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The neighborhood value for cooperative graph games*

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Abstract

We consider cooperative games with a neighborhood structure modeled by a graph. Our approach shares some similarities with the models of graph games (Myerson, 1977) and games with a local permission structure (van den Brink and Dietz, 2014). The value that we study shares the Harsanyi dividend of each coalition equally among the coalition members and their neighbors. We characterize this value by five axioms: Efficiency, Additivity, Null neighborhood out (removing a null player whose neighbors are also null does not affect the remaining players' payoffs), Equal loss in an essential situation (if a single coalition has a non-null Harsanyi dividend and the other players are neighbors of that coalition, removing any player induces the same payoff variation for the remaining players) and Two-player symmetry (In a two-player game, the players obtain equal payoffs if they are symmetric or neighbors).

Keywords: Shapley value, Graph games, Neighborhood, Harsanyi dividends, Axiomatization.

1. Introduction

Cooperative games with transferable utility (simply games) describe the worth that each coalition of players can generate by cooperating. The goal is to find a value that specifies the payoffs obtained by the players from their participation in the game. In this classical model, such a value can only depend on the worths of the coalitions of players and the Shapley value (Shapley, 1953) is the most well-known value. However, in many situations exogenous affinities among players are relevant and can be represented by some social, hierarchical, economical, communicational, or technical structure. For instance, Myerson (1977) models bilateral communication among players by the links of an undirected graph and the so-called Myerson value extends the Shapley value in this framework.

In this article, we retain the same model as in Myerson (1977) but with a totally different interpretation. More specifically, the edges of the graph are interpreted as neighbourhood relations,

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and we hypothesize that each player needs the agreement of each of its neighbours in order to be able to cooperate within a coalition. We introduce a value which shares the Harsanyi dividend (Harsanyi, 1959) of each coalition equally among the extended neighborhood of this coalition, i.e. the coalition members and their neighbors. This is comparable with, albeit different from, the Shapley value and the Myerson value, which shares equally among the coalition's members the Harsanyi dividend of the coalition in the game and graph-restricted game, respectively. We called this value the Neighborhood value. Alternatively, the Neighborhood value can be formulated as the Shapley value of a neighborhood-restricted game in which the worth of a coalition is the worth, in the original game, of the susbset of its members for whom all neighbors are also in the coalition.

Our approach is therefore close to the literature on permission structure initiated by Gilles et al. (1992). It is even the particular case of the class of games with a local permission structure introduced in van den Brink and Dietz (2014) when a player is a direct predecessor of another player if and only if the reciprocal relation also holds. Hence, on this subclass of games with a local permission structure, the Neighborhood value coincides with the local permission value studied by van den Brink and Dietz (2014). The first properties that we exhibit are inherited from this literature even if we keep the demonstrations for completeness.

Our main result is an axiomatic characterization of the Neighborhood value that rests on five axioms. The first two axioms are classical: Efficiency imposes that the sum of distributed payoffs equals the worth achieved by the grand coalition and Additivity requires that the payoffs allocated in the sum of two games equals the sum of the payoffs distributed in these two games. The remaining three axioms are new and take into account the neighborhood structure. Null neighborhood out states that removing a null player whose neighborhood only contains null players as well does not alter the payoffs of the other players. It is similar to the classical null player out axiom (Derks and Haller, 1999). The axiom of Equal loss in an essential situation focuses on a situation in which a single coalition has a non-null Harsanyi dividend and each other player is the neighbor of some member of the aforementioned coalition. The axiom simply requires that removing any player yields for each other player the same payoff variation. It should be noted that these first four axioms are satisfied by the classical Shapley value. The final axiom, called Two-player symmetry, is stated on two-player games only. It imposes equal payoff for the two players if either they are symmetric (i.e. they have the same contributions to coalitions) or if they are neighbors.

The rest of the article is organized as follows. Section 2 introduces cooperative games, graphs, the Neighborhood value and provides examples in which the latter value is relevant. Section 3 presents and proves all results.

2. Preliminaries

2.1. Cooperative games

A cooperative game with transferable utility (simply a **game**) is a pair (N, v) such that, for each coalition $S \subseteq N$, $v(S) \in \mathbb{R}$ is the worth of coalition S, i.e. the best result that the players in S can achieve by cooperating without the help of the other players, and $v(\varnothing) = 0$ by convention. Denote by $\mathcal{G}$ the set of all games in which the player set is finite. For each coalition $S \subseteq N$, s stands for the number of players in s. Players $i, j \in N$ are **symmetric** in (N, v) if for each

$S \subseteq N\backslash\{i,j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$. Player $i \in N$ is **null** in the game (N,v) if for each $S \subseteq N\backslash\{i\}$, $v(S \cup \{i\}) = v(S)$. The sum of two games (N,v) and (N,w) is the game $(N, v+w)$ such that, for each $S \subseteq N$, $(v+w)(S) = v(S) + w(S)$. For any $S \subseteq N$, the **subgame** of (N,v) induced by S is the game $(S, v_{|S})$, where, for each $T \subseteq S$, $v_{|S}(T) = v(T)$. Since Shapley (1953), it is known that any function v can be uniquely expressed as:

$$v = \sum_{S \subseteq N, S \neq \varnothing} \Delta_v(S) u_S, \tag{1}$$

where (N, u_S) is the **unanimity game** on N induced by coalition S given by $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ otherwise, and $\Delta_v(S)$ is called the **Harsanyi dividend** (Harsanyi, 1959) of S. Following Besner (2022), a coalition S is called **essential** is a game (N,v) if $\Delta_v(S) \neq 0$.

The **Shapley value** Sh (Shapley, 1953) shares the dividend of each coalition equally among its members:

$$Sh_i(N,v) = \sum_{S \subseteq N, S \ni i} \frac{\Delta_v(S)}{s}$$

for each $i \in N$ and each $(N,v) \in \mathcal{G}$.

The **Equal Division value** ED is such, for each $(N,v) \in \mathcal{G}$ and each $i \in N$,

$$ED_i(N,v) = \frac{v(N)}{n}.$$

2.2. Graph games

An undirected graph on N is a pair (N,L) such that N is the set of nodes and L is a subset of the set L^N of all unordered pairs $\{\{i,j\} : i,j \in N,\ i \neq j\}$. Each pair $\{i,j\} \in L$ represents the bilateral link between nodes i and j in N. The **complete graph** on N is the the pair (N, L^N). Let $\mathcal{L}$ be the set of all graphs that we can construct from any nonempty and finite set of nodes. The subgraph of $(N,L) \in \mathcal{L}$ induced by $S \subseteq N\backslash\{\varnothing\}$ is the graph $(S, L_{|S}) \in \mathcal{L}$ where $L_{|S} = \{\{i,j\} \in L : i,j \in S\}$. In a graph $(N,L) \in \mathcal{L}$ a sequence of different nodes $(i_1, \ldots, i_r)$, $r \geq 2$, is a **path** from node i_1 to node i_r if for $q = 1, \ldots, r-1$ it holds that $\{i_q, i_{q+1}\} \in L$. Two nodes i and j are **connected** in (N,L) if either $i = j$ or there exists a path between i and j. The graph (N,L) is **connected** if any two nodes of N are connected. A coalition of nodes $S \subseteq N\backslash\{\varnothing\}$ is connected in the graph (N,L) if the induced subgraph $(S, L_{|S})$ is connected. If (N,L) is not connected, then it is partitioned into **components** being maximal connected subsets of nodes with respect to set inclusion. Let N/L be the set of components of (N,L).

A **graph game** is a triple (N,v,L) where $(N,v) \in \mathcal{G}$ and $(N,L) \in \mathcal{L}$. A **value** on $\mathcal{G} \times \mathcal{L}$ is a mapping f on $\mathcal{G} \times \mathcal{L}$ that assigns to each $(N,v,L) \in \mathcal{G} \times \mathcal{L}$ a payoff vector $f(N,v,L) \in \mathbb{R}^N$. The most famous value for graph games is the Myerson value (1977) being the Shapley value of a graph-restricted game constructed from (N,v) and (N,L). Precisely, let $(N,v,L) \in \mathcal{G} \times \mathcal{L}$, the **graph-restricted game** $(N, v^L) \in \mathcal{G}$ is defined as:

$$\forall S \subseteq N, \quad v^L(S) = \sum_{C \in N/L} v(C),$$

that is, the worth of coalition S is defined as the sum of the worths of its connected components in the subgraph induced by this coalition. This means that only connected coalitions are able to cooperate.

The **Myerson value** (Myerson, 1977) on $\mathcal{G} \times \mathcal{L}$, denoted by My, is then defined as:

$$My(N,v,L) = Sh(N,v^L).$$

It is characterized by Component efficiency (each connected component of the associated graph obtains a total payoff equal to its worth) and Fairness (deleting any link from the graph yields the same payoff variations for the two associated players). Since $(N,v^L) = (N,v)$ if $L = L^N$, the Myerson value can be considered as a generalization of the Shapley value.

2.3. The Neighborhood value

For any $(N,L) \in \mathcal{L}$ and any $i \in N$, $L_i = \{j \in N : \{i,j\} \in L\}\}$ is the neighborhood of node $i \in N$. Set $L_i^+ = \{i\} \cup L_i$ and for any nonempty $S \subseteq N$, $L_S^+ = \cup_{i \in S} L_i^+$. In what follows, we assume that a player needs the permission from her neighbors before it can cooperate. For any $(N,L) \in \mathcal{L}$ and any $S \subseteq N$, following Gilles et al. (1992), we can call coalition S **autonomous** if $L_S \subseteq S$. The structure of the set of autonomous is clear: an autonomous coalition is either empty of the union of connected components.[1] For any coalition $S \subseteq N$, and in particular the non-autonomous coalitions, define the **sovereign part** $\sigma_L(S)$ of S in (N,L) and the subset of S for whom all neighbors are also members of S:

$$\sigma_L(S) = \{i \in S : L_i \subseteq S\}.$$

From any $(N,v,L) \in \mathcal{G} \times \mathcal{L}$, define the **neighborhood-restricted game** $(N, r_L(v)) \in \mathcal{G}$ as:

$$\forall S \subseteq N, \quad r_L(v)(S) = v(\sigma_L(S)).$$

In words, the worth of a coalition S in $(N, r_L(v))$ is the worth achieved in (N,v) by the subset of players in S for whom all neighbors are also members of S. The **Neighborhood value** on $\mathcal{G} \times \mathcal{L}$, denoted by NV, is then defined as:

$$NV_(N,v,L) = Sh_(N, r_L(v)).$$

Two comments are in order. If the graph is empty, i.e. if $L = \varnothing$, then $\sigma_\varnothing(S) = S$ for each $S \subseteq N$, which implies that $r_\varnothing(v) = v$ and thus that $NV(N,v,\varnothing) = Sh(N,v)$. If the graph is complete, i.e. if $L = L^N$, then $\sigma_{L^N}(S) = \varnothing$ for each $S \subsetneq N$ and $\sigma_{L^N}(N) = N$, which implies that $r_{L^N}(v)$ is a symmetric game such that $r_L(v)(N) = v(N)$ and thus that $NV(N,v,L^N) = ED(N,v)$. As a consequence, the neighborhood value can be seen as a generalization of both the Shapley value and the Equal Division value.

Furthermore, for each non-empty coalition S, the unique smallest coalition whose sovereign part is S is obviously L_S^+, even if it is often the case that L_S^+ is not autonomous. Hence the role played by L_S^+ is different from the so-called authorizing set of S in Gilles et al. (1992), which represents a first difference with this literature.

[1] Hence the set of autonomous coalition is trivially both closed under union and closed under intersection.

Example 1. ***(Resource pooling)*** A resource pooling problem can be described by a tuple $P = (N, R, (R_i)_{i \in N}, b)$ where:

- O is the finite set of resource owners;
- R is the finite set of resources;
- for each resource owner $j \in O$, $R_j \subseteq R$ is the subset of resources owned by j. Multiple ownership is allowed: it is possible that $R_j \cap R_{j'} \neq \emptyset$ for some distinct $j, j' \in O$;
- $b : 2^R \longrightarrow \mathbb{R}$ is a benefit function which assigns to each subset of resources $Q \subseteq R$ the benefit $b(Q)$ obtained by pooling the resources from Q.

Examples of such resources are patents that must be pooled to construct a smartphone (which is made up of around 250 000 patents), an hotel chain made up of numerous hotels, data that are gathered within a SIEF in order to better understand the impact of a chemical substance in the context of the REACH legislation (Béal and Deschamps, 2016), etc.

To the resource problem P, we can associate a graph game (N_P, v_P, L_P) such that

- the set N_P of players is the set of pairs (j, k) where j is a resource owner and k is a resource owned by j, i.e.
$$N_P = \{(j,k) : j \in O, k \in R_j\}.$$
- for each coalition $S \subseteq N$ of players, $v_P(S)$ is the benefit obtained if resources in S are pooled, i.e.
$$v_P(S) = b\left(\bigcup_{(j,k)\in S} k\right);$$
- L_p contains a link between (j,k) and (j',k') if and only if $j \neq j'$ and $k = k'$, i.e. if j and j' are the co-owners of the same resource.

In this context, the neighborhood restricted game is relevant: it indicates that the agreement of all the owners of a resource is needed to effectively mobilize the resource. For a game (N_P, v_P, L_P), the Neighborhood value will specify a payoff for each pair $(j,k) \in N_P$ describing two things: first, how important is resource k in the pooling of all resources and second, how important is the fact that j owns this resource. □

Example 2. ***(Non-point source pollution)*** Béal et al. (2024) model the problem of sharing the cost of cleaning up non-point source pollution as a triple (N, c, L) where

- N is a set of firms,
- $c = (c_i)_{i \in N}$ is a cost vector specifying, for each firm $i \in N$, the cost $c_i \geq 0$ of cleaning up pollution on the site of firm i,
- L is a set of links such that $\{i, j\} \in L$ means that i's activity can also harm j' site and vice versa. It is therefore considered that pollution is bilateral and L_i^+ indicates the potential polluters of site i.

They construct the polluters game, which assigns to each triple (N, c, L), the game $v_{c,L}$ on N such that, for each $S \subseteq N$, $v_{c,L}(S) = \sum_{i \in L_S^+} c_i$. It is easy to prove that the Polluters game coincides with the neighborhood-restricted game $(N, r_L(v))$ if one sets v as the additive game induced by c, i.e. $v(S) = \sum_{i \in S} c_i$ for each $S \subseteq N$. $\square$

3. Results

We split the results in two parts. Firstly, we provide results regarding the neighborhood-restricted game, which leads to an alternative formulation of the Neighborhood value. Secondly, we prove an axiomatic characterization of the Neighborhood value.

3.1. Properties

The following property connects the neighborhood-restricted game to the dividends of the original game and is inspired by some results obtained by Gilles et al. (1992). First, for any graph (N, L), define

$$A(L) = \{S \subseteq N : S = L_T^+ \text{ for some } T \subseteq N\}.$$

Proposition 1. *For each (N, v, L), it holds that*

$$r_L(v) = \sum_{S \in A(L)} \left(\sum_{T \subseteq N : L_T^+ = S} \Delta_v(T) \right) u_S. \tag{2}$$

Proof. As a start, for each nonempty coalition $S \subseteq N$, we show that

$$r_L(u_S) = u_{L_S^+}. \tag{3}$$

Pick any $T \subseteq N$, Then $r_L(u_S)(T) = u_S(\sigma_L(T)) = u_S(\{i \in T : L_i \subseteq T\})$. Therefore, for each nonempty $T \subseteq N$,

$$r_L(u_S)(T) = \begin{cases} 1 & \text{if } \{i \in T : L_i \subseteq T\} \supseteq S \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 1 & \text{if } L_i^+ \subseteq T \text{ for all } i \in S \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 1 & \text{if } L_S^+ \subseteq T \\ 0 & \text{otherwise,} \end{cases} = u_{L_S^+}(T),$$

as desired. Next, from (3), we can write that

$$\begin{aligned} r_L(v) &= r_L\left(\sum_{S \subseteq N, S \neq \varnothing} \Delta_v(S) \cdot u_S \right) \\ &= \sum_{S \subseteq N, S \neq \varnothing} \Delta_v(S) \cdot r_L(u_S) \\ &= \sum_{S \subseteq N, S \neq \varnothing} \Delta_v(S) \cdot u_{L_S^+} \\ &= \sum_{S \in A(L)} \left(\sum_{T \subseteq N : L_T^+ = S} \Delta_v(T) \right) u_S, \end{aligned}$$

which completes the proof. $\blacksquare$

The previous result allows to provide the following relevant alternative form of the Neibhorhood value in terms of the dividends of the original game.

Proposition 2. *For each* $(N, v, L) \in \mathcal{G} \times \mathcal{L}$ *and each* $i \in N$*, it holds that*

$$NV_i(N, v, L) = \sum_{S \subseteq N : i \in L_S^+} \frac{\Delta_v(S)}{|L_S^+|} \tag{4}$$

Proof. For each $(N, v, L) \in \mathcal{G} \times \mathcal{L}$ and each $i \in N$, using (2), we can write

$$\begin{aligned} NV_i(N, v, L) &= Sh_i(N, r_L(v)) \\ &= Sh_i\left(N, \sum_{S \in A(L)} \left(\sum_{T \subseteq N : L_T^+ = S} \Delta_v(T)\right) u_S\right) \\ &= \sum_{S \in A(L)} Sh_i\left(N, \sum_{T \subseteq N : L_T^+ = S} \Delta_v(T) \cdot u_S\right) \\ &= \sum_{S \in A(L) : S \ni i} \left(\sum_{T \subseteq N : L_T^+ = S} \Delta_v(T)\right) \frac{1}{s}. \end{aligned}$$

Since $S \ni i$ and $L_T^+ = S$ imply that $i \in L_T^+$, the previous expression can rewritten as:

$$NV_i(N, v, L) = \sum_{S \subseteq N : i \in L_S^+} \frac{\Delta_v(S)}{|L_S^+|},$$

as desired. ■

3.2. Axiomatic characterization

We list below axioms for a value f on $\mathcal{G} \times \mathcal{L}$. The first two are classical.

Efficiency (EFF). For each $(N, v, L) \in \mathcal{G} \times \mathcal{L}$, it holds that $\sum_{i \in N} f_i(N, v, L) = v(N)$.

Additivity (ADD). For each $(N, v, L), (N, w, L) \in \mathcal{G} \times \mathcal{L}$, it holds that $f(N, v+w, L) = f(N, v, L) + f(N, w, L)$.

The next two axioms describe the effect of the neighborhood of a player on the allocation process. Both are based on two dimensions of the model: productivity, measured by v, and the neighbourhood role, measured by L. The first one states that if a player and all her neighbors are null in a game, then removing this player from the game should not change the payoff of any other player. Hence, the removed player is inconsequential on both the productivity (it is null) and the neighborhood (its agreement as a neighbor only concerns null players).

Null neighborhood out (NNO). For each $(N, v, L) \in \mathcal{G} \times \mathcal{L}$ and each $i \in N$ such that each $j \in L_i^+$ is a null player in (N, v), it holds that $f_k(N, v, L) = f_k(N \setminus \{i\}, v_{|N \setminus \{i\}}, L(N \setminus \{i\}))$ for each $k \in N \setminus \{i\}$.

The second axiom describing the influence of the neighborhood considers a case in which a unique coalition is essential and in which each player outside this coalition is the neighbor of some player in the coalition. In such a situation, the axiom requires that the departure of any player results in identical payoff variations for the remaining players.

Equal loss in an essential situation (ELES). For each $(N, v, L) \in \mathcal{G} \times \mathcal{L}$ such that there is a unique essential coalition $S \subseteq N$ and each player in $N \backslash S$ is the neighbor of some member of S, for each triple $i, j, k \in N$ of distinct players, it holds that $f_j(N, v, L) - f_j(N\backslash\{i\}, v_{|N\backslash\{i\}}, L_{|N\backslash\{i\}}) = f_k(N, v, L) - f_k(N\backslash\{i\}, v_{|N\backslash\{i\}}, L_{|N\backslash\{i\}})$.

The final axiom that we invoke focuses on two-player games. It requires equal payoffs for the two players if at least one of the two symmetrical conditions is met: the two players are symmetric in the cooperative game and/or they share the same neighborhood.

Two-player symmetry (TPS).) For each $(N, v, L) \in \mathcal{G} \times \mathcal{L}$ such that $N = \{i, j\}$, if i and j are symmetric players in (N, v) or $L = \{\{i, j\}\}$, it holds that $f_i(N, v, L) = f_j(N, v, L)$.

Proposition 3. *The Neighborhood value is the unique value on $\mathcal{G} \times \mathcal{L}$ that satisfies Efficiency (EFF), Additivity (ADD), Null neighborhood out (NNO), Equal loss in an essential situation (ELES) and Two-player symmetry (TPS).*

Proof. Existence. We prove that NV satisfies all axioms on $\mathcal{G} \times \mathcal{L}$.

(EFF). For each $(N, v, L) \in \mathcal{G} \times \mathcal{L}$, we have $r_L(v)(N) = v(\sigma_L(N)) = v(\{i \in N : L_i \subseteq N\}) = v(N)$. Since the Shapley value satisfies **(EFF)** on $\mathcal{G}$, it holds that

$$\sum_{i \in N} NV_i(N, v, L) = \sum_{i \in N} Sh_i(N, r_L(v)) = r_L(v)(N) = v(N),$$

as desired.

(ADD). It is obvious that NV inherits **(ADD)** from the Shapley value and the fact that the mapping r_L is linear.

(NNO). Consider a game $(N, v, L) \in \mathcal{G} \times \mathcal{L}$ and a player $i \in N$ such that i and each of her neighbors $j \in L_i$ are null players in (N, v). It is known that the Harsanyi dividend of a coalition is null if this coalition contains a null player. Hence, it holds that $\Delta_v(S) = 0$ for each $S \subseteq N$ such that $S \cap L_i^+ \neq \emptyset$. Since $i \in L_S^+$ if and only if $S \cap L_i^+ \neq \emptyset$, from (4), we obtain for each $j \in N\backslash\{i\}$,

$$NV_j(N, v, L) = \sum_{S \subseteq N : j \in L_S^+} \frac{\Delta_v(S)}{|L_S^+|} = \sum_{S \subseteq N\backslash\{i\} : j \in L_S^+} \frac{\Delta_v(S)}{|L_S^+|} = NV_j(N\backslash\{i\}, v_{|N\backslash\{i\}}, L_{|N\backslash\{i\}}).$$

(ELES). Consider any $(N, v, L) \in \mathcal{G} \times \mathcal{L}$ such that there is a unique essential coalition $S \subseteq N$ and each player in $N\backslash S$ is the neighbor of some member of S. Hence, (N, v, L) is such that $v = c \cdot u_S$ for some real $c \neq 0$ and $N = L_S^+$. Therefore, there are two types of players: the players in S are necessary in (N, v), i.e. if any $i \in S$ is outside a given coalition, this coalition has a null worth, and each player in $N \backslash S$ is null but the neighbor of some player in S. We consider successively the departure of each type of player. Firstly, pick $i \in S$. Then, for each $j \in N\backslash\{i\}$, we have $NV_j(N,v,L) - NV_j(N\backslash\{i\}, v_{|N\backslash\{i\}}, L_{|N\backslash\{i\}}) = NV_j(N,v,L)$ since $(N\backslash\{i\}, v_{|N\backslash\{i\}})$ is the null game on $N\backslash\{i\}$ and NV assigns null payoffs in the null game. Hence, the equality $NV_j(N,v,L) - NV_j(N\backslash\{i\}, v_{|N\backslash\{i\}}, L_{|N\backslash\{i\}}) = NV_k(N,v,L) - NV_k(N\backslash\{i\}, v_{|N\backslash\{i\}}, L_{|N\backslash\{i\}})$ holds for each $j,k \in N\backslash\{i\}$, as desired. Secondly, pick $i \in N\backslash S$. For each $j \in N\backslash\{i\}$, we have

$$NV_j(N,v,L) - NV_j(N\backslash\{i\}, v_{|N\backslash\{i\}}, L_{|N\backslash\{i\}}) = \frac{c \cdot \Delta_v(S)}{n} - \frac{c \cdot \Delta_v(S)}{n-1},$$

which means that the equality $NV_j(N,v,L) - NV_j(N\backslash\{i\}, v_{|N\backslash\{i\}}, L_{|N\backslash\{i\}}) = NV_k(N,v,L) - NV_j = k(N\backslash\{i\}, v_{|N\backslash\{i\}}, L_{|N\backslash\{i\}})$ also holds for each $j,k \in N\backslash\{i\}$. This proves that NV satisfies **ELES**.

(TPS). Consider $(N,v,L) \in \mathcal{G} \times \mathcal{L}$ such that $N = \{i,j\}$. Firstly, suppose that i and j are symmetric players in (N,v). Then $\Delta_v(\{i\}) = \Delta_v(\{j\})$, which implies that that

$$NV_i(N,v,L) = \Delta_v(\{i\}) + \frac{1}{2}\Delta_v(\{i,j\}) = \Delta_v(\{j\}) + \frac{1}{2}\Delta_v(\{i,j\}) = NV_j(N,v,L)$$

if $L = \varnothing$ or

$$NV_i(N,v,L) = \frac{1}{2}\big(\Delta_v(\{i\}) + \Delta_v(\{j\}) + \Delta_v(\{i,j\})\big) = NV_j(N,v,L) \tag{5}$$

if $L = \{\{i,j\}\}$. Secondly, assume that $L = \{\{i,j\}\}$, which yields $L_i^+ = L_j^+ = \{i,j\}$. Then, whatever v, equality (5) holds again. We proved that NV satisfies **TPS**.

Uniqueness. Consider any value f on $\mathcal{G} \times \mathcal{L}$ that satisfies the five axioms. We proceed by induction on the size of N.

Initialisation. If $|N| = 1$, then $f(N,v,L)$ is uniquely determined by **EFF**. If $|N| = 2$, let $N = \{i,j\}$. We distinguish two cases. If $L = \{\{i,j\}\}$, $f(N,v,L)$ is uniquely determined by TPS and EFF. If $L = \varnothing$, by **ADD**, we have

$$f(N,v,L) = f(N, \Delta_v(\{i\}) \cdot u_{\{i\}}) + f(N, \Delta_v(\{j\}) \cdot u_{\{j\}}) + f(N, \Delta_v(\{i,j\}) \cdot u_{\{i,j\}})$$

Players i and j are symmetric in $(N, \Delta_v(\{i,j\}) \cdot u_{\{i,j\}})$ so that $f(N, \Delta_v(\{i,j\}) \cdot u_{\{i,j\}})$ is uniquely determined by **TPS**. In $(N, \Delta_v(\{i\}) \cdot u_{\{i\}}, L)$, player j is a null player and has no neighbor. Hence, **NNO** can be applied to get $f_i(N, \Delta_v(\{i\}) \cdot u_{\{i\}}, L) = f_i(\{i\}, \Delta_v(\{i\}) \cdot u_{\{i\}}, L) = \Delta_v(\{i\})$. From **EFF** in $(N, \Delta_v(\{i\}) \cdot u_{\{i\}}, L)$, we also obtain $f_j(N, \Delta_v(\{i\}) \cdot u_{\{i\}}, L) = 0$. In $(N, \Delta_v(\{j\}) \cdot u_{\{j\}}, L)$, we proceed as before except that the roles of i and j are inverted. All in all, we proved that $f(N,v,L)$ is uniquely determined.

Induction hypothesis. Assume that $f(N,v,L)$ is uniquely determined for all $(N,v,L) \in \mathcal{G} \times \mathcal{L}$ such that $|N| \leq q$, $q \geq 2$.

Induction step. Consider any $(N,v,L) \in \mathcal{G} \times \mathcal{L}$ such that $|N| = q+1$. Since f satisfies **ADD**, from (1) we have that

$$f(N,v,L) + \sum_{S \subseteq N : \Delta_v(S) < 0} f(N, -\Delta_v(S) \cdot u_S, L) = \sum_{S \subseteq N : \Delta_v(S) \geq 0} f(N, \Delta_v(S) \cdot u_S, L), \tag{6}$$

which implies that, for a given player set N, it is enough to show that f is uniquely determined in any game $(N, c \cdot u_S)$ such that $S \subseteq N$ is nonempty and $c \in \mathbb{R}_+$. If $c = 0$, the additivity of f implies that f is an odd function so that **EFF** yields that $f_i(N, c \cdot u_S, L) = 0$ for each $i \in N$. So suppose that $c \neq 0$. We distinguish two cases.

Firstly, consider any $(N, c \cdot u_S, L)$ such that $L_S^+ \subsetneq N$. Pick any $i \in N \backslash L_S^+$ and note that i and each player in L_i are null players in $(N, c \cdot u_S)$. Therefore, **NNO** can be applied to obtain, for each $j \in N \backslash \{i\}$ that $f_j(N, c \cdot u_S, L) = f_j(N \backslash \{i\}, v_{|N \backslash \{i\}}, L_{|N \backslash \{i\}})$. Since $f_j(N \backslash \{i\}, v_{|N \backslash \{i\}}, L_{|N \backslash \{i\}})$ is uniquely determined by the induction hypothesis, so is $f_j(N, c \cdot u_S, L)$. By **EFF**, $f_i(N, c \cdot u_S, L)$ is uniquely determined as well.

Secondly, consider any $(N, c \cdot u_S, L)$ such that $L_S^+ = N$. Remark that S is the unique essential coalition and since $L_S^+ = N$, each player in $N \backslash S$ is the neighbor of some player in S. Therefore, **ELES** can be applied to get, for each triple $\{i, j, k\}$ of distinct players:

$$f_j(N, c \cdot u_S) - f_k(N, c \cdot u_S) = f_j(N \backslash \{i\}, (c \cdot u_S)_{|N \backslash \{i\}}, L_{|N \backslash \{i\}}) - f_k(N \backslash \{i\}, (c \cdot u_S)_{|N \backslash \{i\}}, L_{|N \backslash \{i\}}). \quad (7)$$

From the induction hypothesis, the right member of (7) is uniquely determined. Hence, the collection of $n^2(n-1)/2$ such equations and the equation provided by **EFF** form a system of linear equations with n unknowns. This system possesses at most one solution and we already proved in the existence part that $NV(N, c \cdot u_S, L)$ is a solution. Conclude that $f(N, c \cdot u_S, L)$ is uniquely determined, which completes the proof. ■

The logical independence of the axioms invoked in Proposition 3 can be demonstrated as follows.

- The null value satisfies all axioms except **EFF**.
- For any game (N, v), denote by $\gamma(N, v)$ the set of non-null players. The value on $\mathcal{G} \times \mathcal{L}$ such that, for each $(N, v, L) \in \mathcal{G} \times \mathcal{L}$,

$$f_i(N, v, L) = \begin{cases} \dfrac{v(N)}{|\gamma(N, v)|} & \text{if } i \in \gamma(N, v), \\ 0 & \text{otherwise,} \end{cases}$$

  satisfies all axioms except **ADD**.
- The Equal Division value (defined on $\mathcal{G} \times \mathcal{L}$) satisfies all axioms except **NNO**.
- The value on $\mathcal{G} \times \mathcal{L}$ such that, for each $(N, v, L) \in \mathcal{G} \times \mathcal{L}$,

$$f_i(N, v, L) = \sum_{S \subseteq N : i \in L_S^+} \frac{|L_i^+|}{\sum_{j \in L_S^+} |L_j^+|} \Delta_v(S)$$

  satisfies all axioms except **ELES**.
- The Shapley value (defined on $\mathcal{G} \times \mathcal{L}$) satisfies all axioms except **TPS**.

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