an individual payoff for participating to this TU-game. This defines a map, called a value, between the set of TU-games and the set of payoff vectors.

TU-games have been widely applied in allocation problems such as environmental economics (Ambec and Sprumont, 2002), operations research (Littlechild and Owen, 1973), queueing theory (Maniquet, 2003) and collusion in auctions (Graham et al., 1990). In the latter case, van den Brink (2007) shows that convex combinations of the equal division solution and equal surplus division solution emerge naturally as a mean for the player who obtains the indivisible good to compensate the other players. In this article we provide three new axioms leading to characterizations of this class of values for a fixed player set.

Firstly, we present a new axiom dealing with player's nullification. Following Béal et al. (2014), a player's nullification in a TU-game is a modification of the characteristic function so that the player loses her productivity, in the sense that she becomes null. Our axiom is called Nullified Equal Loss property. It requires that in case one player is nullified, all other players should deplore the same change in payoff. Thus, this axiom possesses a solidarity flavor.

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Proposition 3 proves that any linear value satisfying this axiom and the classical efficiency axiom only depends on the stand-alone worths and the worth of the grand coalition, setting aside the worths of all other coalitions. Two particular instances of such values, the equal division and the equal surplus division values, will then play a central role. In many of the aforementioned applications, the associated TU-game is fully determined by the worth of these particular coalitions (airport problems for instance, see Thomson, 2007, for a comprehensive literature review). However, only the whole class of TU-games will be considered here so that characterizations will hold within this class.

Secondly, proposition 3 can be obtained in a more general context by weakening linearity for smoothness (proposition 2). The latter axiom imposes that, when the worth of any coalition varies, a value should behave smoothly (i.e. the value is of class $\mathcal{C}^{\infty}$). This is made possible because the worths achieved by the coalitions are perfectly divisible. Smoothness is a technical requirement that implies continuity of the value with respect to the characteristic function. Many microeconomic models impose that the underlying utility functions are at least of class $\mathcal{C}^2$ (see for instance Hurwicz, 1971; Debreu, 1972). Our results are still valid if $\mathcal{C}^{\infty}$ is replaced by $\mathcal{C}^k$ for any fixed $k \geq 2$. Here smoothness will allow to translate other axioms' requirements in terms of differential calculus. More precisely, proposition 1 gives necessary conditions, expressed as differential equations, a value should meet to satisfy particular axioms useful throughout this article. Furthermore smoothness enables to simplify proofs.

Thirdly, in order to characterize the more relevant restricted class of convex mixtures of the equal division value and the equal surplus division value, we introduce the axiom of the pairwise upper bound in monotonic games. This axiom states that, for monotonic TU-games, the payoff difference between any two players is at most the worth achieved by the grand coalition. Adding this axiom and the desirability axiom (Maschler and Peleg, 1966) to efficiency, linearity and the nullified equal loss property yields the previously mentioned class of values. Desirability requires that a player with higher marginal contributions than another player should receive a higher payoff.

As a by-product, we obtain a 3-axiom characterization of the equal surplus division by combining efficiency, nullified equal loss property and inessential game property (proposition 4). The latter axiom specifies that each player obtains his stand-alone worth in an additive game.

This article follows a new literature concerning convex mixture of equal division values initiated by van den Brink (2007). To the best of our knowledge, the only two other articles are van den Brink and Funaki (2009) and Casajus and Hüttner (2012). In the first article, the authors invoke a consistency principle so that the player set can vary. Moreover, our pairwise upper bound in monotonic games axiom is a consequence of the dictator property (see proposition 6.2 in van den Brink and Funaki, 2009). It is used in combination with desirability in order to limit the range of linear mixtures to convex mixtures. In the second article, positivity for weakly cohesive games is used for the same purpose. The latter axiom is independent of pairwise upper bound in monotonic games axiom.

The outline of the article is as follows. Section 2 is devoted to definitions. Section 3 contains all results and their proof. Section 4 provides concluding remarks.

2. Basic definitions and notations

2.1. Cooperative games with transferable utility

The cardinality of any set S is denoted by s. Let N be a finite and fixed set of players such that $n \geq 3$. A **TU-game** v is a map $v : 2^N \longrightarrow \mathbb{R}$ such that $v(\emptyset) = 0$. Define $\mathbb{V}$ as the class of all TU-games on this fixed player set N. $\mathbb{V}$ is endowed with the natural vector space structure. A non-empty subset $S \subseteq N$ is a coalition, and $v(S)$ is the worth of the coalition. For simplicity, we write the singleton $\{i\}$ as i.

Player $i \in N$ is **null** in $v \in \mathbb{V}$ if $v(S) = v(S \backslash i)$ for all $S \subseteq N$ such that $S \ni i$. We denote by $K(v)$ the set of null players in v. For $v \in \mathbb{V}$ and $i \in N$, we denote by v^i the TU-game in which player i is **nullified**: $v^i(S) = v(S\backslash i)$ for all $S \subseteq N$. Remark that $K(v^i) = K(v) \cup i$. Two distinct players $i, j \in N$ are **equal** in $v \in \mathbb{V}$ if $v(S \cup i) = v(S \cup j)$ for all $S \subseteq N\backslash\{i, j\}$.

The **null game** is given by $\mathbf{0}(S) = 0$ for all $S \subseteq N$. A TU-game $v \in \mathbb{V}$ is **inessential** (or additive) if for all $S \subseteq N$, $v(S) = \sum_{i\in S} v(i)$. For any TU-game $v \in \mathbb{V}$, let define the **0-normalized** game v^0 by $v^0(S) = v(S) - \sum_{i\in S} v(i)$ for any $S \subseteq N$ so that any inessential TU-game v is characterized by $v^0 = \mathbf{0}$. A TU-game $v \in \mathbb{V}$ is **monotonic** if $v(S) \leq v(T)$ whenever $S \subseteq T$. For any nonempty $S \in 2^N$, the **unanimity** TU-game induced by S is denoted by u_S and such that $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise. It is well-known that any characteristic function $v : 2^N \longrightarrow \mathbb{R}$ admits a unique decomposition in the unanimity games basis:

$$v = \sum_{S\in 2^N, S\neq\emptyset} \Delta_S(v) u_S,$$

where $\Delta_S(v)$ is called the Harsanyi dividend of S.

2.2. Values

A **value** on $\mathbb{V}$ is a function φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to any $v \in \mathbb{V}$. We consider the following values.

The **Equal division value** is the value ED given by:

$$\mathrm{ED}_i(v) = \frac{v(N)}{n} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

The **Equal surplus division value** is the value ESD given by:

$$\mathrm{ESD}_i(v) = v(i) + \frac{1}{n}\left(v(N) - \sum_{j\in N} v(j)\right) \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

The **Shapley value** (Shapley, 1953) is the value Sh given by:

$$\mathrm{Sh}_i(v) = \sum_{S\ni i} \frac{\Delta_S(v)}{s} \qquad \text{for all } v \in \mathbb{V} \text{ and } i \in N.$$

2.3. Punctual and relational Axioms

In this article, we divide axioms in two categories: punctual axioms if they impose restrictions on the payoff vector of a fixed TU-game $v \in \mathbb{V}$, and relational axioms if they impose a particular relation between the payoff vectors of two different TU-games $v, w \in \mathbb{V}$. The following axioms are punctual:

Efficiency, E. For all $v \in \mathbb{V}$, $\sum_{i\in N} \varphi_i(v) = v(N)$.

Null game, NG. For all $i \in N$, $\varphi_i(\mathbf{0}) = 0$.

Null player, N. For all $v \in \mathbb{V}$, all $i \in K(v)$, $\varphi_i(v) = 0$.

Equal treatment of equals, ET. For all $v \in \mathbb{V}$, all $i, j \in N$ who are equal in v, $\varphi_i(v) = \varphi_j(v)$.

Desirability, D. (Maschler and Peleg, 1966) For all $v \in \mathbb{V}$, all $i, j \in N$ such that, for all $S \subseteq N \backslash \{i, j\}$, $v(S \cup i) \geq v(S \cup j)$, then $\varphi_i(v) \geq \varphi_j(v)$.

Inessential game property, IGP. For all inessential TU-games $v \in \mathbb{V}$, for all $i \in N$, $\varphi_i(v) = v(i)$.

Positivity, P. For all monotonic $v \in \mathbb{V}$, for all $i \in N$, $\varphi_i(v) \geq 0$.

Pairwise upper bound in monotonic games, PUBM. For all monotonic TU-games $v \in \mathbb{V}$, for all $i, j \in N$, $\varphi_i(v) - \varphi_j(v) \leq v(N)$.

This last axiom requires that, compared to any other player, no player can claim a payoff excess greater than $v(N)$. It can be considered as weak as it is implied by the combination of **E** and **P**. Thus it is satisfied by Sh and ED, and also by ESD, even if the latter does not satisfy **P**. Note that **D** implies **ET** and that **NG** is implied by either **N**, either **IGP**, or the combination of **ET** and **E**.

Below is a list of relational axioms containing two new axioms, Smoothness and Nullified equal loss property. The first one allows comparison between payoff vectors of arbitrarily close TU-games, though it does not specify a particular relation at this point. The second links an arbitrary TU-game v to the TU-game v^h in which a player h is nullified, by imposing that the payoff variation should affect all the other players equally, thus preserving payoff differences among them.

Additivity, A. For all $v, w \in \mathbb{V}$, $\varphi(v + w) = \varphi(v) + \varphi(w)$.

Linearity, L. φ is a linear map.

Smoothness, Sm. The map $\varphi : \mathbb{V} \longrightarrow \mathbb{R}^N$ is smooth as a map between $\mathbb{R}^{2^N \backslash \{\emptyset\}}$ and $\mathbb{R}^N$ provided with standard euclidean structure: $\varphi \in \mathcal{C}^\infty(\mathbb{R}^{2^N \backslash \{\emptyset\}}, \mathbb{R}^N)$.

Nullified equal loss property, NEL. For all $v \in \mathbb{V}$, all $h \in N$, all $i, j \in N \backslash h$,

$$\varphi_i(v) - \varphi_i(v^h) = \varphi_j(v) - \varphi_j(v^h).$$

Let us mention that **L** implies **Sm** and **L** implies **A** but both converse implications are false. At last, the combination of **Sm** and **A** implies **L**.

3. Axiomatic study of the Nullified equal loss property

3.1. Smoothness as framework for differential calculus on values

We present here an alternative approach to implement axioms by using differential calculus: imposing the **Sm** axiom as a global framework allows to work within the class of smooth values on $\mathbb{V}$ and to translate other axioms in terms of differential operators, so that a value satisfying the axioms is a solution of the associated (partial) differential equations. In this context, **Sm** can be considered as a technical requirement, similarly as the linearity axiom **L**. The main advantage in this case is that it allows to apply results from linear algebra. To be more precise, $\mathbb{V}$ is endowed with its canonical euclidean structure and elements of $\mathbb{V}$ can be expressed, for instance, as coordinates in the natural Dirac basis so that a smooth value φ is a function of the coordinates $(v(S))_{\emptyset \subsetneq S \subseteq N}$. As a consequence, the expression $\partial \varphi_i / \partial v(S)$ is well-defined. Here is a first example of **Sm** implementation. For the interested reader, some other examples of this "translation" process can be found in the concluding section.

Proposition 1. *Let φ be a value on $\mathbb{V}$ satisfying* **Sm**. *We have the following statements:*

1. *If φ satisfies* **E**, *then:*
$$\sum_{i\in N}\frac{\partial\varphi_i}{\partial v(S)}=\begin{cases}0 \text{ if } S\neq N,\\ 1 \text{ if } S=N.\end{cases}$$
2. *If φ satisfies* **NEL** *then for every $S\subseteq N$, either $S=h$ and for any $i,j\in N\backslash h$, $\partial\varphi_i/\partial v(h)=\partial\varphi_j/\partial v(h)$, or $s\geq 2$ and for any $i,j\in N$, $\partial\varphi_i/\partial v(S)=\partial\varphi_j/\partial v(S)$.*

Proof. Let φ be a smooth value on $\mathbb{V}$.

Suppose that φ satisfies **E**. Then $\sum_{i\in N}\varphi_i(v)=v(N)$ so that, by linearity of the derivative process, for $S\subsetneq N$, $\sum_{i\in N}\partial\varphi_i/\partial v(S)=0$ and $\sum_{i\in N}\partial\varphi_i/\partial v(N)=1$.

Suppose that φ satisfies **NEL**. That is, for every $h\in N$, $\varphi_i(v)-\varphi_i(v^h)$ does not depend on $i\neq h$. Now let $S\ni h$. Recall that $v^h(S)=v(S\backslash h)$ so that v^h does not depend on $v(S)$. As a consequence, the derivative of the composition * $\partial\varphi_i(v^h)/\partial v(S)=0$. Hence $\partial(\varphi_i(v)-\varphi_i(v^h))/\partial v(S)=\partial\varphi_i(v)/\partial v(S)$. By **NEL**:

$$\frac{\partial\varphi_i(v)}{\partial v(S)} \text{ does not depend on } i\neq h. \tag{1}$$

For $S=h$, this means that $\partial\varphi_i/\partial v(h)=\partial\varphi_j/\partial v(h)$ for any $i,j\neq h$ as claimed. For $s\geq 2$, consider the last property (1) for two different $h,h'\in S$. We have proved that $\partial\varphi_i(v)/\partial v(S)$ depends neither on $i\neq h$ nor on $i\neq h'$. As $n\geq 3$, the function $\partial\varphi_i/\partial v(S)$ does not depend on any $i\in N$. ■

A broad range of values satisfies the **Sm** axiom, in particular all linear values, even if some non-smooth values may be interesting (see for instance Béal et al., 2015b). Hence it may legitimately be considered as a very weak axiom. The next result relies on this differential approach and gives consequences of combining **NEL** with **E** in this context.

Proposition 2. *A value φ satisfies Smoothness (***Sm***), Efficiency (***E***) and Nullified equal loss property (***NEL***) if and only if it exists n functions $(\alpha_i)_{i\in N}\in\mathcal{C}^\infty(\mathbb{R})$ and n numbers $(a_i)_{i\in N}\in\mathbb{R}^N$ such that $\sum_{i\in N}a_i=0$ and:*

$$\varphi_i(v)=a_i+\frac{v(N)}{n}+\int_0^{v(i)}\alpha_i(x)\mathrm{d}x-\frac{1}{n-1}\sum_{j\in N\backslash i}\int_0^{v(j)}\alpha_j(x)\mathrm{d}x. \tag{2}$$

*Moreover, if Desirability axiom (***D***) and Pairwise upper bound for monotonic games axiom (***PUBM***) are also required, then φ is given by (2) with, for all $i\in N$, $a_i=0$ and $\alpha_i=\alpha$, α being a positive function such that for all $x\in\mathbb{R}^+$, $\int_0^x\alpha(t)\mathrm{d}t\leq(n-1)x/n$.*

This proposition, though needing a heavy but straightforward proof, provides an explicit formula for the aforementioned class of values. This formula will allow, in proposition 3, to simplify the proof and to easily find a non-additive value, proving the logical independence of the axioms.

*Throughout this article, the notation v^i is (ab)used for both the game v^i obtained from v after i's nullification and the application $v\to v^i$.

Proof. **[Part 1]** Assume that a value φ on $\mathbb{V}$ satisfies **Sm**, **E** and **NEL**. For $S \subseteq N$ such that $s \geq 2$, by the second part of proposition 1 and **NEL**, the function $\partial\varphi_i/\partial v(S)$ does not depend on $i \in N$. By the first part of proposition 1 and **E**, the sum $\sum_{i\in N} \partial\varphi_i/\partial v(S)$ is equal to 0 if $S \neq N$ and 1 if $S = N$. Hence $\partial\varphi_i/\partial v(S) = 0$ if $S \neq N$ and:

$$\frac{\partial\varphi_i}{\partial v(N)} = \frac{1}{n}. \tag{3}$$

We have proved that φ depends only on $v(S)$ for $s \in \{1, n\}$.

Now let us construct expression (2). Firstly, for all $i \in N$, set:

$$\alpha_i = \frac{\partial\varphi_i}{\partial v(i)}. \tag{4}$$

Pick any $j \in N$. By the second part of proposition 1 and **NEL**, we know that $\partial\varphi_i/\partial v(j)$ does not depend on $i \neq j$. By the first part of proposition 1, the sum $\sum_{i\in N} \partial\varphi_i/\partial v(j) = 0$ so that:

$$\frac{\partial\varphi_i}{\partial v(j)} = -\frac{\alpha_j}{n-1}. \tag{5}$$

Let us prove that α_i only depends on $v(i)$. The value φ being a smooth map, Schwarz lemma can be applied: $\partial^2\varphi_i/(\partial v(i)\partial v(N)) = \partial^2\varphi_i/(\partial v(N)\partial v(i))$ where $\partial\varphi_i/\partial v(N)$ is constant by (3). Hence $\partial\alpha_i/\partial v(N) = 0$ and α_i does not depend on $v(N)$. Similarly:

$$\frac{\partial\alpha_i}{\partial v(j)} \overset{\text{Def.}}{=} \frac{\partial^2\varphi_i}{(\partial v(j)\partial v(i))} \overset{\text{Schwarz}}{=} \frac{\partial^2\varphi_i}{(\partial v(i)\partial v(j))} \overset{eq.(5)}{=} \frac{1}{1-n}\cdot\frac{\partial\alpha_j}{\partial v(i)} \tag{6}$$

Interverting i and j in (6), one also gets $\partial\alpha_j/\partial v(i) = (\partial\alpha_i/\partial v(j))/(1-n)$, which finally gives $\partial\alpha_i/\partial v(j) = (1-n)^2 \cdot \partial\alpha_i/\partial v(j)$. Since $n \geq 3$, it follows that $\partial\alpha_i/\partial v(j) = 0$, so that α_i only depends on $v(i)$. Let us finally consider the quantity:

$$Q(v) = \varphi_i(v) - \frac{v(N)}{n} - \int_0^{v(i)} \alpha_i(x)\mathrm{d}x + \frac{1}{n-1}\left(\sum_{j\in N\setminus i}\int_0^{v(j)} \alpha_j(x)\mathrm{d}x\right). \tag{7}$$

With the help of the preceding arguments, it is now easy to show that $\partial Q/\partial v(S) = 0$ for any $S \subseteq N$ and hence is a constant for all $v \in \mathbb{V}$. Thus $Q(v) = Q(\mathbf{0})$, where $Q(\mathbf{0}) = \varphi_i(\mathbf{0})$. Denote by $a_i = \varphi_i(\mathbf{0})$ so that $\sum_{i\in N} a_i = 0$ by **E**. This implies the desired expression (2).

Reciprocally, for any $(\alpha_i)_{i\in N} \in \mathcal{C}^\infty(\mathbb{R})$ and $(a_i)_{i\in N} \in \mathbb{R}^N$, the value φ given by $\varphi_i(v) = a_i + v(N)/n + \int_0^{v(i)} \alpha_i(x)\mathrm{d}x - (\sum_{j\neq i}\int_0^{v(j)} \alpha_j(x)\mathrm{d}x)/(n-1)$ satisfies **Sm** and **E** whenever $\sum_{i\in N} a_i = 0$. For $i \neq h \in N$, $\varphi_i(v) - \varphi_i(v^h) = (v(N) - v(N\backslash h))/n - (\int_0^{v(h)} \alpha_h(x)\mathrm{d}x)/(n-1)$ which does not depend on i so that φ satisfies **NEL**.

[Part 2] Now suppose that φ also satisfies **D**. As **D** implies **ET**, for all $i \in N$, we get $a_i = \varphi_i(\mathbf{0}) = 0$ by **E**. Now, simple computation gives

$$\varphi_i(v) - \varphi_j(v) = \left(1 + \frac{1}{n-1}\right)\left(\int_0^{v(i)} \alpha_i(x)\mathrm{d}x - \int_0^{v(j)} \alpha_j(x)\mathrm{d}x\right) \tag{8}$$

If i and j are equal, $v(i) = v(j)$ and the right hand side of (8) simplifies to $(n/(n-1)) \int_0^{v(i)} (\alpha_i(x) - \alpha_j(x))\text{dx}$. By **ET**, we get $\int_0^{v(i)} (\alpha_i(x) - \alpha_j(x))\text{dx} = 0$ for every $v(i) \in \mathbb{R}$ so that $\alpha_i = \alpha_j$. In this context, denote the smooth function $\alpha = \alpha_i$ for any $i \in N$.

Consider $i, j \in N$ and v a TU-game in which for all $S \subseteq N \backslash \{i, j\}$, $v(S \cup j) \leq v(S \cup i)$. Expression (8) now leads to $\int_{v(j)}^{v(i)} \alpha(x)\text{dx} = (n-1) \cdot (\varphi_i(v) - \varphi_j(v))/n$. By **D**, $\varphi_i(v) - \varphi_j(v) \geq 0$ and we obtain $\int_{v(j)}^{v(i)} \alpha(x)\text{dx} \geq 0$ for every $v(j) \leq v(i)$. Hence α is positive on $\mathbb{R}$.

At last, suppose that φ also satisfies **PUBM**. That is, for any monotonic TU-game v, $\varphi_i(v) - \varphi_j(v) \leq v(N)$. Combined with (8), we get $\int_{v(j)}^{v(i)} \alpha(x)\text{dx} \leq (n-1)v(N)/n$ for every $v(N) \in \mathbb{R}^+$ and $v(i), v(j) \in [0; v(N)]$. In particular, for $v(i) = v(N)$ and $v(j) = 0$, this leads to $\int_0^{v(N)} \alpha(t)\text{dt} \leq (n-1)v(N)/n$ for every $v(N) \in \mathbb{R}^+$.

Reciprocally, given the explicit expression (2) for φ, with $a_i = 0$ and $\alpha_i = \alpha$ for all $i \in N$, α being a positive function one has $\varphi_i(v) - \varphi_j(v) = n(\int_{v(j)}^{v(i)} \alpha(x)\text{dx})/(n-1)$ and φ satisfies **D**. Moreover if for all $x \in \mathbb{R}^+$, $\int_0^x \alpha(t)\text{dt} \leq (n-1)x/n$, then for any monotone TU-game v, $0 \leq v(i), v(j) \leq v(N)$ and $\varphi_i(v) - \varphi_j(v) \leq n(\int_0^{v(N)} \alpha(x)\text{dx})/(n-1) \leq v(N)$.

■

The axioms in proposition 2 are logically independent:

- The value $\varphi = 2 \cdot \text{ESD} - \text{ED}$ satisfies all axioms except **PUBM**. For instance, take $i \neq h$, $\varphi_h(u_h) - \varphi_i(u_h) = 2 - 1/n - (-1/n) > u_h(N)$.
- The value φ given by (2) with $\alpha_i = (n-1)/n$ for some $i \in N$ and $\alpha_j = 0$ for all other $j \neq i$ satisfies all axioms except **D** (as it does not satisfies **ET**). Indeed, in a monotonic TU-game $v \in \mathbb{V}$, $0 \leq v(i) \leq v(N)$ and, for any $j, k \in N$, expression (8) simplifies to $\varphi_k(v) - \varphi_j(v) \leq v(i)$ so that φ satisfies **PUBM**.
- The value Sh satisfies all axioms except **NEL**.
- Given any $(\lambda_s)_{2 \leq s \leq (n-1)} \in \mathbb{R}^{n-2}$, the equal value φ defined for any player $i \in N$ by $\varphi_i(v) = \sum_{S:2 \leq s \leq (n-1)} \lambda_s v(S)$, satisfies all axioms except **E**. Note that it depends on the worth of all coalitions S except those of size 1 and n, contrary to values characterized by (2).
- The value φ defined by $\varphi_i(v) = \max(0, v(i)) + (v(N) - \sum_j \max(0, v(j)))/n$ for all $i \in N$ satisfies all axioms except **Sm** as $\alpha_i(x) = \partial\varphi_i/\partial v(i)(x)$ is either 0 if $x < 0$ or $(n-1)/n$ if $x > 0$ and therefore is not smooth.

Proposition 2 characterizes the class of smooth and efficient values, not necessarily linear, for which the entire loss of productivity of a player does not affect the payoff difference between the other players. These values only rely on the worth of the grand coalition and the stand-alone worth, and carry exogeneous parameters for each player i: an unconditional payoff $a_i = \varphi_i(\mathbf{0})$ and a marginal scaling of individual productivity α_i in the payoffs, corresponding to a nominative and heterogeneous treatment of the players. This class is however compatible with further axioms such as **D** or **PUBM**, which specify the shape of these parameters.

Remark 1. *Requiring* **PUBM** *in addition to* **Sm**, **E**, **NEL** *and* **D** *imposes, in proposition 2, that the positive function α should satisfy $\int_0^x \alpha(t)\text{dt} \leq (n-1)x/n$ for all $x \in \mathbb{R}^+$. This technical* *condition implies that $\alpha(0) \leq (n-1)/n$ and is implied by imposing α to be uniformely bounded by $(n-1)/n$ on $\mathbb{R}^+$. When* **L** *is required instead of* **Sm**, *these two last conditions coincide because α is constant and equal to $\alpha(0)$. This will be used in the following proposition 3.*

Remark 2. *There is no value within the aforementioned class that satifies the null player axiom* **N**. *More generally, it is easy to prove that there is no value on $\mathbb{V}$ that satisfies* **N**, **NEL** *and* **E**: *for $h \in N$, consider the unanimity game $u_{N\backslash h}$. By* **N**, *$\varphi_h(u_{N\backslash h}) = 0$ and for all $i \in N$, $\varphi_i(\mathbf{0}) = 0$. Moreover, $u^j_{N\backslash h} = \mathbf{0}$ for all $j \neq h$ so* **NEL** *leads to $\varphi_i(u_{N\backslash h}) - \varphi_i(u^j_{N\backslash h}) = \varphi_h(u_{N\backslash h}) - \varphi_h(u^j_{N\backslash h})$ hence $\varphi_i(u_{N\backslash h}) = 0$ for any $i \neq j$.* **E** *then implies $\varphi_j(u_{N\backslash h}) = 1$. As $n \geq 3$, these two last equalities contradict* **E** *and* **NEL** *by switching i and j.*

Remark 3. *Assuming the Null Game axiom* **NG** *in addition to* **Sm**, **E** *and* **NEL**, *the formula (2), applied to the 0-normalized game v^0, characterizes the equal division value: for any TU-game $v \in \mathbb{V}$, $\varphi_i(v^0) = v^0(N)/n = \mathrm{ED}_i(v^0)$.*

3.2. Back to Linearity

In this section, the class of smooth values is now restricted to linear values by requiring **L** instead of **Sm**. This leads to a characterization of the class of convex combination of the equal division value and the equal surplus division value.

Proposition 3. *A value φ on $\mathbb{V}$ satisfies Efficiency (***E***), Linearity (***L***) and Nullified equal loss property (***NEL***) if and only if it exists a vector $\beta \in \mathbb{R}^n$ such that*

$$\varphi_i(v) = \beta_i v(i) + \sum_{j \in N\backslash i} \left(\frac{1-\beta_j}{n-1} v(j) \right) + \frac{1}{n} \left(v(N) - \sum_{j \in N} v(j) \right) \tag{9}$$

*Moreover, if Desirability axiom (***D***) and Pairwise upper bound for monotonic games axiom (***PUBM***) are also required, then φ is a convex combination of the equal surplus division value and the equal division value, i.e. there exists a unique $\lambda \in [0,1]$ such that $\varphi = \lambda \mathrm{ESD} + (1-\lambda)\mathrm{ED}$.*

Proof. **[Part 1]** By construction, any value given by (9) satisfies **L**. Moreover, for all $i \in N$, defining $\alpha_i = \beta_i - 1/n$ leads to formula (2) where the functions $(\alpha_i)_{i \in N}$ are constants and $a_i = 0$ for all $i \in N$, so that any value given by (9) also satisfies **E** and **NEL**.

Reciprocally, consider any value φ on $\mathbb{V}$ satisfying **E**, **L** and **NEL**. As **L** implies **Sm**, by proposition 2, there are n constants $(a_i)_{i \in N} \in \mathbb{R}^N$ and n smooth functions $(\alpha_i)_{i \in N} \in \mathcal{C}^\infty(\mathbb{R})$ such that φ is given by (2) with $\alpha_i = \partial\varphi_i/\partial v(i)$ and $a_i = \varphi_i(\mathbf{0})$. By **L**, the functions α_i are constant and $a_i = 0$, leading to formula (9) by defining $\beta_i = \alpha_i + 1/n$.

[Part 2] Next, it is easy to check that both ED and ESD satisfy **E**, **L**, **NEL**, **D** and **PUBM**. Then any convex combination of ED and ESD also satisfies these axioms.

Lastly, consider any value φ satisfying **D** and **PUBM** in addition to **E**, **L** and **NEL**. Again, **L** implies **Sm** so, by the second part of proposition 2, the constants $\alpha_i = \beta_i - 1/n$ are all equals and positive. As precised in remark 1, α is bounded by $(n-1)/n$. This restricts the common value β

to $[1/n;1]$. Straight computation in (9) now leads to:

$$\begin{aligned}\varphi_i(v) &= \left(\beta - \frac{1}{n}\right) v(i) + \sum_{j \in N \setminus i} \left(\left(\frac{1-\beta}{n-1} - \frac{1}{n}\right) v(j)\right) + \frac{v(N)}{n} \\ &= \frac{n\beta - 1}{n-1} \times \frac{n-1}{n} v(i) + \frac{n - n\beta - (n-1)}{n-1} \times \frac{1}{n} \sum_{j \in N \setminus i} v(j) + \frac{v(N)}{n} \\ &= \frac{n\beta - 1}{n-1} \left(v(i) - \frac{1}{n} \sum_{j \in N} v(j)\right) + \frac{v(N)}{n} \\ &= \lambda \mathrm{ESD}_i(v) + (1-\lambda)\mathrm{ED}_i(v)\end{aligned}$$

where $\lambda = (n\beta - 1)/(n-1)$. Clearly $\lambda \in [0;1]$. ■

In order to prove the logical independence of the axioms involved in proposition 3, notice that the first four counter-examples shown in the corresponding part of proposition 2 are linear, and may be switched to expression (9) via defining $\beta_j = \alpha_j + 1/n$ for any $j \in N$. Hence they still apply here. Lastly, the value φ given by:

$$\varphi_i(v) = \frac{v(N)}{n} + \frac{v(i) - \sin(v(i))}{4} - \frac{1}{n-1} \sum_{j \neq i} \frac{v(j) - \sin(v(j))}{4}$$

satisfies all axioms except **L**. The proof is a direct consequence of the second part of proposition 2 with $\alpha(x) = (1 - \cos(x))/4$. Note that $\alpha(x) \leq 2/3 \leq (n-1)/n$ for any $n \geq 3$.

This proposition was expected as a consequence of proposition 2, withal can be established directly through a classic but longer proof, for instance using the decomposition in the unanimity games basis. Béal et al. (2015a,c) introduce weighted equal division values in which the common part to be shared is divided according to individual weights, which sum up to 1 if efficiency is required. These values differ from those resulting from the preceding characterization as the individual weights only affect the stand-alone parts of the payoffs but not the cooperative surplus. Note that efficiency puts no constraint on the weight vector β.

3.3. Characterization of the equal division values

The next result may be considered as a particular case of the classes defined in propositions 2 and 3 with the adjonction of the Inessential game property: this last axiom specifies the payoffs in case of an inessential TU-game, which is completely characterized by the stand-alone worths $v(i)$. Nevertheless, as we will see, no more than Efficiency and Nullified equal loss property are needed in addition to the Inessential game property in order to characterize the equal surplus division. The proof is therefore independent of the preceding propositions and more classic.

Proposition 4. *A value φ on $\mathbb{V}$ satisfies Efficiency (**E**), Nullified equal loss property (**NEL**) and the Inessential game property (**IGP**) if and only if it is the equal surplus division value $\varphi = ESD$.*

Proof. First ESD satisfies **NEL**, **IGP** and **E**.

Now consider any value φ on $\mathbb{V}$ satisfying the three aforementioned axioms. Remind that $n \geq 3$ throughout the article. The proof that $\varphi = ESD$ is done by (descending) induction on the number $k(v)$ of null players in a TU-game v.

INITIALIZATION. If $k(v) \geq n-1$, i.e. it exists at most one non-null player $i \in N$, then v is inessential and **IGP** leads to $\varphi_i(v) = v(i) = ESD_i(v)$.

INDUCTION HYPOTHESIS. Assume that $\varphi(v) = ESD$ for all TU-games $v \in \mathbb{V}$ such that $k(v) \geq k$, $0 < k \leq n-1$.

INDUCTION STEP. Choose any TU-game $v \in \mathbb{V}$ such that $k(v) = k-1$. Because $k(v) < n-1$, there exists at least two distinct non-null players $h, h' \in N \backslash K(v)$. Notice $K(v^h) = K(v) \cup h$ so that $k(v^h) = k(v) + 1 = k$. For all $i, j \neq h$, **NEL** and the induction hypothesis imply:

$$\varphi_i(v) - \varphi_j(v) \overset{\textbf{NEL}}{=} \varphi_i(v^h) - \varphi_j(v^h) = ESD_i(v^h) - ESD_j(v^h) = v(i) - v(j) \tag{10}$$

Let us show that (10) holds for all $i, j \in N$. Indeed, (10) holds for $i, j \neq h'$. Thanks to $n \geq 3$, with the help of an existing $k \neq h, h'$, we have $\varphi_h(v) - \varphi_k(v) = v(h) - v(k)$ and $\varphi_k(v) - \varphi_{h'}(v) = v(k) - v(h')$. Summing up these two last equalities brings $\varphi_h(v) - \varphi_{h'}(v) = v(h) - v(h')$, and so (10) holds for all $i, j \in N$. Now by summing this last equality over $j \in N$ and using **E**, one gets:

$$n \cdot \varphi_i(v) - v(N) = n \cdot v(i) - \sum_{j \in N} v(j)$$

This immediatly leads to $\varphi_i(v) = ESD_i(v)$ for every $i \in N$. ■

The three axioms are logically independent:

- For all $v \in V$, define $\varphi_i(v) = v(i)$ for all $i \in N$. Then φ satisfies all axioms except **E**.
- $\varphi =$ Sh satisfies all axioms except **NEL**.
- $\varphi =$ ED satisfies all axioms except **IGP**.

Remark 4. *This last proof may be mimicked step by step to characterize the equal division value with a similar set of three axioms. For this purpose, only the initialization part of the inductive proof needs to be adapted. We will call this ad hoc axiom* **Equal division for inessential games, EIG** *and define it by the following: for all inessential TU-games $v \in \mathbb{V}$, for all $i \in N$, $\varphi_i(v) = (\sum_{j \in N} v(j))/n$. One easily gets that* **E**, **NEL** *and* **EIG** *characterizes* ED.

4. Conclusion

Our article brings to light a new technique to implement punctual and relational axioms by means of differential equations. Beyond proposition 1, imposing **Sm** also leads, for instance, to the following statements:*

- If φ satisfies **IGP** then: for any inessential TU-game $v \in \mathbb{V}$, for all $i \in N$, $\sum_{S \ni i} (\partial \varphi_i / \partial v(S))(v) = 1$ and for all $i \neq j \in N$, $\sum_{S \ni j} (\partial \varphi_i / \partial v(S))(v) = 0$.

*The proofs are available upon request to the author.

- If φ satisfies **EIG** then: for any inessential TU-game $v \in \mathbb{V}$, $\sum_{S \ni j}(\partial\varphi_i/\partial v(S))(v) = 1/n$, for all $i, j \in N$.
- If φ satisfies **N** then: for all $i \in N$, for all $S \not\ni i$, for all $v \in \mathbb{V}$, $K(v) \ni i \Rightarrow (\partial\varphi_i/\partial v(S))(v) = -(\partial\varphi_i/\partial v(S \cup i))(v)$.
- If φ satisfies **ET** then: for all $i, j \in N$, for all $S \subseteq N$ such that either $\{i, j\} \subseteq S$, or $\{i, j\} \subseteq N \backslash S$, for all $v \in \mathbb{V}$, if i and j are equals in v then $(\partial\varphi_i/\partial v(S))(v) = (\partial\varphi_j/\partial v(S))(v)$. For any $S \subseteq N$, possibly empty, such that $S \cap \{i, j\} = \emptyset$, we also have $(\partial\varphi_i/\partial v(S \cup \{i\}))(v) + (\partial\varphi_i/\partial v(S \cup \{j\}))(v) = (\partial\varphi_j/\partial v(S \cup \{i\}))(v) + (\partial\varphi_j/\partial v(S \cup \{j\}))(v)$.

Note that this process is operational for many axioms defined by equalities but proves difficult to implement axioms defined by inequalities. Let us also point out that we only used game coordinates $(v(S))_{\emptyset \subsetneq S \subseteq N}$ in the Dirac basis to define derivatives $\partial\varphi_i/\partial v(S)$. The unanimity games basis may also be used, changing the game coordinates to $(\Delta_S(v))_{\emptyset \subsetneq S \subseteq N}$ so as to define and compute $\partial\varphi_i/\partial\Delta_S(v)$. More generally, this scouting tool can be used to recover classical results in cooperative game theory and to define classes of non-necessarily linear values, generalizing well-known values by shifting the popular linear axiom to the smoothness axiom.

Another extension of our work, in connection with Béal et al. (2015b), may be to study the impact of translating the Nullified equal loss property to its removal version, changing the player set N by removing a player off the game instead of nullifying her. One may then wonder whether the equal division values still keep the main role. Indeed, the Removal equal loss property would state that, if a player is removed from the game, the payoff difference between any pair of remaining players is preserved: for any $h \in N$, for any pair of players $i, j \neq h$, $\varphi_i(N, v) - \varphi_j(N, v) = \varphi_i(N \backslash h, v|_{N \backslash h}) - \varphi_j(N \backslash h, v|_{N \backslash h})$. Note that φ now depends explicitly on the player set. This last axiom is stronger than the Weak null player out axiom introduced in van den Brink and Funaki (2009) which only requires the preceding preservation of payoff differences when removing a null player $h \in K(N, v)$.

In Béal et al. (2015b), the authors discussed the central role of the Null player out axiom (introduced by Derks and Haller, 1999) in the aforementioned "translation" process. This axiom states that a null player removal does not change the payoff of other players. If a value satisfies the Null player out axiom, then the value satisfies a particular removal axiom if and only if it satisfies the corresponding nullification axiom. None of the values characterized in the present article satisfies the Null player out axiom but all satisfies the Removal equal loss property as soon as all the parameters involved in the value characterization are independent of the player set. These ascertainments raise the question of characterizing these values only by ensuring invariance of the parameters when the player set changes and would probably require other axioms.

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