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Multi-winner rules analogous to the Plurality rule

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Abstract

The aim of this paper is to identify the multi-winner voting rules that can be considered as extensions of the Plurality rule. Multi-winner voting addresses the problem of selecting a fixed-size subset of candidates, called a *committee*, from a larger set of available candidates based on the voters' preferences. In the single-winner setting, where each voter provides a strict ranking of the candidates and the goal is to select a unique candidate, Yeh (2008) characterized the Plurality rule as the only voting rule satisfying five independent axioms: anonymity, neutrality, consistency, efficiency, and top-only. In this paper, we demonstrate that a natural extension of these axioms to the multi-winner framework allows us to identify a class of *top-k counting rules* as multi-winner analogous to the Plurality rule.

Keywords: Multi-winner, voting rules, axioms, Plurality rule, top-k counting rules.
JEL classification: D71, D72.

1 Introduction

Multi-winner elections pose common challenges in social choice theory, where a set of individuals have to aggregate their preferences to select a predefined-sized subset of alternatives from a larger set. This voting scenario is prevalent in real-life situations such as parliamentary elections, candidate shortlisting for competitions, or curating a set of movies for in-flight entertainment. This process is commonly referred to as "committee selection." Formally, a finite set of voters express their preferences over a finite set of alternatives (or candidates) in order to choose a fixed-size subset, known as a "committee." Numerous research endeavors in this field focus on the ordinal setting, where voters possess linear orders over the set of alternatives, ranking them from most to least preferred without ties. The majority of these studies aim to extend single-winner voting rules to the multi-winner framework. Without being exhaustive we refer the reader to the work of Elkind et al.

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(2017), Faliszewski et al. (2018, 2019), Kilgour (2010), Gehrlein (1985), Fishburn (1981) Kilgour and Marshall (2012), and Skowron et al. (2019). Among these contributions, one of most interesting is that of Elkind et al. (2017) introducing the family of committee scoring rules, extending the well-known family of scoring rules for single-winner elections.

In the single-winner setting, the most popular and widely applied rule in many societies is the Plurality rule, which selects the candidate most preferred by the society; that is, the candidate ranked in the top position at least as often as any other candidate. This rule has the advantage of being simple and easy to implement. Given the popularity of the Plurality rule, it is likely that people would be incentivized to apply it in a multi-winner framework. It is therefore crucial to determine which rule(s) can be considered extensions of the Plurality rule in the context of multi-winner elections. In practice, the common way to apply the Plurality rule in this context is to evaluate each candidate individually and select the k candidates who are ranked highest by voters as often as possible, where k is the predetermined size of the committee to be selected. This voting rule is known as the single non-transferable vote (SNTV) rule, or k-Plurality rule, and belongs to the class of (weakly) separable committee scoring rules introduced by Elkind et al. (2017). Each rule in this class evaluates candidates individually using a scoring vector for single-winner elections and selects the k candidates with the highest scores.

The first claim of this paper is that the k-Plurality rule cannot be considered as a true extension of the Plurality rule, since it is simply a natural and straightforward application of the Plurality rule for selecting a committee. It is worth mentioning that most of the contributions cited above propose extensions of single-winner rules based on the definitions of the rules. However, we argue that when a single-winner voting rule has been characterized by certain properties, the appropriate approach to extend it to the multi-winner setting is to first extend these characteristic properties and then to identify the rules that satisfy them. This approach was elegantly followed by Debord (1992), who characterized the extension of the Borda rule for multi-winner elections based on Young (1974)'s characterization of the Borda rule for single-winner elections. A similar methodology was used by Faliszewski et al. (2018), who identified committee scoring rules analogues to the Plurality rule based on the fixed majority property defined by Debord (1993). More precisely, it is well known that the Plurality rule is the only scoring rule for single-winner elections that satisfies the majority principle: if a candidate is ranked first by a strict majority of voters, it should be the unique winning committee. The majority principle has been extended by Debord (1993) to the fixed majority property for multi-winner elections, which states that if all the members of a committee of size k are ranked within the top k positions by a strict majority of voters (regardless of the order), then this committee should be the unique winner. Faliszewski et al. (2018) identify the committee scoring rules that satisfy this fixed majority property as those being analogues to the Plurality rule.

This paper adopts a similar approach for a more general framework, based on the characterization of the Plurality rule. In the single-winner setting, Yeh (2008) characterizes the Plurality rule as the only voting rule that satisfies anonymity, neutrality, consistency, efficiency, and top-only. To identify the multi-winner voting rules that extend the Plurality rule, we consider natural extensions of these properties for multi-winner elections and seek the rules that satisfy them. Note that the extension for anonymity, neutrality, and consistency have already been defined and studied by Elkind et al. (2017). Hence, we introduce the natural extensions of efficiency and top-only, which we refer to as k-efficiency and the top-k only property, respectively. The k-efficiency property requires that any candidate who is Pareto-dominated by k other candidates cannot be part of the selected committee. The top-k only property mandates that the outcome of any rule should depend solely on the top k candidates in each voter's preference. Moreover, through our investigation, we have found that an additional property of the Plurality rule is necessary to fully identify the corresponding class of multi-winner rules. This property is continuity, which asserts that for any preference profile, if a committee is strictly preferred to another committee, then the social comparison of the two committees according to this profile can always dominate the outcome of any other profile, when the former one is repeated sufficiently many times. Our analysis results a class of *top-k counting rules* as extensions of the Plurality rule. These rules compare pairs of committees based on the k best candidates of each voter's preference, using a two-variable function (the *counting decision function*), and aggregate the comparisons across the entire profile.

The remainder of the paper is organized as follows: In Section 2 we present the basic definitions and notations. In Section 3 we provide formal definitions of the axioms and give the main result of the paper. In Section 4 we comment on our result, and, finally, in Section 5 we wrap-up the paper with a conclusion.

2 Definitions and notations

Throughout the paper, for any integer $t \in \mathbb{N}^*$, we write $[t]$ to denote the set $\{1, \cdots, t\}$. For any given finite set X, we denote by $|X|$ the cardinality of X, by 2^X the family of all subsets of X, and by 2^X_k the set of all its subsets of size $k \in [|X|]$.

We consider a set A of available candidates (or alternatives) with $|A| = m \geq 3$, and a set $N = \{1, 2, \cdots\}$ of individuals (or voters). We assume that each voter $i \in N$ is endowed with a linear order on A denoted by p_i; that is, a complete, anti-symmetric and transitive binary relation over the set of candidates A. The set of all linear orders on A is denoted by $\mathcal{L}(A)$ and we have $|\mathcal{L}(A)| = m!$. A preference profile (or simply a profile) with the set of voters N is any tuple $p = (p_1, \cdots, p_{|N|}) \in \mathcal{L}(A)^N$ that specifies the preference relation of all voters. For any two candidates a and b, we write $a \succ_i b$ if voter i strictly prefers a to b; more generally, if a is ranked before b in a linear order $l \in \mathcal{L}(A)$, we write $a \succ_l b$. The ranking $a \succ_i b \succ_i c \cdots$ of voter i will be simply denoted as $p_i = abc \cdots$. The rank of any candidate $a \in A$ in a linear order $l \in \mathcal{L}(A)$ is denoted by $r(l, a)$. We denote by $T(l)$ the top ranked candidate in the order l; that is, the candidate $a = T(l)$ such that $a \succ_l b$ for all $b \in A \setminus \{a\}$ (which means that $r(l, a) = 1$). For any candidate a, we denote $N(a, p)$ the set of all voters who rank a at the top position; that is, $N(a, p) = \{i \in N : T(p_i) = a\}$.

Denote by $n(a,p) = |N(a,p)|$ the number of such voters.

This paper is interested in the multi-winner voting, i.e., given a preference profile over the set of candidates, the goal is to select a fixed-size subset of candidates called a *committee*. Let $k \in [m-1]$,[1] be the fixed size of the committee to be selected. We denote by 2^A_k the set of all committees of size k; i.e., the set of k-element subsets of the set of candidates. Given any linear order l, let $T_k(l)$ be the set of k-top ranked candidates in l; i.e.,

$$T_k(l) = \{a \in A : r(l,a) \leq k\}.$$

Remark that when $k = 1$, $T_k(l)$ consists of a unique candidate which is the top-ranked candidate $T(l)$. Given any committee $W \in 2^A_k$ and any ranking $l \in \mathcal{L}(A)$, $T_k(l) \cap W$ is the set of W members ranked in the top k positions in l, and we simply denote by $t_l(W) = |T_k(l) \cap W|$ as the number of such candidates.

A *multi-winner voting rule* is any mapping F that assigns to any profile $p \in \mathcal{L}(A)^N$ and each committee size $k \in [m-1]$ a weak order $F(p,k) = \succeq^k_p$ on 2^A_k; that is, a complete and transitive binary relation on 2^A_k. As the committee size k is fixed from the start, we simply denote the collective ranking $\succeq^k_p$ by $\succeq_p$ for simplicity. We write $\succ_p$ to refer to the strict part of $\succeq_p$ and $=_p$ to refer to the indifference part of $\succeq_p$. For any two committees W_1 and W_2 of size k, the relation $W_1 \succ_p W_2$ can be interpreted as meaning that, according to the profile p, the society prefers committee W_1 over committee W_2, while the relation $W_1 =_p W_2$ can be interpreted as meaning that, according to the profile p, the society views W_1 and W_2 as equally good. A committee $W \in 2^A_k$ such that $W \succeq_p W'$ for all committees $W' \in 2^A_k \setminus \{W\}$ is called *best committee* according to the profile p.

In the single-winner framework (when $k = 1$), the most well-known and widely applied rule in many societies is the Plurality rule, which assigns to each profile p a weak order $\succeq^1_p = \succeq_p$, such that for any two alternatives a and b, we have $a \succeq_p b$ if and only if $n(a,p) - n(b,p) \geq 0$. This rule has been widely studied in the literature of social choice theory, and among all the contributions devoted to its study, we believe that Yeh (2008) is one of the most remarkable, as it provides a characterization of the Plurality rule as the only single-winner rule satisfying the following five axioms: *anonymity, neutrality, consistency, top-only*, and *efficiency*.[2] Anonymity requires equal treatment of voters, while neutrality requires equal treatment of candidates. A rule satisfying these two axioms is said to be *symmetric*. Consistency requires that if we merge the preferences of two disjoint sets of voters, then the performance of candidates should be combined in such a way that if candidate a is collectively at least as good as candidate b in both profiles, the same comparison should hold when merging the two sets of voters. The top-only axiom asserts that the collective decision should depend solely on the candidates ranked in the top position of

[1] We consider the committee sizes k such that $k \leq m-1$ since the case $k = m$ is straightforward.

[2] Note that Ching (1996) provided a similar characterization, but with a variable set of candidates, which is beyond our scope since we consider a fixed set of candidates.

voters' preferences. Finally, efficiency requires that a candidate who is Pareto dominated[3] by another candidate cannot be considered as the best candidate in the collective decision; that is, it cannot be at least as good as any other candidate.

As mentioned earlier, we believe that an efficient approach to obtaining an extension of a single-winner voting rule to the multi-winner framework is to define natural extensions of its characteristic properties to the multi-winner context and to identify the multi-winner rule(s) satisfying these extended properties. Thus, as we are going to proove in the next section, applying this approach to extend the Plurality rule to the multi-winner framework leads to a class of rules (not only one) called *top-k counting rules*, which we define below.

Let us recall that a numerical function of two real variables f is said to be *anti-symmetric* if it satisfies $f(x, y) = -f(y, x)$ for all real numbers x and y. Moreover, we say that f is *non-decreasing* if $f(x, y) \geq 0$ for all real numbers x and y such that $x \geq y$. Note that the term "non-decreasing" does not refers to the formal mathematical definition.

Definition 1 *A multi-winner voting rule F is a top-k counting rule if for each committee size k, there exists a non-decreasing and anti-symmetric function $f : [k] \times [k] \to \mathbb{R}$ such that for any profile p and any two committees $W_1, W_2 \in 2^A_k$, we have $W_1 \succeq_p W_2$ if and only if $\sum_{i \in N} f\big(t_{p_i}(W_1), t_{p_i}(W_2)\big) \geq 0$, with $\succeq_p = F(p, k)$.*

Literally, the function f allows comparing any committee W_1 to any committee W_2 for each voter's preference, based on the number of candidates from each committee represented in the top-k positions of that preference. The *top-k counting rule* associated with f declares that W_1 is collectively at least as good as W_2, if the aggregation of all comparisons of W_1 versus W_2 across all voters' preferences is non-negative. The non-decreasing assumption can be interpreted as stating that, in any linear order, W_1 cannot be defeated by W_2 , if W_1 is more represented than W_2 in the top-k positions. The anti-symmetry assumption can simply be interpreted as stating that, in any linear order, the comparison of W_2 versus W_1 is the opposite of the comparison of W_1 versus W_2. The *top-k counting rules* defined in Definition 1 above encompass the top-k counting committee scoring rules introduced by Elkind et al. (2017), which form a specific subclass of committee scoring rules introduced in their paper. However, Definition 1 is more general as it may include other rules that do not necessarily belong to the class of committee scoring rules by Elkind et al. (2017).

The function f that defines a top-k counting rule F will be called the *counting decision function* associated with F. Note that according to Definition 1, the counting decision function is dependent on committee size k and should therefore be rigorously denoted as f_k. However, since we assume that the committee size is fixed from the outset, we write f for simplicity. In the single-winner setting, when the committee size is $k = 1$, it can be verified that the Plurality rule is the top-1 counting rule associated with the counting decision function $\beta : \{0, 1\} \times \{0, 1\} \to \mathbb{R}$ defined by $\beta(1, 0) = 1$, $\beta(0, 1) = -1$,

[3] A candidate a is Pareto dominated by candidate b in profile p if $b \succ_i a$ for all $i \in N$.

and $\beta(1,1) = \beta(0,0) = 0$.[4] However, for any committee size $k \geq 1$, extending the Plurality rule to select a committee of size k requires a generalization of the function β, which can be done in different ways. For instance, two natural generalizations of the function β can be considered. The first is the function $f : [k] \times [k] \to \mathbb{R}$ defined by $f(x,y) = x - y$, which compares committee W_1 to W_2 in a given preference by the difference between the number of candidates from W_1 that are ranked in the top k positions of the preference relation and the number of candidates from W_2 that are ranked in the top k positions of the preference. This function defines the well-known *Bloc rule* introduced by Elkind et al. (2017). A second natural generalization of the function β is the function $g : [k] \times [k] \to \mathbb{R}$ defined by $g(x,y) = \delta^k(x) - \delta^k(y)$, where $\delta^k(x) = 1$ if $x = k$ and $\delta^k(x) = 0$ otherwise. This latter function defines the *Perfectionist rule* introduced by Faliszewski et al. (2018). This rule focuses solely on the committee that is fully represented in a given preference relation; thus, for any committees W_1 and W_2 and any linear order l, the comparison value $g\big(t_l(W_1), t_l(W_2)\big)$ equals 1 if $t_l(W_1) = k$, meaning that $T_k(l) = W_1$. It equals -1 if $t_l(W_2) = k$, meaning that $T_k(l) = W_2$, and it equals 0 in all other cases. Therefore, if neither of the two committees is fully represented in the top-k positions of the linear order l, then the two committees are considered indifferent with respect to l. The following example provides an illustration of these rules.

Example 1 *Consider the set five candidates $A = \{a,b,c,d,e\}$ and the following preference profile with five voters, wherein each column represents the ranking of a voter:*

$$p = \begin{bmatrix} a & b & c & a & a \\ b & c & b & d & e \\ c & a & d & b & c \\ d & d & a & c & b \\ e & e & e & e & d \end{bmatrix}$$

Assume that the fixed size of the committee to selected is $k = 2$. Since the counting decision function associated with the Bloc rule compares any two committees based on the difference in the number of members from each committee ranked in the top k positions of each voter's preference, we can determine that the best committee according to the Bloc rule is $\{a,b\}$ as it is the committee most frequently represented in the top k positions across the profile. However, the best committee according to the Perfectionist rule is $\{b,c\}$, since it is the committee that is entirely more represented in the top k positions than any other committee.

We can observe that the Bloc and the Perfectionist rule described above are two natural extensions of the Plurality rule for multi-winner elections, as when the committee size k is set to 1, the functions f and g both coincide with the function β of the Plurality rule. Note that these rules belong to the class of top-k counting rules defined in Definition 1.

[4] Note that in this context, the pair $(1,1)$ cannot be observed since two different candidates cannot both be ranked first in a voter's preference.

However, we cannot immediately assume that every top-k counting rule is an extension of the Plurality rule. Indeed, if we closely examine the counting decision functions associated with these two rules, we find that they both satisfy the following relation: $f(k,x) > 0$ for all $x \leq k-1$. Intuitively, this condition can be interpreted as saying that, for each linear order, the committee whose members are all ranked in the top-k positions defeats any other committee, which seems quite reasonable. In the next section, we will prove that this condition is crucial for a top-k counting rule to be considered as extension of the Plurality rule.

3 Axioms and characterization

This section is devoted to show axiomatically that any top-k counting rule with a counting decision function satisfying $f(k,x) > 0$ for all $x \leq k-1$, is as an extension of the Plurality rule to the multi-winner framework.

3.1 Axioms

The characterization provided by Yeh (2008) for single-winner elections identifies the Plurality rule as the only voting rule that satisfies five independent axioms: *anonymity, neutrality, consistency, efficiency,* and *top-only.* Following the approach presented in this paper, seeking extensions of the Plurality rule for multi-winner elections necessitates defining these axioms in the multi-winner context and determining which rules satisfy the resulting axioms. The axioms of anonymity, neutrality, and consistency have already been defined in the context of multi-winner elections by Elkind et al. (2017) and Skowron et al. (2019) as follows:

Anonymity: Let π be any permutation of the set N of voters, and let $p \in \mathcal{L}(A)^N$ be any preference profile. We denote the preference profile defined by the permutation π as $\pi(p) = \left(p_{\pi(i)}\right)_{i\in N}$. A multi-winner voting rule F satisfies anonymity if, for all profiles $p \in \mathcal{L}(A)^N$ and all permutations π of the set of voters N, we have $F\big(\pi(p),k\big) = \succeq_{\pi(p)} = \succeq_p = F(p,k)$. Clearly, anonymity requires that the collective decision should not depend on the names of the voters.

Neutrality: Given a permutation σ of the set of candidates A and the preference relation p_i of a voter $i \in N$, we define the preference relation $\sigma(p_i)$ such that $\sigma(a) \succ_{\sigma(p_i)} \sigma(b) \Leftrightarrow a \succ_{p_i} b$ for all $a,b \in A$. We denote by $\sigma(p)$ the profile $\Big(\sigma(p_i)\Big)_{i\in N}$. A multi-winner voting rule F satisfies neutrality if, for all preference profiles p and all permutations σ of the candidates such that $F(p,k) = \succeq_p$ and $F(\sigma(p),k) = \succeq_{\sigma(p)}$, we have $\sigma(W_1) \succeq_{\sigma(p)} \sigma(W_2) \Leftrightarrow W_1 \succeq_p W_2$, where $\sigma(W) = \{\sigma(a) \mid a \in W\}$ for all $W \in 2^A_k$. Similar to anonymity, neutrality requires equal treatment of candidates.

A multi-winner rule satisfying both anonymity and neutrality is said to be *symmetric*. Given two profiles p and p' over the same set of candidates, with disjoint sets of voters N and N', we denote by $p + p'$ the profile that gathers all voters from N and N' with their respective preferences.

Consistency: A multi-winner voting rule F satisfies consistency if, for any two profiles p and p' from disjoint sets of voters N and N', such that $F(p,k) = \succeq_p$ and $F(p',k) = \succeq_{p'}$, the following holds for any two committees $W_1, W_2 \in 2^A_k$: (i) if $W_1 \succeq_p W_2$ and $W_1 \succeq_{p'} W_2$, then $W_1 \succeq_{p+p'} W_2$; (ii) if $W_1 \succ_p W_2$ and $W_1 \succeq_{p'} W_2$, then $W_1 \succ_{p+p'} W_2$, where $\succeq_{p+p'} = F(p+p',k)$. Clearly, consistency means that the rule treats small electorates in the same manner as it treats larger ones.

We now adapt the axioms of efficiency and top-only to make them suitable for the multi-winner setting, referring to them as *k-efficiency* and *top-k-only*, respectively.

k-efficiency: A candidate which is Pareto dominated by k candidates in the profile cannot belong to the best committee according to that profile. Formally, a multi-winner voting rule F satisfies k-efficiency if, for any profile p and any candidate b such that there exists $a_1, \dots, a_k \in A \setminus \{b\}$ with $a_j \succ_i b$ for all $i \in N$ and for all $j \in [k]$, we have: if $W \succeq_p W'$ for all $W' \neq W$, then $b \notin W$.

Top-k only: The collective decision depends solely on the candidates ranked in the top-k positions of the voters' preferences. Formally, a multi-winner voting rule satisfies the top-k only axiom if, for all profiles p and p' over the same set of voters N, such that $T_k(p_i) = T_k(p'_i)$ for all $i \in N$, it holds that $F(p,k) = \succeq_p = \succeq_{p'} = F(p',k)$.

When the committee size is $k = 1$, k-efficiency and top-k only align with efficiency and top-only, respectively, used by Yeh (2008) to characterize the Plurality rule.

Another important property satisfied by the Plurality rule is the *continuity* property. Although this property is not necessary to characterize the Plurality rule in the single-winner setting, we find out that following our approach, it is essential for fully identifying the extensions of the Plurality rule to the multi-winner context. The continuity property has already been defined for multi-winner elections as follows:

Continuity: For any profile p and any integer $n \in \mathbb{N}^*$, we denote by np the profile obtained by cloning profile p n times. A multi-winner voting rule F satisfies the continuity property if, for all profiles p and p' such that $F(p',k) = \succeq_{p'}$, if $W_1 \succ_{p'} W_2$ for any two committees W_1 and W_2, then there exists an integer $n \in \mathbb{N}^*$ such that $W_1 \succ_{p+np'} W_2$, where $p + np'$ consists of p and n copies of p'. Clearly, continuity can be understood as a principle of "large enough majority" (see Skowron et al., 2019).

3.2 Characterization

As mentioned earlier, the aim of this paper is to identify which multi-winner selection rules can be considered as extensions of the Plurality rule. The extensions of the characteristic properties of the Plurality rule to the multi-winner setting, as outlined in Section 3.1, enable us to determine the class of multi-winner rules that can be regarded as such extensions. The following lemma is a crucial first step toward our result; it states that every committee is collectively strictly preferred to any other for at least one profile whenever the voting rule satisfies k-efficiency.

Lemma 1 *Let k be the fixed committee size and F be a multi-winner voting rule. If F satisfies k-efficiency, then for any committee $W \in 2_k^A$, there exists a profile p, such that $W \succ_p W'$ for all $W' \in 2_k^A \setminus \{W\}$.*

Proof. Let $W \in 2_k^A$ be any committee. Assume that F satisfies k efficiency and consider any profile p in which all voters rank the members of W in the top k positions; that is, a profile p such that $T_k(p_i) = W$ for every voter i. By k-efficiency, any candidate outside W cannot be in the best committee according to p, since such a candidate is Pareto dominated by all the members of W. This implies that the best committee according to p is W and we have $W \succeq_p W'$ for all $W' \neq W$. Moreover the social indifference $W =_p W'$ cannot holds because in this case, W' would also be a best committee according to p, containing a candidate outside W. Thus, it follows that $W \succ_p W'$ for all $W \neq W'$. ■

Before presenting our result, let us illustrate the necessity of the condition required for the counting decision function associated with a top-k counting rule to be considered as an extension of the Plurality rule.

Example 2 *Consider a set of four candidates $A = \{a, b, c, d\}$ from which we aim to select a committee of size $k = 2$. Consider the top-k counting rule F associated with the counting decision function f defined by $f(x, y) = 1$ if $x \neq 0$ and $y = 0$, and $f(x, y) = 0$ otherwise. This function treats any two committees equally at each voter's preference if at least one (or neither) candidate of each of these committees is ranked in the top-k positions of that preference. Now consider the following unanimous profile with four voters:*

$$p = \begin{bmatrix} a & a & a & a \\ b & b & b & b \\ c & c & c & c \\ d & d & d & d \end{bmatrix}$$

We can verify that according to the rule F, there is a social indifference among all committees of size 2 except the committee $\{c, d\}$. Consequently, the committee $\{b, c\}$ is a best committee with respect to the profile p, while it contains candidate c which is Pareto dominated by two candidates a and b. Therefore, F fails to satisfy k-efficiency and cannot be considered as an extension of the Plurality rule.

The main result of this paper can be stated as follows:

Theorem 1 *A multi-winner voting rule satisfies symmetry, consistency, k-efficiency, top-k only and continuity if and only if it is a top-k counting rule with a counting decision function satisfying $f(k, x) > 0$ for all $x \leq k - 1$.*

Proof. This proof follows a similar approach of Myerson (1995) characterizing scoring rules for single-winner elections.[5]

[5] The same approach can be found in Marchant (2003)

It is not hard to show that every top-k counting rule satisfies symmetry (anonymity and neutrality), consistency, top-k only, and continuity. However, the condition imposed on the counting decision function is necessary to ensure k-efficiency. Indeed, assume that the counting decision function f associated with a top-k counting rule F does not satisfy the condition given in Theorem 1; that is, there exists $x \leq k-1$ such that $f(k,x) \leq 0$. Consider a profile p in which all voters have the same linear order l of the form: $l = a_1 \cdots a_x \cdots a_k b_1 \cdots b_{k-x} \cdots$. Let us set $W = \{a_1, \cdots, a_k\}$ and $W' = \{a_1, \cdots, a_x, b_1, \cdots, b_{k-x}\} = \big(W \setminus \{a_{x+1}, \cdots, a_k\}\big) \cup \{b_1, \cdots, b_{k-x}\}$. Let $\succeq_p = F(p,k)$. At the profile p, we have

$$\sum_{i \in N} f\big(t_{p_i}(W), t_{p_i}(W')\big) = nf\big(t_l(W), t_l(W')\big) = nf(k,x) \leq 0$$

which implies that

$$\sum_{i \in N} f\big(t_{p_i}(W'), t_{p_i}(W)\big) = nf(x,k) \geq 0. \tag{1}$$

Equation (1) implies that $W' \succeq_p W$ and we deduce from Lemma 1 (the proof) that F fails to satisfy k-efficiency. Therefore the condition on the counting decision function is necessary to have k-efficiency. This condition is also sufficient since if the counting decision function satisfies the condition in Theorem 1, then for any candidate b which is Pareto dominated by k different candidates in a profile p, we can always find a committee of size k (from the top k candidates of the voters' preferences) which is collectively strictly preferred to any committee that contains b. This proves that every top-k counting rule with a counting decision function satisfying $f(k,x) > 0$ for all $x \leq k-1$, satisfies all our axioms.

Now let us move to the tricky part of the proof, showing that every multi-winner voting rule that possesses all these axioms is a top-k counting rule whose associated function satisfies the condition in Theorem 1.

Consider a multi-winner rule F having all these axioms. By symmetry, the social outcome of any profile depends only on the number of appearance of each linear order l_j in the the profile, with $j = 1, \cdots, m!$. Thus, from now on, every profile p will be represented by a vector $\alpha = (\alpha_1, \cdots, \alpha_{m!}) \in \mathbb{N}^{m!}$ where for all $j \in [m!]$, α_j is the number of appearances of the order l_j in the profile α.

For any committee $W \in 2^A_k$, denote by $\mathcal{L}(W)$ the set of profiles α, such that $W \succ_\alpha W'$ for all $W' \in 2^A_k \setminus \{W\}$. Remark that $\mathcal{L}(W) \neq \emptyset$ for all W (by Lemma 1). Let W and W' be any two committees (of size k). Let

$$D(W,W') = \{\alpha - \beta : \alpha \in \mathcal{L}(W), \text{ and } \beta \in \mathcal{L}(W')\}.$$

$D(W,W')$ is a subset of $\mathbb{R}^{m!}$ and denote by $C(W,W')$ the convex hull of $D(W,W')$. We claim that the null vector $\tilde{0} = (0, \cdots, 0)$ cannot belong to $C(W,W')$. Indeed, assume that $\tilde{0} \in C(W,W')$. Then it would exist some profiles $\alpha^1, \cdots, \alpha^K \in D(W)$, $\beta^1, \cdots, \beta^K \in$

$D(W')$, and some non-negative real numbers $\lambda_1, \cdots, \lambda_K$ such that

$$\sum_{s=1}^{K} \lambda_s(\alpha^s - \beta^s) = \tilde{0}$$

which implies that

$$\sum_{s=1}^{K} \lambda_s(\alpha_j^s - \beta_j^s) = 0, \ \forall j \in [m!] \tag{2}$$

However, (2) is an homogeneous linear system with integers coefficients $(\alpha_j^s - \beta_j^s)$. So, having a non-negative solution of real numbers implies having a non-negative solution of integers. Then, there exists non-negative integers $\lambda_1, \cdots, \lambda_K$ such that

$$\sum_{s=1}^{K} \lambda_s \alpha^s = \sum_{s=1}^{K} \lambda_s \beta^s \tag{3}$$

Let $\theta = \sum_{s=1}^{K} \lambda_s \alpha^s$ and $\gamma = \sum_{s=1}^{K} \lambda_s \beta^s$. By consistency and by equation (3), we gave

$$\theta = \gamma, W \succ_\theta W' \text{ and } W' \succ_\gamma W$$

which is a contradiction. Therefore, the null vector $\tilde{0}$ does not belong to $C(W, W')$.

Thus, $C(W, W')$ is a convex subset of $\mathbb{R}^{m!}$ whose elements are different form the null vector of $\mathbb{R}^{m!}$. By the Supporting Hyperplane Theorem, there exists a vector $S(W, W') = \big(S_j(W, W')\big)_{1 \leq j \leq m!} \in \mathbb{R}_+^{m!}$ such that $S(W, W') \neq \tilde{0}$ and

$$\sum_{j=1}^{m!} S_j(W, W')(\alpha_j - \beta_j) \geq 0$$

for any $\alpha \in \mathcal{L}(W)$ and $\beta \in \mathcal{L}(W')$. Since these conditions are satisfied when the roles of W and W' are reversed, we have $S_j(W', W) = -S_j(W, W')$ for all $j \in [m!]$.

Now let us show that the rule F is entirely defined by the vectors $\big(S(W, W')\big)_{W, W' \in 2_k^A}$; that is, for all profile α such that $F(\alpha, k) = \succeq_\alpha$, we have

$$W \succeq_\alpha W' \Leftrightarrow \sum_{j=1}^{m!} S_j(W, W')\alpha_j \geq 0 \ \ \forall W, W' \in 2_k^A.$$

Let α be a profile such that $F(\alpha, k) = \succeq_\alpha$, and $W, W' \in 2_k^A$. Assume that $W \succeq_\alpha W'$. Let $\theta \in \mathcal{L}(W)$ and any integer $n \in \mathbb{N}^*$. By consistency, it holds that $n\alpha + \theta \in \mathcal{L}(W)$ because consistency implies that $W \succ_{n\alpha+\theta} W^*$ for all $W^* \neq W$. Therefore, we have

$$\sum_{j=1}^{m!} S_j(W, W')(n\alpha_j + \theta_j) \geq \sum_{j=1}^{m!} S_j(W, W')(n\alpha_j + \theta_j - \beta_j) \geq 0 \ \ \forall \beta \in \mathcal{L}(W')$$

Since the inequality $\sum_{j=1}^{m!} S_j(W, W')(n\alpha_j + \theta_j) \geq 0$ holds for any arbitrary integer n, we deduce that

$$\sum_{j=1}^{m!} S_j(W,W')\alpha_j \geq 0.$$

Moreover, there exists at least one profile $\theta \in \mathcal{L}(W)$ such that $\sum_{j=1}^{m!} S_j(W,W')\theta_j > 0$. Indeed, since $S(W,W') \neq \tilde{0}$, choose a j_0, such that $S_{j_0}(W,W') \neq 0$ and consider the profile θ with a single preference l_{j_0}, with $T_k(l_{j_0}) = W$. It is clear that $\theta \in \mathcal{L}(W)$ (as proved in Lemma 1) and therefore $W \succ_\theta W'$. Assume that $S_{j_0}(W,W') < 0$, then we would have $\sum_{j=1}^{m!} S_j(W,W')\theta_j < 0$ which imply that $W' \succ_\theta W$ and a contradiction holds. Therefore, we have $\sum_{j=1}^{m!} S_j(W,W')\theta_j = S_{j_0}(W,W') > 0$.

Conversely, assume that $\sum_{j=1}^{m!} S_j(W,W')\alpha_j \geq 0$ and let us show that $W \succeq_\alpha W'$. Let $\theta \in \mathcal{L}(W)$ such that $\sum_{j=1}^{m!} S_j(W,W')\theta_j > 0$ (such a theta always exists, as shown above). For any integer $n \in \mathbb{N}^*$, it holds

$$\sum_{j=1}^{m!} S_j(W,W')(n\alpha_j + \theta_j) > 0$$

which implies that

$$\sum_{j=1}^{m!} S_j(W',W)(n\alpha_j + \theta_j) < 0 \text{ since } S_j(W',W) = -S_j(W,W')$$

and it follows that $W \succ_{n\alpha+\theta} W'$. Since the collective decision $W \succ_{n\alpha+\theta} W'$ holds for all positive integers n, then by continuity, we can deduce that $W \succeq_\alpha W'$. Otherwise, there would exist an integer $n_0 > 0$ such that $W' \succ_{n_0\alpha+\theta} W$ (by the continuity hypothesis), which is a contradiction. Hence, we conclude that

$$W \succeq_\alpha W' \Leftrightarrow \sum_{j=1}^{m!} S_j(W,W')\alpha_j \geq 0 \;\; \forall W,W' \in 2^A_k.$$

Furthermore, by top-k only, the collective decision $\succeq_\alpha$ only depends on the sets $\Big(T_k(l_j)\Big)_{j\in[m!]}$ and since F is entirely defined by vectors $\Big(S(W,W')\Big)_{W,W'\in 2^A_k}$, it holds that for all $j \in [m!]$ and for all committees W and W', that $S_j(W,W')$ depends only on the sets $T_k(l_j) \cap W$ and $T_k(l_j) \cap W'$. Moreover, symmetry implies that all the candidates are equally treated and, therefore, $S_j(W,W')$ only depends on $|T_k(l_j) \cap W| = t_{l_j}(W)$ and $|T_k(l_j) \cap W'| = t_{l_j}(W')$. Thus, defining the function $f\big(t_{l_j}(W), t_{l_j}(W')\big) = S_j(W,W')$, we have

$$W \succeq_\alpha W' \Leftrightarrow \sum_{j=1}^{m!} f\Big(t_{l_j}(W), t_{l_j}(W')\Big)\alpha_j \geq 0.$$

This proves that the rule F is a top-k counting rule and as proved earlier, k-efficiency implies the required condition on the function f. ■

4 Comments

It is important to make some comments about the work made in this paper. The first comment pertains to the use of the continuity axiom. Recall that the characterization of the Plurality rule by Yeh (2008) relies on the axioms of symmetry, consistency, efficiency, and top-only. However, the extensions of these axioms to the multi-winner framework not are insufficient to identify all the multi-winner rules that extend the Plurality rule. If there were only one multi-winner rule as an extension of Plurality, it might have been identified by these axioms (see, for instance, Debord, 1992 for the extension of the Borda rule). Since there are multiple multi-winner voting rules that can be viewed as extensions of Plurality (such as the Bloc and the Perfectionist rule), continuity serves as a common property for all of them.

Secondly, we would like to discuss the definition of the top-k only property. Note that the set $T_k(l)$ of the top-k candidates in the ranking l does not take into account the order of these candidates. Thus, top-k only requires that if the top-k candidates of all voters are the same in two profiles, regardless of their ranking, the two profiles should lead to the same collective decision. This is a very natural and flexible top-only extension. However, we could define a more stringent version of this property by requiring that the top-k candidates of each voter be ranked in exactly the same order. We believe that this approach could yield a larger class of voting rules. More precisely, the class of rules characterized in this paper includes the Bloc rule, which is a member of the subclass of (weakly) separable committee scoring rules defined by Elkind et al. (2017). As a separable committee scoring rule, the Bloc rule can be defined using the scoring vector $(\underbrace{1, \cdots, 1}_{k,\text{times}}, 0, \cdots, 0)$, where k is the size of the committee to be selected. By defining the binding version of top-k only with consideration of the ranking of the top-k candidates, we may include other separable committee scoring rules, such as the k-truncated version of the Borda rule, which uses the scoring vector $(m-1, \cdots, m-k, 0, \cdots, 0)$. This presents a challenging opportunity that requires further investigation.

The third comment refers to the combination of axioms used in this paper. This does not necessarily lead to the class of committee scoring rules defined by Elkind et al. (2017). Indeed, one of the axioms employed by Skowron et al. (2019) to characterize the class of committee scoring rules is *committee dominance*, which is crucial for identifying the set of committee scoring rules. Therefore, the class of multi-winner voting rules characterized in this paper may include other rules that are not committee scoring rules.

5 Conclusion

This paper focused on identifying the voting rules that extend the Plurality rule in multi-winner elections. We adopted an axiomatic approach by defining extensions of the properties that characterize the Plurality rule and identifying the set of rules satisfying these extended properties. This method allows us to pinpoint a specific class of top-k counting rules as extensions of the Plurality rule. We believe that a similar approach can be employed to characterize multi-winner approval voting rules, drawing on the characterization of approval voting for single-winner elections provided by Sertel (1988).

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